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Research Article

A Study of Some Problems on the Dirichlet Characters (mod q)

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Abstract: : Consider q to be an integer, and χ to be a Dirichlet character (mod q). The main purpose of this paper is using the analytic methods to study some problems related to the Dirichlet character (mod q).

***Keywords*:** Dirichlet character, principal character, primitive character, analytic methods

MSC: 11L40, 11L20, 11B50, 11B99

1. Introduction

A significant theory regarding to the Dirichlet character by works of Dirichlet [1] and others (see, for instance, [2-10]).

A Dirichlet character can be defined in two ways: axiomatically and constructively. Although Dirichlet himself defined characters constructively (see, [1], p.132-137), we will utilize the first definition in this study.

By Dirichlet character modulo q, we mean a function $\chi : \mathbb{Z} \to \mathbb{C}$ which is periodic with period q (q positive integer), which satisfies the following conditions:

$$\chi(n+q) = \chi(n), \quad \forall n \in \mathbb{Z}, \tag{1}$$

and

$$\chi(nm) = \chi(n)\chi(m), \quad \forall n, m \in \mathbb{Z}. \tag{2}$$

Also, χ satisfies

$$\chi(1) = 1 \tag{3}$$

and

$$\chi(n) = 0 \qquad \text{whenever } (n, q) > 1. \tag{4}$$

The condition (2) is known as "multiplicativity without restrictions" because the term "multiplicativity" in number theory is reserved for a weaker concept: A function $f : \mathbb{Z} \to \mathbb{C}$ is multiplicative if $f(mn) = f(m)f(n)$ holds for all m, n in

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$\mathbb{Z}^+$ with $(m, n) = 1$.

It can readily be proved that if the conditions (1) and (2) are given, the condition (3) is equivalent to the statement that

$$\chi \not\equiv 0 \quad \Leftrightarrow \quad \chi(n) \neq 0, \tag{5}$$

for all n with $(n, q) = 1$, (see [11]).

Before we continue we define the primitive characters (see, for example, Davenport [12], chapter 5): We say that χ is imprimitive if $\chi(n)$ is restricted by $(n, q) = 1$, possesses a period which is less than q. Otherwise, we say that χ is primitive. More precisely if we write in this note that $\chi(n)$ is restricted by $(n, q) = 1$, possesses a period $a \in \mathbb{Z}^+$, this means that for any integer m, n with $(m, q) = (n, q) = 1$ and $m \equiv n \pmod{a}$ we have $\chi(m) = \chi(n)$.

We are seeking in this paper to prove two results. The first one gives us a formula to calculate the number of primitive characters modulo q. The second one shows that for the integers a and b, the sum $\frac{1}{q}\sum_{c=0}^{q-1}\chi(ac+b)$ remains unchanged after multiplication with $\chi(m)$ for the set of $m \in (\mathbb{Z}/q\mathbb{Z})^\times$ provided that $q \nmid a$, while this sum equals $\chi(b)$ if $q \mid a$. These two results play an important role in number theory to solve many other problems. Below we sometimes say just "character or χ" instead of "Dirichlet character".

Theorem 1.1. Let $\varphi^*(q)$ denote the number of primitive characters modulo q, then $\varphi^*(q)$ is given by

$$\varphi^*(q) = q\prod_{p \| q}\left(1 - \frac{2}{p}\right)\prod_{p^2 | q}\left(1 - \frac{1}{p}\right)^2. \tag{6}$$

Here, p is a prime number and q is any integer.

The next theorem provides us with a mathematical tool to tickle some other problems.

Theorem 1.2. Let χ be a Dirichlet character modulo q, then

$$\frac{1}{q}\sum_{c=0}^{q-1}\chi(ac+b) = \begin{cases}\chi(b) & \text{if } q \mid a \\ 0 & \text{if } q \nmid a.\end{cases}$$

for any integer numbers a and b.

We will devote Section 3 and Section 4 to provide the proof for Theorem 1.1 and the proof for Theorem 1.2 respectively. Throughout this work, we assume that the reader is familiar with the fundamental concepts in classical introduction number theory (see, for example, [13-16]), and analytic number theory (see, for instance, [11, 12, 17-20]). Also required are elementary facts about algebra and algebraic number theory (see, for example, [14, 21, 22]).

2. Auxiliary assertions

For our reference, we should collect some facts about the Dirichlet character. We begin by mentioned the following lemma.

Lemma 2.1. [12]

(a) Given an odd prime p, there exists a primitive root modulo p. Moreover, if $\alpha \geq 1$, then there exists a primitive root modulo p^α.

(b) Let g be any primitive integer root modulo p^α (some $\alpha \geq 2$) then, g is also a primitive root modulo p^α for each $1 \leq k \leq \alpha - 1$.

Since $(\mathbb{Z}/q\mathbb{Z})^\times$ is a group, let us first make the general observation that for any $g \in (\mathbb{Z}/q\mathbb{Z})^\times$, if $g^a \equiv g^b \pmod{q}$ with $0 \leq a \leq b$, then $g^{b-a} \equiv 1 \pmod{q}$. So, if v is the smallest positive integer such that $g^v \equiv 1 \pmod{q}$, then all the elements $1 = g^0, g^1, g^2, \ldots, g^{v-1}$ of the group $(\mathbb{Z}/q\mathbb{Z})^\times$ are distinct. Specifically, g is a primitive root if $v = \varphi(q)$. If $q = p$ is a prime, then $(\mathbb{Z}/q\mathbb{Z})^\times$ is cyclic. For more detail and explanation, one can see, for instance, [13, 14, 21]. The next lemma shows how to generate an explicit listing of all Dirichlet characters modulo q.

Lemma 2.2. [11] Let $(\mathbb{Z}/q\mathbb{Z})^{\times}$ be a cyclic group with $q \geq 2$, and let ν be the index function corresponding to a fixed primitive root g in $(\mathbb{Z}/q\mathbb{Z})^{\times}$. Let ω be any $\varphi(q)$th root of unity, i.e., any complex number such that $\omega^{\varphi(q)} = 1$. Then, the formula yields a Dirichlet character χ modulo q

$$\chi(n) = \begin{cases} \omega^{\nu(n)} & \text{if } (n, p) = 1 \\ 0 & \text{if } (n, p) > 1. \end{cases}$$

Conversely, every Dirichlet character modulo q can be expressed in this manner. Different choices of ω gives different χ's, and thus there are exactly $\varphi(q)$ distinct Dirichlet characters of period q.

Let us write X_q for the set of all Dirichlet characters modulo q. If χ is a Dirichlet character of period q, then for all $n \in \mathbb{Z}$ with $(n, q) = 1$, we have by Euler's theorem that $\chi(n)^{\varphi(q)} = 1$. Using this fact and (5), we see that the inverse of any $\chi \in X_q$ is given by the conjugate character χ, defined by $\overline{\chi}(n) = \overline{\chi(n)}$, (see [11]).

Using the Chinese Remainder Theorem, the following lemma provides an exhaustive listing of all Dirichlet characters for a general q.

Lemma 2.3. [11] Let q be an integer ≥ 2, and write its prime factorization as $q = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ (where $p_1, \ldots, p_r$ are distinct primes and $\alpha_1, \ldots, \alpha_r \in \mathbb{Z}^+$). For $j = 1, \ldots, r$, we let $X_{p_{p_j}^{\alpha_j}}$ be the set of all Dirichlet characters modulo $p_j^{\alpha_j}$. Then, for each choice of an r-tuple $< \chi_1, \ldots, \chi_r > \in X_{p_1^{\alpha_1}} \times \ldots \times X_{p_r^{\alpha_r}}$, we get a Dirichlet character χ modulo q by the formula

$$\chi(n) = \prod_{j=1}^{r} \chi_j(n).$$

Conversely, every Dirichlet character modulo q can be expressed in this way. Different choices of r-tuples $< \chi_1, \ldots, \chi_r > \in X_{p_1^{\alpha_1}} \times \ldots \times X_{p_r^{\alpha_r}}$ give different χ's, and hence there are exactly $\prod_{j=1}^{r} \# X_{p_j^{\alpha_j}} = \prod_{j=1}^{r} \varphi(p_j^{\alpha_j}) = \varphi(q)$ Dirichlet characters modulo q.

Note that the above Lemma 2.3 shows, in particular that $\# X_q = \varphi(q)$. So, X_q and $(\mathbb{Z}/q\mathbb{Z})^{\times}$ have the same number of elements. On other hand, one can prove that

Lemma 2.4. [12] If $\chi \in X_q$ and if $q_1 \in \mathbb{Z}^+$ is a period of [$\chi(n)$ for n restricted by $(n, q) = 1$], then $c(\chi) \mid q_1$. Hence, in particular, we have $c(\chi) \mid q$.

3. Proof of Theorem 1.1

To prove Theorem 1.1, we need to prove first that φ^* is multiplicative. For this, we need the help of the following lemma.

Lemma 3.1. Consider the situation in Lemma 2.3. Then, χ is primitive if and only if each χ_j is primitive. More generally, we have $c(\chi) = \prod_{j=1}^{r} c(\chi_j)$.

Proof. We note that the first part of the lemma follows from the second part. So let $q_1 = \prod_{j=1}^{r} c(\chi_j)$. Assume $m, n \in \mathbb{Z}, (m, q) = (n, q) = 1$ and $m \equiv n (\bmod q_1)$. Then, for each $j \in \{1, \ldots, r\}$, we have $(m, p_j^{\alpha_j}) = (n, p_j^{\alpha_j}) = 1$, and hence $m \equiv n \bmod c(\chi_j)$ since $c(\chi_j) | q_1$; hence $\chi_j(m) = \chi_j(n)$. Hence, $\chi(m) = \prod_{j=1}^{r} \chi_j(m) = \prod_{j=1}^{r} \chi_j(n) = \chi(n)$. This proves that $\chi(n)$ for n restricted by $(n, q) = 1$ has period q_1, and hence by Lemma 2.4, we have $c(\chi) \mid q$.

Next, for every $k \in \{1, \ldots, r\}$, we can argue as follows. Since $\chi_k \in X_{p_k^{\alpha_k}}$, we have $c(\chi_k) = p^{\beta}$ for some $\beta \in \{0, 1, \ldots, \alpha_k\}$, by Lemma 2.4. Suppose that $\beta > 0$. Then, [$\chi_k(n)$ restricted by $(n, p_k^{\alpha_k}) = 1$] does not have period $p_k^{\beta-1}$ and hence there are some $m, n \in$ with $(m, p_k) = (n, p_k) = 1$ and $m \equiv n (\bmod p_k^{\beta-1})$ and $\chi_k(m) \neq \chi_k(n)$. Now, by the Chinese Remainder Theorem, there exist $m', n' \in \mathbb{Z}$ such that $m' \equiv m (\bmod p^{\alpha})$ and $m' \equiv 1 (\bmod p_j^{\alpha_j})$ for all $j \neq k$, and

$m' \equiv n(\bmod\ p_k^{\alpha_k})$ and $n' \equiv 1(\bmod\ p_j^{\alpha_j})$ for all $j \neq k$, Now,

$$\chi(m') = \prod_{j=1}^{r} \chi(m') = 1 \cdots 1 \cdot \chi_k(m') \cdot 1 \cdots 1 = \chi_k(m') = \chi_k(m),$$

and similarly $\chi(n') = \chi_k(n)$; thus $\chi(m') \neq \chi(n')$. But, we also have $(m', q) = (n', q) = 1$ and $m' \equiv n'(\bmod q_1 / p_k)$; hence this proves that [$\chi(n)$ for n restricted by $(n, q) = 1$] does not have period q_1 / p_k, and thus $c(\chi) \nmid \frac{q_1}{p_k}$.

We thus have $c(\chi) \nmid q_1$ but for each prime $p \mid q_1$ we also have $c(\chi) \nmid \frac{q_1}{p_k}$. This implies that $c(\chi) \nmid q_1$.

Proof of Theorem 1.1. Let q have the prime factorization $q = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$; then by Lemma 2.3 combined with Lemma 3.1, we have $\varphi^*(q) = \prod_{j=1}^{r} \varphi^*(p_j^{\alpha_j})$. Thus, φ^* is multiplicative.

It now only remains to compute $\varphi^*(p^\alpha)$ for any prime p and $\alpha \geq 1$. First, assume $p \geq 3$. Then, by Lemma 2.2 (combined with Lemma 2.1 (b)) the Dirichlet characters modulo p^α are in 1-1-correspondence with the $\varphi^*(p^\alpha)$ roots of unity ω, i.e., (since $\varphi(p^\alpha) = p^{\alpha-1}(p-1)$) the numbers $\omega = e(\frac{k}{p^{\alpha-1}(p-1)})$ for $k = 0, 1, \ldots, p^{\alpha-1}(p-1) - 1$. The Dirichlet character corresponding to ω is given by $\chi(g^j) = w^j$ for all $j \geq 0$, where g is our fixed primitive root modulo p^α.

Now, χ is primitive if and only if $c(\chi) \mid p^\alpha$; and we know that $c(\chi) \mid p^\alpha$; hence χ is primitive if and only if $c(\chi) \nmid p^{\alpha-1}$, i.e., if and only if [$\chi(n)$ for n restricted by $(n, q) = 1$] does not have period $p^{\alpha-1}$. In other words, χ is primitive if and only if there are two $j_1, j_2 \geq 0$ such that $g^{j_1} \equiv g^{j_2} \ (\bmod\ p^{\alpha-1})$ and $\chi(g^{j_1}) \neq \chi(g^{j_2})$. If $\alpha = 1$, then this means that χ is primitive if and only if there are any two $j_1, j_2 \geq 0$ with $\chi(g^{j_1}) \neq \chi(g^{j_2})$, and this clearly holds if and only if hence there are $p-2$ primitive characters modulo p. If $\alpha \geq 2$, then we know that g is also a primitive root modulo $p^{\alpha-1}$ (see Lemma 2.1 (b)); hence $g^{j_1} \equiv g^{j_2} \ (\bmod\ p^{\alpha-1})$ holds if and only if $j_1 \equiv j_2 \ (\bmod\ p^{\alpha-2}(p-1))$. Hence, χ corresponding to ω is primitive if and only if there are two $j_1, j_2 \geq 0$ with $j_1 \equiv j_2 \ (\bmod\ p^{\alpha-2}(p-1))$ such that $\chi(g^{j_1}) \neq \chi(g^{j_2})$. But

$$\chi(g^{j_1}) \neq \chi(g^{j_2}) \quad \Leftrightarrow \quad \omega^{j_1} \neq \omega^{j_2} \quad \Leftrightarrow \quad \omega^{j_1 - j_2} \neq 0.$$

Hence, χ is primitive if and only if ω is not a $p^{\alpha-1}(p-1)$th root of unity; that is, if ω is of the form

$$\omega = e\left(\frac{k}{p^{\alpha-1}(p-1)}\right), \quad k \in \{0, 1, \ldots, p^{\alpha-1}(p-1) - 1\}, \quad p^{\alpha-2}(p-1) \nmid k.$$

There are $p^{\alpha-1}(p-1)(1-p^{-1}) = p^{\alpha-2}(p-1)^2 = p^\alpha(1-p^{-1})^2$ distinct such ω's. Hence, the number of primitive roots modulo p^α is:

$$\varphi^*(p^\alpha) = p^\alpha \begin{cases} 1 - 2p^{-1} & \text{if } \alpha = 1 \\ (1-p^{-1})^2 & \text{if } \alpha \geq 2. \end{cases} \tag{7}$$

We proved this for $p \geq 3$ but we claim that the same formula also holds for $p = 2$. Indeed, one checks directly that there are no primitive characters modulo 2 and that there is exactly one primitive character modulo 4. Finally, if $q = 2^\alpha$ with $\alpha \geq 3$, then the Dirichlet characters modulo q are in bijective correspondence with pairs $< \omega, \omega' >$ as in (Lemma 4.11, [11]) and one checks by a similar argument as above that $< \omega, \omega' >$ corresponds to a primitive character if and only if ω is not a $2^{\alpha-3}$th root of unity. There are $2^{\alpha-3} \cdot 2$ distinct such pairs, and this agrees with (7).

Now (6) follows using $\varphi^*(q) = \prod_{j=1}^{r} \varphi^*(p_j^{\alpha_j})$ and (7).

4. Proof of Theorem 1.2

For the proof of Theorem 1.2, we start by proving the following useful lemma.

Lemma 4.1. Let χ be a Dirichlet character modulo q, then for a given positive integer q_1 is a period of $\chi(n)$ restricted by $(n, q) = 1$ if and only if $\chi(n) = 1$ holds for all integers n satisfying $n \equiv 1 \pmod{q_1}$ and $(n, q) = 1$.

Proof. If q_1 is a period of $\chi(n)$ restricted by $(n, q) = 1$ then for all integers m, n with $(m, q) = (n, q)$ and $m \equiv n \pmod{q_1}$, we have $\chi(m) = \chi(n)$. In particular, taking $m = 1$, it follows that $\chi(n) = 1$ for all integers n satisfying $(n, q) = 1$ and $n \equiv 1 \pmod{q_1}$.

Conversely, suppose that q_1 is a positive integer and that $\chi(n) = 1$ holds for all integers n satisfying $n \equiv 1 \pmod{q_1}$ and $(n, q) = 1$. Let $q_2 = (q, q_1)$; then we know that there are some integers x, y such that $q_2 = xq + yq_1$. Now, if n is any integer satisfying $n \equiv 1 \pmod{q_2}$ and $(n, q) = 1$, then we have $n = 1 + hq_2$ for some integer h, and hence $n = 1 + h(xq + yq_1) \equiv 1 + hyq_1 \pmod{q}$ so that $\chi(n) = \chi(1 + hyq_1)$ and $(1 + hyq_1, q) = 1$. Furthermore, $1 + hyq_1 \equiv 1 \pmod{q_1}$ and thus by our assumption $\chi(1 + hyq_1) = 1$. Hence, q_2 has exactly the same property as q_1, i.e., $\chi(n) = 1$ holds for all integers n satisfying $n \equiv 1 \pmod{q_2}$ and $(n, q) = 1$. The advantage is that q_2 also divides q by construction.

Now, take any two integers m_1, m_2 with $(m_1, q) = (m_2, q) = 1$ and $m_1 \equiv m_2 \pmod{q_2}$. Then, m_1, m_2 correspond to two elements in $(\mathbb{Z}/q\mathbb{Z})^{\times}$ and hence there is a unique $n \in (\mathbb{Z}/q\mathbb{Z})^{\times}$ such that $m_1 \equiv nm_2 \pmod{q}$. Since $q_2 \mid q$, this implies which forces $n \equiv 1 \pmod{q_2}$ (since $(m_1, q_2) = (m_2, q_2) = 1$). Hence, by what we proved in last paragraph, $\chi(n) = 1$. Hence, $\chi(m_1) = \chi(nm_2) = \chi(n)\chi(m_2) = \chi(m_2)$.

This proves that $\chi(n)$ restricted by $(n, q) = 1$ has period q_2. Since $q_2 \mid q_1$, it follows that $\chi(n)$ restricted by $(n, q) = 1$ also has period q_1.

Sketch the proof of Theorem 1.2. Since χ has period q the sum $\frac{1}{q}\sum_{c=0}^{q-1}\chi(ac+b)$ only depends on a and b modulo q, and equals $\frac{1}{q}\sum_{x\in\mathbb{Z}/q\mathbb{Z}}\chi(ax+b)$. For a given $d \in \mathbb{Z}/q\mathbb{Z}$, the congruence equation $ax + b \equiv d \pmod{q}$ (viewed as an equation in $x \in \mathbb{Z}/q\mathbb{Z}$) is solvable if and only if $(a, q) \mid d - b$, and in this case it is equivalent with $\frac{a}{(a,q)}x \equiv \frac{d-b}{(a,q)} \left(\mathrm{mod}\, \frac{q}{(a,q)}\right)$, which has a unique solution $x \left(\mathrm{mod}\, \frac{q}{(a,q)}\right)$ since $\left(\frac{a}{(a,q)}, \frac{q}{(a,q)}\right)$ 1. Hence, when x runs through $\mathbb{Z}/q\mathbb{Z}$ in our sum, $ax + b$ visits exactly those congruence classes modulo q which are $\equiv b$ (mod a, q)) and each such congruence class is visited exactly (a, q) times. In other words, our sum equals

$$S := \frac{1}{q}\sum_{c=0}^{q-1}\chi(ac+b) = \frac{(a,q)}{q}\sum_{\substack{y\in\mathbb{Z}/q\mathbb{Z} \\ y\equiv b(\mathrm{mod}\,(a,q))}}\chi(y).$$

Now, fix some $m \in (\mathbb{Z}/q\mathbb{Z})^{\times}$; we wish to understand the product $\chi(m)S$. For each y, as in the above sum, we have $\chi(m)\chi(y) = \chi(my)$ and $my \equiv mb$ (mod a, q)). Thus, let us from now on assume that $m \equiv 1$ (mod a, q). Then, $my \equiv mb$ (mod a, q)) for all y as in the above sum. Since $m \in (\mathbb{Z}/q\mathbb{Z})^{\times}$; we also know that the map $y \mapsto my$ is a permutation of $\mathbb{Z}/q\mathbb{Z}$, hence it also restricts to a permutation of the set which we are interested in, i.e., $\{y \in \mathbb{Z}/q\mathbb{Z} : y \equiv b(\mathrm{mod}\,(a, q))\}$.Thus,

$$\chi(m)S = \frac{(a,q)}{q}\sum_{\substack{y\in\mathbb{Z}/q\mathbb{Z} \\ y\equiv b(\mathrm{mod}\,(a,q))}}\chi(my) = \frac{(a,q)}{q}\sum_{\substack{y\in\mathbb{Z}/q\mathbb{Z} \\ y\equiv b(\mathrm{mod}\,(a,q))}}\chi(y) = S.$$

Hence, if $S \neq 0$ then for all $m \in (\mathbb{Z}/q\mathbb{Z})^{\times}$; with $m \equiv 1$ (mod a, q) we must have $\chi(m) = 1$. In other words, for all integers m with $(m, q) = 1$ and $m \equiv 1\,(\mathrm{mod}\,(a, q))$, we must have $\chi(m) = 1$. By Lemma 4.1, this implies that (a, q) is a period of $\chi(n)$ restricted by $(n, q) = 1$, and thus $(a, q) = q$ since χ is primitive. We have thus proved that $S = 0$ unless $(a, q) = 1$, i.e., unless $q \mid a$.

Finally, if $q \mid a$, then all terms in the sum equals $\chi(b)$ and hence the whole sum also $\chi(b)$.

Remark 4.1. The formula in Theorem 1.2 is in general not valid if χ is not primitive. If χ is the principal character modulo q and $a = 1$, then $\frac{1}{q}\sum_{c=0}^{q-1}\chi(ac+b) = \frac{\varphi(q)}{q}$, and this does not agree with the stated formula for any $q \geq 2$.

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Conflict of interest

The author does not have any competing interest to declare.

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