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Research Article

Efficacy of the Single Server Markovian Two-Phase Symmetric Queue with Encouraged Stationary-Queue-Size Analysis

R. Jeyachandhiran, P. Rajendran[*] ID

Department of Mathematics, School of advanced sciences, Vellore Institute of Technology, Vellore-632014, India
E-mail: prajendran@vit.ac.in

Abstract: The efficacy of the Markovian single server symmetric queue is examined in this paper together with the suggested encouraged stationary queue-size analysis. First, we propose the stochastic Markov renewal process with encouraged arrival see time averages and encouraged arrival-stationary-queue-size analysis. The server takes further vacations after two extra service phases or until the server discovers a new set of clients in the service. Finally, we discuss the utility of the Bernoulli two-vacation symmetric queue, and it is clearly shown that the degree of efficiency rises when the encouraged arrival is taken into account.

Keywords: encouraged arrival, stochastic Markov renewal process, vacation time, Bernoulli two vacation symmetric queue

MSC: 60K20, 60K25, 90B36, 90B22, 60K30

1. Introduction

This paper's primary contribution is to demonstrate the efficiency and utility of Markovian single-channel Bernoulli two-vacation symmetric queues with encouraged stationary queue-size analysis and stochastic Markov renewal processes. The generalized Bernoulli system suggested in [1] states that a single server can take up to k more breaks with probability p if the line is empty when it returns. The Bernoulli vacation models have indeed been the subject of current research by several writers in many fields. Recently, several works [2-6] that implement Bernoulli-Schedule-Vacations (BSV) in accordance with different vacation rules have been developed. The research [7-9] is related with various queues with two-phase. Utilization of free time in $M/G/1$ queue thoroughly examined in [10]. First developed by [11], the $M/G/1$ type queue with generalized vacations. Encouraged arrival concept in queueing system is mainly used to attract more customers by giving them some amount of discounts to gain positive response from customers, so this will surely increase amount of clients in the system and business profit. [12, 29] developed an $M/M/1/N$ queueing model with encouraged arrival. In [13], a steady state study of an $MX/G/1$ queue with a two-phase service and Bernoulli vacation schedule was discovered. The author of [14] developed the busy and free hours of controllable $M/G/1$ queue. [15] studied the system size for the bulk service queuing system under the T-policies. [16] investigated the busy period distribution of the T-policy model using the entropy enlargement principle. The enormous body of research on queuing system optimization covers a wide variety of subjects. Different transit systems offer examples of these kinds of

situations. Additionally, [17] discussed its applications in a dual-access network. The dynamic of the $M/G/1$ Bernoulli vacation model and blocking probability were studied in [18-19]. When the $M/G/1$ system server is off or on studied in [20]. working vacation and interruption of $M/G/1$ vacation queue investigated in [21]. The k phases service, Bernoulli's feedback T-policies for $M/G/1$ line model investigated in [22-23]. $M^{[X]}/G/1$ queuing model with time-dependent solution instead of Bernoulli k vacation and balking consumers was examined in [23]. [24] Studied the expected delay of the $M/G/1$ cyclic system service line using Bernoulli scheduling. The busy period of vacation $M/G/1$ queue l with a Bernoulli schedule in [25]. The period of peak activity for the $M/G/1$ vacation model was studied in [26]. Queues, basic stochastic model and the equation studied in [27-28, 30-32]. Proof for Poisson observe-time- averages has been developed in [33]. A single service provider retrial queue with disappointed consumers, vacations, and balking consumers developed in [34]. An encouraged arrival batch queue with secondary optional consumer service, breakdown and several vacations studied in [35]. In [36] the authors developed an encouraged arrival with various (W-V) working vacations.

There are many real-world point of sight this sort of model can be used model construction of a manufacture technique. For example, consider a manufacture process, where the engine fabricating certain substances may need two stages of service such as initial checking (1st phase of service) trailed by normal processing (2nd stage of service) to finish the processing of resources. It may so occur that the procedure either wants to be stopped for fixing and preservation of the organization after these 2-stages of service or may stay the extra processing of the resources if no fault in the organization. This repairing can be used as a vacation in our (organization) system. To be additional truthful, we further undertake that the resources arrive in bunches (batch) of arbitrary size instead of solitary units. There might be several other circumstances such as digital-communication or data communication organizations which contain 2-stages or phases of service.

The structure of this essay is as follows: portion 1 talks about the introductory portion. Section 2 provides an explanation of the Notation and Model. Section 3 presents the stochastic Markov renewal theory for the stationary-queue-size distribution. Section 4 provides the embedded Markov chain and the stochastic Markov renewal mechanism at exit epochs. Delivered in Section 5 are the embedded Markov process and stationary queue size (S-Q-S) distribution. Bernoulli model's efficiency and application in Section 6. Section 7 contains the conclusion and future scope.

2. Notations and $M^x/(G_1, G_2)/1$ model description

The following notations are used in this paper.

$\lambda \times (1 + \iota)$	Encouraged arrival rate and represent the discount value
ζ_n	Probability-Mass-Function (PMF)
$\alpha(o)$	Probability-Generating-Function (PGF)
$Y_c(t)$	Probability-Density-Function (PDF)
$\beta^*(\varphi)$ & $Y_c^*(\varphi)$	Laplace-Stieltjes-Transform-Functions
$U(\Pi)$	General-Probability (G-P) with Density-Function
c, d	phases
Υ_k	Busy time distribution
$Exp(T_0)$	Expected-length-of-idle-period
T_β and $T_\beta^*(\varphi)$	Busy-time random-variable
ψ	Utilization-component of the-system
T	Length of the-period
ϖ_2	matrix variances
A	Markov process

We investigate an $M^x/(G_1, G_2)/1$ queuing system, where the Encouraged-arrival rate is given by "$\lambda \times (1 + \iota)$". Size of the successive batch arrivals are, $\Pi_1, \Pi_2 \ldots$, distributed with PMF $\zeta_n = P[\Pi_c = k]$; $k \geq 1$ and PGF $\alpha(o) = E[o^{\Pi_1}]$. The 1st and 2nd moments are $\alpha^{(1)} = E[\Pi_1]$ and $\zeta^{(2)} = E[\Pi_1^2]$ respectively.

All unit observing, multiple phases of heterogeneous system service from the system are given the First-Phase-Service (F-P-S) and then the Second-Phase-Service (S-P-S). It is assumed that the service discipline is First Come First Service (F-C-F-S). Further, the exponential time Y_c of the cth phase of service follow General-Probability-Law (G-P-L) with PDF. $Y_c(t)$ with the condition that $(Y_c(0) = 0)$ Laplace-Stieltjes-Transform [L-S-T], $Y_c^*(\varphi) = E[e^{-\varphi Y_c}]$ with finite 1st and 2nd moments as $b_c^{(1)} = E(Y_c)$ and $b_c^{(2)} = E(Y_c^2)$ respectively, for $c = 1$ and 2.

In which the sub-index $c = 1, 2$ denotes F-P-S and S-P-S respectively. The server will decide to a vacation random length "U" with probability p $(0 \leq p \leq 1)$. When the S-P-S of a unit is finished, the server could keep providing them with services with probability $q(1 - p)$. The-vacation period random-variable "U" of the system has a General-Probability (G-P) with density function $U(\Pi)$, [L-S-T], $U^*(\varphi) = Exp[e^{-\varphi U}]$ and the 1st and 2nd factorials of free time moments are $\vartheta^{(1)} = Exp(U)$ and $\vartheta^{(2)} = Exp(U^2)$ are finite. This kind of model is called as single vacation, batch arrival Bernoulli vacation queue with two-phases of heterogeneous service studied by [3] and [4].

We now propose the idea of multiple vacation policy for further development of this model:

It is to be emphasized that the same server handles both stages of the process. A modified-service is defined as follows:

$$Y_1 + Y_2 + U \text{ with probability "} p \text{"}$$

$$Y_1 + Y_2 \text{ with probability "} q \text{" } (= 1 - p).$$

Now, we denote $f_{c,d}$ and h_d as the-probability d independent arrival with c^{th} phases of service for "c" belongs to 1 and 2 with $d = \{0, 1, 2, \ldots\}$; and is defined by $f_{c,d} = \int_{0.}^{\infty} p_d(t)dY_c(t)$ and $h_d = \int_{0.}^{\infty} p_d(t)dU(t)$; respectively, where $p_d(t) = [e^{-\lambda \times (1 + \iota)t}(\lambda \times (1 + \iota)t)^d]/d!$ is the probability that d arrivals occur during $[0, t]$ and therefore we write $f_{c,d} = \int_{0.}^{\infty} \frac{e^{-\lambda \times (1+\iota)t}\left(\lambda \times (1+\iota)t\right)^j}{d!}dY_c(t)$ and $h_d = \int_{0.}^{\infty} \frac{e^{-\lambda \times (1+\iota)t}\left(\lambda \times (1+\iota)t\right)^d}{d!}dU(t)$; respectively. So, the probability that d customers are accepted under server-on vacation period is represented by g_d and is given by

$$g_d = \sum_{c=0.}^{\infty} (h_0)^c h_d = \frac{h_d}{(1 - h_0)}; d = 1, 2, 3, \ldots.$$

Now, we elaborate the queue-size-distribution at busy-vacation-period-initiation-epoch, defined as Υ_k ($k \geq 1$) as the steady-state probability that random customers observe a various batches 'n' customer in the queue (including customer already in service, if any) at busy-period (working) initiation epoch. Then conditioning number of customer arrivals through the free period and an appeal to the P-A-S-T-A property (see [33]) we observe that

$$\Upsilon_k = \sum_{d=1}^{k} \zeta_k^{(d)} g_d; k \geq 1, \tag{1}$$

where $\zeta_k^{(d)} = Prob\{(\Pi_1 + \Pi_2 + \cdots + \Pi_k) = d\}$ is the k-fold of $\{\zeta_d\}$ with itself and $\zeta_0^{[0]} = 1$.

Now, we define $\Upsilon(o) = Exp[o^{\Upsilon}]$ be the P-G-F of $\{\Upsilon_k$ for $k \geq 1\}$, from Equation [1], we know that

$$\Upsilon(o) = \frac{\left[U^*\left(\lambda \times (1+\iota) - \lambda \times (1+\iota)\zeta(o)\right) - U^*\left(\lambda \times (1+\iota)\right)\right]}{\left(1 - U^*\left(\lambda \times (1+\iota)\right)\right)}, \tag{2}$$

where $U^*\left(\lambda \times (1+\iota) - \lambda \times (1+\iota)\zeta(o)\right)$ is the Z-transform of U.

The 1st two-factorials moments of free time equations are,

$$\text{(i). } Exp(\Upsilon) = \Upsilon'(1) = \frac{\lambda \times (1+\iota)\zeta^1 \vartheta^1}{\left(1 - U^*\left(\lambda \times (1+\iota)\right)\right)}, \tag{3}$$

and

$$\text{(ii). } Exp\left[\Upsilon(\Upsilon - 1)\right] = \Upsilon''(1) = \frac{\left(\lambda \times (1+\iota)\zeta^1\right)^2 \vartheta^2 + \lambda \times (1+\iota)\vartheta^1\left(\zeta^2 - \zeta^1\right)}{\left(1 - U^*\left(\lambda \times (1+\iota)\right)\right)}. \tag{4}$$

We note equation (3) expression is denoted by average number of arrivals in the free time and equation (4) represent the second derivative of arrival in the system.

2.1 *The T-policy*

The T-policy model is applicable, when the period is deterministic and has a set length of period (T).
Let $Exp(T_0)$ be the Expected-Length-of the Idle-Period [ELIP], by using the Little's law in (3),
We obtain,

$$Exp(T_0) = \frac{E(\Upsilon)}{\lambda \times (1+\iota)\zeta^1} = \frac{\vartheta^1}{\left(1 - U^*\left(\lambda \times (1+\iota)\right)\right)}. \tag{5}$$

Now, we define T_β, the busy-time-random-variable and its L-S-T as $T_\beta^*(\varphi) = E[e^{-T_\beta \varphi}]$ given by

$$T_\beta^*(\varphi) = \sum_{k=1.}^{\infty} \left[\beta^*(\varphi)\right]^k a_k = \frac{\left[U^*\left(\lambda \times (1+\iota) - \lambda \times (1+\iota)\zeta\left(\beta^*(\varphi)\right)\right) - U^*\left(\lambda \times (1+\iota)\right)\right]}{\left(1 - U^*\left(\lambda \times (1+\iota)\right)\right)},$$

where $\beta^*(\varphi) = G^*\left(\varphi + \lambda \times (1+\iota) - \lambda \times (1+\iota)\zeta\left(\beta^*(\varphi)\right)\right)$ represents the well-known L-S-T of an $M^x/(G_1, G_2)/1$ queue's busy period that began with one unit utilizing the upgraded service period as the real service period. The anticipated busy time is defined by,

$$E\left(T_\beta\right) = \frac{-dT_\beta^*\left(\varphi\right)}{d\varphi}\Bigg]_{\varphi=0}$$

$$= \frac{\lambda \times (1+\iota)\zeta^1\left(b_1^1 + b_2^1\right)\vartheta^1}{(1-\psi)\left(1 - U^*\left(\lambda \times (1+\iota)\right)\right)} + \frac{p\lambda \times (1+\iota)\zeta^1\vartheta^2}{(1-\psi)\left(1 - U^*\left(\lambda \times (1+\iota)\right)\right)}, \tag{6}$$

where $\psi = \lambda \times (1+\iota)\zeta^1\left[b_1^1 + b_2^1 p\vartheta^1\right]$.

Let $Exp(T_1)$ be the [ELIP] and so,

$$Exp(T_I) = Exp(T_0) + Exp(T_\beta) = \frac{\vartheta^1}{(1-\psi)\left(1 - U^*\left(\lambda \times (1+\iota)\right)\right)}. \tag{7}$$

In [13], we know that, $U^*\left(\lambda \times (1+\iota)\right) = e^{-\lambda \times (1+\iota)T}$, $\vartheta^1 = T$, $\psi = \lambda \times (1+\iota)\zeta^1\left[b_1^1 + b_2^1 pT\right]$, $Exp(T_0) = \dfrac{T}{\left(1 - e^{-\lambda \times (1+\iota)T}\right)}$
and

$$Exp(T_\beta) = \frac{\lambda \times (1+\iota)\zeta^1\left(b_1^1 + b_2^1\right)T}{(1-\psi)\left(1 - e^{-\lambda \times (1+\iota)T}\right)} + \frac{p\lambda \times (1+\iota)\zeta^1 T^2}{(1-\psi)\left(1 - e^{-\lambda \times (1+\iota)T}\right)}. \tag{8}$$

2.2 *Working model*
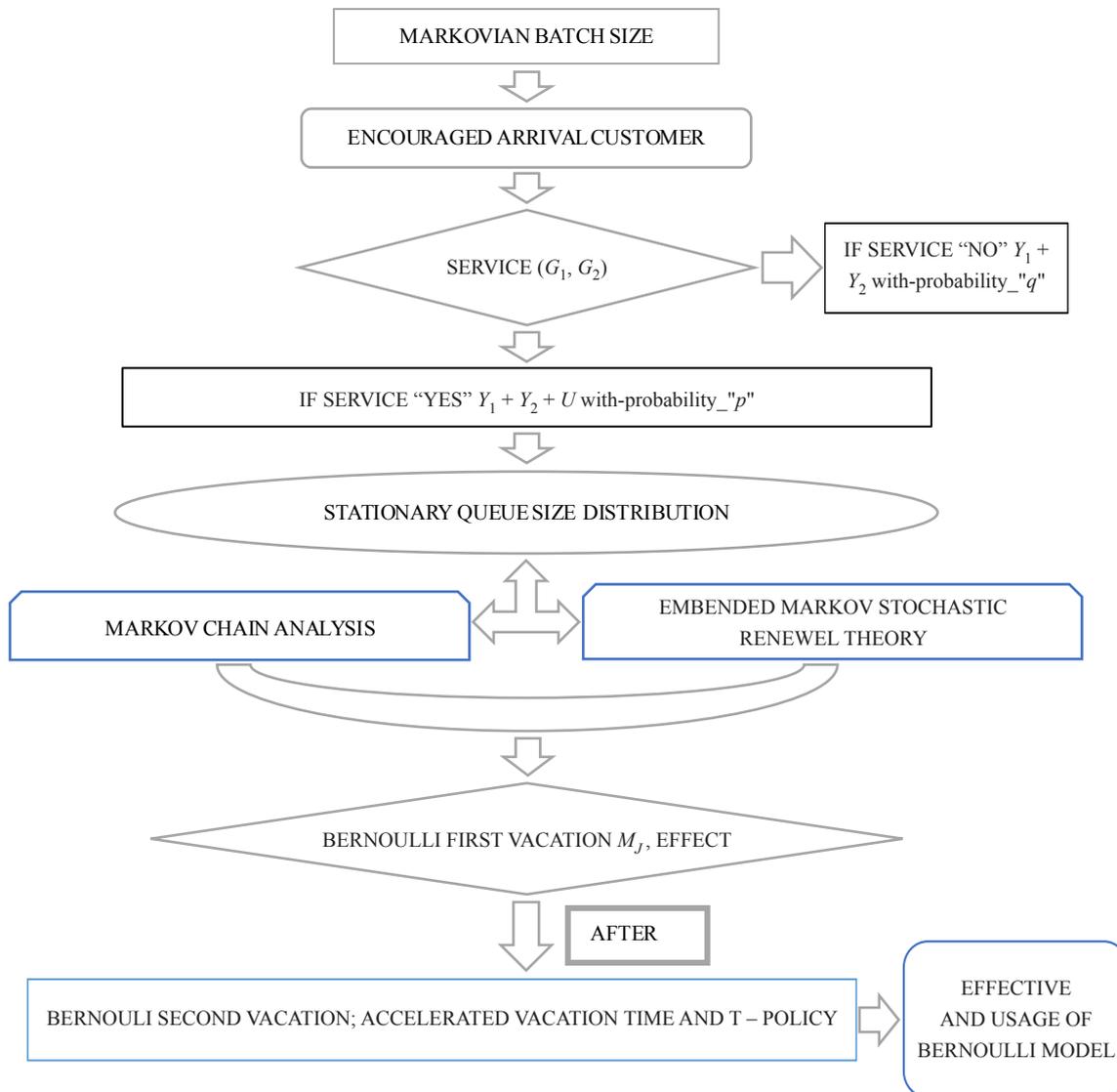
3. Stochastic Markov renewal theory of stationary-queue-size distribution

In this section, we examine the stationary queue size distribution.

Let $\{\rho_k; k \geq 1\}$, the steady-state-probability that several batches totalling 'k' customers will arrive before an identified customer during the idle period's residual life, where the identifiable customer is randomly chosen from the arriving groups at the end of the idle period and by "Stationary-Renewal-Process", we can write

$$\rho_k = \sum_{K=k+1}^{\infty} \frac{\gamma_K}{K}; k = 0, 1, 2, \ldots, \tag{9}$$

where $\{\gamma_K; K \geq 1\}$ is the identified arrival belong to the kth batch of an idle time, which is picked at random with probability $(1/k)$.

From (1) by implementing the Length of-Biasing-Argument [L-B-A] of Markovian renewal theory, we-have

$$\gamma_k = \frac{k\Upsilon_k}{\sum_{k=1.}^{\infty} k\Upsilon_k} = \frac{k(1-h_0)\Upsilon_n}{\lambda \times (1+\iota)\zeta^1 \vartheta^1}, \ k = 1, 2, \ldots \quad (10)$$

We define $\psi(o)$ represent the utilizing-component of the system given by

$$\psi(o) = \frac{\left[1 - U^*\left(\lambda \times (1+\iota) - \lambda \times (1+\iota)\zeta(o)\right)\right]}{\lambda \times (1+\iota)\zeta^1 \vartheta^1 (1-o)}. \quad (11)$$

By the encouraged arrival see time averages property, the average queue-size due to idle period, M_0 is given by

$$M_0 = \rho'(1) = \frac{\lambda \times (1+\iota)\zeta^1 \vartheta^2}{2\vartheta^1} + \frac{(\zeta^2 - \zeta^1)}{2\zeta^1}. \quad (12)$$

Now, we discuss how the *T*-policy method is incorporated:
We have the vacation period random variable *U*. The *Z*-transform of *U* is given by

$$U^*\left(\lambda \times (1+\iota)\right) - \lambda \times (1+\iota)\zeta(o) = e^{-\lambda \times (1+\iota)T(1-\zeta(o))} \ \text{and} \ \vartheta^2 = T^2.$$

Equation (11) and (12) we get

$$\rho(o) = \frac{\left[1 - e^{-\lambda \times (1+\iota)T(1-\zeta(o))}\right]}{\lambda \times (1+\iota)T\zeta^1 (1-o)}. \quad (13)$$

Let $V_0^*(\varphi)$ be the L-S-T of the P-D-F for the unfinished work at free time. Then, by applying classic queuing theory approach [28], we get

$$V_0^*(\varphi) = \rho\left[G^*(\varphi)\right] = \frac{\left[1 - U^*\left(\lambda \times (1+\iota) - \lambda \times (1+\iota)\zeta\left(G^*(\varphi)\right)\right)\right]}{\lambda \times (1+\iota)\zeta^1 \vartheta^1 \left[1 - G^*(\varphi)\right]}, \quad (14)$$

where $G^*(\varphi)$ denotes the adjusted accelerated time distribution of *G*. In order to make it simple to derive the L-S-T of this model's probability distribution function for the unfinished task V^* (say), the following decomposition result is used:

$$V^*(\varphi) = V^* M^x / (G_1, G_2)/1,$$

where

$$V^* M^x / (G_1, G_2) / 1 = \frac{(1 - \psi) \varphi}{\left[\varphi - \lambda \times (1 + \iota) + \lambda \times (1 + \iota) \zeta \left(G^* (\varphi) \right) \right]}$$

is the L-S-T of the W-T-D of the 1st batch with $M^x / (M^x / (G_1, G_2) / 1)$ line and upgraded accelerated period.

Remark 1.

We note that equation (12) is same as the equation (15) of the reference [19] for $\zeta(o) = 0$ and $\zeta^1 = 1$.

4. The embedded-Markov-chain and the stochastic Markov-renewal process at exit epoch:

In this section, the distribution of the Embedded-Markov-chain and the Stochastic Markov-renewal process at exit epoch is discussed:

When we talk about the epoch, at which a customer's entire service request expires, let δ_l be the period of the lth Service-completion-epoch. Then $K_l = K(\delta_l + 0)$ forms a Markov-chain. This is an embedded-stochastic-Markov-renewal process for a continuous-time-Markov process.

A Markov chain is present in $\{K_l; l \geq 0\}$ and the transition that causes this is responsible.

$$K_{l+1} = \begin{cases} Y_{l+1} + U_{l+1} - 1 & \text{and } K_l = 0 \\ K_l + Y_{l+1} - 1 & \text{and } K_l > 0 \end{cases}, \tag{15}$$

where Y_l is no. of arrival due to the lth upgraded service time, U_l is the number of arrival due to the lth vacation time. Then the transition-probability-matrix $(P_{c,d})$ to be a ϖ_2 matrix in [29]. The ϖ_2 matrix-variances from that of $M^x / (G_1, G_2) / 1$ queue in the 1st row only. Now, we consider that $\psi < 1$ will assure $\{K_l; l \geq 0\}$ is a positive-recurrent function, where K_l represent the embedded-stochastic Markov-renewal process for a continuous time-Markov process. Which is limiting-probabilities are,

$$x_d = \lim_{l \to \infty} \text{Prob} \left[K_l = d \right]; \ d \geq 0 \text{ exists.}$$

The Kolmogorov-govern equation is,

$$\varpi_d = \sum_{c := 1}^{d+1} (\varpi_0 a_c + \varpi_c) \{ q m_{d-c+1} + p n_{d-c+1} \} : d \geq 0, \tag{16}$$

where, for $d \geq 0$, $m_d = \sum_{c := 0}^{d} m_{1,c} m_{2,d-c}$, $n_d = \sum_{c := 0}^{d} m_c \beta_{d-c}$.

Let $m_{c,d} = \sum_{k := 0}^{d} \zeta_d^{(k)} f_{c,k} = \sum_{k := 0}^{d} \int_0^\infty \frac{e^{-ct} \left(\lambda \times (1 + \iota) t \right)^k}{k!} \zeta_d^k dY_c(t)$ is many-batches totaling "d" units due to the cth phase for $\{1, 2\}$.

Let $\beta_d = \sum_{k := 0}^{d} \zeta_d^{(k)} h_k = \sum_{k := 0}^{d} \int_0^\infty \frac{e^{-ct} \left(\lambda \times (1 + \iota) t \right)^k}{k!} \zeta_d^{(k)} dU(t)$ represent that many batches-totaling d units due to be a vacation respectively.

Now, the following z-transform can be used to convert equation (15)

$$x(o) = \sum_{d.=0}^{\infty} o^d x_d, \; M_c(o) = \sum_{d.=0}^{\infty} o^d m_{c,d}, \; M(o) = \sum_{d.=0}^{\infty} o^d m_d \; \text{and} \; N(o) = \sum_{d.=0}^{\infty} o^d n_d.$$

Note that, $M_c(o) = Y_c^* \left(\lambda \times (1+\iota) - \lambda \times (1+\iota) \zeta(o) \right)$, for $c = \{1, 2\}$; $M(o) = M_1(o) . \otimes M_2(o)$ and $N(o) = M(o) \otimes U^* \left(\lambda \times (1+\iota) - \lambda \times (1+\iota) \zeta(o) \right)$.

Using (2) in (16), we get,

$$x(o) = \frac{x_0 \left[1 - U^* \left(\lambda \times (1+\iota) - \lambda \times (1+\iota) \zeta(o) \right) \right] \left[q + pU^* \left(\lambda \times (1+\iota) - \lambda \times (1+\iota) \zeta(o) \right) \right]}{(1-h_0) \left[q + pU^* \lambda \times (1+\iota) - \lambda \times (1+\iota) \zeta(o) \times Y_1^* \left(\begin{array}{c} \lambda \times (1+\iota) - \\ \lambda \times (1+\iota) \zeta(o) \end{array} \right) Y_2^* \left(\begin{array}{c} \lambda \times (1+\iota) - \\ \lambda \times (1+\iota) \zeta(o) \end{array} \right) - o \right]} \tag{17}$$

$$x_0 = \left[\lambda \times (1+\iota) \zeta^1 E(T_1) \right]^{-1}. \tag{18}$$

In reality, this is comparable to the ordinary $(M^x/(G_1, G_2)/1)$ queue, $x(o)$ average number of units in the-system thus we have,

$$x(o) = \left[\frac{(1-\psi)(1-o) \left[q + pU^* \left(\lambda \times (1+\iota) - \lambda \times (1+\iota) \zeta(o) \right) \right] Y_1^* \left(\begin{array}{c} \lambda \times (1+\iota) - \\ \lambda \times (1+\iota) \zeta(o) \end{array} \right) Y_2^* \left(\begin{array}{c} \lambda \times (1+\iota) - \\ \lambda \times (1+\iota) \zeta(o) \end{array} \right)}{\left[\left\{ q + pU^* \left(\lambda \times (1+\iota) - \lambda \times (1+\iota) \zeta(o) \right) \right\} Y_1^* \left(\begin{array}{c} \lambda \times (1+\iota) - \\ \lambda \times (1+\iota) \zeta(o) \end{array} \right) Y_2^* \left(\begin{array}{c} \lambda \times (1+\iota) - \\ \lambda \times (1+\iota) \zeta(o) \end{array} \right) \right]} \right]$$

$$\times \left[\frac{1 - U^* \left(\lambda \times (1+\iota) - \lambda \times (1+\iota) \zeta(o) \right)}{\zeta^1 \chi^1 \left(\lambda \times (1+\iota) - \lambda \times (1+\iota) o \right)} \right]$$

$$= x_0(o) \rho(o); \tag{19}$$

$$x_0(o) = \frac{(1-\psi)(1-o) \left[q + pU^* \left(\lambda \times (1+\iota) - \lambda \times (1+\iota) \zeta(o) \right) \right] Y_1^* \left(\begin{array}{c} \lambda \times (1+\iota) - \\ \lambda \times (1+\iota) \zeta(o) \end{array} \right) Y_2^* \left(\begin{array}{c} \lambda \times (1+\iota) - \\ \lambda \times (1+\iota) \zeta(o) \end{array} \right)}{\left[\left\{ q + pU^* \left(\lambda \times (1+\iota) - \lambda \times (1+\iota) \zeta(o) \right) \right\} Y_1^* \left(\begin{array}{c} \lambda \times (1+\iota) - \\ \lambda \times (1+\iota) \zeta(o) \end{array} \right) Y_2^* \left(\begin{array}{c} \lambda \times (1+\iota) - \\ \lambda \times (1+\iota) \zeta(o) \end{array} \right) - o \right]}, \tag{20}$$

is the PGF of the queue-size distribution at Random-Process (RP) of period of an $M^x/(G_1, G_2)/1$ queue.

If M_J is found to be

$$M_J = x^{'}(1)$$

$$= \psi + \frac{\left(\lambda \times (1+\iota)\zeta^1\right)^2 \left[b_1^2 + b_2^2 + p\vartheta^2\right]}{2(1-\psi)} + \frac{\left(\lambda \times (1+\iota)\zeta^1\right)^2 \left[b_1^1 b_2^1 + p\vartheta^1 \left\{b_1^1 + b_2^1\right\}\right]}{(1-\psi)}$$

$$+ \frac{\psi\left(\zeta^2 - \zeta^1\right)}{2(1-\psi)\zeta^1} + M_0. \tag{21}$$

In equation (21) corresponds with the outcome found in [26]. We obtained that $\beta_2^1 = \beta_2^2 = 0$. $\Upsilon^1 = 1$ and $\Upsilon^2 = 0$. Similarly, $\beta_2^1 = \beta_2^2 = 0$, $p = 0$ in equation (21) with the results will be obtained by [3].

Remark 2. The $M^x/(G_1, G_2)/1$ line provided in equation (19). Decomposes the service into the two different probabilities (Random Variable). An $M^x/(G_1, G_2)/1$ queue's stationary queue size distribution, where V_S stands for one vacation [produced by our first-term of equation (19)]. The queue-size distribution outcome from the not used of vacation period, which happens due to the free time process [produced by second-term of equation (19)].

Remark 3. The gained result is fairly broad & applies to a variety of real-world scenarios. For example, consider the T-policy model. Hence outcome for the T-policy method with a two-phase service under a Bernoulli schedule, equations, (19) and (21) obtained

$$x(o) = \frac{(1-\psi)\left[1 - e^{-\lambda \times (1+\iota)T(1-\zeta(o))}\right]\left[q + pe^{-\lambda \times (1+\iota)T(1-\zeta(o))}\right]Y_1^*\left(\begin{matrix}\lambda \times (1+\iota)-\\ \lambda \times (1+\iota)\zeta(o)\end{matrix}\right)Y_2^*\left(\begin{matrix}\lambda \times (1+\iota)-\\ \lambda \times (1+\iota)\zeta(o)\end{matrix}\right)}{\lambda \times (1+\iota)\zeta^1 T\left[\left\{q + pe^{-\lambda \times (1+\iota)T(1-\zeta(o))}\right\}Y_1^*\left(\begin{matrix}\lambda \times (1+\iota)-\\ \lambda \times (1+\iota)\zeta(o)\end{matrix}\right)Y_2^*\left(\begin{matrix}\lambda \times (1+\iota)-\\ \lambda \times (1+\iota)\zeta(o)\end{matrix}\right) - o\right]} \tag{22}$$

and

$$M_J = \psi + \frac{\left(\lambda \times (1+\iota)\zeta\right)^2 \left[b_1^2 + b_2^2 + pT^2\right]}{2(1-\psi)} + \frac{\left(\lambda \times (1+\iota)\zeta^1\right)^2 \left[b_1^1 b_2^1 + pT\left\{b_1^1 + b_2^1\right\}\right]}{(1-\psi)} + \frac{\left(\zeta^2 - \zeta^1\right)}{2(1-\psi)}$$

$$+ \frac{\lambda \times (1+\iota)\zeta^1 T}{2}. \tag{23}$$

Assume that, we have $Y_2^*\left(\lambda \times (1+\iota) - \lambda \times (1+\iota)\zeta(o)\right) = 1$ and $b_2^1 = b_2^2 = 0$, equation (22) and (23) obtained

$$x(o) = \frac{(1-\psi)\left[1 - e^{-\lambda \times (1+\iota)T(1-\zeta(o))}\right]\left[q + pe^{-\lambda \times (1+\iota)T(1-\zeta(o))}\right]Y_1^*\left(\lambda \times (1+\iota) - \lambda \times (1+\iota)\zeta(o)\right)}{\lambda \times (1+\iota)\zeta^{(1)} T\left[\left\{q + pe^{-\lambda \times (1+\iota)T(1-\zeta(o))}\right\}Y_1^*\left(\lambda \times (1+\iota) - \lambda \times (1+\iota)\zeta(o)\right) - o\right]}, \tag{24}$$

and

$$M_J = \psi + \frac{\left(\lambda \times (1+\iota)\varsigma^1\right)^2 \left[b_1^2 + pT^2 + 2pb_1^1 T\right]}{2(1-\psi)} + \frac{\left(\varsigma^2 - \varsigma^1\right)}{2(1-\psi)\varsigma^1} + \frac{\lambda \times (1+\iota)\varsigma^1 T}{2}$$

respectively, where M_J is an anticipated service in this-system.

Remark 4. We obtained that these two results are consistent with formulas for $p = 0$ and $\zeta(o) = o$. These two outcomes support the formula in [15].

5. Embedded-Markov-Process and Stationary-Queue-Size (S-Q-S) distribution

In this section, the S-Q-S of $M^x/(G_1, G_2)/1$ queue is examined. Given that the [P-D] of the embedded-Markov-process is known, we use regenerative process approach to this model. A Markov process is used to define the system at period t. $\pi(t) = \{S(t), K(t), \varsigma(t)\}$, where $S(t) = 0, 1, 2$ or 3 occurs to the system is free, the server is work on F-P-S, the-server is work on S-P-S, or the-server is on vacation at period t respectively. $K(t)$ indicates the number-of customers in the line at period 't', $S(t) \in \{0, 1, 2, 3\}$, then we obtained relevant vacation period is represented by $\varsigma(t)$.

Subsequently, the Markov regenerative process theory ensures that for (c, d) belongs to ϑ; where $\vartheta = \{(c, d) : d \geq 0; 0 \leq c \leq 3\}$ bounded probabilities

$$A_{c,d} = \lim_{t \to \infty} \Pr\{(S(t), K(t)) = (c, d)\},$$

where $A_{c, d}$ is represent the Markov process with phases and given the same-limitations on probability, are positive $\{x_d; d \geq 0\}$.

We define $\{\Pi(t); t \geq 0.\}$ is an [E-M-R-P] that is a Markov-Regenerative-Process, $\{K_l; l \geq 0.\}$. Therefore, we will be using the proved Classical-limiting-Theorems [C-L-T] in [30].

We get

$$A_{c,d} = \frac{\sum_{k=0}^{\infty} x_k \delta_k (c, d)}{\sum_{k=0}^{\infty} x_k \mu_k}; \ (c, d) \in \vartheta, \tag{25}$$

where $\delta_k(c, d)$ = the anticipated duration of the processed, $\{\Pi(t); t \geq 0.\}$ in the state (c, d) due to a service-cycle, given starting queue-size is 'k' & μ_k Considering that the starting queue size is "k".

Now, we have

$$\mu_k = \begin{cases} \dfrac{\vartheta^1}{(1-h_0)} B_1^1 + B_2^1 + p\vartheta^1; & \text{for } k = 0, \\[3mm] B_1^1 + B_2^1 + p\vartheta^1; & \text{for } k \geq 1, \end{cases} \tag{26}$$

and therefore $\sum_{k=0}^{\infty} \mu_k x_k = \left[\lambda \times (1+\iota)\varsigma^1\right]^{-1}$; is mean anticipated cycle.

Let's now assume the scenario of a [S-Q-S] distribution at an idle-period before-vacation period.

Now, $c = 0$ is simply-probability-argument results in (25), we have

$$A_{0,d} = \lambda \times (1 + \iota) \zeta^1 x_0 \sum_{k=0}^{d} \zeta_d^k e'_k; \, d \geq 0, \tag{27}$$

where $e'_k = (1 - h_0)^{-1} \int_{.0}^{\infty} p_d(t)(1 - U(t)) \, jt; \, d \geq 0$.

Let's now assume that the F-P-S accelerated-time finishes with k units still in the line. We are able to independent between two-instances based location of the customer who will observe the following F-P-S.

Assuming k point is the basically one, let's say that the F-P-S begins at period $t = 0$.

Next, we find that the period span $(t, t + \varpi t)$ contributes to $\delta_k(1, d)$ if,

(i) The F-P-S not finish in period t, and

(ii) $(d. - k + 1)$ the peak period for customers $[0, t]$.

Then, we obtain

$$\delta_k(1, d) = l'_{1, d-k+1}; \, \text{for } d \geq l(0, k-1), \tag{28}$$

where $l'_{1,d} = \sum_{c=0}^{d} \int_{0.}^{\infty} p_c(t) \zeta_d^c (1 - Y_1(t)) \, jt; \, d \geq 0$.

We define that, $\lambda \times (1 + \iota) l'_{1,d} = \left(1 - \sum_{c=0}^{d} l_{1,c} \right); \, \text{for } d \geq 0$.

Using (28) in (25), on simplification, we obtain

$$A_{1,d} = \lambda \times (1 + \iota) \zeta^1 \left[\sum_{c=1.}^{d+1} \{ x_0 \Upsilon_c + x_c \} m'_{1,d-c+1} \right]; \, d \geq 0. \tag{29}$$

Similarly, we obtain,

$$A_{2,d} = \lambda \times (1 + \iota) \zeta^1 \left[\sum_{c=1.}^{d+1} \{ x_0 \Upsilon_c + x_c \} \sum_{k=1.}^{d-c+1} m_{1,k} l'_{2,d-k-c+1} \right]; \, d \geq 0, \tag{30}$$

and

$$A_{3,d} = \lambda \times (1 + \iota) p \zeta^1 \left[\sum_{c=1.}^{d+1} \{ x_0 \Upsilon_c + x_c \} \sum_{k=1.}^{d-c+1} m_k \beta'_{d-k-c+1} \right]; \, d \geq 0, \tag{31}$$

where $\lambda \times (1 + \iota) l'_{2,d} = \left(1 - \sum_{.c=0}^{d} m_{2,c} \right)$ and $\lambda \times (1 + \iota) \beta'_d = \left(1 - \sum_{.c=0}^{d} \beta_c \right); \, \text{for } d \geq 0$.

Using the efficient recursive method for calculating the bounded probabilities of $\{ A_{c,d}; \, d \geq 0 \}$ for $c \in \{1, 2, 3\}$ in terms of $\{ x_d; \, d \geq 0 \}$ then combines (6) with equation (31)-(29).

Now we consider markov process with two phase $A_c(o)$. Then the P-G-F of $A_c(o) = \sum_{d=0}^{\infty} o^d A_{c,d}$ for $\in \{0, 1, 2, 3\}$, are given by

$$A_0(o) = \frac{(1-\psi)\left[1 - U^*\left(\lambda \times (1+\iota) - \lambda \times (1+\iota)\zeta(o)\right)\right]}{\vartheta^1\left(\lambda \times (1+\iota) - \lambda \times (1+\iota)\zeta(o)\right)},$$

$$A_1(o) = \frac{(1-\psi)\left[1 - U^*\left(\lambda \times (1+\iota) - \lambda \times (1+\iota)\zeta(o)\right)\right]\left[1 - Y_1^*\left(\lambda \times (1+\iota) - \lambda \times (1+\iota)\zeta(o)\right)\right]}{\vartheta^1\left(\lambda \times (1+\iota) - \lambda \times (1+\iota)\zeta(o)\right)\left[\begin{array}{c}\left\{q + pU^*\left(\lambda \times (1+\iota) - \lambda \times (1+\iota)\zeta(o)\right)\right\}Y_1^*\left(\begin{array}{c}\lambda \times (1+\iota) - \\ \lambda \times (1+\iota)\zeta(o)\end{array}\right) \\ Y_2^*\left(\lambda \times (1+\iota) - \lambda \times (1+\iota)\zeta(o)\right) - o\end{array}\right]},$$

$$A_2(o) = \frac{(1-\psi)\left[1 - U^*\left(\lambda \times (1+\iota) - \lambda \times (1+\iota)\zeta(o)\right)\right]\left[1 - Y_2^*\left(\begin{array}{c}\lambda \times (1+\iota) - \\ \lambda \times (1+\iota)\zeta(o)\end{array}\right)\right]Y_1^*\left(\begin{array}{c}\lambda \times (1+\iota) - \\ \lambda \times (1+\iota)\zeta(o)\end{array}\right)}{\vartheta^1\left(\lambda \times (1+\iota) - \lambda \times (1+\iota)\zeta(o)\right)\left[\begin{array}{c}\left\{q + pU^*\left(\lambda \times (1+\iota) - \lambda \times (1+\iota)\zeta(o)\right)\right\}Y_1^*\left(\begin{array}{c}\lambda \times (1+\iota) - \\ \lambda \times (1+\iota)\zeta(o)\end{array}\right) \\ Y_2^*\left(\lambda \times (1+\iota) - \lambda \times (1+\iota)\zeta(o)\right) - o\end{array}\right]}, \text{ and}$$

$$A_3(o) = \frac{p(1-\psi)\left[1 - U^*\left(\lambda \times (1+\iota) - \lambda \times (1+\iota)\zeta(o)\right)\right]^2\left[Y_1^*\left(\begin{array}{c}\lambda \times (1+\iota) - \\ \lambda \times (1+\iota)\zeta(o)\end{array}\right)\right]Y_2^*\left(\begin{array}{c}\lambda \times (1+\iota) - \\ \lambda \times (1+\iota)\zeta(o)\end{array}\right)}{\vartheta^1\left(\lambda \times (1+\iota) - \lambda \times (1+\iota)\zeta(o)\right)\left[\begin{array}{c}\left\{q + pU^*\left(\lambda \times (1+\iota) - \lambda \times (1+\iota)\zeta(o)\right)\right\}Y_1^*\left(\begin{array}{c}\lambda \times (1+\iota) - \\ \lambda \times (1+\iota)\zeta(o)\end{array}\right) \\ Y_2^*\left(\lambda \times (1+\iota) - \lambda \times (1+\iota)\zeta(o)\right) - o\end{array}\right]}.$$

The probability of the system's states is given by

Pr {The system is not in use} $= A_0(1) = (1 - \psi)$,

Pr {The server is occupied by FPS} $= A_1(1) = \lambda \times (1 + \iota)\zeta^1 B_1^1$

Pr {SPS is occupying the server} $= A_2(1) = \lambda \times (1 + \iota)\zeta^1 B_2^1$, and

Pr {The -server is away on-vacation} $= A_3(1) = p\lambda \times (1 + \iota)\zeta^1\vartheta^1$. Now, we determine the anticipated number of unit in the line due to the free period, F-P-S-time, S-P-S-time-and-vacation-time-as-follows:

$$E(K_0) = A_0'(1) = \frac{(1-\psi)\lambda \times (1+\iota)\zeta^1\vartheta^2}{2\chi^1},$$

$$E\left(K_{Y_1}\right) = A_1'(1) = \frac{\psi B_1^1\left(\zeta^2 - \zeta^1\right)}{(1-\psi)} + \frac{\left[\lambda \times (1+\iota)\zeta^1\right]^Y B_1^1\left[\beta^2 + p\left\{\chi^2 + \vartheta^1\beta^1\right\}\right]}{2(1-\psi)}$$

$$+ \frac{\left[\lambda \times (1+\iota)\zeta^1\right]^2\left[B_1^1\chi^2 + \vartheta^1 B_1^2\right]}{2\chi^1} + \lambda \times (1+\iota)B_1^1\left(\zeta^2 - \zeta^1\right),$$

$$E\left(K_{Y_2}\right) = A_2'(1) = \frac{\psi B_2^1\left(\zeta^2 - \zeta^1\right)}{(1-\psi)} + \frac{\left[\lambda \times (1+\iota)\zeta^1\right]^Y B_2^1\left[\beta^2 + p\left\{\chi^2 + \vartheta^1\beta^1\right\}\right]}{2(1-\psi)}$$

$$+ \frac{\left[\lambda \times (1+\iota)\zeta^1\right]^2\left[B_2^1\vartheta^2 + \vartheta^1\left\{B_2^2 + 2B_1^1 B_2^1\right\}\right]}{2\chi^1} + \lambda \times (1+\iota)B_2^1\left(\zeta^2 - \zeta^1\right),$$

$$E\left(K_U\right) = A_3'(1) = \frac{p \times \psi \times \vartheta^1\left(\zeta^2 - \zeta^1\right)}{(1-\psi)} + \frac{\left[\lambda \times (1+\iota)\zeta^1\right]^3 p \times \vartheta^1\left[\beta^2 + p\left\{\vartheta^2 + \vartheta^1\beta^1\right\}\right]}{2(1-\psi)}$$

$$+ \frac{\left[\lambda \times (1+\iota)\zeta^1\right]^2 p\vartheta^1\left[\vartheta^1\beta^1 + 2\vartheta^2\right]}{2\vartheta^1} + p\lambda \times (1+\iota)\vartheta^1\left(\zeta^2 - \zeta^1\right),$$

where $\beta^1 = B_1^1 + B_2^1$ and $\beta^2 = B_1^2 + B_2^2 + 2B_1^1 B_2^1$.

The quantity of units in the queue L_A the length of unit in the-system is found to be

$$L_A = E\left(K_0\right) + E\left(K_{Y_1}\right) + E\left(K_{Y_2}\right) + E\left(K_U\right) = P'(1),$$

$$= M_J + \frac{\left(\zeta^1 - \zeta^2\right)}{2\zeta^1} \text{ as obtained,}$$

where M_J represent the length of-anticipated service.

6. Effectiveness and usage of Bernoulli model:

In this section provided by service-periods follows accelerated distributions with $B_c^1 = 1/\mu_c$, $B_c^2 = 2/\mu_c^2$; for $c = \{1, 2\}$ and vacation period is distributed, with $\chi^{(1)} = 1/u$, $\chi^2 = 2/u^2$ for in model-I. In [13], when the distribution of vacation period is predicted with $\vartheta^1 = T$, $\vartheta^2 = T^2$ correspondingly for model-II, in [13]. Additionally, we used batch-size G-D (Geometrical-Distribution) with average $Exp(\Pi) = 1/\Delta$.

Thus, the equation (21) for case 1 becomes

$$M_J = \psi + \frac{\left(\lambda \times (1+\iota)/\Delta\right)^2 \left[\left\{1/\mu_1^2 + 1/\mu_2^2 + p/\mu^2\right\} + \left\{1/\mu_1\mu_2 + (p/u)(1/\mu_1 + 1/\mu_2)\right\}\right]}{(1-\psi)}$$

$$+ \frac{(1-\Delta)}{\Delta(1-\rho)} + \frac{\lambda \times (1+\iota)}{\Delta u}, \tag{32}$$

where $\psi = \left(\dfrac{\lambda \times (1+\iota)}{x}\right)\left(\dfrac{1}{\mu_1} + \dfrac{1}{\mu_2} + \dfrac{p}{u}\right) < 1$ and M_J is an anticipated-service in this system and ψ represent the utilization component of the system.

Now, we investigate the scenario when $p = 0$ as equation (32) simplifies that,

$$M_{J_0} = \psi_0 + \frac{\left(\lambda \times (1+\iota)/\Delta\right)^2 \left[\mu_1^{-2} + \mu_1^{-1}\mu_2^{-1} + \mu_2^{-2}\right]}{(1-\psi_0)} + \frac{(1-\Delta)}{\Delta(1-\psi)} + \frac{\lambda \times (1+\iota)}{\Delta u}, \tag{33}$$

where $\psi_0 = \left(\dfrac{\lambda \times (1+\iota)}{x}\right)\left(\dfrac{1}{\mu_1} + \dfrac{1}{\mu_2}\right) < 1,$ therefore, the impact of using the Bernoulli schedule on the system's predicted number at a departure epoch will be

$$M_J - \text{impact} = \left|M_J - M_{J_0}\right|. \tag{34}$$

We consider the following four system parameters in order to examine the M_J-effect in different encouraged arrival-rates and the Bernoulli-symbol represented "p" – $(\mu_1, \mu_2, u, \Delta, \iota) = (2.0, 1.5, 1.5, 0.25, 0.1 \text{ to } 0.3)$.

We Take Table 1 to Table 3 of our results.
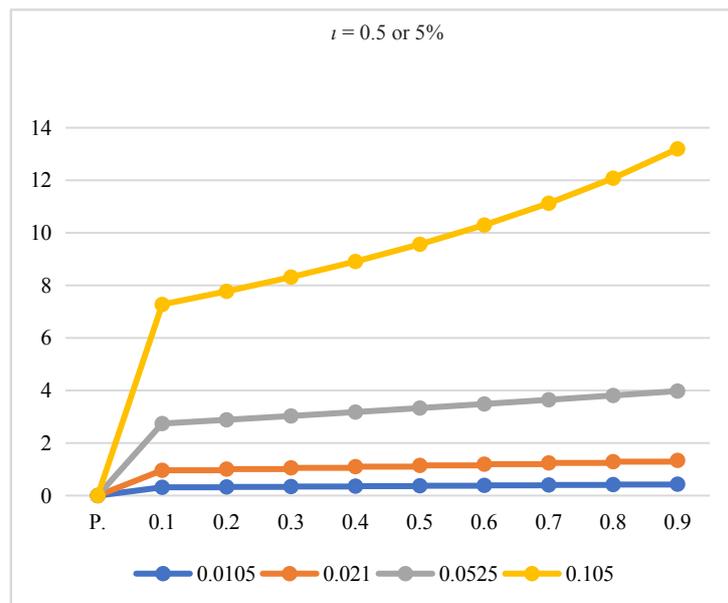
Figure 1. M_J-impact value with encouraged arrival ($\iota = 0.5$).

Table 1. Calculated M_J-effect value with encouraged arrival $\lambda = 0.01$, 0.02, 0.05, 0.1, $\iota = 0.05$, $\lambda \times (1 + \iota)$: 0.105, 0.021, 0.525, 0.105 for accelerated vacation period

P.	0.0105	0.021	0.0525	0.105
0.1	0.3155	0.6476	1.7784	4.5322
0.2	0.3302	0.6788	1.8751	4.8892
0.3	0.3450	0.7103	1.9746	5.2858
0.4	0.3598	0.7420	2.0769	5.7303
0.5	0.3747	0.7739	2.1823	6.2336
0.6	0.3897	0.8062	2.2908	6.8101
0.7	0.4047	0.8387	2.4029	7.4795
0.8	0.4198	0.8715	2.5185	8.2690
0.9	0.4349	0.9045	2.6381	9.2177

Table 2. Calculated M_J-impact value with encouraged arrival $\lambda = 0.01$, 0.02, 0.05, 0.1, $\iota = 0.10$, $\lambda \times (1 + \iota)$: 0.11, 0.022, 0.055, 0.11 for accelerated vacation period

P.	0.011	0.022	0.055	0.11
0.1	0.3309	0.6802	1.8793	4.9041
0.2	0.3464	0.7131	1.9828	5.3116
0.3	0.3619	0.7463	2.0894	5.7704
0.4	0.3775	0.7797	2.1993	6.2928
0.5	0.3932	0.8134	2.3127	6.8953
0.6	0.4089	0.8474	2.4299	7.6004
0.7	0.4247	0.8818	2.5511	8.4403
0.8	0.4405	0.9164	2.6766	9.4621
0.9	0.4564	0.9514	2.8068	10.7376
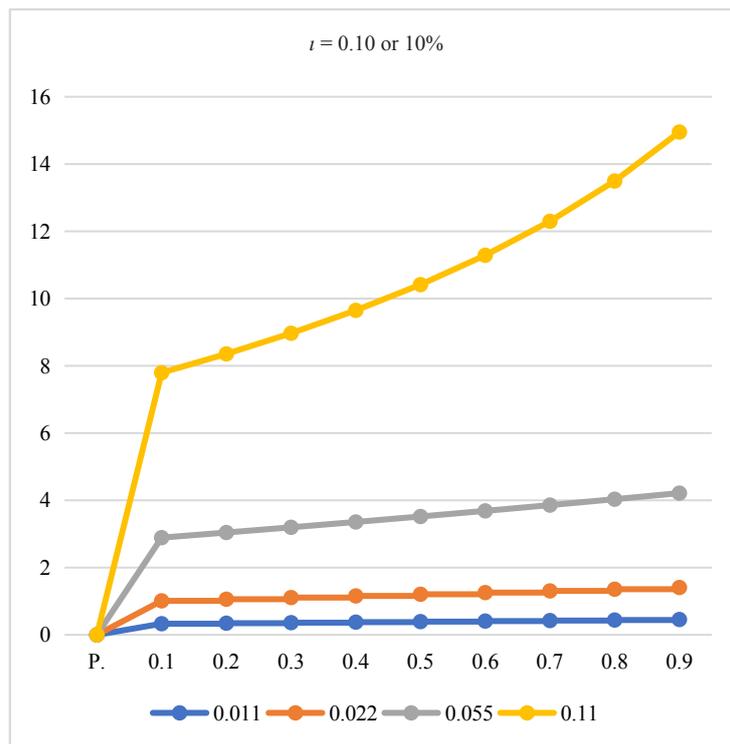
Figure 2. M_J-impact value with encouraged arrival ($\iota = 0.10$)

Table 3. Calculated M_J-impact value with encouraged arrival $\lambda = 0.01, 0.02, 0.05, 0.1, \iota = 0.15, \lambda \times (1 + \iota)$: 0.115, 0.023, 0.575, 0.115 for accelerated vacation period

P.	0.0115	0.023	0.0575	0.115
0.1	0.3464	0.7130	1.9823	5.3083
0.2	0.3626	0.7476	2.0928	5.7758
0.3	0.3789	0.7825	2.2069	6.3103
0.4	0.3952	0.8177	2.3248	6.9301
0.5	0.4116	0.8532	2.4468	7.6604
0.6	0.4281	0.8890	2.5732	8.5375
0.7	0.4447	0.9252	2.7042	9.6159
0.8	0.4613	0.9618	2.8403	10.9804
0.9	0.4780	0.9987	2.9819	12.7716
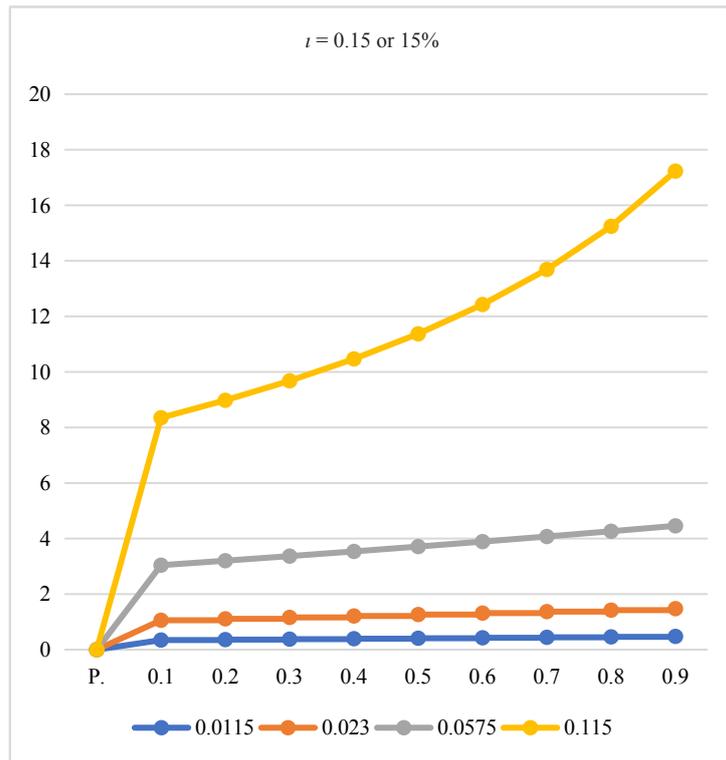
Figure 3. M_J-impact value with encouraged arrival ($\iota = 0.15$)

Table 4. M_J-impact in the T-policy value computed for the Encouraged arrival $\lambda = 0.01, 0.02, 0.05, 0.1, \iota = 0.05, \lambda \times (1 + \iota)$: 0.105, 0.021, 0.525, 0.105

P.	0.0105	0.021	0.0525	0.105
0.1	0.0078	0.0192	0.0839	0.4354
0.2	0.0149	0.0352	0.1460	0.8016
0.3	0.0220	0.0514	0.2110	1.2867
0.4	0.0291	0.0676	0.2794	1.9781
0.5	0.0362	0.0839	0.3517	3.0754
0.6	0.0433	0.1004	0.4285	5.1601
0.7	0.0504	0.1170	0.5107	10.9365
0.8	0.0576	0.1337	0.5992	36.6952
0.9	0.0647	0.1505	0.6953	15.3997
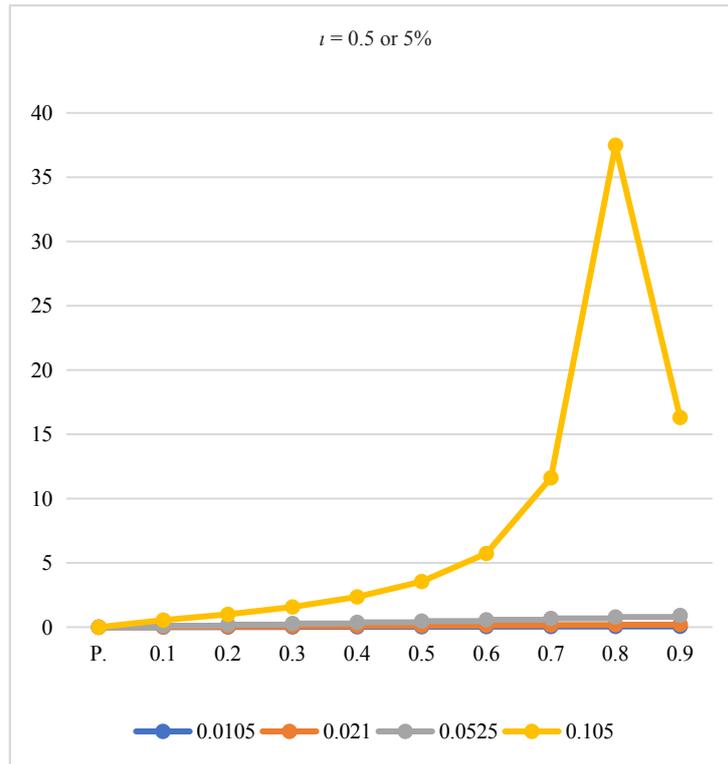
Figure 4. M_J-impact in the T-policy value computed for the Encouraged arrival ($\iota = 0.5$).

Table 5. M_J-impact in the T-policy value computed for the Encouraged arrival $\lambda = 0.01, 0.02, 0.05, 0.1, \iota = 0.10, \lambda \times (1 + \iota)$: 0.11, 0.022, 0.055, 0.11

P.	0.011	0.022	0.055	0.11
0.1	0.0083	0.0205	0.0918	0.5060
0.2	0.0157	0.0375	0.1595	0.9483
0.3	0.0232	0.0546	0.2307	1.5627
0.4	0.0307	0.0718	0.3061	2.5022
0.5	0.0382	0.0892	0.3863	4.1776
0.6	0.0457	0.1067	0.4723	8.1962
0.7	0.0532	0.1244	0.5651	33.3663
0.8	0.0607	0.1422	0.6660	21.6348
0.9	0.0683	0.1601	0.7771	8.9137
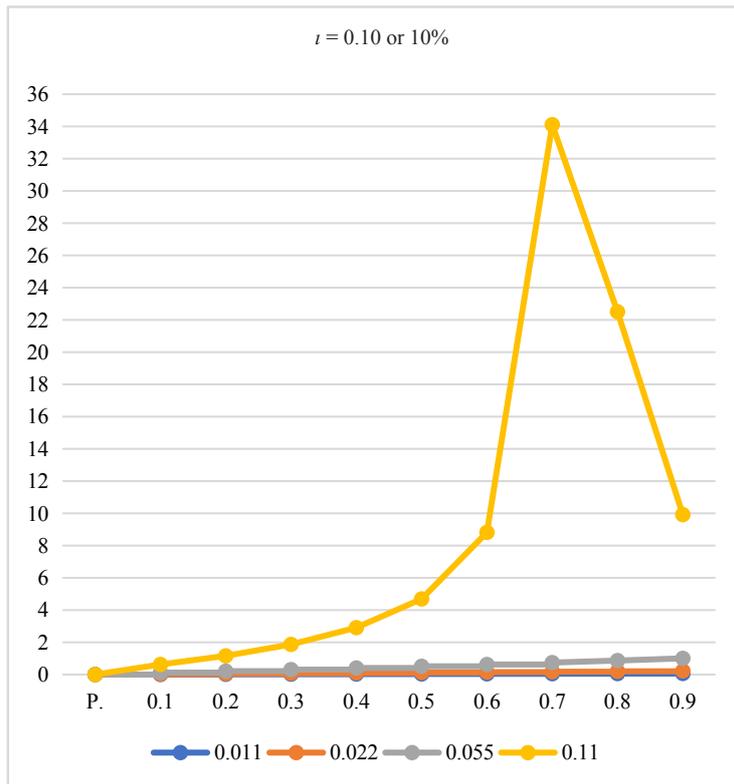
Figure 5. M_J-impact in the T-policy value computed for the Encouraged arrival ($\iota = 0.10$).

Table 6. M_J-impact in the T-policy value computed for the Encouraged arrival $\lambda = 0.01, 0.02, 0.05, 0.1, \iota = 0.15, \lambda \times (1 + \iota)$: 0.115, 0.023, 0.575, 0.115

P.	0.0115	0.023	0.0575	0.115
0.1	0.0088	0.0218	0.1001	0.5894
0.2	0.0166	0.098	0.1739	1.1284
0.3	0.0244	0.0579	0.2519	1.9206
0.4	0.0323	0.0762	0.3350	3.2454
0.5	0.0402	0.0947	0.4241	6.0349
0.6	0.0481	0.1132	0.5203	16.4342
0.7	0.0560	0.1320	0.6253	45.6537
0.8	0.0639	0.1509	0.7409	11.0012
0.9	0.0719	0.1700	0.8698	6.6184
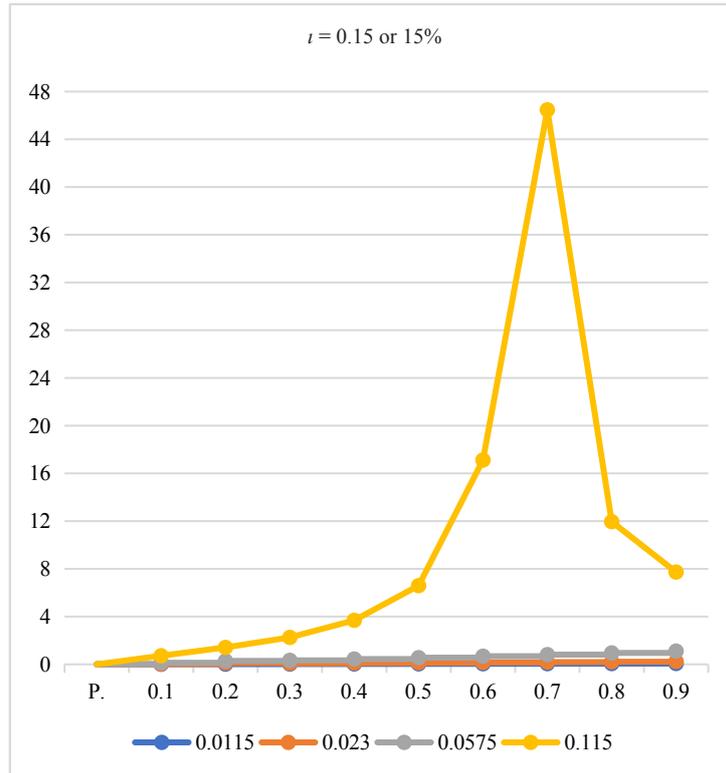
Figure 6. M_J-impact in the T-policy value computed for the Encouraged arrival ($\iota = 0.15$).

Produced in equation (34), We provide some quantitative findings for M_J-effect for $\lambda = 0.01, 0.02, 0.05$ and 0.1; $\iota = 0.05, 0.10, 0.15$; $p = 0.1$ to 0.9 (in Table 1).

The expression for (21) in the T-policy model, which we will now investigate, will be

$$M_J = \psi + \frac{\left(\lambda \times (1+\iota)/\Delta\right)^2 \left[\left\{1/\mu_1^2 + 1/\mu_2^2 + pT^2\right\} + \left\{1/\mu_1\mu_2 + pT\left(1/\mu_1 + 1/\mu_2\right)\right\}\right]}{\left(1-\psi\right)}$$

$$+ \frac{\left(1-\Delta\right)}{\Delta\left(1-\psi\right)} + \frac{\lambda \times (1+\iota)T}{\Delta}, \tag{35}$$

where $\psi = \left(\dfrac{\lambda \times (1+\iota)}{x}\right)\left(\dfrac{1}{\mu_1} + \dfrac{1}{\mu_2} + pT\right) < 1$, and when $p = 0$, then equation (25) decreases to

$$M_{J_0} = \psi_0 + \frac{\left(\lambda \times (1+\iota)/\Delta\right)^2 \left[\mu_1^{-2} + \mu_1^{-1}\mu_2^{-1} + \mu_2^{-2}\right]}{\left(1-\psi_0\right)} + \frac{\left(1-\Delta\right)}{\Delta\left(1-\psi\right)} + \frac{\lambda \times (1+\iota)T}{\Delta},$$

where $\psi_0 = \left(\dfrac{\lambda \times (1+\iota)}{\Delta}\right)\left(\dfrac{1}{\mu_1} + \dfrac{1}{\mu_2}\right) < 1$ and M_J is Length of the anticipated-service in this-system and ψ represent utlization component of the system.

$$\text{Result} = \left| M_J - M_{J_0} \right|. \tag{36}$$

We now numerical findings regarding the same data's M_J impact. i.e., $(\mu_1, \mu_2, T, \varpi) = (2.0, 1.5, 1.5, 0.25)$ for $\lambda = 0.01, 0.02, 0.05$ and 0.1; $\iota = 0.05, 0.10, 0.15$; $p = 0.1$ (in Table 4 to Table 6). Figure 1, 2 and 3 represents M_J-impact value with encouraged arrival ($\iota = 0.05, 0.10$ and 0.15) respectively. Figure 4, 5 and 6 represents M_J-impact in the T-policy value computed for the Encouraged arrival ($\iota = 0.05, 0.10$ and 0.15) respectively.

Every Table has its corresponding Figures. The effect of using the Bernoulli schedule on the no. of the system at an exit epoch is now compulsory insignificant for the reasons shown in Tables 1 and 2 above. $\lambda = 0.01, 0.02$ and 0.05, $\iota = 0.05, 0.10, 0.15$ (in model I), and $\lambda = 0.01, 0.02$, $\iota = 0.05, 0.10, 0.15$ (in model II i.e., in case of T-policy model). However, for bigger values of λ i.e., for $\lambda = 0.1$ in both models, the implications on the predicted number in the system are large. More so than the accelerated model, the model without T-policy has greater effect on the projected number in the system. It is evident that the ($\iota = 0.15$) 15% offer shown more effective results of M_J compared to 5% and 10% in both the model I and II. On the other hand, the values of Table 1 (Figure 1) to Table 3 (Figure 3) has high effect on the projected number in the system when compared with the Poisson process method Table 1 provided by [13]. As a result, our intuition was correct and it was discovered that Bernoulli schedules are essential, especially in overcrowded queues when arrivals happen more quickly than departures.

7. Conclusion

In this paper, we have investigated the effectiveness and the usefulness of Markovian single-channel Bernoulli two-vacation symmetric queue with encouraged stationary-queue-size analysis and Stochastic Markov-renewal process. Also, we have shown the effectiveness of the accelerated vacation time with the T-policy. We have introduced the encouraged arrival stationary-queue-size analysis and Stochastic Markov-renewal process with encouraged arrival see time averages. We have provided the usefulness of the Bernoulli two-vacation symmetric queue. It is evidently identified that the efficiency level increased while the encouraged arrival is incorporated. It is found that the discount 15% shown more efficient results. We gave numerical examples to show the effectiveness and the usefulness of Markovian single-channel Bernoulli two-vacation symmetric queue with encouraged stationary-queue-size analysis and Stochastic Markov-renewal process. In future, this study be extended to multi-channel with Bernoulli vacation symmetric queue along with encouraged arrival see time averages.

Conflict of interest

The authors declare no competing financial interest.

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