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Research Article

Odd and Even q-Type Lidstone Polynomial Sequences

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Abstract: In this paper, we introduce two general classes of q-type Lidstone polynomial sequences. We give some characterizations of these classes, including matrix and determinate representations, generating functions, recurrence relations, and conjugate sequences. Some illustrative examples are included.

***Keywords*:** q-Lidstone polynomials, q-difference equations, special sequences of q-polynomials, infinite matrices

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1. Introduction

The Lidstone series approximates an entire function f of exponential type less than π in a neighborhood of two points instead of one, that is

$$f(z)=\sum_{n=0}^{\infty}\left[f^{(2n)}(1)\Lambda_n(z)-f^{(2n)}(0)\Lambda_n(z-1)\right],$$

where $\{\Lambda_n(z)\}_n$ is a set of polynomials that called Lidstone polynomials (see [1, 2]). This expansion has played a key role in the theoretical and computational studies related to entire functions and applied it to high order boundary value problems (see [3–10]).

Recently, Ismail and Mansour [11] constructed a q-analog of Lidstone expansion theorem. They proved that, under certain conditions, an entire function $f(z)$ has a convergent representation as

$$f(z)=\sum_{n=0}^{\infty}\left[(D_{q^{-1}}^{2n}f)(1)A_n(z)-(D_{q^{-1}}^{2n}f)(0)B_n(z)\right],$$

where $\{A_n(z)\}_n$ and $\{B_n(z)\}_n$ are two sets of q-Lidstone polynomials defined by the generating functions

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$$\frac{E_q(zw) - E_q(-zw)}{E_q(w) - E_q(-w)} = \sum_{n=0}^{\infty} A_n(z) w^{2n}, \tag{1}$$

$$\frac{E_q(zw)E_q(-w) - E_q(-zw)E_q(w)}{E_q(w) - E_q(-w)} = \sum_{n=0}^{\infty} B_n(z) \frac{w^n}{[n]_q!}, \tag{2}$$

respectively. Moreover, they proved that the polynomial $B_n(z)$ is a constant multiplier of the q-Bernoulli polynomial of order $2n+1$. More precisely,

$$B_n(z) = \frac{2^{2n+1}}{[2n+1]_q!} B_{2n+1}(z/2;\, q), \tag{3}$$

where $\{B_n(z;\, q)\}_n$ is a set of q-Bernoulli polynomials defined by the generating function

$$\frac{t\, E_q(zt)}{E_q(t/2)e_q(t/2) - 1} = \sum_{n=0}^{\infty} B_n(z;\, q) \frac{t^n}{[n]_q!}. \tag{4}$$

Here, $E_q(z)$ and $e_q(z)$ are the q-exponential functions defined by Jackson, cf. e.g., [12, 13],

$$\begin{aligned} E_q(z) &:= \sum_{j=0}^{\infty} q^{j(j-1)/2} \frac{z^j}{[j]_q!};\ z \in \mathbb{C} \\ e_q(z) &:= \sum_{j=0}^{\infty} \frac{z^j}{[j]_q!};\ |z| < 1. \end{aligned} \tag{5}$$

In [14], Al-Towailb introduced another set of q-Lidstone polynomials $\{N_{n+1}(z)\}_n$ defined by

$$N_{n+1}(z) = \frac{2^{2n+1}}{[2n+1]_q!} E_{2n+1}(z/2;\, q), \tag{6}$$

where $\{E_n(z;\, q)\}_n$ is a set of q-Euler polynomials defined by the generating function

$$\frac{2\, E_q(zt)}{e_q(t/2)E_q(t/2) + 1} = \sum_{n=0}^{\infty} E_n(z;\, q) \frac{t^n}{[n]_q!}. \tag{7}$$

Also, the two sets $\{\nu_n(z)\}_n$ and $\{\tau_n(z)\}_n$ of complementary q-Lidstone polynomials was studied by Mansour and AL-Towailb [15], where

$$
\begin{cases}
\nu_0(z) = 1 = \tau_0(z), \\
D_{q^{-1}}\nu_n(0) = D_{q^{-1}}\tau_n(1) = 0, \\
D^2_{q^{-1}}\tau_n(z) = \tau_{n-1}(z) \text{ and } D^2_{q^{-1}}\nu_n(z) = \nu_{n-1}(z).
\end{cases} \tag{8}
$$

For more details about a q-Lidstone expansion theorem, properties, and applications of q-Lidstone polynomials, readers may refer to the literature (see [11, 14–18]).

Throughout this paper, we assume that q is a positive number less than one and $\mathbb{N}$ is the set of positive integers. We follow the notations and terminologies in [12, 19].

Our aim is to present and study general classes of polynomial sequences including (3), (6) and (8), called odd and even q-type Lidstone polynomials, respectively. For this, we consider a sequence of q-polynomials $\{L_n(z; q)\}_n$ $(n \in \mathbb{N}_0)$, which satisfies one of the following q-difference equations:

$$
D^2_q L_n(z; q) = a_n L_{n-1}(z; q) \quad \text{or} \quad D^2_{q^{-1}} L_n(z; q) = a_n L_{n-1}(z; q), \tag{9}
$$

where $a_n \in \mathbb{R}$. We say that $\{L_n(z; q)\}_n$ is a q-type Lidstone polynomial sequence, and in this situation, we write qLPS shortly.

Note that when $q \to 1$, Equation (9) reduces to

$$
\frac{d^2}{dz^2} L_n(z) = a_n L_{n-1}(z),
$$

so, we may think of qLPS as a generalization of Lidstone-type polynomial sequences studied in [20–22].

This article is organized as follows: in the next section, we define the class of odd q-type I Lidstone polynomial sequences, and we give some characterizations of this type, including matrix and determinate representations, the generating function, recurrence relation, and conjugate sequences. In Section 3, we study the class of even q-type I Lidstone polynomial sequences. We give some properties, and theoretical results related to them. In Section 4, an analogy with q-type I Lidstone polynomial sequences, we introduce odd and even q-type II Lidstone polynomial sequences, respectively. Finally, in Section 5, we give some illustrative examples.

2. Odd q-type I Lidstone polynomial sequences

In this section, we define and study the first class of q-type Lidstone polynomial sequences which satisfy the q-difference equation:

$$
D^2_q L_n(z; q) = a_n L_{n-1}(z; q), \quad a_n \in \mathbb{R}\,(n \in \mathbb{N}). \tag{10}
$$

Definition 1 The odd q-type I Lidstone sequences (qOLS-I) is a set of polynomial sequences which satisfy

$$\begin{cases} D_q^2 p_n(z;\, q) = [2n+1]_q\, [2n]_q\, p_{n-1}(z;\, q),\ n \in \mathbb{N}; \\ \\ p_n(0) = 0,\ p_0(z;\, q) = \alpha_0 z,\ \alpha_0 \in \mathbb{R}\backslash\{0\},\ n \in \mathbb{N}_0. \end{cases} \tag{11}$$

Notice, one can verify that $p_n(z;\, q)$ is a polynomial of degree $2n+1$ for each $n \in \mathbb{N}_0$.

Proposition 1 A q-type Lidstone polynomial sequence $\{p_n(z;\, q)\}_n$ is an element of the class qOLS-I if and only if there exists a sequence $(\alpha_{2k})_k$ of real numbers such that $\alpha_0 \neq 0$, and

$$\begin{aligned} p_n(z;\, q) &= \sum_{k=0}^{n} \begin{bmatrix} 2n+1 \\ 2k+1 \end{bmatrix}_q \frac{\alpha_{2k}}{[2(n-k)+1]_q} z^{2(n-k)+1} \\ &= \sum_{k=0}^{n} \begin{bmatrix} 2n+1 \\ 2k+1 \end{bmatrix}_q \frac{\alpha_{2(n-k)}}{[2(n-k)+1]_q} z^{2k+1}. \end{aligned} \tag{12}$$

Proof. Let $\{p_n(z;\, q)\}_n \in q$OLS-I. Then, there exists a constant $\alpha_0 \neq 0$ such that $p_0(z;\, q) = \alpha_0 z$. Therefore,

$$D_q^2 p_1(z;\, q) = c_1 z, \quad c_1 = [2]_q [3]_q\, \alpha_0,$$

and then $D_q p_1(z;\, q) = c_2 z^2 + \alpha_2\, (c_2,\, \alpha_2 \in \mathbb{R})$. This implies $D_q p_1(0) = \alpha_2$, and by induction we can set

$$D_q p_n(0;\, q) = \alpha_{2n}, \quad \alpha_{2n} \in \mathbb{R}. \tag{13}$$

Now, assume that $p_n(z;\, q) = \sum_{k=0}^{n} \alpha_k^{(n)} z^{2(n-k)+1}$. Then,

$$D_q^2 p_n(z;\, q) = \sum_{k=0}^{n-1} \alpha_k^{(n)} [2(n-k)+1]_q [2(n-k)]_q\, z^{2(n-k)-1}. \tag{14}$$

According to the q-difference equation in (11), we have

$$D_q^2 p_n(z;\, q) = [2n+1]_q\, [2n]_q \sum_{k=0}^{n-1} \alpha_k^{(n-1)}\, z^{2(n-k)-1}. \tag{15}$$

From (14) and (15), we get

$$\prod_{n=k+1}^{m} \frac{\alpha_k^{(n)}}{\alpha_k^{(n-1)}} = \prod_{n=k+1}^{m} \frac{[2n+1]_q [2n]_q}{[2(n-k)+1]_q [2(n-k)]_q}.$$

Consequently, we obtain

$$\alpha_k^{(m)} = \frac{[2m+1]_q!}{[2(m-k)+1]_q![2k+1]_q!}\,\alpha_k^{(k)} = \begin{bmatrix} 2m+1 \\ 2k+1 \end{bmatrix}_q \frac{\alpha_k^{(k)}}{[2(m-k)+1]_q}, \tag{16}$$

where $\alpha_k^{(k)}$ is the coefficient of z in $p_k(z;\,q)$. Using (13), we can replace $\alpha_k^{(k)}$ by α_{2k} and then we get the result in (12). On the other hand, if (12) is satisfied and $\alpha_0 \neq 0$, we get easily (11) which complete the proof. □

Remark 1 From (12), we obtain

i. $p_n(z;\,q)$ an odd function for each $n \in \mathbb{N}$;

ii. $\{z^{2n+1}\}_n \in q$OLS-I and qOLS-I$\subset \tilde{P}$, where $\tilde{P} = \text{span}\{z^{2j+1} \mid j \in \mathbb{N}\}$;

iii. $\int_0^1 p_n(z;\,q)\,d_q z = [2n+1]_q! \sum_{k=0}^{n} \frac{\alpha_{2(n-k)}}{[2k+2]!\,[2(n-k)+1]_q!},\ n \in \mathbb{N}.$

Proposition 2 Let $n \in \mathbb{N}$ and $\{p_n(z;\,q)\}_n \in q$OLS-I. Then

(1) $D_q^{2m} p_n(z;\,q) = \frac{[2n+1]_q!}{[2(n-m)+1]_q!}\, p_{n-m}(z;\,q), \quad m = 1, 2, \ldots, n;$

(2) $D_q^{2m+1} p_n(z;\,q) = \frac{[2n+1]_q!}{[2(n-m)+1]_q!}\, D_q p_{n-m}(z;\,q), \quad m = 1, 2, \ldots, n;$

(3) $D_q^{2m} p_n(0) = 0;\ D_q^{2m+1} p_n(0) = \frac{[2n+1]_q!}{[2(n-m)+1]_q!}\,\alpha_{2(n-m)}, \quad m = 1, 2, \ldots, n.$

Proof. The proof follows immediately from (11) and (12) by induction. □

2.1 *Matrix form*

Recall that a matrix $M = [m_{ij}]_{i,\,j\geq 0}$ is infinite lower triangular if $m_{ij} = 0$ whenever $j > i$. We denote by $\mathscr{L}$ the set of all lower triangular matrices.

A matrix $T = [a_{ij}]_{i,\,j\geq 0}$ is a Toeplitz if and only if $T \in \mathscr{L}$ and $a_{ij} = a_{i-j}$ for $i \geq j$. That is,

$$T = \begin{bmatrix} & \vdots & \vdots & \vdots & \vdots & \\ \cdots & a_0 & 0 & 0 & 0 & \cdots \\ \cdots & a_1 & a_0 & 0 & 0 & \cdots \\ \cdots & a_2 & a_1 & a_0 & 0 & \cdots \\ \cdots & a_3 & a_2 & a_1 & a_0 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}.$$

We need the following results from [23].

Lemma 1 Let $A,\,B \in \mathscr{L}$. Then the product $AB = [c_{ij}]_{i,\,j\geq 0}$ is well-defined, and

$$c_{ik} = \sum_{j=k}^{i} a_{ij} b_{jk},\ i \geq k.$$

Lemma 2 Let T_a be a Toeplitz matrix in $\mathscr{L}$ defined by

$$T_a = [a_{ij}]_{i,\, j\geq 0} := \begin{cases} a_{i-j}, & i \geq j; \\ 0, & i < j. \end{cases}$$

If $(T_a)^{-1} = T_b := [b_n]\,(n = i - j)$, then

$$b_n = \frac{(-1)^n}{a_0^{n+1}} \det \begin{bmatrix} a_1 & a_0 & & & \\ a_2 & a_1 & a_0 & & \\ \vdots & \vdots & \vdots & \ddots & \\ a_{n-1} & a_{n-2} & a_{n-3} & \dots & a_0 \\ a_n & a_{n-1} & a_{n-2} & \dots & a_1 \end{bmatrix}.$$

Definition 2 The odd q-type I Lidstone matrix $A_q = [a_{ij}]_{i,\, j\geq 0}$ is an infinite lower triangular matrix with

$$a_{ij} = \begin{bmatrix} 2i+1 \\ 2j+1 \end{bmatrix}_q \frac{\alpha_{2(i-j)}}{[2(i-j)+1]_q}\ (i \geq j), \tag{17}$$

where $(\alpha_{2k})_k$ a sequence of real numbers and $\alpha_0 \neq 0$.

Remark 2 According to Definition 2, Formula (12) can be written in the following matrix form

$$P_q = A_q Z_q, \tag{18}$$

where P_q and Z_q are two vectors defined by

$$P_q = [p_0(z;\, q),\, p_1(z;\, q),\, \dots,\, p_n(z;\, q),\, \dots]^T, \quad Z_q = [z,\, z^3,\, \dots,\, z^{2n+1},\, \dots]^T.$$

Moreover, if we set $A_{q,\, n} = [a_{ij}]$ such that $j = 0, 1, ..., i$, $i = 0, 1, ..., n$, $n \in \mathbb{N}$, and a_{ij} defined in (17), then we have a sequence $(A_{q,\, n})_n$ of the principle submatrices of order n of A_q which satisfy

$$P_{q,\, n} = A_{q,\, n} Z_{q,\, n}, \tag{19}$$

where

$$P_{q,\, n} = [p_0(z;\, q),\, p_1(z;\, q),\, \dots,\, p_n(z;\, q)]^T, \quad Z_{q,\, n} = [z,\, z^3,\, \dots,\, z^{2n+1}]^T. \tag{20}$$

Remark 3 From (20), we get

$$D_q^2 P_{q,\,n} = [D_q^2 p_0,\, D_q^2 p_1,\, \ldots,\, D_q^2 p_n]^T.$$

So, if we denote by $\mathscr{C} = [c_{ij}]$ $(i,\, j = 0,\, 1,\, 2,\, \ldots,\, n)$ to the derivation matrix for $P_{q,\,n}$, i.e,

$$D_q^2 P_{q,\,n} = \mathscr{C} P_{q,\,n} \quad (n \in \mathbb{N}_0),$$

then, according to (11), we obtain

$$c_{ij} = \begin{cases} [2i+1]_q [2i]_q, & i = j+1; \\ \\ 0, & \text{otherwise}. \end{cases}$$

We define a q-type Toeplitz matrix $T_{q\alpha}$ as the matrix in $\mathscr{L}$ whose $(i,\, j)$ entry is

$$t_{ij}^{\alpha} = \frac{\alpha_{2(i-j)}}{[2(i-j)+1]_q!} \quad \text{for} \quad i \geq j,$$

and zero otherwise. Also, we denote by $\mathscr{D}$ the diagonal matrix with entries $d_{ii} = [2i+1]_q!$.

Proposition 3 The odd q-type I Lidstone matrix can be factorized as

$$A_q = \mathscr{D} T_{q\alpha} \mathscr{D}^{-1}. \tag{21}$$

Proof. According to Lemma 1, the product $\mathscr{D} T_{q\alpha} \mathscr{D}^{-1}$ is well-defined and we easily get the result. □

Proposition 4 The odd q-type I Lidstone matrix A_q is invertible and

$$(A_q)^{-1} = \mathscr{D} T_{q\beta} D^{-1}, \tag{22}$$

where $(\beta_{2n})_n$ is the numerical sequence satisfying

$$\sum_{j=0}^{n} \frac{\beta_{2j}\alpha_{2(n-j)}}{[2j+1]_q![2(n-j)+1]_q!} = \delta_{n0}, \tag{23}$$

with δ_{nj} the Kronecker's delta.

Proof. It follows directly by using (21) and calculating $(T_{q\alpha})^{-1}$ from the result in Lemma 2. □

Remark 4 Equation (23) can be considered as an infinite linear system which determines the numerical sequence $(\beta_{2n})_n$. According to Cramer's rule, the first $m+1$ equations in (23) give

$$\beta_0 = \frac{1}{\alpha_0},$$

$$\beta_{2n} = (-1)^n \frac{[3]_q![5]_q!\dots[2n+1]_q!}{\alpha_0^{n+1}}$$

$$\times \det \begin{bmatrix} \frac{\alpha_2}{[3]_q!} & \frac{\alpha_0}{[3]_q!} & 0 & \dots & 0 \\ \frac{\alpha_4}{[5]_q!} & \frac{\alpha_2}{[3]_q![3]_q!} & \frac{\alpha_0}{[5]_q!} & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\alpha_{2(n-1)}}{[2n-1]_q!} & \frac{\alpha_{2(n-2)}}{[2n-3]_q![3]_q!} & \frac{\alpha_{2(n-3)}}{[2n-5]_q![5]_q!} & \dots & \frac{\alpha_0}{[2n-1]_q!} \\ \frac{\alpha_{2n}}{[2n+1]_q!} & \frac{\alpha_{2(n-1)}}{[2n-1]_q![3]_q!} & \frac{\alpha_{2(n-2)}}{[2n-3]_q![5]_q!} & \dots & \frac{\alpha_2}{[2n-1]_q![3]_q!} \end{bmatrix}, \tag{24}$$

$n = 1, 2, \dots, m$.

Now, we consider the polynomials

$$\hat{p}_n(z;\, q) = \sum_{k=0}^{n} \begin{bmatrix} 2n+1 \\ 2k+1 \end{bmatrix}_q \frac{\beta_{2(n-k)}}{[2(n-k)+1]_q} z^{2k+1} \ (n \in \mathbb{N}_0), \tag{25}$$

where $(\beta_{2n})_n$ is defined as in (24). Note that $\{\hat{p}_n(z;\, q)\}_n \in q$OLS-I.

Definition 3 The two sequences $\{p_n(z;\, q)\}_n$ and $\{\hat{p}_n(z;\, q)\}_n$ defined in (12) and (25), respectively, are called conjugate odd q-type I Lidstone sequences.

We denote $B_q = [b_{ij}]_{i,\, j\geq 0}$ the infinite lower triangular matrix with

$$b_{ij} = \begin{bmatrix} 2i+1 \\ 2j+1 \end{bmatrix}_q \frac{\beta_{2(i-j)}}{[2(i-j)+1]_q}, \quad i \geq j.$$

Set $\widehat{P}_q = [\hat{p}_0(z;\, q),\, \hat{p}_1(z;\, q),\, \dots,\, \hat{p}_n(z;\, q),\, \dots]^T$. Then, we have the matrix forms

$$\widehat{P}_q = B_q Z_q, \tag{26}$$

and for $n \in \mathbb{N}$,

$$\widehat{P}_{q,\, n} = B_{q,\, n} Z_{q,\, n}. \tag{27}$$

Proposition 5 The two sequences $\{p_n(z;\, q)\}_n$ and $\{\hat{p}_n(z;\, q)\}_n$ are conjugate odd q-type I Lidstone sequences if and only if

$$P_q = A_q^2 \widehat{P}_q, \quad \widehat{P}_q = B_q^2 P_q,$$

and for $n \in \mathbb{N}$,

$$P_{q,\,n} = A_{q,\,n}^2 \widehat{P}_{q,\,n}, \quad \widehat{P}_{q,\,n} = B_{q,\,n}^2 P_{q,\,n}.$$

Proof. The proof follows directly from (18), (19), (26) and (27), and taking into account $A_q^{-1} = B_q$. □

Remark 5 From Proposition 5, we can write

$$p_n(z;\,q) = \sum_{k=0}^{n} \tilde{a}_{nk}\, \hat{p}_k(z;\,q) \quad \text{and} \quad \hat{p}_n(z;\,q) = \sum_{k=0}^{n} \tilde{b}_{nk}\, p_k(z;\,q),$$

where $\tilde{a}_{nk}$ and $\tilde{b}_{nk}$ $(k = 0, \ldots, n,\ n \in \mathbb{N}_0)$ are elements of the matrices $A_{q,\,n}^2$ and $B_{q,\,n}^2$, respectively.

2.2 *Recurrence relations and q-difference equations*

We start by deriving some recurrence relations for a sequence of odd q-type I Lidstone polynomials.

Theorem 1 Let $\{p_n(z;\,q)\}_n \in q$OLS-I. Then

$$p_n(z;\,q) = \frac{1}{\beta_0}\left[z^{2n+1} - \sum_{k=0}^{n-1} \begin{bmatrix} 2n+1 \\ 2k+1 \end{bmatrix}_q \frac{\beta_{2(n-k)}}{[2(n-k)+1]_q}\, p_k(z;\,q)\right], \tag{28}$$

where $(\beta_{2n})_n$ is defined as in (24).

Proof. Assume that A_q is an odd q-type I Lidstone matrix. Then, from Equation (19) we have

$$Z_{q,\,n} = B_{q,\,n} P_{q,\,n}, \tag{29}$$

where $Z_{q,\,n}$ and $P_{q,\,n}$ are defined as in (20), and $B_{q,\,n} = A_{q,\,n}^{-1}$. Therefore, we obtain

$$z^{2n+1} = \sum_{k=0}^{n} \begin{bmatrix} 2n+1 \\ 2k+1 \end{bmatrix}_q \frac{\beta_{2(n-k)}}{[2(n-k)+1]_q}\, p_k(z;\,q)\ (k = 0,\, 1,\, \ldots,\, n), \tag{30}$$

and then we get the result. □

Note that Relation (30) can be considered as infinite linear system in unknowns polynomials $p_n(z;\,q)$ $(n \in \mathbb{N}_0)$. In the following theorem, we use Cramer's rule to solve the first $m+1$ equations $(m = 0,\, 1,\, \ldots,\, n)$ of (30). Then, we obtain a first determinate form of odd q-type I Lidstone sequences.

Theorem 2 Let $\{p_n(z;\,q)\}_n \in q$OLS-I. Then

$$p_0(z;\, q) = \frac{1}{\beta_0} z,$$

$$p_n(z;\, q) = \frac{(-1)^n}{[3]_q![5]_q!\dots[2n-1]_q!\,\beta_0^{n+1}}$$

$$\times \det \begin{bmatrix} z & z^3 & z^5 & \dots & z^{2n-1} & z^{2n+1} \\ \beta_0 & \beta_2 & \beta_4 & \dots & \beta_{2(n-1)} & \beta_{2n} \\ 0 & [3]_q!\beta_0 & \frac{[5]_q!}{[3]_q!}\beta_2 & \dots & \frac{[2n-1]_q!}{[2n-3]_q!}\beta_{2(n-2)} & \frac{[2n+1]_q!}{[2n+3]_q!}\beta_{2(n-1)} \\ \vdots & \ddots & \ddots & & \vdots & \vdots \\ \vdots & \ddots & \ddots & & \vdots & \vdots \\ 0 & \dots & \dots & & [2n-1]_q!\beta_0 & \frac{[2n+1]_q!}{[3]_q!}\beta_2 \end{bmatrix}, \tag{31}$$

where $(\beta_{2n})_n$ is defined as in (24).

The following theorem gives a recurrence relation for the sequence $\{\hat{p}_n(z;\, q)\}_n$.

Theorem 3 Let $\{p_n(z;\, q)\}_n$ be an odd q-type I Lidstone sequence. Then, the conjugate sequence $\{\hat{p}_n(z;\, q)\}_n$ satisfies the recursive relation

$$\hat{p}_n(z;\, q) = \frac{1}{\alpha_0}\left[z^{2n+1} - \sum_{k=0}^{n-1} \begin{bmatrix} 2n+1 \\ 2k+1 \end{bmatrix}_q \frac{\alpha_{2(n-k)}}{[2(n-k)+1]_q} \hat{p}_k(z;\, q)\right], \tag{32}$$

where $(\alpha_{2n})_n$ is defined as in (12). Moreover, $\{\hat{p}_n(z;\, q)\}_n$ can be expressed in a determinate form similar to (31) with α_{2k} instead of β_{2k}, for $k = 0,\, 1,\, ...,\, n$ and $n \in \mathbb{N}_0$.

Proof. The proof is similar to the proof of Theorem 1 and is omitted. □

Now, we determine another recurrence relation by using the production matrix. Recall that the production matrix Π_A of a nonsingular infinite lower triangular matrix A, defined by

$$\Pi_A = A^{-1}\bar{A},$$

where $\bar{A}$ is the matrix A with its first row removed (see [24]).

Proposition 6 Let A be an infinite lower triangular matrix, and B be the inverse matrix of A. Then, the production matrices Π_A and Π_B of A and B, respectively, satisfy

$$\Pi_B A = AD \quad \text{and} \quad \Pi_A B = BD, \tag{33}$$

where $D = [\delta_{(i+1)j}]_{i,\, j\geq 0}$ and δ_{ij} is the Kronecker's delta.

Lemma 3 Let $A_q = [a_{ij}]_{i,\, j\geq 0}$ be an odd q-type I Lidstone matrix, $B_q = [b_{ij}]_{i,\, j\geq 0}$ be the inverse of A_q, and $\Pi_B = [\pi_{ij}]_{i,\, j\geq 0}$ be the production matrix of B_q. Then

$$\pi_{ij} = \sum_{n=0}^{i} a_{in} b_{(n+1)j}$$

$$= \begin{cases} \alpha_0 \beta_2, & i = j = 0, \\ 0, & j > i+1, \\ \displaystyle\sum_{n=0}^{i-j+1} \begin{bmatrix} 2i+1 \\ 2(n+j)-1 \end{bmatrix}_q \frac{\beta_{2n} \alpha_{2(i-j-n)+2} [2(n+j)+1]_q!}{(2(i-j-n)+3)[2j+1]_q! [2n+1]_q!}, & \text{otherwise}, \end{cases} \tag{34}$$

where $(\alpha_{2n})_n$ and $(\beta_{2n})_n$ are numerical sequences defined as in (23).

Proof. From (33), we have $\Pi_B = ADB$. Thus, $\pi_{ij} = \sum_{n=0}^{i} a_{in} b_{(n+1)j}$, and by Proposition 4, we get the result. □

Theorem 4 Let $\{p_n(z;\, q)\}_n \in q$OLS-I. If A_q an odd q-type I Lidstone matrix related to $\{p_n(z;\, q)\}_n$, and $\Pi_q = [\pi_{ij}]_{i,\, j \geq 0}$ is the production matrix of A_q^{-1}, then

$$\begin{aligned} p_0(z;\, q) &= \frac{1}{\beta_0} z, \\ p_{n+1}(z;\, q) &= \frac{1}{\pi_{n(n+1)}} \left[z^2 p_n(z;\, q) - \sum_{k=0}^{n} \pi_{nk} p_k(z;\, q) \right] \; (n \in \mathbb{N}_0). \end{aligned} \tag{35}$$

Proof. From (18) and (33), we have $\Pi_q P_q = A_q (D Z_q)$. Since $D Z_q = [z^3,\, z^5,\, \ldots]^T = z^2 Z_q$, we obtain

$$\Pi_q P_q = z^2 A_q Z_q = z^2 P_q. \tag{36}$$

Consider the $(n+1)$th equations of (36), we have $\sum_{k=0}^{n+1} \pi_{nk} p_k(z;\, q) = z^2 p_n(z;\, q)$. Hence, after some calculations, we get (35). □

Theorem 5 Let $\{p_n(z;\, q)\}_n \in q$OLS-I. Then

$$p_0(z;\, q) = \frac{1}{\beta_0} z,$$

$$p_{n+1}(z;q) = \frac{(-1)^{n+1} p_0(z;q)}{\pi_{01}\pi_{12}\dots\pi_{n(n+1)}}$$

$$\times \det \begin{bmatrix} \pi_{00} - z^2 & \pi_{01} & 0 & \dots & \dots & 0 \\ \pi_{10} & \pi_{11} - z^2 & \pi_{12} & \dots & \dots & 0 \\ \pi_{20} & \pi_{21} & \pi_{22} - z^2 & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \pi_{(n-1)n} \\ \pi_{n0} & \pi_{n1} & \pi_{n2} & \dots & \dots & \pi_{nn} - z^2 \end{bmatrix},$$

where π_{ij} are defined as in (34).

Proof. According to Theorem 4, we have the linear system (36) which can be expressed in a matrix form as

$$\begin{bmatrix} \pi_{01} & 0 & 0 & 0 & \dots \\ \pi_{11} - z^2 & \pi_{12} & 0 & 0 & \dots \\ \pi_{21} & \pi_{22} - z^2 & \pi_{23} & 0 & \dots \\ \pi_{31} & \pi_{32} & \pi_{33} - z^2 & \pi_{34} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ \vdots \end{bmatrix} = p_0 \begin{bmatrix} z^2 - \pi_{00} \\ -\pi_{10} \\ -\pi_{20} \\ -\pi_{30} \\ \vdots \end{bmatrix}.$$

By using Cramer's rule, we get the solution of first $n+1$ equations and then, we obtain the result. □

We end this section by proving that the odd q-type I Lidstone sequences satisfy some of q-difference equations.

Theorem 6 Let $\{p_n(z;q)\}_n$ be an odd q-type I Lidstone sequence. Then, it satisfies the following linear q-difference equations

$$\sum_{k=0}^{n} \frac{\beta_{2k}}{[2k+1]_q!} D_q^{2k} u(z) - z^{2n+1} = 0, \tag{37}$$

$$\sum_{k=1}^{n} \frac{[2k+1]_q!}{[2n+1]_q!} \pi_{nk} D_q^{2(n-k+1)} u(z) - D_q^2(z^2 u(z)) + [2n+3]_q [2n+2]_q \pi_{n(n+1)} u(z) = 0. \tag{38}$$

Proof. From Proposition 2, we have

$$D_q^{2k} p_n(z;q) = \frac{[2n+1]_q!}{[2(n-k)+1]_q!} p_{n-k}(z;q). \tag{39}$$

Substituting (39) into (28), we obtain

$$p_{n+1}(z;q) = \frac{1}{\beta_0} \Big[z^{2n+3} - \sum_{k=0}^{n} \frac{[2n+3]_q [2n+2]_q}{[2k+3]_q!} \beta_{2k+2} D_q^{2k} p_n(z;q) \Big].$$

Therefore,

$$D_q^2 p_{n+1}(z;q) = \frac{1}{\beta_0}\Big[[2n+3]_q[2n+2]_q z^{2n+1} - \sum_{k=0}^{n} \frac{[2n+3]_q[2n+2]_q}{[2k+3]_q!}\beta_{2k+2} D_q^{2k+2} p_n(z;q)\Big].$$

Since

$$D_q^2 p_{n+1}(z;q) = [2n+2]_q[2n+3]_q p_n(z;q) \text{ and } D_q^{2n+2} p_n(z;q) = 0,$$

we obtain (37). On the other hand, from Theorem 4, we have

$$\pi_{n(n+1)} p_{n+1}(z;q) = z^2 p_n(z;q) - \sum_{k=0}^{n} \pi_{nk} p_k(z;q). \tag{40}$$

By taking the second q-derivative of (40), we obtain

$$[2n+3]_q[2n+2]_q \pi_{n(n+1)} p_n(z;q) = D_q^2(z^2 p_n(z;q)) - \sum_{k=1}^{n} [2k+1]_q[2k]_q \pi_{nk} p_{k-1}(z;q). \tag{41}$$

According to Proposition 2, we get

$$D_q^{2(n-k+1)} p_n(z;q) = \frac{[2n+1]_q!}{[2k-1]_q!} p_{k-1}(z;q). \tag{42}$$

Substituting (42) into (41) yields (38) and completes the proof. □

2.3 *Generating function*

Our aim here is to get the generating function of odd q-type I Lidstone sequences.
Recall that the q-trigonometric functions

$$\mathrm{Sin}_q(z) := \frac{E_q(iz) - E_q(-iz)}{2i}, \quad \sin_q(z) = \frac{e_q(iz) - e_q(-iz)}{2i},$$

$$\mathrm{Cos}_q(z) := \frac{E_q(iz) + E_q(-iz)}{2}, \quad \cos_q(z) = \frac{e_q(iz) + e_q(-iz)}{2i},$$

where $E_q(z)$ and $e_q(z)$ are the q-exponential functions defined in (5). The q-analog of hyperbolic functions $\sinh z$ and $\cosh z$ are defined by

$$\sinh_q(z) := -i\sin_q(iz) = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{[2n+1]_q!},$$

$$\cosh_q(z) := \cos_q(iz) = \sum_{n=0}^{\infty} \frac{z^{2n}}{[2n]_q!}. \tag{43}$$

Moreover, $\mathrm{Sinh}_q(z) = \sinh_{1/q}(z)$ and $\mathrm{Cosh}_q(z) = \cosh_{1/q}(z)$.

Let $\{p_n(z;\, q)\}_n$ be the odd q-type I Lidstone sequence related to the numerical sequence $(\alpha_{2n})_n$, and consider the following power series

$$g_q(t) = \sum_{n=0}^{\infty} \frac{\alpha_{2n}}{[2n+1]_q!} t^{2n}. \tag{44}$$

Lemma 4 Let $g_q(t)$ be the power series defined in (44). Then $\frac{1}{g_q(t)}$ is a well-defined function, and it has the series representation

$$\frac{1}{g_q(t)} = \sum_{n=0}^{\infty} \frac{\beta_{2n}}{[2n+1]_q!} t^{2n}, \tag{45}$$

where $(\beta_{2n})_n$ is defined as in (24).

Proof. Since $\alpha_0 \neq 0$, $g_q(t)$ is invertible. To get Equation (45), we prove that

$$\left(\sum_{n=0}^{\infty} \frac{\alpha_{2n}}{[2n+1]_q!} t^{2n}\right) \left(\sum_{n=0}^{\infty} \frac{\beta_{2n}}{[2n+1]_q!} t^{2n}\right) = 1.$$

By using the Cauchy product for power series and Equation (23), we obtain

$$\left(\sum_{n=0}^{\infty} \frac{\alpha_{2n}}{[2n+1]_q!} t^{2n}\right) \left(\sum_{n=0}^{\infty} \frac{\beta_{2n}}{[2n+1]_q!} t^{2n}\right)$$

$$= \sum_{n=0}^{\infty} t^{2n} \sum_{k=0}^{n} \frac{\beta_{2k}}{[2k+1]_q!} \frac{\alpha_{2(n-k)}}{[2(n-k)+1]_q!}$$

$$= 1,$$

which implies the result. □

Theorem 7 Let $\{p_n(z;\, q)\}_n$ and $\{\hat{p}_n(z;\, q)\}_n$ be the conjugate odd q-type I Lidstone sequences. Then

$$g_q(t) \frac{\sinh_q(zt)}{t} = \sum_{n=0}^{\infty} p_n(z;\, q) \frac{t^{2n}}{[2n+1]_q!}, \tag{46}$$

$$\frac{1}{g_q(t)}\frac{\sinh_q(zt)}{t}=\sum_{n=0}^{\infty}\hat{p}_n(z;\,q)\frac{t^{2n}}{[2n+1]_q!}, \tag{47}$$

where $g_q(t)$ is defined as in (44).

Proof. From Equation (30), we have

$$\sum_{k=0}^{n}\begin{bmatrix}2n+1\\2k+1\end{bmatrix}_q\frac{\beta_{2(n-k)}}{[2(n-k)+1]_q}p_k(z;\,q)=z^{2n+1}. \tag{48}$$

Multiplying both sides of (48) by $\frac{t^{2n+1}}{[2n+1]_q!}$ and adding on n, we obtain

$$\sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n}\begin{bmatrix}2n+1\\2k+1\end{bmatrix}_q\frac{\beta_{2(n-k)}}{[2(n-k)+1]_q}p_k(z;\,q)\Big)\frac{t^{2n+1}}{[2n+1]_q!}=\sum_{n=0}^{\infty}\frac{(zt)^{2n+1}}{[2n+1]_q!}.$$

Therefore,

$$t\sum_{n=0}^{\infty}\beta_{2n}\frac{t^{2n}}{[2n+1]_q!}\sum_{n=0}^{\infty}p_n(z;\,q)\frac{t^{2n}}{[2n+1]_q!}=\sinh_q(zt).$$

By using Lemma 4, we obtain (46).

Similarly, from Equation (32), we can derive the generating function of $\{\hat{p}_n(z;\,q)\}_n$ and get Equation (47). □

Example 1 Ismail and Mansour [11, Eq.(3.37)] introduced the identity

$$\sum_{k=0}^{n}(-1)^k2^{2k}\frac{\beta_{2k}(q)}{[2k]_q!}\frac{T_{2n-2k+1}(q)}{[2n-2k+1]_q!}=\delta_{n0}, \tag{49}$$

where δ_{n0} the Kroncker's delta function, $\beta_n(q)$ the q-Bernoulli number, and $(T_n)_n$ a sequence of tangent numbers that defined by

$$\mathrm{Tan}_qt=\tan_q t=\sum_{n=0}^{\infty}T_{2n+1}\frac{t^{2n+1}}{[2n+1]_q!}. \tag{50}$$

It is worth noting that there was a small typo in [11] [Eq.(3.37)], which we have corrected in (49).

We define $(\alpha_{2j})_j$ and $(\beta_{2j})_j$ by

$$\alpha_{2j}=T_{2j+1},\quad \beta_{2j}=(-1)^j2^{2j}[2j+1]_q\beta_{2j}(q).$$

Then, the two sequences $\{p_n(z;\,q)\}_n$ and $\{\hat{p}_n(z;\,q)\}_n$:

$$p_n(z;q) \quad := \quad \sum_{k=0}^{n} \begin{bmatrix} 2n \\ 2k \end{bmatrix}_q \frac{T_{2n-2k+1}}{[2n-2k+1]_q} z^{2k+1},$$

$$\hat{p}_n(z;q) \quad := \quad \sum_{k=0}^{n} \begin{bmatrix} 2n+1 \\ 2k+1 \end{bmatrix}_q (-1)^k 2^{2k} \beta_{2k} z^{2n-2k+1}$$

are conjugate odd q-type I Lidstone sequences. By using (46) and (50), one can verify that

$$\frac{\tan_q t}{t} \frac{\sinh_q(zt)}{t} = \sum_{n=0}^{\infty} p_n(z;q) \frac{t^{2n}}{[2n+1]_q!}.$$

On the other hand, since

$$t \operatorname{Cot}_q t = t \cot_q t = \sum_{n=0}^{\infty} (-1)^n \beta_{2n}(q) \frac{t^{2n}}{[2n]_q!}$$

(see [11]), we obtain

$$t \coth_q t \frac{\sinh_q(zt)}{t} = \sum_{n=0}^{\infty} p_n(z;q) \frac{t^{2n}}{[2n+1]_q!}.$$

2.4 *Relationship with q-Appell polynomial sequences*

Recall that, for $n \in \mathbb{N}_0$, an Appell polynomial $a_n(z)$ is a polynomial that has the following series representation

$$a_n(z) = \sum_{k=0}^{n} \binom{n}{k} a_{n-k}(0) z^k.$$

Or equivalently, it defined by the generating function

$$g(t) e^{zt} = \sum_{n=0}^{\infty} a_n(z) \frac{t^n}{n!},$$

where $g(t) := \sum_{n=0}^{\infty} a_n(0) \frac{t^n}{n!}$, $|t| \leq M$ for some $M > 0$ (see [25]).

The sequence $\{a_n(z;q)\}_n$ of q-Appell polynomials was introduced by Sharma, Chak and Al-Salam [26, 27]. They defined $a_n(z;q)$ by the generating function

$$A_q(t) E_q(zt) = \sum_{n=0}^{\infty} a_n(z;q) \frac{t^n}{[n]_q!},$$

where $E_q(z)$ the exponential function defined in (5), and $A_q(t)$ is the determining function for $\{a_n(z;\, q)\}_n$ which given by

$$A_q(t) = \sum_{n=0}^{\infty} a_n \frac{t^n}{[n]_q!} \; (a_n \in \mathbb{R}).$$

Furthermore, the q-Appell polynomials satisfy the following properties:

1. $a_0(z;\, q) \neq 0$;
2. $D_q a_n(z;\, q) = [n]_q a_{n-1}(z;\, q)$;
3. $a_n(0;\, q) = a_n,\; n \in \mathbb{N}_0$.

The following result gives a characterization of q-Appell polynomial sequence (see [28]):

Proposition 7 The sequence $\{a_n(z;\, q)\}_n$ is a q-Appell polynomial sequence if and only if there exists a numerical sequence $(a_k)_k$, independent of n, such that $a_0 \neq 0$ and

$$a_n(z;\, q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{n-k}{2}} a_k z^{n-k}.$$

In the following, we establish a relationship between odd q-type I Lidstone sequences and q-Appell polynomial sequences.

Theorem 8 Let $\{a_n(z;\, q)\}_n$ be a sequence of q-Appell polynomials. If $a_{2n+1}(0;\, q) = 0$ $(n \in \mathbb{N})$, then the sequence $\{f_n(z;\, q)\}_n$, where

$$f_n(z;\, q) \colon = 2^{2n+1} a_{2n+1}(\frac{z}{2};\, q) \quad (n \in \mathbb{N}) \tag{51}$$

is an odd q-type I Lidstone sequence.

Proof. Since $\{a_n(z;\, q)\}_n$ is a sequence of q-Appell polynomials, it satisfies

$$D_q a_n(z;\, q) = [n]_q a_{n-1}(z;\, q) \tag{52}$$

By using the assumption $a_{2n+1}(0;\, q) = 0$ with Equation (52), we conclude that the functions $f_n(z;\, q)$ satisfy (11). This implies $\{f_n(z;\, q)\}_n \in q$OLS-I. □

3. Even q-type I Lidstone polynomial sequences

We study the sequence of polynomials $\{\omega_n(z;\, q)\}_n$ which satisfy

$$\begin{cases} D_q^2 \omega_n(z;\, q) = [2n]_q\, [2n-1]_q\, \omega_{n-1}(z;\, q), \\ D_q \omega_n(0;\, q) = 0 \; (n \in \mathbb{N}_0), \quad \omega_0(z;\, q) = \gamma_0,\; \gamma_0 \in \mathbb{R} \backslash \{0\}. \end{cases} \tag{53}$$

In this case, $\{\omega_n(z;\, q)\}_n$ is an element of even q-type I Lidstone sequences (qELS-I).

Remark 6 From (53), one can verify that $\omega_n(z;\, q)$ is a polynomial of degree $2n$ for each $n \in \mathbb{N}_0$.

Throughout this section, the proofs are omitted because they follow along similar lines as the corresponding proofs of Section 2.

Proposition 8 The sequence $\{\omega_n(z; q)\}_n \in q$ELS-I if and only if there exists a sequence $(\gamma_{2k})_k$ of real numbers such that $\gamma_0 \neq 0$ and

$$\omega_n(z; q) = \sum_{k=0}^{n} \begin{bmatrix} 2n \\ 2k \end{bmatrix}_q \gamma_{2(n-k)} z^{2k}, \quad n \in \mathbb{N}. \tag{54}$$

Remark 7 From (54), we obtain

i. $\omega_n(z; q)$ is an even function for all $n \in \mathbb{N}$;

ii. $\{z^{2n}\}_n \in qELS$ and $qELS \subset \hat{P}$, where $\hat{P} := \text{span}\{z^{2j} | j \in \mathbb{N}\}$;

iii. $\displaystyle\int_0^1 \omega_n(z; q)\, d_q z = \sum_{k=0}^{n} \begin{bmatrix} 2n \\ 2k \end{bmatrix}_q \frac{\gamma_{2(n-k)}}{[2k+1]_q}, \quad n \in \mathbb{N}.$

Proposition 9 Let $n \in \mathbb{N}$ and $\{\omega_n(z; q)\}_n \in q$ELS-I. Then

1. $D_q^{2m} \omega_n(z; q) = \frac{[2n]_q!}{[2(n-m)]_q!} \omega_{n-m}(z; q), \ m = 0, 1, \ldots, n;$
2. $D_q^{2m+1} \omega_n(z; q) = \frac{[2n]_q!}{[2(n-m)]_q!} D_q \omega_{n-m}(z; q), \ m = 0, 1, ..., n-1;$
3. $D_q^{2m+1} \omega_n(0) = 0$ and $D_q^{2m} \omega_n(0) = \frac{[2n]_q!}{[2(n-m)]_q!} \gamma_{2(n-m)}, \ m = 1, 2, \ldots, n.$

3.1 *Matrix form*

Definition 4 The even q-type I Lidstone matrix is an infinite lower triangular matrix $F_q = [f_{ij}]_{i, j \geq 0}$ with

$$f_{ij} = \begin{bmatrix} 2i \\ 2j \end{bmatrix}_q \gamma_{2(i-j)}, \quad i \geq j, \tag{55}$$

where $(\gamma_{2k})_k$ a sequence of real numbers and $\gamma_0 \neq 0$.

Remark 8 The polynomials (54) can be written in the matrix form

$$\Omega_q = F_q \hat{Z}_q, \tag{56}$$

where Ω_q and $\hat{Z}$ are two vectors defined by

$$\Omega_q = [\omega_0(z; q), \omega_1(z; q), \ldots, \omega_n(z; q), \ldots]^T, \quad \hat{Z} = [1, z^2, \ldots, z^{2n}, \ldots]^T.$$

Moreover, if we set $F_{q, n} = [f_{ij}]$ such that $j = 0, 1, ..., i$, $i = 0, 1, ..., n$ for $n \in \mathbb{N}$ and f_{ij} defined in (55), then we have a sequence $(F_{q, n})_n$ of the principle submatrices of order n of F_q which satisfy $\Omega_{q, n} = F_{q, n} \hat{Z}_n$, where

$$\Omega_{q, n} = [\omega_0(z; q), \omega_1(z; q), \ldots, \omega_n(z; q)]^T, \quad \hat{Z}_n = [1, z^2, \ldots, z^{2n}]^T. \tag{57}$$

In the following, we assume that $T_{q\gamma} = [t_{ij}^{\gamma}]_{i, j \geq 0}$ is a q-Toeplitz matrix whose (i, j) entry defined by

$$t_{ij}^{\gamma} = \begin{cases} \frac{\gamma_{2(i-j)}}{[2(i-j)]_q!}, & i \geq j; \\ 0, & \text{otherwise,} \end{cases}$$

and $\hat{\mathscr{D}}$ is the diagonal matrix with entries $\hat{d}_{ii} = [2i]_q!$.

Proposition 10 An even q-type I Lidstone matrix F_q can be factorized as

$$F_q = \hat{\mathscr{D}} T_{q\gamma} \hat{\mathscr{D}}^{-1}. \tag{58}$$

Proposition 11 The even q-type I Lidstone matrix is invertible and

$$(F_q)^{-1} = \hat{\mathscr{D}} T_{q\xi} \hat{D}^{-1}, \tag{59}$$

where $(\xi_{2n})_n$ is a numerical sequence satisfying

$$\sum_{j=0}^{n} \frac{\gamma_{2j} \xi_{2(n-j)}}{[2j]_q! [2(n-j)]_q!} = \delta_{n0} \quad (n \in \mathbb{N}_0), \tag{60}$$

with δ_{nj} is the Kronecker's delta.

Remark 9 Equation (60) describes an infinite linear system which determines the numerical sequence $(\xi_{2n})_n$. According to Cramer's rule, the first $n+1$ equations give

$$\xi_0 = \frac{1}{\gamma_0},$$

$$\xi_{2n} = (-1)^n \frac{[2]_q! [4]_q! \ldots [2n]_q!}{\gamma_0^{n+1}}$$

$$\times \det \begin{bmatrix} \frac{\gamma_2}{[2]_q!} & \frac{\gamma_0}{[2]_q!} & 0 & \cdots & 0 \\ \frac{\gamma_4}{[4]_q!} & \frac{\gamma_2}{[2]_q![2]_q!} & \frac{\gamma_0}{[4]_q!} & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\gamma_{2(n-1)}}{[2n-2]_q!} & \frac{\gamma_{2(n-2)}}{[2n-4]_q![2]_q!} & \frac{\gamma_{2(n-3)}}{[2n-6]_q![4]_q!} & \cdots & \frac{\gamma_0}{[2n-2]_q!} \\ \frac{\gamma_{2n}}{[2n]_q!} & \frac{\gamma_{2(n-1)}}{[2n-2]_q![2]_q!} & \frac{\gamma_{2(n-2)}}{[2n-4]_q![4]_q!} & \cdots & \frac{\gamma_2}{[2n-2]_q![2]_q!} \end{bmatrix}. \tag{61}$$

As in the odd q-type I Lidstone sequences, we consider the polynomials

$$\hat{\omega}_n(z;\, q) = \sum_{k=0}^{n} \begin{bmatrix} 2n \\ 2k \end{bmatrix}_q \xi_{2(n-k)}\, z^{2k}, \tag{62}$$

where $(\xi_{2n})_n$ is defined as in (61). The two sequences $\{\omega_n(z;\, q)\}_n$ and $\{\hat{\omega}_n(z;\, q)\}_n$ are called conjugate even q-type I Lidstone sequences.

We denote $G_q = [g_{ij}]_{i,\, j\geq 0}$ the infinite lower triangular matrix with

$$g_{ij} = \begin{bmatrix} 2i \\ 2j \end{bmatrix}_q \xi_{2(i-j)}, \quad i \geq j,$$

and set $\widehat{\Omega}_q = [\hat{\omega}_0(z;\, q),\, \hat{\omega}_1(z;\, q),\, \ldots,\, \hat{\omega}_n(z;\, q),\, \ldots]^T$. Then,

$$\widehat{\Omega}_q = G_q \hat{Z} \;\text{ and }\; \widehat{\Omega}_{q,\, n} = G_{q,\, n} \hat{Z}_n \quad (n \in \mathbb{N}). \tag{63}$$

Proposition 12 The two sequences $\{\omega_n(z;\, q)\}_n$ and $\{\hat{\omega}_n(z;\, q)\}_n$ are conjugate even q-type I Lidstone sequences if and only if

$$\Omega_q = F_q^2 \widehat{\Omega}_q, \quad \widehat{\Omega}_q = G_q^2 \Omega_q,$$

and for $n \in \mathbb{N}$,

$$\Omega_{q,\, n} = F_{q,\, n}^2 \widehat{\Omega}_{q,\, n}, \quad \widehat{\Omega}_{q,\, n} = G_{q,\, n}^2 \Omega_{q,\, n}.$$

Remark 10 From Proposition 12, we can write

$$\omega_n(z;\, q) = \sum_{k=0}^{n} \tilde{f}_{nk}\, \hat{\omega}_n(z;\, q) \quad \text{and} \quad \hat{\omega}_n(z;\, q) = \sum_{k=0}^{n} \tilde{g}_{nk}\, \omega_n(z;\, q)\ (n \in \mathbb{N}_0),$$

where $\tilde{f}_{nk}$ and $\tilde{g}_{nk}$ $(k = 0,\, \ldots,\, n)$ are elements of the matrices F^2 and G^2, respectively.

3.2 *Recurrence relations and determinant form*

Theorem 9 *(First recurrence relation)* Let $\{\omega_n(z;\, q)\}_n$ and $\{\hat{\omega}_n(z;\, q)\}_n$ be the conjugate even q-type I Lidstone sequences. Then

$$\omega_n(z;\, q) = \frac{1}{\xi_0} \left[z^{2n} - \sum_{k=0}^{n-1} \begin{bmatrix} 2n \\ 2k \end{bmatrix}_q \xi_{2(n-k)}\, \omega_k(z;\, q) \right],$$

$$\hat{\omega}_n(z;\, q) = \frac{1}{\gamma_0}\left[z^{2n} - \sum_{k=0}^{n-1}\begin{bmatrix}2n\\2k\end{bmatrix}_q \gamma_{2(n-k)}\,\hat{\omega}_k(z;\, q)\right], \tag{64}$$

where $(\gamma_{2k})_n$ and $(\xi_{2n})_n$ are the numerical sequences satisfying (61).

Theorem 10 Let $\{\omega_n(z;\, q)\}_n \in q$ELS-I. Then

$$\omega_0(z;\, q) = \frac{1}{\xi_0} z,$$

$$\omega_n(z;\, q) = \frac{(-1)^n}{\xi_0^{n+1}}\begin{vmatrix} 1 & z^2 & z^4 & \cdots & z^{2n-2} & z^{2n} \\ \xi_0 & \xi_2 & \xi_4 & \cdots & \xi_{2(n-2)} & \xi_{2n} \\ 0 & \xi_0 & \begin{bmatrix}4\\2\end{bmatrix}_q \xi_2 & \cdots & \begin{bmatrix}2n-2\\2\end{bmatrix}_q \xi_{2(n-2)} & \begin{bmatrix}2n\\2\end{bmatrix}_q \xi_{2(n-1)} \\ \vdots & \ddots & \ddots & & \vdots & \vdots \\ \vdots & \ddots & \ddots & & \vdots & \vdots \\ 0 & \cdots & \cdots & & \xi_0 & \begin{bmatrix}2n\\2(k-1)\end{bmatrix}_q \xi_2 \end{vmatrix}, \tag{65}$$

where $(\xi_{2n})_n$ is defined as in (61). Moreover, $\{\hat{\omega}_n(z;\, q)\}_n$ can be expressed in a determinate form similar to (65) with γ_{2k} instead of ξ_{2k}, for $k = 0, 1, \ldots, n$, and $n \in \mathbb{N}_0$.

Lemma 5 Let $F_q = [f_{ij}]_{i,\, j\geq 0}$ be an even q-type I Lidstone matrix, $G_q = [g_{ij}]_{i,\, j\geq 0}$ be the inverse matrix of F_q, and $\bar{\Pi}_G = [\bar{\pi}_{ij}]_{i,\, j\geq 0}$ be the production matrix of G_q. Then

$$\begin{aligned}\bar{\pi}_{ij} &= \sum_{n=0}^{i} f_{in}\, g_{(n+1)j} \\ &= \begin{cases} 0, & j > i+1; \\ \displaystyle\sum_{n=0}^{i}\begin{bmatrix}2i\\2n\end{bmatrix}_q \begin{bmatrix}2(n+1)\\2j\end{bmatrix}_q \gamma_{2(i-n)}\xi_{2(n-j+1)}, & \text{otherwise}, \end{cases}\end{aligned} \tag{66}$$

where $(\gamma_{2n})_n$ and $(\xi_{2n})_n$ are the numerical sequences defined as in (60).

Theorem 11 (Second recurrence relation) Let $\{\omega_n(z;\, q)\}_n \in q$ELS-I. Assume that F_q is the even q-type I Lidstone matrix related to $\{\omega_n(z;\, q)\}_n$, and $\bar{\Pi}_G = [\bar{\pi}_{ij}]_{i,\, j\geq 0}$ is the production matrix of F_q^{-1}. Then $\omega_0(z;\, q) = \frac{1}{\xi_0}$, and for $n \in \mathbb{N}$

$$\omega_{n+1}(z;\, q) = \frac{1}{\bar{\pi}_{n(n+1)}}\left[z^2\omega_n(z;\, q) - \sum_{k=0}^{n}\bar{\pi}_{nk}\,\omega_k(z;\, q)\right]. \tag{67}$$

Theorem 12 Let $\{\omega_n(z;\, q)\}_n \in q$ELS-I. Then

$$\omega_0(z;\, q) = \frac{1}{\xi_0},$$

$$\omega_{n+1}(z;\, q) = \frac{(-1)^{n+1}\omega_0(z;\, q)}{\bar{\pi}_{01}\bar{\pi}_{12}\ldots\bar{\pi}_{n(n+1)}}$$

$$\times \det \begin{bmatrix} \bar{\pi}_{00} - z^2 & \bar{\pi}_{01} & 0 & \ldots & \ldots & 0 \\ \bar{\pi}_{10} & \bar{\pi}_{11} - z^2 & \bar{\pi}_{12} & \ldots & \ldots & 0 \\ \bar{\pi}_{20} & \bar{\pi}_{21} & \bar{\pi}_{22} - z^2 & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \bar{\pi}_{(n-1)n} \\ \bar{\pi}_{n0} & \bar{\pi}_{n1} & \bar{\pi}_{n2} & \ldots & \ldots & \bar{\pi}_{nn} - z^2 \end{bmatrix},$$

where $\bar{\pi}_{ij}$ are defined as in (66).

3.3 *The Generating function and relationship with q-Appell polynomial sequences*

Let $\{\omega_n(z;\, q)\}_n$ be an even q-type I Lidstone polynomial sequence related to the numerical sequence $(\gamma_{2n})_n$. Consider the following power series

$$h_q(t) = \sum_{n=0}^{\infty} \frac{\gamma_{2n}}{[2n]_q!} t^{2n} \tag{68}$$

Lemma 6 Let $h_q(t)$ be the power series defined in (68). Then, $\frac{1}{h_q(t)}$ is a well-defined function and

$$\frac{1}{h_q(t)} = \sum_{n=0}^{\infty} \frac{\xi_{2n}}{[2n]_q!} t^{2n}, \tag{69}$$

where $(\xi_{2n})_n$ is defined as in (61).

Theorem 13 Let $\{\omega_n(z;\, q)\}_n$ and $\{\hat{\omega}_n(z;\, q)\}_n$ be the conjugate even q-type I Lidstone sequences. Then

$$h_q(t)\cosh_q(zt) = \sum_{n=0}^{\infty} \omega_n(z;\, q)\frac{t^{2n}}{[2n]_q!},$$

$$\frac{1}{h_q(t)}\cosh_q(zt) = \sum_{n=0}^{\infty} \hat{\omega}_n(z;\, q)\frac{t^{2n}}{[2n]_q!},$$

where $h_q(t)$ is defined as in (68).

Example 2 Consider Equation (60). From Identity (49), we can take

$$\gamma_{2j} = (-1)^j 2^{2j} \beta_{2j}(q) \text{ and } \xi_{2j} = \frac{T_{2j+1}}{[2j+1]_q}.$$

Hence, the two sequences $\{w_n(z;\, q)\}_n$ and $\{\hat{w}_n(z;\, q)\}_n$ which defined by

$$w_n(z;\, q) \quad := \quad \sum_{k=0}^{n} \begin{bmatrix} 2n \\ 2k \end{bmatrix}_q (-1)^k 2^{2k} \beta_{2k}(q)\, z^{2n-2k},$$

$$\hat{w}_n(z;\, q) \quad := \quad \sum_{k=0}^{n} \begin{bmatrix} 2n \\ 2k \end{bmatrix}_q \frac{T_{2n-2k+1}}{[2n-2k+1]_q} z^{2k},$$

are conjugate even q-type I Lidstone sequences. Furthermore, since

$$\mathrm{Tan}_q t = \tan_q t = \sum_{n=0}^{\infty} T_{2n+1} \frac{t^{2n+1}}{[2n+1]_q!} \quad \text{and} \quad t\,\mathrm{Cot}_q t = t \cot_q t = \sum_{n=0}^{\infty} (-1)^n \beta_{2n}(q) \frac{t^{2n}}{[2n]_q!},$$

we can verify that

$$\frac{\tan_q t}{t} \cosh_q(zt) = \sum_{n=0}^{\infty} w_n(z;\, q) \frac{t^{2n}}{[2n]_q!} \quad \text{and} \quad t \coth_q t \cosh_q(zt) = \sum_{n=0}^{\infty} \hat{w}_n(z;\, q) \frac{t^{2n}}{[2n]_q!},$$

which coincide with the results of Theorem 13.

The following result gives a relationship between even q-type I Lidstone polynomial sequences and q-Appell polynomial sequences.

Theorem 14 Let $\{a_n(z;\, q)\}_n$ be a sequence of q-Appell polynomials. Assume that $a_{2n+1}(0;\, q) = 0$ for $n \in \mathbb{N}$. Then, the sequence $\{\omega_n(z;\, q)\}_n$ which defined by

$$\omega_n(z;\, q) := 2^{2n} a_{2n}\left(\frac{z}{2};\, q\right) \tag{70}$$

is an even q-type I Lidstone polynomial sequence.

4. Odd and even q-type II Lidstone polynomial sequences

In this section, we consider two general classes of q-type Lidstone polynomial sequences (called odd and even q-type II Lidstone polynomial sequences, respectively).

In this type, a sequence of polynomials $\{L_n(z;\, q)\}_n$ satisfies the q-difference equation

$$D^2_{q^{-1}} L_n(z;\, q) = a_n L_{n-1}(z;\, q), \quad a_n \in \mathbb{R}\, (n \in \mathbb{N}).$$

4.1 *Odd q-type II Lidstone polynomial sequences*

Definition 5 The odd q-type II Lidstone sequences (qOLS-II) is the set of polynomial sequences satisfying

$$\begin{cases} D^2_{q^{-1}}\tilde{p}_n(z;\,q) = [2n]_{q^{-1}}\,[2n+1]_{q^{-1}}\,\tilde{p}_{n-1}(z;\,q), \\ \tilde{p}_n(0;\,q) = 0\,(n \in \mathbb{N}_0), \quad \tilde{p}_0(z;\,q) = \tilde{\alpha}_0 z,\ \tilde{\alpha}_0 \in \mathbb{R}\backslash\{0\}. \end{cases} \tag{71}$$

Without loss of generality, we may assume that the sequence $\{\tilde{p}_n(z;\,q)\}_n$ satisfies the q-difference equation

$$D^2_{q^{-1}}\tilde{p}_n(z;\,q) = [2n]_q\,[2n+1]_q\,\tilde{p}_{n-1}(z;\,q).$$

The following result gives a characterization of qOLS-II.

Proposition 13 The sequence $\{\tilde{p}_n(z;\,q)\}_n$ is an element of qOLS-II if and only if there exists a numerical sequence $(\tilde{\alpha}_{2k})_k$ such that $\tilde{\alpha}_0 \neq 0$, and

$$\begin{aligned} \tilde{p}_n(z;\,q) &= \sum_{k=0}^{n} \begin{bmatrix} 2n+1 \\ 2k+1 \end{bmatrix}_q \frac{q^{2(n-k)^2+(n-k)}\,\tilde{\alpha}_{2k}}{[2(n-k)+1]_q} z^{2(n-k)+1} \\ &= \sum_{k=0}^{n} \begin{bmatrix} 2n+1 \\ 2k+1 \end{bmatrix}_q \frac{q^{k(2k+1)}\,\tilde{\alpha}_{2(n-k)}}{[2(n-k)+1]_q} z^{2k+1}. \end{aligned} \tag{72}$$

An odd q-type II Lidstone matrix is an infinite lower triangular matrix $\tilde{A}_q = [\tilde{a}_{ij}]_{i,\,j\geq 0}$ with

$$\tilde{a}_{ij} = \begin{bmatrix} 2i+1 \\ 2j+1 \end{bmatrix}_q \frac{q^{j(2j+1)}\,\tilde{\alpha}_{2(i-j)}}{[2(i-j)+1]_q},\ i \geq j,$$

where $(\tilde{\alpha}_{2k})_k$ is a numerical sequence, and $\tilde{\alpha}_0 \neq 0$.

Notice, Formula (72) can be written in the matrix form $\tilde{P}_q = \tilde{A}_q Z$, where $\tilde{P}_q$ and Z are two vectors defined by

$$\tilde{P}_q = [\tilde{p}_0(z;\,q),\,\tilde{p}_1(z;\,q),\,\ldots,\,\tilde{p}_n(z;\,q),\,\ldots]^T, \quad Z = [z,\,z^3,\,\ldots,\,z^{2n+1},\,\ldots]^T.$$

Proposition 14 The odd q-type II Lidstone matrix $\tilde{A}_q$ can be factorized as

$$\tilde{A}_q = \mathscr{D}T_{q\tilde{\alpha}}\mathscr{D}^{-1},$$

where $T_{q\tilde{\alpha}} = [t^{\tilde{\alpha}}_{ij}]_{i,\,j\geq 0}$ with

$$t_{ij}^{\tilde{\alpha}} = \begin{cases} \frac{q^{j(2j+1)}\,\tilde{\alpha}_{2(i-j)}}{[2(i-j)+1]_q!}, & i \geq j; \\ \\ 0, & \text{otherwise,} \end{cases}$$

and $\mathscr{D}$ a diagonal matrix with entries $d_{ii} = [2i+1]_q!$. Moreover, the matrix $\tilde{A}_q$ is invertible and

$$(\tilde{A}_q)^{-1} = \mathscr{D} T_{q\tilde{\beta}} D^{-1},$$

where $(\tilde{\beta}_{2n})_n$ is a numerical sequence satisfying

$$\sum_{j=0}^{n} \frac{\tilde{\beta}_{2j}\tilde{\alpha}_{2(n-j)}}{[2j+1]_q![2(n-j)+1]_q!} = \delta_{n0} \quad (n \in \mathbb{N}_0). \tag{73}$$

Definition 6 Let $(\tilde{\alpha}_{2n})_n$ and $(\tilde{\beta}_{2n})_n$ be two numerical sequences which satisfy Equation (73). If

$$\hat{p}_n(z;\,q) = \sum_{k=0}^{n} \begin{bmatrix} 2n+1 \\ 2k+1 \end{bmatrix}_q \frac{q^{k(2k+1)}\,\tilde{\beta}_{2(n-k)}}{[2(n-k)+1]_q} z^{2k+1} \quad \text{for} \quad n \in \mathbb{N}_0,$$

then, the two sequences $\{\tilde{p}_n(z;\,q)\}_n$ and $\{\hat{p}_n(z;\,q)\}_n$ are called conjugate odd q-type II Lidstone sequences.

Remark 11 If $\tilde{B}_q = [\tilde{b}_{ij}]_{i,\,j \geq 0}$ is the infinite lower triangular matrix with entities

$$\tilde{b}_{ij} = \begin{bmatrix} 2i+1 \\ 2j+1 \end{bmatrix}_q \frac{q^{j(2j+1)}\,\tilde{\beta}_{2(i-j)}}{[2(i-j)+1]_q}, \quad i \geq j,$$

and $\tilde{P}_q^* = [\hat{p}_0(z;\,q),\, \hat{p}_1(z;\,q),\, \ldots,\, \hat{p}_n(z;\,q),\, \ldots]^T$, then we have the matrix form

$$\tilde{P}_q^* = \tilde{B}_q Z.$$

Proposition 15 The two sequences $\{\tilde{p}_n(z;\,q)\}_n$ and $\{\hat{p}_n(z;\,q)\}_n$ are conjugate odd q-type II Lidstone sequences if and only if

$$\tilde{P}_q = \tilde{A}_q^2 \tilde{P}_q^* \quad \text{and} \quad \tilde{P}_q^* = \tilde{B}_q^2 \tilde{P}_q.$$

In the following theorem, we determine recurrence relations for the two sequences $\{\tilde{p}_n(z;\,q)\}_n$ and $\{\hat{p}_n(z;\,q)\}_n$.

Theorem 15 Let $\{\tilde{p}_n(z;\,q)\}_n$ and $\{\hat{p}_n(z;\,q)\}_n$ be conjugate odd q-type II Lidstone sequences. Then,

$$\tilde{p}_n(z;\, q) = \frac{q^{-n(2n+1)}}{\tilde{\beta}_0}\left[z^{2n+1} - \sum_{k=0}^{n-1}\begin{bmatrix}2n+1\\2k+1\end{bmatrix}_q q^{k(2k+1)}\frac{\tilde{\beta}_{2(n-k)}}{[2(n-k)+1]_q}\,\tilde{p}_k(z;\, q)\right];$$

$$\hat{p}_n(z;\, q) = \frac{q^{-n(2n+1)}}{\tilde{\alpha}_0}\left[z^{2n+1} - \sum_{k=0}^{n-1}\begin{bmatrix}2n+1\\2k+1\end{bmatrix}_q q^{k(2k+1)}\frac{\tilde{\alpha}_{2(n-k)}}{[2(n-k)+1]_q}\,\hat{p}_k(z;\, q)\right],$$

where $(\tilde{\alpha}_{2n})_n$ and $(\tilde{\beta}_{2n})_n$ are the numerical sequences which satisfy Equation (73).

Corollary 1 The conjugate sequences $\{\tilde{p}_n(z;\, q)\}_n$ and $\{\hat{p}_n(z;\, q)\}_n$ satisfy the q-difference equations

$$\sum_{k=0}^{n}\frac{\tilde{\beta}_{2k}}{[2k+1]_q!}\,q^{(n-k+1)(2n-2k+3)}D_{q^{-1}}^{2k}u(z) - z^{2n+1} = 0,$$

$$\sum_{k=0}^{n}\frac{\tilde{\alpha}_{2k}}{[2k+1]_q!}\,q^{(n-k+1)(2n-2k+3)}D_{q^{-1}}^{2k}u(z) - z^{2n+1} = 0.$$

Lemma 7 Let $\tilde{A}_q = [\tilde{a}_{ij}]_{i,\, j\geq 0}$ be an odd q-type II Lidstone matrix, $\tilde{B}_q = [\tilde{b}_{ij}]_{i,\, j\geq 0}$ be the inverse of $\tilde{A}_q$, and $\tilde{\Pi}_B = [\tilde{\pi}_{ij}]_{i,\, j\geq 0}$ be the production matrix of $\tilde{B}_q$. Then

$$\tilde{\pi}_{ij} = \begin{cases} \tilde{\alpha}_0\tilde{\beta}_2, & i = j = 0, \\ 0, & j > i+1, \\ \sum_{n=0}^{i-j+1}\begin{bmatrix}2i+1\\2(n+j)-1\end{bmatrix}_q q^{(n+j-1)(2(n+j)-1}\frac{\tilde{\beta}_{2n}\tilde{\alpha}_{2(i-j-n)+2}[2(n+j)+1]_q!}{(2(i-j-n)+3)[2j+1]_q![2n+1]_q!}, & \text{otherwise}, \end{cases} \tag{74}$$

where $(\tilde{\alpha}_{2n})_n$ and $(\tilde{\beta}_{2n})_n$ are defined as in (73).

Theorem 16 Let $\{\tilde{p}_n(z;\, q)\}_n \in q$OLS-II. Assume that $\tilde{A}_q$ is the odd q-type II Lidstone matrix related to $\{\tilde{p}_n(z;\, q)\}_n$, and $\tilde{\Pi}_q = [\tilde{\pi}_{ij}]_{i,\, j\geq 0}$ is the production matrix of $\tilde{A}_q^{-1}$. Then

$$\begin{aligned} \tilde{p}_0(z;\, q) &= \frac{1}{\tilde{\beta}_0}z, \\ \tilde{p}_{n+1}(z;\, q) &= \frac{1}{\tilde{\pi}_{n(n+1)}}\left[z^2\tilde{p}_n(z;\, q) - \sum_{k=0}^{n}\tilde{\pi}_{nk}\,\tilde{p}_k(z;\, q)\right] \quad (n\in\mathbb{N}). \end{aligned} \tag{75}$$

Moreover, the conjugate sequence $\{\hat{p}_n(z;\, q)\}_n$ has a relation similar to (75) with $\tilde{\alpha}_0$ instead $\tilde{\beta}_0$, and $\tilde{\Pi}_q$ is the production matrix of $\tilde{A}_q$ instead of $\tilde{A}_q^{-1}$.

Theorem 17 *(Generating functions)* Let $\{\tilde{p}_n(z;\, q)\}_n$ and $\{\hat{p}_n(z;\, q)\}_n$ be the conjugate odd q-type II Lidstone sequences. Then

$$\tilde{g}_q(t)\frac{Sinh_q(zt)}{t}=\sum_{n=0}^{\infty}\tilde{p}_n(z;q)\frac{t^{2n}}{[2n+1]_q!}, \tag{76}$$

$$\frac{1}{t\tilde{g}_q(t)}\frac{Sinh_q(zt)}{t}=\sum_{n=0}^{\infty}\hat{p}_n(z;q)\frac{t^{2n}}{[2n+1]_q!}, \tag{77}$$

where $\tilde{g}_q(t)$ the power series defined by

$$\tilde{g}_q(t)=\sum_{n=0}^{\infty}\frac{\tilde{\alpha}_{2n}}{[2n+1]_q!}t^{2n}.$$

Consider the q-Appell polynomials that satisfy

$$D_{q^{-1}}\tilde{a}_n(z;q)=[n]_q\tilde{a}_{n-1}(z;q)\ (n\in\mathbb{N}_0). \tag{78}$$

Theorem 18 Let $\{\tilde{a}_n(z;q)\}_n$ be a sequence of q-Appell polynomials. Assume that $\tilde{a}_{2n+1}(0;q)=0$, and

$$f_n(z;q):=2^{2n+1}\tilde{a}_{2n+1}(\frac{z}{2};q)\quad \text{for}\quad n\in\mathbb{N}. \tag{79}$$

Then, $\{f_n(z;q)\}_n\in q$OLS-II.

4.2 *Even q-type II Lidstone polynomial sequences*

Definition 7 An even q-type II Lidstone sequences (qELS-II) is a set of polynomial sequences satisfying

$$\begin{cases} D^2_{q^{-1}}\tilde{\omega}_n(z;q)=[2n]_q\,[2n-1]_q\,\tilde{\omega}_{n-1}(z;q), \\ \tilde{\omega}_n(0;q)=0\,(n\in\mathbb{N}_0),\quad \tilde{\omega}_0(z;q)=\tilde{\gamma}_0 z,\ \tilde{\gamma}_0\in\mathbb{R}\backslash\{0\}. \end{cases} \tag{80}$$

Proposition 16 The sequence $\{\tilde{\omega}_n(z;q)\}_n\in q$ELS-II if and only if there exists a sequence $(\tilde{\gamma}_{2k})_k$ of real numbers such that $\tilde{\gamma}_0\neq 0$, and

$$\tilde{\omega}_n(z;q)=\sum_{k=0}^{n}\begin{bmatrix}2n\\2k\end{bmatrix}_q q^{k(2k-1)}\tilde{\gamma}_{2(n-k)}\,z^{2k}. \tag{81}$$

Note that Equation (81) can be written in the matrix form $\tilde{\Omega}_q=\tilde{F}_q\hat{Z}$, where

$$\tilde{\Omega}_q=[\tilde{\omega}_0(z;q),\,\tilde{\omega}_1(z;q),\,\ldots,\,\tilde{\omega}_n(z;q),\,\ldots]^T,\quad \hat{Z}=[1,\,z^2,\,\ldots,\,z^{2n},\,\ldots]^T,$$

and $\tilde{F}_q = [\tilde{f}_{ij}]_{i,\, j\geq 0}$ with

$$\tilde{f}_{ij} = \begin{bmatrix} 2i \\ 2j \end{bmatrix}_q q^{j(2j-1)} \frac{\tilde{\gamma}_{2(i-j)}}{[2(i-j)+1]_q},\ i \geq j.$$

Proposition 17 The even q-type II Lidstone matrix $\tilde{F}_q$ can be factorized as

$$\tilde{F}_q = \hat{\mathscr{D}} T_{q\tilde{\gamma}} \hat{\mathscr{D}}^{-1},$$

where $T_{q\tilde{\gamma}} = [t_{ij}^{\tilde{\gamma}}]_{i,\, j\geq 0}$ with

$$t_{ij}^{\tilde{\gamma}} = \begin{cases} \frac{q^{j(2j-1)}\, \tilde{\gamma}_{2(i-j)}}{[2(i-j)]!}, & i \geq j, \\ 0, & \text{otherwise}, \end{cases}$$

and $\hat{\mathscr{D}}$ is the diagonal matrix with entries $\hat{d}_{ii} = [2i]_q!$. Moreover, the matrix $\tilde{F}_q$ is invertible and

$$(\tilde{F}_q)^{-1} = \hat{\mathscr{D}} T_{q\tilde{\xi}} \hat{D}^{-1},$$

where $(\tilde{\xi}_{2n})_n$ is a numerical sequence satisfying

$$\sum_{j=0}^{n} \frac{\tilde{\xi}_{2j}\, \tilde{\gamma}_{2(n-j)}}{[2j]_q![2(n-j)]!} = \delta_{n0} \quad (n \in \mathbb{N}_0), \tag{82}$$

and δ_{n0} is the Kronecker's delta.

Definition 8 Let $(\tilde{\gamma}_{2n})_n$ and $(\tilde{\xi}_{2n})_n$ be two numerical sequences satisfying Equation (82), and $\{\hat{\omega}_n(z;\, q)\}_n$ be a sequence of polynomials that satisfy

$$\hat{\omega}_n(z;\, q) = \sum_{k=0}^{n} \begin{bmatrix} 2n \\ 2k \end{bmatrix}_q q^{k(2k-1)}\, \tilde{\xi}_{2(n-k)}\, z^{2k} \quad (n \in \mathbb{N}_0).$$

Then, the two sequences $\{\tilde{\omega}_n(z;\, q)\}_n$ and $\{\hat{\omega}_n(z;\, q)\}_n$ are called conjugate even q-type II Lidstone sequences.

Remark 12 If $\tilde{G}_q = [\tilde{g}_{ij}]_{i,\, j\geq 0}$ is the infinite lower triangular matrix with

$$\tilde{g}_{ij} = \begin{bmatrix} 2i \\ 2j \end{bmatrix}_q q^{j(2j-1)}\, \tilde{\xi}_{2(i-j)}, \quad i \geq j,$$

and $\tilde{\Omega}_q^* = [\hat{\omega}_0(z;\, q),\, \hat{\omega}_1(z;\, q),\, \ldots,\, \hat{\omega}_n(z;\, q),\, \ldots]^T$, then we have

$$\tilde{\Omega}_q^* = \tilde{G}_q \hat{Z}.$$

Proposition 18 The two sequences $\{\tilde{\omega}_n(z;\, q)\}_n$ and $\{\hat{\omega}_n(z;\, q)\}_n$ are conjugate even q-type II Lidstone sequences if and only if

$$\tilde{\Omega}_q = \tilde{F}_q^2 \tilde{\Omega}_q^* \quad \text{and} \quad \tilde{\Omega}_q^* = G_q^2 \tilde{\Omega}_q.$$

Theorem 19 Let $\{\tilde{\omega}_n(z;\, q)\}_n$ and $\{\hat{\omega}_n(z;\, q)\}_n$ be conjugate even q-type II Lidstone sequences. Then

$$\tilde{\omega}_n(z;\, q) = \frac{q^{n(1-2n)}}{\tilde{\tilde{\xi}}_0} \left[z^{2n} - \sum_{k=0}^{n-1} \begin{bmatrix} 2n \\ 2k \end{bmatrix}_q q^{k(2k-1)} \tilde{\tilde{\xi}}_{2(n-k)}\, \tilde{\omega}_k(z;\, q) \right];$$

$$\hat{\omega}_n(z;\, q) = \frac{q^{n(1-2n)}}{\tilde{\gamma}_0} \left[z^{2n} - \sum_{k=0}^{n-1} \begin{bmatrix} 2n \\ 2k \end{bmatrix}_q q^{k(2k-1)} \tilde{\gamma}_{2(n-k)}\, \hat{\omega}_k(z;\, q) \right],$$

where $(\tilde{\gamma}_{2n})_n$ and $(\tilde{\tilde{\xi}}_{2n})_n$ are defined as in Equation (82).

The following theorem gives the generating functions of an even q-type II Lidstone sequences.

Theorem 20 Let $\{\tilde{\omega}_n(z;\, q)\}_n$ and $\{\hat{\omega}_n(z;\, q)\}_n$ be the conjugate even q-type II Lidstone sequences. Then

$$\tilde{h}_q(t) Cosh_q(zt) = \sum_{n=0}^{\infty} \tilde{\omega}_n(z;\, q) \frac{t^{2n}}{[2n]_q!}, \tag{83}$$

$$\frac{1}{\tilde{h}_q(t)} Cosh_q(zt) = \sum_{n=0}^{\infty} \hat{\omega}_n(z;\, q) \frac{t^{2n}}{[2n]_q!}, \tag{84}$$

where $\tilde{h}_q(t)$ is a power series defined by

$$\tilde{h}_q(t) = \sum_{n=0}^{\infty} \frac{\tilde{\gamma}_{2n}}{[2n]_q!} t^{2n}.$$

Example 3 From Equation (49), we can take

$$\widetilde{\gamma}_{2j} = (-1)^j 2^{2j} \beta_{2j}(q) \quad \text{and} \quad \widetilde{\xi}_{2j} = \frac{T_{2j+1}}{[2j+1]_q}.$$

Then, the sequences $\{\widetilde{w}_n(z;\, q)\}_n$ and $\{\hat{w}_n(z;\, q)\}_n$ defined by

$$\widetilde{w}_n(z;\, q) := \sum_{k=0}^{n} \begin{bmatrix} 2n \\ 2k \end{bmatrix}_q q^{k(2k-1)} (-1)^{n-k} 2^{2n-2k} \beta_{2n-2k}(q) z^{2k},$$

$$\hat{w}_n(z;\, q) := \sum_{k=0}^{n} \begin{bmatrix} 2n \\ 2k \end{bmatrix}_q q^{k(2k-1)} \frac{T_{2n-2k+1}}{[2n-2k+1]_q} z^{2k},$$

are conjugate even q-type II Lidstone sequences. Moreover,

$$\begin{aligned} \sum_{n=0}^{\infty} w_n(z;\, q) \frac{t^{2n}}{[2n]_q!} &= \sum_{n=0}^{\infty} \frac{t^{2n}}{[2n]_q!} \sum_{k=0}^{n} \begin{bmatrix} 2n \\ 2k \end{bmatrix}_q q^{k(2k-1)} (-1)^{n-k} 2^{2n-2k} \beta_{2n-2k} z^{2k} \\ &= \left(\sum_{k=0}^{\infty} \frac{T_{2n+1}}{[2n+1]_q!} t^{2n} \right) \left(\sum_{n=0}^{\infty} \frac{q^{n(2n-1)}}{[2n]_q!} (zt)^{2k} \right) \\ &= \frac{\tan_q t}{t} \mathrm{Cosh}_q(zt). \end{aligned}$$

Similarly, we obtain

$$\sum_{n=0}^{\infty} \hat{w}_n(z;\, q) \frac{t^{2n}}{[2n]_q!} = (t \cot_q t) \mathrm{Cosh}_q(zt).$$

Theorem 21 Let $\{\tilde{a}_n(z;\, q)\}_n$ be a sequence of q-Appell polynomials that satisfy (78). If $\tilde{a}_{2n+1}(0;\, q) = 0$ $(n \in \mathbb{N})$, then the function

$$\tilde{\omega}_n(z;\, q) = 2^{2n} \tilde{a}_{2n}\left(\frac{z}{2};\, q\right)$$

is in the class of even q-type II Lidstone sequences.

5. Examples

We consider some illustrative examples of odd and even q-type Lidstone polynomial sequences. These sequences are associated with q-Bernoulli and q-Euler's polynomials generated by the first and second Jackson q-Bessel functions (see [11]).

Example 4 Let $0 < q < 1$ and $\{b_n(z;\, q)\}_n$ be a set of q-Bernoulli polynomials which defined by the generating function

$$\frac{t\, e_q(zt)}{e_q(t/2) E_q(t/2) - 1} = \sum_{n=0}^{\infty} b_n(z;\, q) \frac{t^n}{[n]_q!},$$

where $E_q(z)$ and $e_q(z)$ are the q-exponential functions defined as in (5). We define the sequence $\{p_n(z;q)\}_n$ by

$$p_n(z;q) := 2^{2n+1} b_{2n+1}(\frac{z}{2};q), \quad n \in \mathbb{N}. \tag{85}$$

Since $D_q b_n(z;q) = [n]_q b_{n-1}(z;q)$ (see [11]), $\{p_n(z;q)\}_n$ is an odd q-type I Lidstone sequence. Therefore, it satisfies (11). By using Equation (13), we have $D_q p_n(0;q) = \alpha_{2n}$. This implies

$$\alpha_{2n} = 2^{2n}[2n+1]_q \beta_{2n}(q) \quad (n \in \mathbb{N}_0), \tag{86}$$

where $\beta_{2n}(q)$ denotes the q-Bernoulli numbers, i.e., $\beta_n(q) := b_n(0;q)$.

Ismail and Mansour in [11] introduced the expansion

$$t \coth_q t = \sum_{n=0}^{\infty} \beta_{2n} \frac{(2t)^n}{[2n]_q!}.$$

Therefore, by using (44), we get

$$\frac{g_q(t)}{t} = \frac{1}{t} \sum_{n=0}^{\infty} \alpha_{2n} \frac{t^{2n}}{[2n+1]_q!} = \frac{1}{t} \sum_{n=0}^{\infty} 2^{2n} \frac{\beta_{2n}(q)}{[2n]_q!} t^{2n} = \coth_q(t).$$

Consequently, from (46), the generating function of the sequence $\{p_n(z;q)\}_n$ is $\frac{g_q(t)}{t} \sinh_q zt$, i.e.,

$$\coth_q(t) \sinh_q(tz) = \sum_{n=0}^{\infty} p_n(z;q) \frac{t^{2n}}{[2n+1]_q!}.$$

Moreover, from (47), the generating function for the conjugate sequence $\{\hat{p}_n(z;q)\}_n$ is

$$\tanh_q(t) \sinh_q(tz) = \sum_{n=0}^{\infty} \hat{p}_n(z;q) \frac{t^{2n+2}}{[2n+1]_q!}.$$

Since

$$\tanh_q t = \mathrm{Tanh}_q t = -\sum_{n=0}^{\infty} \frac{\widetilde{E}_{2n+1}(q)}{[2n+1]_q!} 2^{2n+1} t^{2n+1}$$

(see [11] [Eq. (3.36)]), we obtain

$$\hat{p}_n(z;q) = -\sum_{k=0}^{n} \begin{bmatrix} 2n+1 \\ 2k+1 \end{bmatrix}_q \frac{\widetilde{E}_{2k+1}(q)}{[2n-2k+1]_q} 2^{2k+1} z^{2n-2k+1} \quad (n \in \mathbb{N}_0).$$

Example 5 Let $\{B_n(z;\, q)\}_n$ be a set of q-Bernoulli polynomials generated by the second Jackson q-Bessel functions defined as in (4). Consider the sequence $\{\tilde{p}_n(z;\, q)\}_n$ defined by

$$\tilde{p}_n(z;\, q) = 2^{2n+1} B_{2n+1}(\frac{z}{2};\, q).$$

By the same argument as in Example 4, the sequence $\{p_n(z;\, q)\}_n$ is an odd q-type II Lidstone sequence with $\tilde{\alpha}_{2n} = 2^{2n}[2n+1]_q \beta_{2n}(q)$ for every $n \in \mathbb{N}_0$. According to Equation (72), we get

$$\tilde{p}_n(z;\, q) = \sum_{k=0}^{n} \begin{bmatrix} 2n+1 \\ 2k+1 \end{bmatrix}_q q^{k(2k+1)} 2^{2(n-k)} \beta_{2(n-k)}(q)\, z^{2k+1}.$$

Also, the conjugate sequences $\{\tilde{p}_n(z;\, q)\}_n$ and $\{\hat{p}_n(z;\, q)\}_n$ have the following generating functions:

$$\mathrm{Coth}_q(t)\,\mathrm{Sinh}_q(tz) = \sum_{n=0}^{\infty} \tilde{p}_n(z;\, q) \frac{t^{2n}}{[2n+1]_q!},$$

$$\mathrm{Tanh}_q(t)\,\mathrm{Sinh}_q(tz) = \sum_{n=0}^{\infty} \hat{p}_n(z;\, q) \frac{t^{2n}}{[2n+1]_q!}.$$

Thus,

$$\hat{p}_n(z;\, q) = -\sum_{k=0}^{n} \begin{bmatrix} 2n+1 \\ 2k+1 \end{bmatrix}_q \frac{\widetilde{E}_{2k+1}(q)}{[2n-2k+1]_q} 2^{2n-2k+1} q^{k(2k+1)} z^{2k+1} \quad (n \in \mathbb{N}_0).$$

Remark 13 In Example 5, the sequence $\{p_n(z;\, q)\}_n$ up to a constant $[2n+1]_q!$ coincides with the set of q-Lidstone polynomials $\{B_n(z)\}_n$ which defined in (3).

Example 6 Let $\{E_n(z;\, q)\}_n$ be the set of q-Euler polynomials generated by the second Jackson q-Bessel functions, and defined as in (7). Consider the sequence $\{f_n(z;\, q)\}_n$ defined by

$$f_n(z;\, q) = 2^{2n+1} E_{2n+1}(\frac{z}{2};\, q).$$

Then, the sequence $\{f_n(z;\, q)\}_n$ is an odd q-type II Lidstone sequence with

$$\tilde{\alpha}_{2n} = 2^{2n}[2n+1]_q \tilde{E}_{2n}(q),$$

where $\tilde{E}_{2n}(q) = E_{2n}(0;\, q)$. Taking into account that $\tilde{E}_{2n}(q) = \delta_{n0}$ where δ_{n0} is the Kronecker's delta, we get $\tilde{\alpha}_0 = 1$ and $\tilde{\alpha}_{2n} = 0$ for every $n \in \mathbb{N}$. This implies $h_q(t) = 1$ and then the conjugate sequences $\{f_n(z;\, q)\}_n$ and $\{\hat{f}_n(z;\, q)\}_n$ have the following generating functions:

$$\frac{1}{t}\mathrm{Sinh}_q(tz) = \sum_{n=0}^{\infty} f_n(z;\, q)\frac{t^{2n}}{[2n+1]_q!},$$

$$t\,\mathrm{Sinh}_q(tz) = \sum_{n=0}^{\infty} \hat{f}_n(z;\, q)\frac{t^{2n}}{[2n+1]_q!}.$$

Remark 14 In Example 6, the sequence $\{f_n(z;\, q)\}_n$ multiplied by $\frac{2}{[2n+1]_q!}$ coincides with the set of q-Lidstone polynomials $\{N_{n+1}(z)\}_n$ which defined in (6).

Example 7 Let $\{\omega_n(z;\, q)\}_n$ be a sequence of polynomials defined by

$$\omega_n(z;\, q) = 2^{2n} e_{2n}(\frac{z}{2};\, q), \quad n \in \mathbb{N}_0, \tag{87}$$

where $\{e_n(z;\, q)\}_n$ is the set of q-Euler polynomials defined by the generating function

$$\frac{2\, e_q(zt)}{e_q(t/2)E_q(t/2)+1} = \sum_{n=0}^{\infty} e_n(z;\, q)\frac{t^n}{[n]_q!}.$$

By Theorem 14, the sequence $\{\omega_n(z;\, q)\}_n$ which defined in (87) is an even q-type I Lidstone sequence, and then it satisfies Equation (54). According to Proposition 9, we get

$$\gamma_{2n} = \omega_n(0) = 2^{2n}\tilde{E}_{2n}(q) = 2^{2n}\delta_{n0} \quad (n \in \mathbb{N}_0). \tag{88}$$

This implies

$$h_q(t) = \sum_{n=0}^{\infty} \gamma_{2n}\frac{t^{2n}}{[2n]_q!} = 1.$$

So, the generating function of the sequence $\{\omega_n(z;\, q)\}_n$ is

$$\cosh_q(tz) = \sum_{n=0}^{\infty} \omega_n(z;\, q)\frac{t^{2n}}{[2n]_q!}. \tag{89}$$

Example 8 Let $\{E_n(z;\, q)\}$ be the set of q-Euler polynomials generated by the second Jackson q-Bessel functions which defined in (7). Consider the sequence $\{\tilde{\omega}_n(z;\, q)\}_n$:

$$\tilde{\omega}_n(z;\, q) = 2^{2n} E_{2n}(\frac{z}{2};\, q) \quad (n \in \mathbb{N}_0).$$

By the same argument as in Example 7, the sequence $\{\tilde{\omega}_n(z;\,q)\}_n$ is an even q-type II Lidstone sequence with $\tilde{\gamma}_{2n} = \tilde{E}_{2n}(q)$ for every $n \in \mathbb{N}_0$. Here, the conjugate sequences $\{\tilde{\omega}_n(z;\,q)\}_n$ and $\{\hat{\omega}_n(z;\,q)\}_n$ have the same generating functions:

$$\mathrm{Cosh}_q(tz) = \sum_{n=0}^{\infty} \tilde{\omega}_n(z;\,q)\frac{t^{2n}}{[2n]_q!} = \sum_{n=0}^{\infty} \hat{\omega}_n(z;\,q)\frac{t^{2n}}{[2n]_q!}.$$

6. Conclusion

We considered a sequence $\{L_n(z;\,q)\}_n$ of polynomials which satisfies one of the following q-difference equations:

$$D_q^2 L_n(z;\,q) = a_n L_{n-1}(z;\,q) \quad \text{or} \quad D_{q^{-1}}^2 L_n(z;\,q) = a_n L_{n-1}(z;\,q),$$

where $a_n \in \mathbb{R}$. The sequence $\{L_n(z;\,q)\}_n$ is a generalization of Lidstone-type polynomial sequence defined in [20–22] and called the q-type Lidstone polynomial sequence.

We studied some classes of these sequences. More precisely, we considered the q-Lidstone polynomials of odd and even degrees in two types (I and II), called odd and even q-type Lidstone polynomial sequences, respectively. Then, we gave some characterizations of these classes including matrix form, generating function, recurrence relations, and conjugate sequences.

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Conflict of interest

The authors declare no competing financial interest.

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