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Research Article

Some Orthogonal Combinations of Legendre Polynomials

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Abstract: The main purpose of this study is to introduce and study certain orthogonal polynomials (OPs) that are written as combinations of Legendre polynomials. These polynomials can be viewed as generalized Jacobi polynomials (GJPs) since they are Jacobi polynomials (JPs) of certain negative parameters. The analytic and inversion formulas of the GJPs are established. New expressions of the derivatives of these polynomials are derived in detail as combinations of their original ones. Other derivative expressions for these polynomials are found, but as combinations of some orthogonal and non-orthogonal polynomials. Some product formulas with some other polynomials are also obtained. Certain definite and weighted definite integrals are obtained using the newly introduced connection and product formulas.

***Keywords*:** orthogonal polynomials, Jacobi polynomials, symmetric and non-symmetric polynomials, generalized hypergeometric functions

MSC: 33C20, 33C45

1. Introduction

Various classical problems in physics are solved using special functions, which are mathematical functions. The importance of special functions is not limited to physics. They also arise in other disciplines such as statistics, number theory, engineering, and numerical analysis; see, for instance, ([1–5]). Due to their wide applicability in a variety of disciplines, investigations of various OPs have become more crucial. Their applications in the scope of numerical analysis reflect their significance (see, for instance, [6, 7]). In addition, OPs have been shown to play a crucial role in both mathematical statistics and quantum physics. The JPs are some of the most significant OPs. Numerous authors have researched these polynomials from a theoretical perspective; see, for instance, [8–10]. The fact that numerous well-known polynomials are special JPs is one of their benefits. The four different types of Chebyshev polynomials (CPs), Legendre polynomials, and Gegenbauer polynomials are specific JPs. For some articles regarding JPs and their applications, one can be directed, for instance, to [11–16], while for some others regarding the applications of the different families of JPs, one can consult [17–21].

There are many investigations regarding the combinations of OPs; from a theoretical point of view, one can refer to [22–24]. In his published works [25, 26], Shen explored the idea of constructing OPs for the solution of differential equations (DEs). He constructed suitable combinations of Legendre and CPs and then used these combinations and followed a spectral Galerkin approach to solve numerically the BVPs of the second and fourth orders. By choosing such

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combinations, it is possible to transform the DEs with their governing constraints into solvable algebraic systems of equations. In [27], the authors generalized the combinations of Shen [25, 26] to have the ability to deal with the general even-order BVPs. Furthermore, the same polynomials in [28] were theoretically explored in depth.

Special functions have two fundamental problems called connection and linearization problems. Numerous articles have been dedicated to the study of these issues; see [29–32]. Several applications require these problems; for example, see [33]. In [34], one can find a study of CPs of the fifth kind. Here is an example of how linearization coefficients can be useful in treating non-linear problems numerically. Check out [35], for example.

An important goal for many authors is to establish explicit expressions for the derivatives of OPs and a variety of special functions in general. Due to their role in numerically solving DEs of many kinds, these expressions are important. Some high-order derivatives of JPs were developed in [36] and they are applied to the solution of BVPs of even- orders. The sixth-kind CPs high-order derivative formulas were derived in [37] to treat a specific type of non-linear DEs.

Here are the main objectives of the current article:

- Introducing a type of GJPs.
- Establishing some essential relations regarding the GJPs.
- Derivation of some new derivative expressions of the GJPs as combinations of some well-known polynomials.
- Utilizing some of the introduced formulas to find closed forms for some definite and weighted definite integrals.

The paper is structured as follows: Section 2 gives some fundamentals regarding some celebrated polynomials. In addition, the class of polynomials, GJPs is also accounted for. Section 3 establishes two basic formulas for the GJPs that serve to derive our results. New expressions for the GJPs derivatives are derived in Section 4. Other derivative formulas of these polynomials are given as combinations of different polynomials in Section 5. Some product formulas involving these polynomials are derived in Section 6. Section 7 reports on some of the findings.

2. Some basics of certain celebrated polynomials

In this part, we will look at some of the fundamental characteristics shared by Legendre polynomials and some combinations of them. Furthermore, some properties of some orthogonal and non-OPs are presented.

2.1 *An account of some polynomials related to legendre polynomials*

The orthogonality relation of Legendre polynomials on $[-1, 1]$ is

$$\int_{-1}^{1} P_r(x)\, P_\ell(x)\, dx = \begin{cases} \dfrac{2}{2\ell+1}, & r = \ell, \\ 0, & r \neq \ell. \end{cases} \tag{1}$$

$P_\ell(x),\ \ell \geq 0$ can be represented as ([38]):

$$P_\ell(x) = 2^{-\ell} \sum_{m=0}^{\left\lfloor \frac{\ell}{2} \right\rfloor} \frac{(-1)^m\,(2\ell-2m)!}{m!\,(\ell-2m)!\,(\ell-m)!}\, x^{\ell-2m}, \tag{2}$$

where $\lfloor z \rfloor$ is the floor function.

Also, x^ℓ has the following form

$$x^{\ell}=2^{-\ell}\sqrt{\pi}\,\ell!\sum_{r=0}^{\left\lfloor\frac{\ell}{2}\right\rfloor}\frac{\ell-2r+\frac{1}{2}}{r!\,\Gamma\left(\ell-r+\frac{3}{2}\right)}P_{\ell-2r}(x),\quad \ell\geq 0. \tag{3}$$

Here, we present the following two combinations of Legendre polynomials

$$G_r^1(x)=P_r(x)+\frac{2r+3}{2r+5}P_{r+1}(x)-P_{r+2}(x)-\frac{2r+3}{2r+5}P_{r+3}(x),\quad r\geq 0, \tag{4}$$

$$\begin{aligned}G_r^2(x)=&P_r(x)+\frac{2r+3}{2r+7}P_{r+1}(x)-\frac{2(2r+5)}{2r+7}P_{2+r}(x)-\frac{2(2r+3)}{2r+9}P_{r+3}(x)+\frac{2r+3}{2r+7}P_{r+4}(x)\\&+\frac{(2r+3)(2r+5)}{(2r+7)(2r+9)}P_{r+5}(x),\quad r\geq 0.\end{aligned} \tag{5}$$

The above two combinations of the third- and fifth degrees were previously used to solve certain third- and fifth-order BVPs in [39]. In fact, these combinations can be written in terms of certain non-symmetric JPs as follows:

$$\phi_r(x)=\frac{2r+3}{2}\,(1-x^2)\,(1+x)\,V_r^{(1,\,2)}(x),\quad r\geq 0, \tag{6}$$

$$\chi_r(x)=\frac{(2r+3)\,(2r+5)}{8}\,(1-x^2)^2\,(1+x)\,V_r^{(2,\,3)}(x),\quad r\geq 0, \tag{7}$$

where $V_r^{(\alpha,\,\beta)}(x)$ are the normalized JPs that were defined in [28]. Therefore, the two identities (4) and (5) indicate that $\phi_r(x)$ and $\chi_r(x)$ are respectively OPs regarding: $w_1(x)=(1-x^2)^{-1}(1+x)^{-1}$ and $w_2(x)=(1-x^2)^{-2}(1+x)^{-1}$. We have the following two orthogonality relations:

$$\int_{-1}^{1} w_1(x)\,\phi_r(x)\,\phi_s(x)\,dx=\begin{cases}\dfrac{2(2r+3)^2}{(r+1)_3}, & r=s,\\ 0, & r\neq s,\end{cases} \tag{8}$$

$$\int_{-1}^{1} w_2(x)\,\chi_r(x)\,\chi_s(x)\,dx=\begin{cases}\dfrac{2(2r+3)^2(2r+5)^2}{(r+1)_5}, & r=s,\\ 0, & r\neq s,\end{cases} \tag{9}$$

where $(z)_m$ is the standard Pochhammer symbol whose definition is given as $(z)_m=\dfrac{\Gamma(z+m)}{\Gamma(z)}$.

2.2 *An account on normalized JPs*

The normalized JPs $V_{\ell}^{(\mu,\,\nu)}(x)$ (see, [28]) are defined as

$$V_\ell^{(\mu,\,\nu)}(x) = {}_2F_1\left(\begin{array}{c} -\ell,\, \ell+\mu+\nu+1 \\ \nu+1 \end{array} \middle| \frac{1-x}{2}\right).$$

We refer here to the ultraspherical polynomials as the symmetric normalized JPs defined as

$$U_\ell^{(\nu)}(x) = V_\ell^{(\nu-\frac{1}{2},\,\nu-\frac{1}{2})}(x). \tag{10}$$

These polynomials have the following representation (see, [38]):

$$U_\ell^{(\nu)}(x) = \frac{\ell!\,\Gamma(2\nu+1)}{2\Gamma(\nu+1)\,\Gamma(\ell+2\nu)} \sum_{i=0}^{\left\lfloor \frac{\ell}{2} \right\rfloor} \frac{(-1)^i\, 2^{\ell-2i}\,\Gamma(\ell-i+\nu)}{i!\,(\ell-2i)!} x^{\ell-2i}, \quad \ell \geq 0. \tag{11}$$

Moreover, x^ℓ can be represented as:

$$x^\ell = \frac{2^{1-\ell}\,\Gamma(\nu+1)}{\Gamma(2\nu+1)} \sum_{i=0}^{\left\lfloor \frac{\ell}{2} \right\rfloor} \frac{\ell!\,(\ell-2i+\nu)\,\Gamma(\ell-2i+2\nu)}{i!\,(\ell-2i)!\,\Gamma(\ell-i+\nu+1)} U_{\ell-2i}^{(\nu)}(x), \quad \ell \geq 0. \tag{12}$$

We refer here also to the fact that there are four kinds of CPs that are considered JPs of particular parameters; see [28]. In addition, if $\psi_i(x)$ is any Chebyshev polynomial, then they are characterized by having the next unified recursive formula:

$$\psi_\ell(t) = 2t\,\psi_{\ell-1}(t) - \psi_{\ell-2}(t), \quad \ell \geq 2, \tag{13}$$

with distinct initials.

2.3 *An account on GJPs*

A family of GJPs is proposed by Guo et al. in their intriguing paper [40], Other GJPs were also investigated in [41]. Let r, s be two integers. The authors in [40] defined the following GJPs $G_i^{(r,\,s)}(x)$ as:

$$G_i^{(r,\,s)}(x) = \begin{cases} (1-x)^{-r}\,(1+x)^{-s}\,P_{i-i_0}^{(-r,\,-s)}(x), & i_0 = -(r+s),\ r,\,s \leq -1, \\ (1-x)^{-r}\,P_{i-i_0}^{(-r,\,s)}(x), & i_0 = -r,\ r \leq -1,\ s > -1, \\ (1+x)^{-s}\,P_{i-i_0}^{(r,\,-s)}(x), & i_0 = -s,\ r > -1,\ s \leq -1, \\ P_{i-i_0}^{(r,\,s)}(x), & i_0 = 0,\ r,\,s > -1, \end{cases} \tag{14}$$

where $P_i^{(\alpha,\,\beta)}(x)$ are the classical JPs.

According to [40], these polynomials were called “generalized Jacobi polynomials”, and they are abbreviated by GJPs.

GJPs possess an important property for $\ell,\ j \in \mathbb{Z}^+$,

$$D^\ell J_i^{(-r,\,-s)}(1) = 0, \qquad 0 \le \ell \le r-1;$$

$$D^j J_i^{(-r,\,-s)}(-1) = 0, \qquad 0 \le j \le s-1.$$

In [42], Abd-Elhameed showed that $G_i^{(-m,\,-m-1)}(x),\ m \in \mathbb{Z}^+$ are non-symmetric polynomials that can be written in terms of Legendre polynomials. The following formula was proven:

$$\begin{aligned} G_r^{(-m,\,-m-1)}(x) =&(1-x^2)^m\,(1+x)\,V_r^{(m,\,m+1)}(x) \\ =&\frac{1}{2}\left(\frac{3}{2}+r\right)_m\left(\sum_{j=0}^{m}\frac{(-1)^j(3+4j+2r)\binom{m}{j}}{\left(\frac{3}{2}+j+r\right)_{m+1}}P_{2j+r+1}(x)+\sum_{j=0}^{m}\frac{(-1)^j(1+4j+2r)\binom{m}{j}}{\left(\frac{1}{2}+j+r\right)_{m+1}}P_{2j+r}(x)\right). \end{aligned} \tag{15}$$

The general combination in (15) generalizes the combinations in (4) and (5). More precisely, we have

$$\phi_i(x) = G_i^{-1,\,-2}(x), \qquad \chi_i(x) = G_i^{-2,\,-3}(x).$$

Relation (15) indicates that $G_r^m(x) = G_r^{(-m,\,-m-1)}(x)$ are OPs on $[-1,\,1]$ regarding: $w(x) = (1-x^2)^{-m}(1+x)^{-1}$. We have

$$\int_{-1}^{1} w(x)\,G_r^m(x)\,G_s^m(x)\,dx = \begin{cases} \dfrac{2^{2m+1}r!\,\left(\left(\frac{3}{2}+r\right)_m\right)^2}{(2m+r+1)!}, & r=s, \\ 0, & r \neq s. \end{cases} \tag{16}$$

Remark 1 There is another type of GJPs that has been investigated in [28], however, these polynomials are symmetric ones since the two parameters of the JPs are identical. This type was convenient to treat even-order BVPs; see [43].

Remark 2 We expect that our introduced polynomials in this paper will be convenient to treat odd-order BVPs.

2.4 *An account on some celebrated polynomials*

In this part, we consider two types of polynomials. Assume that $\phi_k(x)$, and $\psi_k(x)$ are respectively two symmetric and non-symmetric polynomials that have the following forms:

$$\phi_\ell(x) = \sum_{t=0}^{\left\lfloor \frac{\ell}{2} \right\rfloor} A_{t,\,\ell}\,x^{\ell-2t}, \tag{17}$$

$$\bar{\phi}_\ell(x) = \sum_{t=0}^{\ell} B_{t,\,\ell}\,x^{\ell-t}. \tag{18}$$

There are important classes of symmetric polynomials. The Hermite polynomials $\{H_i(x)\}_{i\geq 0}$ are the classical OPs that satisfy the following orthogonality relation

$$\int_{-\infty}^{\infty} e^{-x^2} H_\ell(x) H_m(x)\, dx = \sqrt{\pi}\, 2^m\, m!\, \delta_{\ell,\, m}, \tag{19}$$

and $\delta_{\ell,\, m}$ is the well-known Kronecker delta function.

In addition $H_i(x)$ can be represented as

$$H_i(x) = i! \sum_{\ell=0}^{\left\lfloor \frac{i}{2} \right\rfloor} \frac{(-1)^\ell\, 2^{i-2\ell}}{\ell!\,(i-2\ell)!}\, x^{i-2\ell}, \quad i \geq 0. \tag{20}$$

In addition, x^i has the following expression

$$x^i = \frac{i!}{2^i} \sum_{\ell=0}^{\left\lfloor \frac{i}{2} \right\rfloor} \frac{1}{\ell!(i-2\ell)!} H_{i-2\ell}(x), \quad i \geq 0. \tag{21}$$

There are two classes that generalize Fibonacci and Lucas polynomials. These two classes can be constructed using the two following recursive formulas:

$$F_m^{a,\, b}(x) = a\, x\, F_{m-1}^{a,\, b}(x) + b\, F_{m-2}^{a,\, b}(x), \quad F_0^{a,\, b}(x) = 1,\ F_1^{a,\, b}(x) = a\, x, \quad m \geq 2, \tag{22}$$

$$L_m^{\bar{a},\, \bar{b}}(x) = \bar{a}\, x\, L_{m-1}^{\bar{a},\, \bar{b}}(x) + \bar{b}\, L_{m-2}^{\bar{a},\, \bar{b}}(x), \quad L_0^{\bar{a},\, \bar{b}}(x) = 2,\ L_1^{\bar{a},\, \bar{b}}(x) = \bar{a}\, x, \quad m \geq 2, \tag{23}$$

with non-zro constant a, b, $\bar{a}$, $\bar{b}$.

$F_r^{a,\, b}(x)$ and $L_r^{\bar{a},\, \bar{b}}(x)$ can be represented respectively as ([28]):

$$F_r^{a,\, b}(x) = \sum_{\ell=0}^{\left\lfloor \frac{r}{2} \right\rfloor} \binom{r-\ell}{\ell}\, b^\ell\, a^{r-2\ell}\, x^{r-2\ell}, \quad r \geq 0, \tag{24}$$

and

$$L_r^{\bar{a},\, \bar{b}}(x) = r \sum_{\ell=0}^{\left\lfloor \frac{r}{2} \right\rfloor} \frac{\bar{b}^\ell\, \bar{a}^{r-2\ell} \binom{r-\ell}{\ell}}{r-\ell}\, x^{r-2\ell}, \quad r \geq 1. \tag{25}$$

In addition, we have ([28])

$$x^r = a^{-r} \sum_{m=0}^{\lfloor \frac{r}{2} \rfloor} \frac{(-b)^m \binom{r}{m} (r-2m+1)}{r-m+1} F_{r-2m}^{a,\, b}(x), \quad r \geq 0, \tag{26}$$

$$x^r = \bar{a}^{-r} \sum_{m=0}^{\lfloor \frac{r}{2} \rfloor} (-\bar{b})^m c_{r-2m} \binom{r}{m} L_{r-2m}^{\bar{a},\, \bar{b}}(x), \quad r \geq 0, \tag{27}$$

where

$$c_m = \begin{cases} \frac{1}{2}, & m = 0, \\ 1, & m \geq 1. \end{cases} \tag{28}$$

3. Two basic relations of the GJPs

In this section, we state and prove two basic formulae about the GJPs that form the foundation of most of the results. The analytic and inversion formulae of these polynomials will be derived. The next lemma will help get started.

Lemma 1 Consider three non-negative integers m, r, and s. The next two identities apply:

$$\sum_{\ell=0}^{s} \frac{(-3+4\ell-4m-2r)\binom{m}{m-\ell} \Gamma\left(\frac{3}{2}-\ell+2m+r-s\right)}{(s-\ell)!\, \left(\frac{3}{2}-\ell+m+r\right)_{m+1}} = -\frac{2\Gamma\left(\frac{3}{2}+m+r-s\right)}{s!}, \tag{29}$$

$$\sum_{\ell=0}^{s} \frac{(-1+4\ell-4m-2r)\binom{m}{m-\ell} \Gamma\left(\frac{1}{2}-\ell+2m+r-s\right)}{(s-\ell)!\, \left(\frac{1}{2}-\ell+m+r\right)_{m+1}} = -\frac{2\Gamma\left(\frac{1}{2}+m+r-s\right)}{s!}. \tag{30}$$

Proof. Let

$$T_{s,\, r,\, m} = \sum_{\ell=0}^{s} \frac{(-3+4\ell-4m-2r)\binom{m}{m-\ell} \Gamma\left(\frac{3}{2}-\ell+2m+r-s\right)}{(s-\ell)!\, \left(\frac{3}{2}-\ell+m+r\right)_{m+1}}, \tag{31}$$

$$\bar{T}_{s,\, r,\, m} = \sum_{\ell=0}^{s} \frac{(-1+4\ell-4m-2r)\binom{m}{m-\ell} \Gamma\left(\frac{1}{2}-\ell+2m+r-s\right)}{(s-\ell)!\, \left(\frac{1}{2}-\ell+m+r\right)_{m+1}}. \tag{32}$$

Zeilberger's algorithm (see, [44]) aids in finding the recursive formulas satisfied by $T_{s,\, r,\, m}$ and $\bar{T}_{s,\, r,\, m}$. They are given, respectively, as

$$T_{s+1,\,r,\,m}-\frac{2}{(s+1)(-2s+2m+2r+1)}T_{s,\,r,\,m}=0,\quad T_{0,\,r,\,m}=-2\Gamma\left(m+r+\frac{3}{2}\right), \tag{33}$$

$$\bar{T}_{s+1,\,r,\,m}-\frac{2}{(s+1)(-2s+2m+2r-1)}\bar{T}_{s,\,r,\,m}=0,\quad \bar{T}_{0,\,r,\,m}=-2\Gamma\left(m+r+\frac{1}{2}\right). \tag{34}$$

With a little effort, it can be demonstrated that

$$T_{s,\,r,\,m}=-\frac{2\Gamma\left(\frac{3}{2}+m+r-s\right)}{s!}, \tag{35}$$

$$\bar{T}_{s,\,r,\,m}=-\frac{2\Gamma\left(\frac{1}{2}+m+r-s\right)}{s!}. \tag{36}$$

This completes the proof. □

The two key theorems listed below can now be stated and proved.

Theorem 1 The polynomials $G_r^m(x)$ have the following analytic form

$$G_r^m(x)=\frac{\left(\frac{3}{2}+r\right)_m}{\sqrt{\pi}}\left(\sum_{s=0}^{\left\lfloor\frac{r+1}{2}\right\rfloor+m}\frac{(-1)^{m+s}\,2^{1+2m+r-2s}\,\Gamma\left(\frac{3}{2}+m+r-s\right)}{s!\,(1+2m+r-2s)!}x^{r+2m-2s+1}\right.$$
$$\left.+\sum_{s=0}^{\left\lfloor\frac{r}{2}\right\rfloor+m}\frac{(-1)^{m+s}2^{2m+r-2s}\Gamma\left(\frac{1}{2}+m+r-s\right)}{s!\,(2m+r-2s)!}x^{r+2m-2s}\right). \tag{37}$$

Proof. If we start with the expression of $G_r^m(x)$ in (15) along with (3), then we can write

$$G_r^m(x)=\frac{\left(\frac{3}{2}+r\right)_m}{2\sqrt{\pi}}\sum_{j=0}^{m}\frac{(-1)^j\,(3+4j+2r)\binom{m}{j}}{\left(\frac{3}{2}+j+r\right)_{m+1}}\sum_{\ell=0}^{\left\lfloor\frac{1}{2}(2j+r+1)\right\rfloor}\frac{(-1)^{\ell}2^{1+2j-2\ell+r}\Gamma\left(\frac{3}{2}+2j-\ell+r\right)}{\ell!\,\Gamma(2+2j-2\ell+r)}x^{2j+r+1-2\ell}$$
$$+\frac{\left(\frac{3}{2}+r\right)_m}{2\sqrt{\pi}}\sum_{j=0}^{m}\frac{(-1)^j(1+4j+2r)\binom{m}{j}}{\left(\frac{1}{2}+j+r\right)_{m+1}}\sum_{\ell=0}^{\left\lfloor\frac{1}{2}(2j+r)\right\rfloor}\frac{(-1)^{\ell}2^{2j-2\ell+r}\Gamma\left(\frac{1}{2}+2j-\ell+r\right)}{\ell!\,(2j-2\ell+r)!}x^{2j+r-2\ell}. \tag{38}$$

It is convenient to turn the last formula into the following one:

$$G_r^m(x) = \frac{\left(\frac{3}{2}+r\right)_m}{\sqrt{\pi}} \sum_{s=0}^{\left\lfloor \frac{r+1}{2} \right\rfloor + m} \frac{(-1)^{m+s+1} 2^{2m+r-2s}}{(1+2m+r-2s)!} \times$$

$$\sum_{\ell=0}^{s} \frac{(-3+4\ell-4m-2r)\binom{m}{m-\ell}\Gamma\left(\frac{3}{2}-\ell+2m+r-s\right)}{(s-\ell)!\left(\frac{3}{2}-\ell+m+r\right)_{m+1}} x^{r+2m-2s+1}$$

$$+ \frac{\left(\frac{3}{2}+r\right)_m}{\sqrt{\pi}} \sum_{s=0}^{\left\lfloor \frac{r}{2} \right\rfloor + m} \frac{(-1)^{m+s+1} 2^{2m+r-2s-1}}{(2m+r-2s)!} \times \tag{39}$$

$$\sum_{\ell=0}^{s} \frac{(-1+4\ell-4m-2r)\binom{m}{m-\ell}\Gamma\left(\frac{1}{2}-\ell+2m+r-s\right)}{(s-\ell)!\left(\frac{1}{2}-\ell+m+r\right)_{m+1}} x^{r+2m-2s}.$$

Making use of the closed forms of the two sums in Lemma 1, Formula (39) can be simplified to give the analytic formula in (37). □

Now, we give the inversion to Formula (37).

Theorem 2 If we choose any two non-negative integers j and m, then this formula is valid:

$$x^{j+2m+1} = (-1)^m 2^{-1-j-2m} \sqrt{\pi}\, (j+2m+1)! \left(\sum_{\ell=0}^{\left\lfloor \frac{j}{2} \right\rfloor} \frac{\Gamma\left(\frac{3}{2}+j-2\ell\right)}{\ell!\,\Gamma\left(\frac{3}{2}+j+m-2\ell\right)\Gamma\left(\frac{3}{2}+j+m-\ell\right)} G_{j-2\ell}^m(x) \right.$$

$$\left. - \sum_{\ell=0}^{\left\lfloor \frac{j-1}{2} \right\rfloor} \frac{\Gamma\left(\frac{1}{2}+j-2\ell\right)}{\ell!\,\Gamma\left(\frac{1}{2}+j+m-2\ell\right)\Gamma\left(\frac{3}{2}+j+m-\ell\right)} G_{j-2\ell-1}^m(x) \right) + \varepsilon_{j,m}(x), \tag{40}$$

and $\varepsilon_{j,m}(x)$ is given by

$$\varepsilon_{j,m}(x) = \begin{cases} -\dfrac{\left(\frac{(-1+x^2)^m}{m} - \frac{2x^{-1+2m}\,{}_2F_1\left(1-m,\, \frac{2+j}{2};\, \frac{4+j}{2};\, \frac{1}{x^2}\right)}{2+j} \right) \left(1+\frac{j}{2}\right)_m}{(m-1)!}, & j \text{ even}, \\[2ex] \dfrac{2x^{2m}\,{}_2F_1\left(-m,\, \frac{1+j}{2};\, \frac{3+j}{2};\, \frac{1}{x^2}\right)\left(\frac{1+j}{2}\right)_{1+m}}{(1+j)m!}, & j \text{ odd}. \end{cases} \tag{41}$$

Proof. The proof can be done via lengthy manipulations similar to that given in Theorem 2 in [28]. □

4. Derivatives of the GJPs

We present in this section new derivatives expressions for $G_r^m(x)$. These formulas can be obtained through the analytic form of $G_r^m(x)$ and its inversion formula.

Theorem 3 Consider p, r, and m to be positive integers with $r+2m \geq p$. The pth-derivative of $G_r^m(x)$ can be expressed as combinations of their original ones as

$$D^p G_r^m(x) = \frac{(-2)^p \Gamma\left(\frac{1}{2}+r+m\right)\Gamma\left(\frac{3}{2}+r+m\right)}{\Gamma\left(\frac{3}{2}+r\right)} \times$$

$$\sum_{s=0}^{\left\lfloor \frac{1}{2}(r-p-1)\right\rfloor} \frac{(p)_{s+1}\Gamma\left(\frac{1}{2}-p+r-2s\right)}{\Gamma\left(\frac{1}{2}+m-p+r-2s\right)\Gamma\left(\frac{3}{2}+m-p+r-s\right) s! \left(\frac{1}{2}+m+r-s\right)_s} G_{r-p-2s-1}^m(x)$$

$$+ \frac{2^{p-1}\Gamma\left(\frac{1}{2}+r+m\right)\Gamma\left(\frac{3}{2}+r+m\right)}{\Gamma\left(\frac{3}{2}+r\right)} \times \tag{42}$$

$$\sum_{s=0}^{\left\lfloor \frac{r-p}{2}\right\rfloor} \frac{(p)_s (1+2m+2r-2s)\Gamma\left(\frac{3}{2}-p+r-2s\right)}{s!\Gamma\left(\frac{3}{2}+m-p+r-2s\right)\Gamma\left(\frac{3}{2}+m-p+r-s\right)\left(\frac{1}{2}+m+r-s\right)_s} G_{r-p-2s}^m(x)$$

$$+ \sum_{j=0}^{\lfloor \frac{r}{2}\rfloor+m} F_{j,\,r,\,m,\,p}\, \varepsilon_{r-2j-2p-1,\,m}(x) + \sum_{j=0}^{\lfloor \frac{r}{2}\rfloor+m} \bar{F}_{j,\,r,\,m,\,p}\, \varepsilon_{r-2j-2p,\,m}(x),$$

where $F_{j,\,r,\,m,\,p}$ and $\bar{F}_{j,\,r,\,m,\,p}$ are as follows

$$F_{j,\,r,\,m,\,p} = \frac{(-1)^{j+m} 2^{-2j+2m+r}\Gamma\left(\frac{3}{2}+m+r\right)\Gamma\left(\frac{1}{2}-j+m+r\right)}{\sqrt{\pi}\, j!\Gamma\left(\frac{3}{2}+r\right)(-2j+2m-p+r)!}, \tag{43}$$

$$\bar{F}_{j,\,r,\,m,\,p} = \frac{(-1)^{j+m} 2^{1-2j+2m+r}\Gamma\left(\frac{3}{2}+m+r\right)\Gamma\left(\frac{3}{2}-j+m+r\right)}{\sqrt{\pi}\, j!\Gamma\left(\frac{3}{2}+r\right)(1-2j+2m-p+r)!}, \tag{44}$$

and $\varepsilon_{j,\,m}(x)$ is as given in (41).

Proof. If we differentiate the analytic form of the polynomials $G_r^m(x)$ in (37) with respect to x, then we get

$$D^p G_r^m(x) = \sum_{j=0}^{\lfloor \frac{r}{2}\rfloor+m} F_{j,\,r,\,m,\,p} x^{r-2j+2m-p} + \sum_{j=0}^{\left\lfloor \frac{r+1}{2}\right\rfloor+m} \bar{F}_{j,\,r,\,m,\,p} x^{r-2j+2m-p+1}, \tag{45}$$

where $F_{j,\,r,\,m,\,p}$ and $\bar{F}_{j,\,r,\,m,\,p}$ are respectively given by (43) and (44).

The inversion formula in (40) yields the following formula

$$
\begin{aligned}
D^p G_r^m(x)) = & \sum_{j=0}^{\lfloor \frac{r}{2} \rfloor + m} F_{j,\, r,\, m,\, p} \left(\sum_{\ell=0}^{\lfloor \frac{1}{2}(r-2j-p-1) \rfloor} M_{\ell,\, r-2j-p-1}\, G_{r-2j-p-2\ell-1}^m(x) \right. \\
& \left. + \sum_{\ell=0}^{\lfloor \frac{1}{2}(r-2j-p-2) \rfloor} R_{\ell,\, r-2j-p-1} G_{r-2j-p-2\ell-2}^m(x) + \varepsilon_{r-2j-2p-1,\, m}(x) \right) \\
& + \sum_{j=0}^{\lfloor \frac{r+1}{2} \rfloor + m} \bar{F}_{j,\, r,\, m,\, p} \left(\sum_{\ell=0}^{\lfloor \frac{1}{2}(r-2j-p) \rfloor} M_{\ell,\, r-2j-p}\, G_{r-2j-p-2\ell}^m(x) \right. \\
& \left. + \sum_{\ell=0}^{\lfloor \frac{1}{2}(r-2j-p-1) \rfloor} R_{\ell,\, r-2j-p}\, G_{r-2j-p-2\ell-1}^m(x) + \varepsilon_{r-2j-2p,\, m}(x) \right),
\end{aligned} \tag{46}
$$

where $M_{r,\, i}$ and $R_{r,\, i}$ have the following forms:

$$
M_{r,\, i} = \frac{(-1)^m 2^{-1-i-2m} \sqrt{\pi}(i+2m+1)!\Gamma\left(\frac{3}{2}+i-2r\right)}{\Gamma\left(\frac{3}{2}+i+m-2r\right)\Gamma\left(\frac{3}{2}+i+m-r\right) r!},
$$

$$
R_{r,\, i} = \frac{(-1)^{m+1} 2^{-1-i-2m} \sqrt{\pi}(i+2m+1)!\Gamma\left(\frac{1}{2}+i-2r\right)}{\Gamma\left(\frac{1}{2}+i+m-2r\right)\Gamma\left(\frac{3}{2}+i+m-r\right) r!}.
$$

The last formula can be written as

$$
D^p G_r^m(x) = \sum_1 + \sum_2 + \sum_{j=0}^{\lfloor \frac{r}{2} \rfloor + m} F_{j,\, r,\, m,\, p}\, \varepsilon_{r-2j-2p-1,\, m}(x) + \sum_{j=0}^{\lfloor \frac{r}{2} \rfloor + m} \bar{F}_{j,\, r,\, m,\, p}\, \varepsilon_{r-2j-2p,\, m}(x), \tag{47}
$$

where

$$
\begin{aligned}
\sum_1 = & \sum_{j=0}^{\lfloor \frac{r}{2} \rfloor + m} F_{j,\, r,\, m,\, p}(x) \sum_{\ell=0}^{\lfloor \frac{1}{2}(r-2j-p-1) \rfloor} M_{\ell,\, r-2j-p-1}(x)\, G_{r-2j-p-2\ell-1}^m(x) \\
& + \sum_{j=0}^{\lfloor \frac{r+1}{2} \rfloor + m} \bar{F}_{j,\, r,\, m,\, p}(x) \sum_{\ell=0}^{\lfloor \frac{1}{2}(r-2j-p-1) \rfloor} R_{\ell,\, r-2j-p}\, G_{r-2j-p-2\ell-1}^m(x),
\end{aligned} \tag{48}
$$

$$\sum_{2}=\sum_{j=0}^{\left\lfloor\frac{r}{2}\right\rfloor+m} F_{j,\,r,\,m,\,p} \sum_{\ell=0}^{\left\lfloor\frac{1}{2}(r-2j-p-2)\right\rfloor} R_{\ell,\,r-2j-p-1}(x)\, G^{m}_{r-2j-p-2\ell-2}(x)$$
$$+\sum_{j=0}^{\left\lfloor\frac{r+1}{2}\right\rfloor+m} \bar{F}_{j,\,r,\,m,\,p} \sum_{\ell=0}^{\left\lfloor\frac{1}{2}(r-2j-p)\right\rfloor} M_{\ell,\,r-2j-p}\, G^{m}_{r-2j-p-2\ell}(x). \tag{49}$$

Some lengthy algebraic computations convert Formula (47) into the following one:

$$D^{p} G^{m}_{r}(x)=\sum_{s=0}^{\left\lfloor\frac{1}{2}(r-p-1)\right\rfloor} V_{s,\,r,\,m,\,p}\, G^{m}_{r-p-2s-1}(x)+\sum_{s=0}^{\left\lfloor\frac{r-p}{2}\right\rfloor} \bar{V}_{s,\,r,\,m,\,p}\, G^{m}_{r-p-2s}(x)$$
$$+\sum_{j=0}^{\left\lfloor\frac{r}{2}\right\rfloor+m} F_{j,\,r,\,m,\,p}\, \varepsilon_{r-2j-2p-1,\,m}(x)+\sum_{j=0}^{\left\lfloor\frac{r}{2}\right\rfloor+m} \bar{F}_{j,\,r,\,m,\,p}\, \varepsilon_{r-2j-2p,\,m}(x), \tag{50}$$

where

$$V_{s,\,r,\,m,\,p}=\sum_{\ell=0}^{s}\left(F_{\ell,\,r,\,m,\,p} M_{s-\ell,\,r-p-2\ell-1}+\bar{F}_{\ell,\,r,\,m,\,p} R_{s-\ell,\,r-p-2\ell}\right), \tag{51}$$

$$\bar{V}_{s,\,r,\,m,\,p}=\sum_{\ell=0}^{s}\left(F_{\ell,\,r,\,m,\,p} R_{s-\ell-1,\,r-p-2\ell-1}+\bar{F}_{\ell,\,r,\,m,\,p} M_{s-\ell,\,r-p-2\ell}\right). \tag{52}$$

After some computations, it can be demonstrated that

$$V_{s,\,r,\,m,\,p}=\sum_{\ell=0}^{s} \frac{(-1)^{\ell+1}\, 2^{p}\,(p+s)\,\Gamma\left(\frac{3}{2}+m+r\right)\Gamma\left(\frac{1}{2}-\ell+m+r\right)\Gamma\left(\frac{1}{2}-p+r-2s\right)}{\ell!\,\Gamma\left(\frac{3}{2}+r\right)\Gamma\left(\frac{1}{2}+m-p+r-2s\right)\Gamma\left(\frac{3}{2}-\ell+m-p+r-s\right)(s-\ell)!}, \tag{53}$$

$$\bar{V}_{s,\,r,\,m,\,p}=\sum_{\ell=0}^{s} \frac{(-1)^{\ell}\, 2^{p-1}(1+2m+2r-2s)\Gamma\left(\frac{3}{2}+m+r\right)\Gamma\left(\frac{1}{2}-\ell+m+r\right)\Gamma\left(\frac{3}{2}-p+r-2s\right)}{\ell!\,(s-\ell)!\Gamma\left(\frac{3}{2}+r\right)\Gamma\left(\frac{3}{2}+m-p+r-2s\right)\Gamma\left(\frac{3}{2}-\ell+m-p+r-s\right)}. \tag{54}$$

Zeilberger's algorithm ([44]) aids in finding closed forms for $V_{r,\,m,\,p,\,s}$ and $\bar{V}_{r,\,m,\,p,\,s}$. They satisfy, respectively, the following two recursive formulas:

$$(s+1)(4s+2p-2r+1)(4s+2p-2r+3)(-2s+2m+2r-1)\, V_{s+1,\, r,\, m,\, p}$$

$$+(s+p+1)(2s-2m+2p-2r-1)(4s-2m+2p-2r+1)(4s-2m+2p-2r+3)\, V_{s,\, r,\, m,\, p} = 0, \tag{55}$$

$$V_{0,\, r,\, m,\, p} = 1,$$

$$(s+1)(4s+2p-2r-1)(4s+2p-2r+1)(-2s+2m+2r+1)\bar{V}_{s+1,\, r,\, m,\, p}$$

$$+(s+p)(2s-2m+2p-2r-1)(4s-2m+2p-2r-1)(4s-2m+2p-2r+1)\bar{V}_{s,\, r,\, m,\, p} = 0, \tag{56}$$

$$\bar{V}_{0,\, r,\, m,\, p} = 1,$$

The above two recursive formulas can be solved to give

$$V_{s,\, r,\, m,\, p} = -\frac{2^p \Gamma\left(\frac{3}{2}+m+r\right)\Gamma\left(\frac{1}{2}-p+r-2s\right)\Gamma\left(\frac{1}{2}+m+r-s\right)(p)_{s+1}}{s!\Gamma\left(\frac{3}{2}+r\right)\Gamma\left(\frac{1}{2}+m-p+r-2s\right)\Gamma\left(\frac{3}{2}+m-p+r-s\right)},$$

$$\bar{V}_{s,\, r,\, m,\, p} = \frac{2^p (p)_s \Gamma\left(\frac{3}{2}+m+r\right)\Gamma\left(\frac{3}{2}-p+r-2s\right)\Gamma\left(\frac{3}{2}+m+r-s\right)}{s!\Gamma\left(\frac{3}{2}+r\right)\Gamma\left(\frac{3}{2}+m-p+r-2s\right)\Gamma\left(\frac{3}{2}+m-p+r-s\right)}.$$

Inserting the above two formulas in (50) yields the desired formula (42). □

Remark 3 There is a great deal of importance in finding the derivatives of OPs as combinations of their original ones. In this regard, some types of linear and non-linear even-order BVPs can be handled using derivative formulas of certain JPs in [36].

5. Some other expressions involving the GJPs

We give in this section some derivatives formulas of the GJPs but in terms of different polynomials. These formulas will yield connection formulas for these polynomials and different kinds of polynomials.

5.1 *Various relations for the derivatives of* $G_r^m(x)$

Theorem 4 The pth-derivative of $G_r^m(x)$ can be written as combinations of ultraspherical polynomials $U_k^{(\lambda)}(x)$ as:

$$
\begin{aligned}
D^p G_r^m(x) =& \frac{2^{1+p-2\nu}\,\Gamma\left(\frac{1}{2}+m+r\right)\,\Gamma\left(\frac{3}{2}+m+r\right)}{\Gamma\left(\frac{3}{2}+r\right)\,\Gamma\left(\frac{1}{2}+\nu\right)}\times \\
& \sum_{\ell=0}^{\left\lfloor \frac{r-p}{2}\right\rfloor+m} \frac{(-1)^{\ell+m}\,(-2\ell+2m-p+r+\nu)\,\Gamma(-2\ell+2m-p+r+2\nu)\,\left(\frac{1}{2}-\ell+m-p+\nu\right)_\ell}{\ell!\,(-2\ell+2m-p+r)!\,\Gamma(1-\ell+2m-p+r+\nu)\,\left(\frac{1}{2}-\ell+m+r\right)_\ell}\times \\
& U^{(\nu)}_{r+2m-p-2\ell}(x) \\
& +\frac{2^{1+p-2\nu}\,\left(\Gamma\left(\frac{3}{2}+m+r\right)\right)^2}{\Gamma\left(\frac{3}{2}+r\right)\,\Gamma\left(\frac{1}{2}+\nu\right)}\times \\
& \sum_{\ell=0}^{\left\lfloor \frac{1}{2}(r-p+1)\right\rfloor+m} \frac{(-1)^{\ell+m}\,(1-2\ell+2m-p+r+\nu)\,\Gamma(1-2\ell+2m-p+r+2\nu)\,\left(\frac{1}{2}-\ell+m-p+\nu\right)_\ell}{\ell!\,(1-2\ell+2m-p+r)!\,\Gamma(2-\ell+2m-p+r+\nu)\,\left(\frac{3}{2}-\ell+m+r\right)_\ell}\times \\
& U^{(\nu)}_{r+2m-p-2\ell+1}(x).
\end{aligned}
\tag{57}
$$

Proof. Differentiating the analytic form in (37) with respect to x yields the following formula

$$
D^p G_r^m(x) = \sum_{\ell=0}^{\left\lfloor \frac{r}{2}\right\rfloor+m} F_{\ell,\,r,\,m.p}\, x^{r-2\ell+2m-p} + \sum_{\ell=0}^{\left\lfloor \frac{r+1}{2}\right\rfloor+m} \bar{F}_{\ell,\,r,\,m,\,b}\, x^{r-2\ell+2m-p+1}, \tag{58}
$$

where $F_{\ell,\,r,\,m,\,p}$ and $\bar{F}_{\ell,\,r,\,m,\,p}$ are respectively given in (43) and (44). Thanks to Formula (12), we get

$$
\begin{aligned}
D^p G_r^m(x) =& \sum_{\ell=0}^{\left\lfloor \frac{r}{2}\right\rfloor+m} F_{\ell,\,r,\,m,\,p} \sum_{s=0}^{\left\lfloor \frac{1}{2}(r-2\ell+2m-p)\right\rfloor} W_{s,\,r-2\ell+2m-p}\, U^{(\nu)}_{r-2\ell+2m-p-2s}(x) \\
& + \sum_{\ell=0}^{\left\lfloor \frac{r+1}{2}\right\rfloor+m} \bar{F}_{\ell,\,r,\,m,\,p} \sum_{s=0}^{\left\lfloor \frac{1}{2}(r-2\ell+2m-p+1)\right\rfloor} \bar{W}_{s,\,r-2\ell+2m-p+1}\, U^{(\nu)}_{r-2\ell+2m-p-2s+1}(x),
\end{aligned}
\tag{59}
$$

with

$$
W_{r,\,j} = \frac{2^{-j+1}\,(j-2r+\nu)\,j!\,\Gamma(\nu+1)\,\Gamma(j-2r+2\nu)}{(j-2r)!\,r!\,\Gamma(2\nu+1)\,\Gamma(1+j-r+\nu)}.
$$

This formula, after rearranging the terms, can be written as follows:

$$
\begin{aligned}
D^p G_r^m(x) = & \sum_{\ell=0}^{\left\lfloor \frac{r-p}{2} \right\rfloor + m} \sum_{s=0}^{\ell} F_{s,\, r,\, m,\, p}\, W_{\ell-s,\, r+2m-p-2s}\, U_{r+2m-p-2\ell}^{(\nu)}(x) \\
& + \sum_{\ell=0}^{\left\lfloor \frac{1}{2}(r-p+1) \right\rfloor + m} \sum_{s=0}^{\ell} \bar{F}_{s,\, r,\, m,\, p}\, \bar{W}_{\ell-s,\, r+2m-p-2s+1}\, U_{r+2m-p-2\ell+1}^{(\nu)}(x),
\end{aligned} \tag{60}
$$

which can be written as

$$
D^p G_r^m(x) = \sum_{\ell=0}^{\left\lfloor \frac{r-p}{2} \right\rfloor + m} \sum_{s=0}^{\ell} \theta_{s,\, \ell,\, r,\, m,\, p}\, U_{r+2m-p-2\ell}^{(\nu)}(x) + \sum_{\ell=0}^{\left\lfloor \frac{1}{2}(r-p+1) \right\rfloor + m} \sum_{s=0}^{\ell} \bar{\theta}_{s,\, \ell,\, r,\, m,\, p}\, U_{r+2m-p-2\ell+1}^{(\nu)}(x), \tag{61}
$$

where

$$
\begin{aligned}
\theta_{s,\, \ell,\, r,\, m,\, p} = & \frac{(-1)^{m+s}\, 2^{1+p-2\nu}\, (-2\ell+2m-p+r+\nu)\, \Gamma\left(\frac{3}{2}+m+r\right)\, \Gamma\left(\frac{1}{2}+m+r-s\right)}{\Gamma\left(\frac{3}{2}+r\right)\, (-2\ell+2m-p+r)!\, (\ell-s)!\, s!\, \Gamma\left(\frac{1}{2}+\nu\right)} \times \\
& \frac{\Gamma(-2\ell+2m-p+r+2\nu)}{\Gamma(1-\ell+2m-p+r-s+\nu)},
\end{aligned}
$$

$$
\begin{aligned}
\bar{\theta}_{s,\, \ell,\, r,\, m,\, p} = & \frac{(-1)^{m+s}\, 2^{1+p-2\nu}\, (1-2\ell+2m-p+r+\nu)\, \Gamma\left(\frac{3}{2}+m+r\right)\, \Gamma\left(\frac{3}{2}+m+r-s\right))}{\Gamma\left(\frac{3}{2}+r\right)\, (1-2\ell+2m-p+r)!\, (\ell-s)!\, s!\, \Gamma\left(\frac{1}{2}+\nu\right)} \times \\
& \frac{\Gamma(1-2\ell+2m-p+r+2\nu)}{\Gamma(2-\ell+2m-p+r-s+\nu)}.
\end{aligned}
$$

Now, we use the following two transformation formulas:

$$
\begin{aligned}
\sum_{s=0}^{\ell} \theta_{s,\, \ell,\, r,\, m,\, p} = & \frac{(-1)^{m}\, 2^{1+p-2\nu}\, (-2\ell+2m-p+r+\nu)\, \Gamma\left(\frac{1}{2}+m+r\right)\, \Gamma\left(\frac{3}{2}+m+r\right)}{\ell!\, \Gamma\left(\frac{3}{2}+r\right)\, (-2\ell+2m-p+r)!\, \Gamma\left(\frac{1}{2}+\nu\right)} \times \\
& \frac{\Gamma(-2\ell+2m-p+r+2\nu)}{\Gamma(1-\ell+2m-p+r+\nu)}\ {}_2F_1\left(\left. \begin{array}{c} -\ell,\, \ell-2m+p-r-\nu \\ \frac{1}{2}-m-r \end{array} \right| 1 \right),
\end{aligned} \tag{62}
$$

$$\sum_{s=0}^{\ell}\bar{\theta}_{s,\,\ell,\,r,\,m,\,p}=\frac{(-1)^m\,2^{1+p-2\nu}\,(1-2\ell+2m-p+r+\nu)\,\left(\Gamma\left(\frac{3}{2}+m+r\right)\right)^2}{\ell!\,\Gamma\left(\frac{3}{2}+r\right)\,(1-2\ell+2m-p+r)!\,\Gamma\left(\frac{1}{2}+\nu\right)}\times$$

$$\frac{\Gamma(1-2\ell+2m-p+r+2\nu)}{\Gamma(2-\ell+2m-p+r+\nu)}\,{}_2F_1\left(\begin{array}{c}-\ell,\,-1+\ell-2m+p-r-\nu\\-\frac{1}{2}-m-r\end{array}\middle|\,1\right). \tag{63}$$

Thanks to the celebrated Chu Vandermond's identity ([44]), we have

$$\sum_{s=0}^{\ell}\theta_{s,\,\ell,\,r,\,m,\,p}=\frac{(-1)^{\ell+m}2^{1+p-2\nu}(-2\ell+2m-p+r+\nu)\left(\frac{1}{2}-\ell+m-p+\nu\right)_{\ell}}{\ell!\,\Gamma\left(\frac{3}{2}+r\right)(-2\ell+2m-p+r)!\Gamma\left(\frac{1}{2}+\nu\right)\left(\frac{1}{2}-\ell+m+r\right)_{\ell}}\times$$

$$\frac{\Gamma\left(\frac{1}{2}+m+r\right)\Gamma\left(\frac{3}{2}+m+r\right)\Gamma(-2\ell+2m-p+r+2\nu)}{\Gamma(1-\ell+2m-p+r+\nu)}, \tag{64}$$

$$\sum_{s=0}^{\ell}\bar{\theta}_{s,\,\ell,\,r,\,m,\,p}=\frac{(-1)^{\ell+m}2^{1+p-2\nu}(1-2\ell+2m-p+r+\nu)\left(\frac{1}{2}-\ell+m-p+\nu\right)_{\ell}}{\ell!\,\Gamma\left(\frac{3}{2}+r\right)(1-2\ell+2m-p+r)!\left(\frac{3}{2}-\ell+m+r\right)_{\ell}\Gamma\left(\frac{1}{2}+\nu\right)}\times$$

$$\frac{\left(\Gamma\left(\frac{3}{2}+m+r\right)\right)^2\Gamma(1-2\ell+2m-p+r+2\nu)}{\Gamma(2-\ell+2m-p+r+\nu)}, \tag{65}$$

and hence, some computations lead to (57). This ends the proof of Theorem 4. □

Theorem 5 In terms of the Hermite polynomials $H_k(x)$, the pth-derivative of $G_r^m(x)$ can be represented as:

$$D^pG_r^m(x)=\frac{(-1)^m\,2^p\,\Gamma\left(\frac{1}{2}+m+r\right)\Gamma\left(\frac{3}{2}+m+r\right)}{\sqrt{\pi}\,\Gamma\left(\frac{3}{2}+r\right)}\sum_{\ell=0}^{\left\lfloor\frac{r-p}{2}\right\rfloor+m}\frac{{}_1F_1\left(-\ell;\,\frac{1}{2}-m-r;\,-1\right)}{\ell!\,(-2\ell+2m-p+r)!}\,H_{r+2m-p-2\ell}(x)$$

$$+\frac{(-1)^m\,2^p\,\left(\Gamma(\frac{3}{2}+m+r\right)^2}{\sqrt{\pi}\,\Gamma\left(\frac{3}{2}+r\right)}\sum_{\ell=0}^{\left\lfloor\frac{1}{2}(r-p+1)\right\rfloor+m}\frac{{}_1F_1\left(-\ell;\,-\frac{1}{2}-m-r;\,-1\right)}{\ell!\,(-2\ell+2m-p+r+1)!}\,H_{r+2m-p-2\ell+1}(x). \tag{66}$$

Proof. The proof can be done using the analytic form in (37) along with the inversion formula in (21). □

Theorem 6 In terms of the polynomials $F_k^{a,\,b}(x)$, the pth-derivative of $G_r^m(x)$ can be represented as

$$
\begin{aligned}
D^{q} G_{r}^{m}(x) = & \frac{2^{2m+r}\, a^{p-r-2m}\, \Gamma\left(\frac{1}{2}+m+r\right)\Gamma\left(\frac{3}{2}+m+r\right)}{\sqrt{\pi}\,\Gamma\left(\frac{3}{2}+r\right)} \times \\
& \sum_{\ell=0}^{\left\lfloor \frac{r-p}{2} \right\rfloor + m} \frac{(-1)^{\ell+m+1}\, b^{\ell}\, (-1+2\ell-2m+p-r)}{\ell!\,(1-\ell+2m-p+r)!} \times \\
& {}_2F_1\left(\left.\begin{array}{c} -\ell,\ -1+\ell-2m+p-r \\ \frac{1}{2}-m-r \end{array}\right| -\frac{a^2}{4b}\right) F_{r+2m-p-2\ell}^{a,\,b}(x) \\
& + \frac{2^{1+2m+r}\, a^{p-r-2m-1}\, \left(\Gamma\left(\frac{3}{2}+m+r\right)\right)^{2}}{\sqrt{\pi}\,\Gamma\left(\frac{3}{2}+r\right)} \times \\
& \sum_{\ell=0}^{\left\lfloor \frac{1}{2}(r-p+1) \right\rfloor + m} \frac{(-1)^{\ell+m+1}\, b^{\ell}\, (-2+2\ell-2m+p-r)}{\ell!\,(2-\ell+2m-p+r)!} \times \\
& {}_2F_1\left(\left.\begin{array}{c} -\ell,\ -2+\ell-2m+p-r \\ -\frac{1}{2}-m-r \end{array}\right| -\frac{a^2}{4b}\right) F_{r+2m-p-2\ell+1}^{a,\,b}(x).
\end{aligned}
\tag{67}
$$

Proof. The proof can be done using the analytic form in (37) along with the inversion formula in (26). □

Theorem 7 In terms of the polynomials $L_{k}^{\bar{a},\,\bar{b}}(x)$, the pth-derivative of $G_{r}^{m}(x)$ can be represented as

$$
\begin{aligned}
D^q G_r^m(x) = &\frac{2^{2m+r}\,\bar{a}^{p-r-2m}\,\Gamma\left(\frac{1}{2}(1+2m+2r)\right)\Gamma\left(\frac{1}{2}(3+2m+2r)\right)}{\sqrt{\pi}\,\Gamma\left(\frac{3}{2}+r\right)} \\
&\times \sum_{\ell=0}^{\left\lfloor \frac{r-p}{2} \right\rfloor + m} \frac{(-1)^{\ell+m}\, c_{-2\ell+2m-p+r}\, \bar{b}^{\ell}}{\ell!\,(-\ell+2m-p+r)!} \times \\
&{}_2F_1\left(\begin{matrix} -\ell,\ \ell-2m+p-r \\ \frac{1}{2}-m-r \end{matrix} \,\middle|\, -\frac{\bar{a}^2}{4\bar{b}} \right) L_{r+2m-p-2\ell}^{\bar{a},\bar{b}}(x) \\
&+ \frac{2^{1+2m+r}\,\bar{a}^{p-r-2m-1}\left(\Gamma\left(\frac{3}{2}+m+r\right)\right)^2}{\sqrt{\pi}\,\Gamma\left(\frac{3}{2}+r\right)} \times \\
&\times \sum_{\ell=0}^{\left\lfloor \frac{1}{2}(r-p+1) \right\rfloor + m} \frac{(-1)^{\ell+m}\, \bar{b}^{\ell}\, c_{-2\ell+2m-p+r+1}}{\ell!\,(-\ell+2m-p+r+1)!} \times \\
&{}_2F_1\left(\begin{matrix} -\ell,\ -1+\ell-2m+p-r \\ -\frac{1}{2}-m-r \end{matrix} \,\middle|\, -\frac{\bar{a}^2}{4\bar{b}} \right) L_{r+2m-p-2\ell+1}^{\bar{a},\bar{b}}(x),
\end{aligned}
\tag{68}
$$

with

$$
c_i = \begin{cases} \frac{1}{2}, & i = 0, \\ 1, & \text{otherwise}. \end{cases} \tag{69}
$$

Proof. The proof can be done using the analytic form in (37) along with the inversion formula in (27). □

5.2 *Some connection formulas between $G_r^m(x)$ and other polynomials with applications*

This section provides some connection formulas between $G_r^m(x)$ and some other polynomials. These formulas are direct consequences of the expressions of the derivatives of $G_r^m(x)$ as combinations of different polynomials. Additionally, some definite integrals and definite weighted integrals will be introduced.

Corollary 1 The GJPs-ultraspherical connection formula is

$$G_r^m(x) = \frac{2^{1-2\nu}\,\Gamma\left(\frac{1}{2}+m+r\right)\Gamma\left(\frac{3}{2}+m+r\right)}{\Gamma\left(\frac{3}{2}+r\right)\Gamma\left(\frac{1}{2}+\nu\right)}\times$$

$$\sum_{\ell=0}^{\lfloor\frac{r}{2}\rfloor+m} \frac{(-1)^{\ell+m}(-2\ell+2m+r+\nu)\Gamma(-2\ell+2m+r+2\nu)\left(\frac{1}{2}-\ell+m+\nu\right)_\ell}{\ell!\,(-2\ell+2m+r)!\,\Gamma(1-\ell+2m+r+\nu)\left(\frac{1}{2}-\ell+m+r\right)_\ell} U_{r+2m-2\ell}^{(\nu)}(x)$$

$$+\frac{2^{1-2\nu}\left(\Gamma\left(\frac{3}{2}+m+r\right)\right)^2}{\Gamma\left(\frac{3}{2}+r\right)\Gamma\left(\frac{1}{2}+\nu\right)}\times \tag{70}$$

$$\sum_{\ell=0}^{\left\lfloor\frac{r+1}{2}\right\rfloor+m} \frac{(-1)^{\ell+m}(1-2\ell+2m+r+\nu)\Gamma(1-2\ell+2m+r+2\nu)\left(\frac{1}{2}-\ell+m+\nu\right)_\ell}{\ell!\,(1-2\ell+2m+r)!\,\Gamma(2-\ell+2m+r+\nu)\left(\frac{3}{2}-\ell+m+r\right)_\ell} U_{r+2m-2\ell+1}^{(\nu)}(x).$$

Proof. Substitution by $p=0$ in (57) yields Formula (70) □

Remark 4 Three special formulas of the connection formula (70) can be deduced taking into consideration the particular classes of the $U_r^{(\nu)}(x)$. These connection formulas are given as follows:

Corollary 2 The GJPs-first kind Chebyshev connection formula is

$$G_r^m(x) = \frac{2\Gamma\left(\frac{1}{2}+m\right)\Gamma\left(\frac{3}{2}+m+r\right)}{\sqrt{\pi}\,\Gamma\left(\frac{3}{2}+r\right)} \sum_{\ell=0}^{\lfloor\frac{r}{2}\rfloor+m} \frac{c_{r-2\ell+2m}(-1)^{\ell+m}\Gamma\left(\frac{1}{2}-\ell+m+r\right)}{\ell!\,\Gamma\left(\frac{1}{2}-\ell+m\right)(-\ell+2m+r)!} T_{r+2m-2\ell}(x)$$

$$+\frac{2\Gamma\left(\frac{1}{2}+m\right)\left(\Gamma\left(\frac{3}{2}+m+r\right)\right)^2}{\sqrt{\pi}\,\Gamma\left(\frac{3}{2}+r\right)} \sum_{\ell=0}^{\left\lfloor\frac{r+1}{2}\right\rfloor+m} \frac{c_{r-2\ell+2m+1}(-1)^{\ell+m}}{\ell!\,\Gamma\left(\frac{1}{2}-\ell+m\right)(1-\ell+2m+r)!\left(\frac{3}{2}-\ell+m+r\right)_\ell} T_{r+2m-2\ell+1}(x), \tag{71}$$

where c_i is defined in (69).

Proof. Substitution by $\nu=0$ in (70) yields Formula (71). □

Corollary 3 The GJPs-second kind Chebyshev connection formula is

$$G_r^m(x) = \frac{\Gamma\left(\frac{3}{2}+m\right)\Gamma\left(\frac{3}{2}+m+r\right)}{\sqrt{\pi}\,\Gamma\left(\frac{3}{2}+r\right)} \left(\sum_{\ell=0}^{\lfloor\frac{r}{2}\rfloor+m} \frac{(-1)^{\ell+m}(1-2\ell+2m+r)\Gamma\left(\frac{1}{2}-\ell+m+r\right)}{\ell!\,(1-\ell+2m+r)!\,\Gamma\left(\frac{3}{2}-\ell+m\right)} U_{r+2m-2\ell}(x)\right.$$

$$\left.+\sum_{\ell=0}^{\left\lfloor\frac{r+1}{2}\right\rfloor+m} \frac{(-1)^{\ell+m}(2-2\ell+2m+r)\Gamma\left(\frac{3}{2}-\ell+m+r\right)}{\ell!\,(2-\ell+2m+r)!\,\Gamma\left(\frac{3}{2}-\ell+m\right)} U_{r+2m-2\ell+1}(x)\right). \tag{72}$$

Proof. Substitution by $\nu=1$ in (70) yields Formula (72). □

Corollary 4 The GJPs-Legendre connection formula is

$$G_r^m(x) = \frac{m!\,\Gamma\left(\frac{3}{2}+m+r\right)}{\Gamma\left(\frac{3}{2}+r\right)} \sum_{\ell=0}^{\lfloor \frac{r}{2}\rfloor+m} \frac{(-1)^{\ell+m}\left(\frac{1}{2}-2\ell+2m+r\right)\Gamma\left(\frac{1}{2}-\ell+m+r\right)}{\ell!\,(m-\ell)!\,\Gamma\left(\frac{3}{2}-\ell+2m+r\right)} P_{r+2m-2\ell}(x)$$
$$+ \frac{\Gamma\left(\frac{3}{2}+m+r\right)^2}{\Gamma\left(\frac{3}{2}+r\right)} \sum_{\ell=0}^{\lfloor \frac{r+1}{2}\rfloor+m} \frac{(-1)^{\ell+m}\left(\frac{3}{2}-2\ell+2m+r\right)(1-\ell+m)_\ell}{\ell!\,\Gamma\left(\frac{5}{2}-\ell+2m+r\right)\left(\frac{3}{2}-\ell+m+r\right)_\ell} P_{r+2m-2\ell+1}(x). \tag{73}$$

Proof. Setting $\nu = \frac{1}{2}$ in (70) gives Formula (73). □

Now, we are going to give an explicit formula for a certain definite integral of the polynomials $G_r^m(x)$.

Corollary 5 If r is a non-negative integer, then the following integral formula applies:

$$\int_0^1 G_r^m(x)\,dx = \frac{\Gamma\left(\frac{3}{2}+m\right)\Gamma\left(\frac{3}{2}+m+r\right)}{\sqrt{\pi}\,\Gamma\left(\frac{3}{2}+r\right)} \times$$
$$\begin{cases} \displaystyle\sum_{\ell=0}^{\lfloor \frac{r}{2}\rfloor+m} \frac{(-1)^{\ell+m}\left(2-\ell+2m+r+\frac{1}{2}\left(1+(-1)^{-\ell+m+\frac{r}{2}}\right)(1-2\ell+2m+2r)\right)\Gamma\left(\frac{1}{2}-\ell+m+r\right)}{\ell!\,\Gamma\left(\frac{3}{2}-\ell+m\right)(2-\ell+2m+r)!}, & r \text{ even}, \\ \displaystyle\sum_{\ell=0}^{\lfloor \frac{r-1}{2}\rfloor+m} \frac{(-1)^{m}\left((-1)^{\ell}-(-1)^{m+\frac{r+1}{2}}\right)\Gamma\left(\frac{1}{2}-\ell+m+r\right)}{\ell!\,\Gamma\left(\frac{3}{2}-\ell+m\right)(1-\ell+2m+r)!} & \\ \displaystyle+\Gamma\left(\frac{3}{2}+m\right)\Gamma\left(\frac{3}{2}+m+r\right) \sum_{\ell=0}^{\lfloor \frac{r+1}{2}\rfloor+m} \frac{(-1)^{\ell+m}\Gamma\left(\frac{3}{2}-\ell+m+r\right)}{\ell!\,\Gamma\left(\frac{3}{2}-\ell+m\right)(2-\ell+2m+r)!}, & r \text{ odd}. \end{cases} \tag{74}$$

Proof. Based on the connection formula (72), we get

$$\int_0^1 G_r^m(x)\,dx = \frac{\Gamma\left(\frac{3}{2}+m\right)\Gamma\left(\frac{3}{2}+m+r\right)}{\sqrt{\pi}\,\Gamma\left(\frac{3}{2}+r\right)} \times$$
$$\left(\sum_{\ell=0}^{\lfloor \frac{r}{2}\rfloor+m} \frac{(-1)^{\ell+m}(1-2\ell+2m+r)\Gamma\left(\frac{1}{2}-\ell+m+r\right)}{\ell!\,(1-\ell+2m+r)!\,\Gamma\left(\frac{3}{2}-\ell+m\right)} \int_0^1 U_{r+2m-2\ell}(x)\,dx \right.$$
$$\left. + \sum_{\ell=0}^{\lfloor \frac{r+1}{2}\rfloor+m} \frac{(-1)^{\ell+m}(2-2\ell+2m+r)\Gamma\left(\frac{3}{2}-\ell+m+r\right)}{\ell!\,(2-\ell+2m+r)!\,\Gamma\left(\frac{3}{2}-\ell+m\right)} \int_0^1 U_{r+2m-2\ell+1}(x)\,dx \right). \tag{75}$$

Based on the integral (see, [45])

$$\int_0^1 U_\ell(x)\,dx = \frac{1}{\ell+1} \begin{cases} 1, & \ell \text{ even}, \\ 1+(-1)^{\frac{\ell+3}{2}}, & \ell \text{ odd}, \end{cases} \tag{76}$$

we can obtain the following two integral formulas

$$\int_0^1 G_{2r}^m(x)\,dx = \frac{\Gamma\left(\frac{3}{2}+m\right)\Gamma\left(\frac{3}{2}+m+2r\right)}{\sqrt{\pi}\,\Gamma\left(\frac{3}{2}+2r\right)} \times$$

$$\sum_{\ell=0}^{r+m} \frac{(-1)^{\ell+m}\left(2-\ell+2m+2r+\frac{1}{2}\left(1+(-1)^{-\ell+m+r}\right)(1-2\ell+2m+4r)\right)\Gamma\left(\frac{1}{2}-\ell+m+2r\right)}{\ell!\,\Gamma\left(\frac{3}{2}-\ell+m\right)(2-\ell+2m+2r)!},$$

$$\int_0^1 G_{2r+1}^m(x)\,dx = \frac{\Gamma\left(\frac{3}{2}+m\right)\Gamma\left(\frac{5}{2}+m+2r\right)}{\sqrt{\pi}\,\Gamma\left(\frac{5}{2}+2r\right)} \times$$

$$\left(\sum_{\ell=0}^{r+m} \frac{(-1)^{\ell+m}\left(1+(-1)^{-\ell+m+r}\right)\Gamma\left(\frac{3}{2}-\ell+m+2r\right)}{\ell!\,(-\ell+2(1+m+r))!\,\Gamma\left(\frac{3}{2}-\ell+m\right)}\right.$$

$$\left.+\Gamma\left(\frac{3}{2}+m\right)\Gamma\left(\frac{5}{2}+m+2r\right)\sum_{\ell=0}^{r+m+1} \frac{(-1)^{\ell+m}\,\Gamma\left(\frac{5}{2}-\ell+m+2r\right)}{\ell!\,\Gamma\left(\frac{3}{2}-\ell+m\right)\Gamma(-\ell+2(2+m+r))}\right).$$

If the above two formulas are merged, then the integral formula (74) can be acquired. □

Remark 5 Every connection formula for any polynomial and an orthogonal polynomial $\theta_j(x)$ yields a weighted integral formula based on the orthogonality relation of $\theta_j(x)$. In this concern, we present the following corollary.

Corollary 6 For $\nu > -\frac{1}{2}$ and non-negative integers i, r, and m, the following integral formula applies:

$$\int_{-1}^1 \left(1-x^2\right)^{\nu-\frac{1}{2}} U_i^{(\nu)}(x)\, G_r^m(x)\,dx = \frac{\Gamma\left(\frac{3}{2}+m+r\right)\Gamma\left(\frac{1}{2}+\nu\right)\Gamma\left(\frac{1}{2}+m+\nu\right)}{\Gamma\left(\frac{3}{2}+r\right)} \times$$

$$\begin{cases} \dfrac{(-1)^{\frac{r-i}{2}}\,\Gamma\left(\frac{1}{2}(1+i+r)\right)}{\left(m+\frac{r-i}{2}\right)!\,\Gamma\left(\frac{1}{2}(1+i-r)+\nu\right)\Gamma\left(\frac{1}{2}(2+i+2m+r)+\nu\right)}, & (r+i)\,\text{even},\\ \dfrac{(-1)^{\frac{1}{2}(r-i+1)}\,\Gamma\left(\frac{1}{2}(2+i+r)\right)}{\left(m+\frac{1}{2}(r-i+1)\right)!\,\Gamma\left(\frac{i-r}{2}+\nu\right)\Gamma\left(\frac{1}{2}(3+i+2m+r)+\nu\right)}, & (r+i)\,\text{odd}. \end{cases} \tag{77}$$

Proof. Making use of the connection formula (70), it can be shown that

$$\begin{aligned} \int_{-1}^1 \left(1-x^2\right)^{\nu-\frac{1}{2}} U_i^{(\nu)}(x)\, G_r^m(x)\,dx =& \sum_{\ell=0}^{\lfloor\frac{r}{2}\rfloor+m} Q_{\ell,\,r,\,m} \int_{-1}^1 \left(1-x^2\right)^{\nu-\frac{1}{2}} U_i^{(\nu)}(x)\, U_{r+2m-2\ell}^{(\nu)}(x)\,dx \\ &+ \sum_{\ell=0}^{\lfloor\frac{r+1}{2}\rfloor+m} \bar{Q}_{\ell,\,r,\,m} \int_{-1}^1 \left(1-x^2\right)^{\nu-\frac{1}{2}} U_i^{(\nu)}(x)\, U_{r+2m-2\ell+1}^{(\nu)}(x)\,dx, \end{aligned} \tag{78}$$

where $Q_{\ell,\,r,\,,\,m}$ and $\bar{Q}_{\ell,\,r,\,,\,m}$ are given as follows:

$$Q_{\ell,\, r,\, ,\, m} = \frac{(-1)^{\ell+m}\, 2^{1-2\nu}\, (-2\ell+2m+r+\nu)\, \Gamma\left(\frac{1}{2}+m+r\right)\, \Gamma\left(\frac{3}{2}+m+r\right)\, \Gamma(-2\ell+2m+r+2\nu)}{\ell!\, (-2\ell+2m+r)!\, \Gamma\left(\frac{3}{2}+r\right)\, \Gamma\left(\frac{1}{2}+\nu\right)\, \Gamma(1-\ell+2m+r+\nu)} \times$$

$$\frac{\left(\frac{1}{2}-\ell+m+\nu\right)_{\ell}}{\left(\frac{1}{2}-\ell+m+r\right)_{\ell}},$$

$$\bar{Q}_{\ell,\, r,\, ,\, m} = \frac{(-1)^{\ell+m}\, 2^{1-2\nu}\, (1-2\ell+2m+r+\nu)\, \left(\Gamma\left(\frac{3}{2}+m+r\right)\right)^2\, \Gamma(1-2\ell+2m+r+2\nu)}{\ell!\, \Gamma\left(\frac{3}{2}+r\right)\, (1-2\ell+2m+r)!\, \Gamma\left(\frac{1}{2}+\nu\right)\, \Gamma(2-\ell+2m+r+\nu)} \times$$

$$\frac{\left(\frac{1}{2}-\ell+m+\nu\right)_{\ell}}{\left(\frac{3}{2}-\ell+m+r\right)_{\ell}}.$$

The orthogonality relation of the ultraspherical polynomials (see, [38]) turns (78) into

$$\int_{-1}^{1} \left(1-x^2\right)^{\nu-\frac{1}{2}} U_i^{(\nu)}(x)\, G_r^m(x)\, dx = \sum_{\ell=0}^{\left\lfloor \frac{r}{2} \right\rfloor + m} Q_{\ell,\, r,\, m} h_i\, \delta_{i,\, r+2m-2\ell} + \sum_{\ell=0}^{\left\lfloor \frac{r+1}{2} \right\rfloor + m} \bar{Q}_{\ell,\, r,\, m}\, h_i\, \delta_{i,\, r+2m-2\ell+1}, \tag{79}$$

and this results in the following integral formula:

$$\int_{-1}^{1} \left(1-x^2\right)^{\nu-\frac{1}{2}} U_i^{(\nu)}(x)\, G_r^m(x)\, dx = h_i \begin{cases} Q_{\frac{r-i}{2}+m,\, r,\, m}, & (r+i)\,\text{even}, \\ \bar{Q}_{\frac{r-i+1}{2}+m,\, r,\, m}, & (r+i)\,\text{odd}. \end{cases} \tag{80}$$

Thus, we obtain Formula (77). □

Corollary 7 The GJPs-Hermite connection formula is

$$G_r^m(x) = \frac{(-1)^m \Gamma\left(\frac{1}{2}+m+r\right) \Gamma\left(\frac{3}{2}+m+r\right)}{\sqrt{\pi}\, \Gamma\left(\frac{3}{2}+r\right)} \times$$

$$\left(\sum_{\ell=0}^{\left\lfloor \frac{r}{2} \right\rfloor + m} \frac{1}{\ell!\, (-2\ell+2m+r)!}\, {}_1F_1\left(-\ell;\, \frac{1}{2}-m-r;\, -1\right) H_{r+2m-2\ell}(x) \right. \tag{81}$$

$$\left. + \sum_{\ell=0}^{\left\lfloor \frac{r+1}{2} \right\rfloor + m} \frac{\frac{1}{2}+m+r}{\ell!\, (-2\ell+2m+r+1)!}\, {}_1F_1\left(-\ell;\, -\frac{1}{2}-m-r;\, -1\right) H_{r+2m-2\ell+1}(x) \right).$$

Proof. Substitution by $p=0$ in (66) yields Formula (81). □

Corollary 8 The next integral formula applies:

$$\int_{-\infty}^{\infty} e^{-x^2} H_i(x)\, G_r^m(x)\, dx =$$

$$\begin{cases} \dfrac{(-1)^m\, 2^i\, \Gamma\left(\frac{1}{2}+m+r\right)\, \Gamma\left(\frac{3}{2}+m+r\right)}{\left(\frac{1}{2}(r-i+2m)\right)!\, \Gamma\left(\frac{3}{2}+r\right)}\, {}_1F_1\left(\dfrac{1}{2}(i-2m-r);\, \dfrac{1}{2}-m-r;\, -1\right), & (r+i)\,\text{even}, \\ \dfrac{(-1)^m\, 2^i\, \left(\Gamma\left(\frac{3}{2}+m+r\right)\right)^2}{\left(\frac{1}{2}(r-i+2m+1)\right)!\, \Gamma\left(\frac{3}{2}+r\right)}\, {}_1F_1\left(\dfrac{1}{2}(i-2m-r-1);\, -\dfrac{1}{2}-m-r;\, -1\right), & (r+i)\,\text{odd}. \end{cases} \tag{82}$$

Proof. The connection formula (81) along with the orthogonality relation of Hermite polynomials ([1]) yields the desired integral formula (82). □

Corollary 9 The GJPs-generalized Fibonacci connection formula is:

$$\begin{aligned} G_r^m(x) = & \frac{2^{2m+r}\, a^{-2m-r}\, \Gamma\left(\frac{1}{2}+m+r\right)\, \Gamma\left(\frac{3}{2}+m+r\right)}{\sqrt{\pi}\, \Gamma\left(\frac{3}{2}+r\right)} \sum_{\ell=0}^{\left\lfloor \frac{r}{2} \right\rfloor + m} \frac{(-1)^{1+\ell+m}\, b^{\ell}\, (-1+2\ell-2m-r)}{\ell!\, (1-\ell+2m+r)!} \times \\ & {}_2F_1\left(\left. \begin{array}{c} -\ell,\, -1+\ell-2m-r \\ \frac{1}{2}-m-r \end{array} \right| -\frac{a^2}{4b} \right) F_{r+2m-2\ell}^{a,\, b}(x) \\ & + \frac{2^{1+2m+r}\, a^{-1-2m-r}\, \left(\Gamma\left(\frac{3}{2}+m+r\right)\right)^2}{\sqrt{\pi}\, \Gamma\left(\frac{3}{2}+r\right)} \sum_{\ell=0}^{\left\lfloor \frac{r+1}{2} \right\rfloor + m} \frac{(-1)^{1+\ell+m}\, b^{\ell}\, (-2+2\ell-2m-r)}{\ell!\, (2-\ell+2m+r)!} \times \\ & {}_2F_1\left(\left. \begin{array}{c} -\ell,\, -2+\ell-2m-r \\ -\frac{1}{2}-m-r \end{array} \right| -\frac{a^2}{4b} \right) F_{r+2m-2\ell+1}^{a,\, b}(x). \end{aligned} \tag{83}$$

Proof. Substitution by $p = 0$ in (67) yields Formula (83). □

Corollary 10 The GJPs-generalized Lucas connection formula is:

$$G_r^m(x) = \frac{2^{2m+r}\bar{a}^{-2m-r}\Gamma\left(\frac{1}{2}+m+r\right)\Gamma\left(\frac{3}{2}+m+r\right)}{\sqrt{\pi}\Gamma\left(\frac{1}{2}(3+2r)\right)} \sum_{\ell=0}^{\lfloor \frac{r}{2} \rfloor+m} \frac{(-1)^{\ell+m} c_{-2\ell+2m+r}\bar{b}^{\ell}}{\ell!(-\ell+2m+r)!} \times$$

$${}_2F_1\left(\begin{matrix} -\ell, \ell-2m-r \\ \frac{1}{2}-m-r \end{matrix} \middle| -\frac{\bar{a}^2}{4\bar{b}}\right) L_{r+2m-2\ell}^{\bar{a},\bar{b}}$$

$$+ \frac{2^{1+2m+r}\bar{a}^{-1-2m-r}\left(\Gamma\left(\frac{3}{2}+m+r\right)\right)^2}{\sqrt{\pi}\Gamma\left(\frac{3}{2}+r\right)} \sum_{\ell=0}^{\lfloor \frac{r+1}{2} \rfloor+m} \frac{(-1)^{\ell+m} c_{-2\ell+2m+r+1}\bar{b}^{\ell}}{\ell!(1-\ell+2m+r)!} \times \tag{84}$$

$${}_2F_1\left(\begin{matrix} -\ell, -1+\ell-2m-r \\ -\frac{1}{2}-m-r \end{matrix} \middle| -\frac{\bar{a}^2}{4\bar{b}}\right) L_{r+2m-2\ell+1}^{\bar{a},\bar{b}}.$$

Proof. Substitution by $p=0$ in (68) yields Formula (84). □

6. Some linearization formulas involving GJPs

This section introduces some new linearization formulas for the GJPs and some other polynomials. In addition, based on these formulas, some new definite integrals will be deduced.

Theorem 8 Let $\psi_j(x)$ be any kind of Chebyshev polynomial of the four kinds. The next linearization formula (LF) holds:

$$G_r^m(x)\psi_j(x) = \frac{\left(r+\frac{3}{2}\right)_m}{\sqrt{\pi}} \left(\sum_{i=0}^{r+2m} \frac{(-1)^{m+i}\left(m-i+\frac{1}{2}\right)_i \Gamma\left(r+m-i+\frac{1}{2}\right)}{i!(-i+r+2m)!} \psi_{r+2m+j-2i}(x)\right.$$

$$\left. + \sum_{i=0}^{r+2m+1} \frac{(-1)^{m+i}\left(m-i+\frac{1}{2}\right)_i \Gamma\left(r+m-i+\frac{3}{2}\right)}{i!(-i+r+2m+1)!} \psi_{r+2m+j-2i+1}(x)\right). \tag{85}$$

Proof. From the analytic form in (37), we have

$$G_r^m(x)\psi_j(x) = \frac{(2r+3)\left(\frac{5}{2}+r\right)_{m-1}}{\sqrt{\pi}} \left(\sum_{\ell=0}^{\lfloor \frac{r}{2} \rfloor+m} \frac{(-1)^{\ell+m} 2^{-1+r-2\ell+2m}\Gamma\left(\frac{1}{2}+r-\ell+m\right)}{\ell!(r-2\ell+2m)!} x^{r-2\ell+2m}\psi_j(x)\right.$$

$$\left. + \sum_{\ell=0}^{\lfloor \frac{r+1}{2} \rfloor+m} \frac{(-1)^{\ell+m} 2^{r-2\ell+2m}\Gamma\left(\frac{3}{2}+r-\ell+m\right)}{\ell!(r-2\ell+2m+1)!} x^{r-2\ell+2m+1}\psi_j(x)\right). \tag{86}$$

The application to the moment formula of $\psi_j(x)$ given by

$$x^m\,\psi_r(x)=\frac{1}{2^r}\sum_{s=0}^{r}\binom{r}{s}\psi_{m+r-2s}(x),$$

yields the following formula

$$G_r^m(x)\,\psi_j(x)=\frac{(2r+3)\,\left(\frac{5}{2}+r\right)_{m-1}}{\sqrt{\pi}}\times$$

$$\left(\sum_{\ell=0}^{\lfloor\frac{r}{2}\rfloor+m}\frac{(-1)^{\ell+m}\,2^{-1+r-2\ell+2m}\,\Gamma\left(\frac{1}{2}+r-\ell+m\right)}{\ell!\,(r-2\ell+2m)!}\sum_{s=0}^{r-2\ell+2m}2^{-r+2\ell-2m}\binom{r-2\ell+2m}{s}\psi_{r-2\ell+2m+j-2s}(x)\right. \tag{87}$$

$$\left.+\sum_{\ell=0}^{\lfloor\frac{r+1}{2}\rfloor+m}\frac{(-1)^{\ell+m}\,2^{r-2\ell+2m}\,\Gamma\left(\frac{3}{2}+r-\ell+m\right)}{\ell!\,(r-2\ell+2m+1)!}\sum_{s=0}^{r-2\ell+2m+1}2^{-1-r+2\ell-2m}\binom{1+r-2\ell+2m}{s}\psi_{r-2\ell+2m+j-2s+1}(x)\right).$$

Formula (87) can be alternatively rewritten as

$$\begin{aligned}G_r^m(x)\,\psi_j(x)=&\frac{\left(r+\frac{3}{2}\right)_m}{\sqrt{\pi}}\left(\sum_{i=0}^{r+2m}\sum_{\ell=0}^{i}\frac{(-1)^{\ell+m}\,\Gamma\left(\frac{1}{2}+r-\ell+m\right)}{\ell!\,(r-\ell+2m-i)!\,(i-\ell)!}\psi_{r+2m+j-2i}(x)\right.\\&\left.+\sum_{i=0}^{r+2m+1}\sum_{\ell=0}^{i}\frac{(-1)^{\ell+m}\,\Gamma\left(\frac{3}{2}+r-\ell+m\right)}{\ell!\,(r-\ell+2m-i+1)!\,(i-\ell)!}\psi_{r+2m+j-2i+1}(x)\right).\end{aligned} \tag{88}$$

Based on the algorithm of Zeilberger (see, [44]), it can be shown that

$$\sum_{\ell=0}^{i}\frac{(-1)^{\ell+m}\,\Gamma\left(\frac{1}{2}+r-\ell+m\right)}{\ell!\,(r-\ell+2m-i)!\,(i-\ell)!}=\frac{(-1)^{m+i}\,\left(m-i+\frac{1}{2}\right)_i\,\Gamma\left(r+m-i+\frac{1}{2}\right)}{(-i+r+2m)!\,i!}, \tag{89}$$

$$\sum_{\ell=0}^{i}\frac{(-1)^{\ell+m}\,\Gamma\left(\frac{3}{2}+r-\ell+m\right)}{\ell!\,(r-\ell+2m-i+1)!\,(i-\ell)!}=\frac{(-1)^{m+i}\,\left(m-i+\frac{1}{2}\right)_i\,\Gamma\left(r+m-i+\frac{3}{2}\right)}{(-i+r+2m+1)!\,i!}, \tag{90}$$

and hence the linearization formula (88) transforms into the following one:

$$G_r^m(x)\,\psi_j(x) = \frac{\left(r+\frac{3}{2}\right)_m}{\sqrt{\pi}} \left(\sum_{i=0}^{r+2m} \frac{(-1)^{m+i}\,\left(m-i+\frac{1}{2}\right)_i \Gamma\left(r+m-i+\frac{1}{2}\right)}{i!\,(-i+r+2m)!}\,\psi_{r+2m+j-2i}(x) \right.$$

$$\left. + \sum_{i=0}^{r+2m+1} \frac{(-1)^{m+i}\,\left(m-i+\frac{1}{2}\right)_i \Gamma\left(r+m-i+\frac{3}{2}\right)}{i!\,(-i+r+2m+1)!}\,\psi_{r+2m+j-2i+1}(x) \right).$$

This proves Theorem 8. □

Corollary 11 For all positive integers $i,\ j$, r and m, the next integral formula applies:

$$\int_0^1 G_r^m(x)\,T_i(x)\,T_j(x)\,dx = \frac{1}{2}\left(\sum_{p=0}^{r+2m} A_{p,\,r,\,m}\,\left(Z_{i+r+2m+j-2p} + Z_{r+2m+j-2p-i}\right) \right.$$
$$\left. + \sum_{p=0}^{r+2m+1} B_{p,\,r,\,m}\,\left(Z_{i+r+2m+j-2p+1} + Z_{r+2m+j-2p-i+1}\right) \right), \tag{91}$$

where

$$A_{p,\,r,\,m} = \frac{(-1)^{m+p}\,\Gamma\left(\frac{1}{2}+m\right)\Gamma\left(\frac{3}{2}+r+m\right)\Gamma\left(\frac{1}{2}+r+m-p\right)}{\sqrt{\pi}\,\Gamma\left(\frac{3}{2}+r\right)\Gamma\left(\frac{1}{2}+m-p\right)\,p!\,(r+2m-p)!}, \tag{92}$$

$$B_{p,\,r,\,m} = \frac{(-1)^{m+p}\,\left(\Gamma\left(\frac{3}{2}+r+m\right)\right)^2\,\left(\frac{1}{2}+m-p\right)_p}{\sqrt{\pi}\,\Gamma\left(\frac{3}{2}+r\right)\,p!\,(1+r+2m-p)!\,\left(\frac{3}{2}+r+m-p\right)_p}, \tag{93}$$

$$Z_j = \begin{cases} \frac{1}{1-j^2}, & j \text{ even}, \\ \frac{-1-(-1)^{\frac{j+1}{2}}j}{(j-1)(j+1)}, & j \text{ odd},\ j^2 \neq 1, \\ \frac{1}{2}, & j^2 = 1. \end{cases} \tag{94}$$

Proof. The linearization formula (85) for $\psi_j(x) = T_j(x)$ yields the following triple product:

$$\phi_r(x)\,T_i(x)T_j(x) = \sum_{p=0}^{r+2m} A_{p,\,r,\,m}T_i(x)T_{r+2m+j-2p}(x) + \sum_{p=0}^{r+2m+1} B_{p,\,r,\,m}T_i(x)T_{r+2m+j-2p+1}(x), \tag{95}$$

where $A_{p,\,r,\,m}$ and $B_{p,\,r,\,m}$ are as given in (92) and (93). Based on the well-known linearization formulas:

$$T_r(x)\,T_s(x) = \frac{1}{2}\left(T_{r+s}(x) + T_{r-s}(x)\right),$$

we get

$$\int_0^1 \phi_r(x)\,T_i(x)\,T_j(x)\,dx = \frac{1}{2}\left(\sum_{p=0}^{r+2m} A_{p,\,r,\,m}\int_0^1 \left(T_{i+r+2m+j-2p}(x)+T_{r+2m+j-2p-i}(x)\right)dx \right.$$
$$\left. + \sum_{p=0}^{r+2m+1} B_{p,\,r,\,m}\int_0^1 \left(T_{i+r+2m+j-2p+1}(x)+T_{r+2m+j-2p-i+1}(x)\right)dx\right). \tag{96}$$

The last formula can be turned into Formula (91) using the definite integral (see, [45])

$$\int_0^1 T_j(x)\,dx = \begin{cases} \frac{1}{1-j^2}, & j \text{ even},\\ \frac{-1-(-1)^{\frac{j+1}{2}}j}{(j-1)(j+1)}, & j \text{ odd},\ j^2 \neq 1,\\ \frac{1}{2}, & j^2 = 1. \end{cases}$$

It is thus possible to obtain Formula (91). □

Theorem 9 For all non-negative integers $r,\ j$, and m. the next LF applies:

$$G_r^m(x)\,F_j^{a,\,b}(x) = \frac{(-1)^m\,2^{r+2m}\,a^{-r-2m}\,\Gamma\left(\frac{1}{2}+r+m\right)\Gamma\left(\frac{3}{2}+r+m\right)}{\sqrt{\pi}\,\Gamma\left(\frac{3}{2}+r\right)\,(r+2m)!}\times$$
$$\sum_{i=0}^{r+2m}(-b)^i\binom{r+2m}{i}\,{}_2F_1\left(\left.\begin{array}{c}-i,\,-r-2m+i\\ \frac{1}{2}-r-m\end{array}\right|-\frac{a^2}{4b}\right)F^{a,\,b}_{r+2m+j-2i}(x)$$
$$+\frac{(-1)^m\,2^{r+2m}\,a^{-1-r-2m}\,(3+2r)\,\Gamma\left(\frac{3}{2}+r+m\right)^2}{\sqrt{\pi}\,\Gamma\left(\frac{5}{2}+r\right)\,(r+2m+1)!}\times$$
$$\sum_{i=0}^{r+2m+1}(-b)^i\binom{1+r+2m}{i}\,{}_2F_1\left(\left.\begin{array}{c}-i,\,-1-r-2m+i\\ -\frac{1}{2}-r-m\end{array}\right|-\frac{a^2}{4b}\right)F^{a,\,b}_{r+2m+j-2i+1}(x). \tag{97}$$

Proof. The proof can be done using the moment formula of the $F_i^{a,\,b}(x)$ (see, [28]). □

Theorem 10 For all non-negative integers $r,\ j$, and m, the following LF holds

$$G_r^m(x)\,L_j^{\bar{a},\bar{b}}(x) = \frac{(-1)^m\,2^{r+2m}\bar{a}^{-r-2m}\,\Gamma\left(\frac{1}{2}+r+m\right)\Gamma\left(\frac{3}{2}+r+m\right)}{\sqrt{\pi}\,\Gamma\left(\frac{3}{2}+r\right)(r+2m)!}\times$$

$$\sum_{i=0}^{r+2m}(-\bar{b})^i\binom{r+2m}{i}\,{}_2F_1\left(\left.\begin{array}{c}-i,\,-r-2m+i\\ \frac{1}{2}-r-m\end{array}\right|-\frac{\bar{a}^2}{4\bar{b}}\right)L_{r+2m+j-2i}^{\bar{a},\bar{b}}(x)$$

$$+\frac{(-1)^m\,2^{r+2m}\,\bar{a}^{-1-r-2m}\,(3+2r)\,\Gamma\left(\frac{3}{2}+r+m\right)^2}{\sqrt{\pi}\,\Gamma\left(\frac{5}{2}+r\right)(r+2m+1)!}\times \tag{98}$$

$$\sum_{i=0}^{r+2m+1}(-\bar{b})^i\binom{1+r+2m}{i}\,{}_2F_1\left(\left.\begin{array}{c}-i,\,-1-r-2m+i\\ -\frac{1}{2}-r-m\end{array}\right|-\frac{\bar{a}^2}{4\bar{b}}\right)L_{r+2m+j-2i+1}^{\bar{a},\bar{b}}(x).$$

Proof. The proof can be done using the moment formula of the $L_i^{\bar{a},\bar{b}}(x)$ (see, [28]). □

7. Conclusions

This article was devoted to investigating the JPs of some particular negative integers. We showed that this kind of polynomial can be represented as a combination of Legendre polynomials. The basic formulas related to these polynomials were derived, which helped establish new results concerning these polynomials. Derivative formulas for the generalized polynomials are found in terms of some other orthogonal and non-orthogonal polynomials. Some product formulas of these polynomials with some other polynomials were derived. Some definite and weighted definite integrals were deduced according to the derived formulas in this paper. We anticipate further research into other orthogonal combinations of polynomials in order to apply them to the solution of additional classes of differential equations.

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Conflict of interest

The authors declare no competing financial interest.

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