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Research Article

Exploring the Hidden Symmetry of Quaternions: Dual Balancing and Cobalancing Numbers and Their Corresponding Quaternions

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Abstract: This paper unveils the hidden symmetry of quaternions through the introduction of Dual Balancing and Dual Cobalancing numbers, along with their corresponding quaternions. We provide Binet's formulas, generating functions, and a multitude of intriguing properties to foster a comprehensive understanding of these concepts. Furthermore, we present matrix representations to offer a fresh perspective on the Dual Balancing and Dual Cobalancing quaternions. The implications of this study go beyond theoretical applications and may have practical implications in diverse fields.

***Keywords*:** dual balancing numbers, dual cobalancing numbers, dual balancing and cobalancing quaternions, recurrence relations, matrix representations

MSC: 11B37, 20G20, 11R52

1. Introduction

In recent years, a new concept in sequences of integer of Balancing numbers was first introduced in the year 1999 by Behera and Panda [1] as solutions of the equation

$$1+2+\ldots+(n-1)=(n+1)+(n+2)+\ldots+(n+r) \tag{1}$$

They call $n \in \mathbb{Z}^+$ a balancing number and r, the balancer corresponding to n. For example: the corresponding of the balancing numbers are 6, 35 and 204 with 2, 14 and 84, respectively.

In [1], Behera and Panda obtained the recurrence relation

$$B_{n+1}=6B_n-B_{n-1} \tag{2}$$

and then developed in [1–7] the Binet formula by solving this recurrence relation as a second order linear homogeneous difference equation. Also, they studied the generating functions and some interesting results.

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By slightly modifying equation (1), Panda and Ray [3, 4, 8] introduced $n \in \mathbb{Z}^+$ a cobalancing number if

$$1+2+\ldots+n=(n+1)+(n+2)+\ldots+(n+r)$$

for some $r \in \mathbb{Z}^+$. r is called cobalancer corresponding to the cobalancing number n. The first three cobalancing numbers are 2, 14 and 84 with cobalancers 1, 6 and 35, respectively. They developed the recurrence relation

$$b_{n+1}=6b_n-b_{n-1}+2 \tag{3}$$

for cobalancing numbers. Using this recurrence relation, they obtained the generating function for cobalancing numbers and proved some interesting results between balancing and cobalancing numbers.

The family of quaternion arithmetic plays an important role in mathematics such as algebraic systems, skew field or non commutative division algebras and matrices in commutative rings. We can find studying areas in mathematics in [9].

The real quaternions were first introduced by Irish Mathematician William Rowan Hamilton in 1843 [10]. Hamilton defined the set of real quaternions which can be represented as

$$H=\{q=q_0e_0+q_1e_1+q_2e_2+q_3e_3 : q_i \in \mathbb{R},\ i=0,\ 1,\ 2,\ 3\}$$

as the four-dimensional vector space over $\mathbb{R}$ having a basis $\{e_0,\ e_1,\ e_2,\ e_3\}$ where

$$e_1^2=e_2^2=e_3^2=-1$$

$$e_1e_2=-e_2e_1=e_3, \quad e_2e_3=-e_3e_2=e_1, \quad e_3e_1=-e_1e_3=e_2$$

Note that the set of real quaternions form an associative but non commutative algebra.

The quaternion q is a hyper-complex number and can be written as

$$q=q_0e_0+q_1e_1+q_2e_2+q_3e_3=\sum_{i=0}^{3} q_ie_i \in H.$$

Also, q is shown in two parts as $q=S_q+\overrightarrow{V_q}$ where $S_q=q_0e_0$ and $\overrightarrow{V_q}=q_1e_1+q_2e_2+q_3e_3$. Here, the first part S_q is called the scalar part and the second part $\overrightarrow{V_q}$ is called the vector part of the quaternion q. If two quaternions as $q=S_q+\overrightarrow{V_q}=q_0e_0+\sum_{i=1}^{3} q_ie_i$ and $p=S_p+\overrightarrow{V_p}=p_0e_0+\sum_{i=1}^{3} p_ie_i$ then the addition and subtraction of them are

$$q \pm p=(q_0 \pm p_0)e_0+(q_1 \pm p_1)e_1+(q_2 \pm p_2)e_2+(q_3 \pm p_3)e_3$$

$$=\sum_{i=0}^{3}(q_i \pm p_i)\,e_i$$

and the multiplication of q and p is defined by

$$q \cdot p = S_q \cdot S_p + S_q \cdot \overrightarrow{V_p} + \overrightarrow{V_q} \cdot S_p - \overrightarrow{V_q} \cdot \overrightarrow{V_p} + \overrightarrow{V_q} \times \overrightarrow{V_p}$$

where $\overrightarrow{V_q} \cdot \overrightarrow{V_p} = q_1 p_1 + q_2 p_2 + q_3 p_3$ and $\overrightarrow{V_q} \cdot \overrightarrow{V_p} = (q_2 p_3 - q_3 p_2)e_1 - (q_1 p_3 - q_3 p_1)e_2 + (q_1 p_2 - q_2 p_1)e_3$.

The conjugate of the quaternion q is defined by $\overline{q}$,

$$\overline{q} = S_q - \overrightarrow{V_q} = q_0 e_0 - q_1 e_1 - q_2 e_2 - q_3 e_3$$

$$= q_0 e_0 - \sum_{i=1}^{3} q_i e_i$$

Also, let q and p be two quaternions, then $\overline{\overline{q}} = q$, $(\overline{qp}) = \overline{pq}$.

The norm of q is defined by $N(q)$,

$$\|q\| = N(q) = q \cdot \overline{q} = q_0^2 + q_1^2 + q_2^3 = \sum_{i=0}^{3} q_i^2$$

If $N(q) = 1$, the quaternion of q is called a unit quaternion.

The inverse of q is denoted in [11] by q^{-1} as

$$q^{-1} = \frac{\overline{q}}{N(q)} = \frac{\overline{q}}{q \cdot \overline{q}}.$$

William Clifford introduced dual numbers in 1873 when dealing with the theory of engines which used a nilpotent noted ε. The dual numbers extend to the real numbers has the form in [12, 13]

$$d = a + \varepsilon a^* \ \text{ with } \ a,\ a^* \in \mathbb{R}$$

where ε is the dual unit and $\varepsilon^2 = 0$, $\varepsilon \neq 0$. Their application to the study of kinematics of rigid articulated bodies was developed by Kotelnikov in [14, 15].

The dual quaternion is shown in the form as

$$DQ = q + \varepsilon q^*$$

where q and q^* are quaternions and ε is a dual unit.

Let q and q^* be two quaternions such that $q = q_0 e_0 + q_1 e_1 + q_2 e_2 + q_3 e_3$ and $q^* = q_0^* e_0 + q_1^* e_1 + q_2^* e_2 + q_3^* e_3$, then the dual quaternion DQ can be described as:

$$
\begin{aligned}
DQ &= q+\varepsilon q^{*} \\
&= (q_0e_0+q_1e_1+q_2e_2+q_3e_3)+\varepsilon(q_0^{*}e_0+q_1^{*}e_1+q_2^{*}e_2+q_3^{*}e_3) \\
&= (q_0+\varepsilon q_{0^{*}})e_0+(q_1+\varepsilon q_{1^{*}})e_1+(q_2+\varepsilon q_{2^{*}})e_2+(q_3+\varepsilon q_{3^{*}})e_3 \\
&= \sum_{i=0}^{3}(q_i+\varepsilon q_i^{*})e_i.
\end{aligned}
$$

Hence, the dual quaternion DQ has eight real parameters. So, DQ can be written as

$$DQ = S_{DQ}+\overrightarrow{V_{DQ}}$$

where S_{DQ} is called scalar part as $S_{DQ}=q_0+\varepsilon q_0^{*}=S_q+\varepsilon S_{q^{*}}$ and $\overrightarrow{V_{DQ}}$ is called vectoral part as $\overrightarrow{V_{DQ}}=(q_1+\varepsilon q_{1^{*}})e_1+(q_2+\varepsilon q_{2^{*}})e_2+(q_3+\varepsilon q_{3^{*}})e_3=\overrightarrow{V_q}+\varepsilon\overrightarrow{V_{q^{*}}}$, respectively.

If two dual quaternions are $DQ=q+\varepsilon q^{*}$ and $DP=p+\varepsilon p^{*}$, then the addition and subtraction are defined by the following:

$$DQ\pm DP=(q\pm p)+\varepsilon(q^{*}\pm p^{*})$$

and multiplication is given by

$$DQ\cdot DP=q\cdot p+\varepsilon(q\cdot p^{*}+q^{*}\cdot p)$$

where $q=\sum_{i=0}^{3}q_ie_i,\ q^{*}=\sum_{i=0}^{3}q_i^{*}e_i,\ p=\sum_{i=0}^{3}p_ie_i$ and $p^{*}=\sum_{i=0}^{3}p_i^{*}e_i$.

The conjugate of the dual quaternion $DQ=q+\varepsilon q^{*}$ is defined as

$$
\begin{aligned}
\overline{DQ} &= \overline{q}+\varepsilon\overline{q^{*}} \\
&= (q_0+\varepsilon q_0^{*})e_0-(q_1+\varepsilon q_1^{*})e_1-(q_2+\varepsilon q_2^{*})e_2-(q_3+\varepsilon q_3^{*})e_3.
\end{aligned}
$$

The norm of DQ is written as

$$\|DQ\|=N(DQ)=DQ\cdot\overline{DQ}=A^2+B^2+C^2+D^2$$

where $A=q_0+\varepsilon q_0^{*}$, $B=q_1+\varepsilon q_1^{*}$, $C=q_2+\varepsilon q_2^{*}$ and $D=q_3+\varepsilon q_3^{*}$. If $N(DQ)=1$, the dual quaternion of DQ is called a unit dual quaternion.

To explore the vast landscape of quaternion and dual quaternion theory, readers are encouraged to refer to [9, 10, 16], where a wealth of results and insights on these fascinating mathematical constructs can be found.

Quaternions and dual quaternions have been the subject of numerous studies. Horadam's pioneering work in [17] introduced nth Fibonacci and Lucas quaternions in 1963, and later in [18] examined the recurrence relations of quaternions and defined Pell quaternions and generalized Pell quaternions. In [19], many intriguing properties of Fibonacci and Lucas quaternions were presented. Halıcı further explored these concepts in [20], providing Binet's formulas, generating functions, and various properties. In [21], Halıcı extended the research to complex Fibonacci quaternions, and presented their generating function and Binet formula. More recently, Torunbalcı and Yüce introduced the dual Pell quaternions in [22], and Torunbalcı and Yüce defined generalized dual Pell quaternions in [23]. Patel and Ray's work in [24] focused on Balancing and Lucas-Balancing Quaternions. Collectively, these studies have significantly advanced our understanding of quaternions and their various applications.

This paper delves into the unexplored territory of Dual Balancing and Cobalancing numbers, introducing their corresponding quaternions. By providing Binet's formulas, generating functions, and various properties, we present a comprehensive study of these concepts. Additionally, we showcase matrix representations to offer new insights into the Dual Balancing and Cobalancing quaternions. The results of this study are poised to make significant contributions to the field and open up new avenues for research and practical applications.

2. Dual balancing quaternions and dual cobalancing quaternions

Horadam in [17] introduced complex fibonacci numbers as

$$C_n = F_n + iF_{n+1}, \;\; i^2 = -1$$

where F_n is the nth Fibonacci number.

Also Ascı and Aydinyuz in [24] defined the nth Balancing and Cobalancing quaternions as follows:

$$QB_n = B_n e_0 + B_{n+1} e_1 + B_{n+2} e_2 + B_{n+3} e_3 \tag{4}$$

$$Qb_n = b_n e_0 + b_{n+1} e_1 + b_{n+2} e_2 + b_{n+3} e_3 \tag{5}$$

where B_n and b_n are nth Balancing and Cobalancing numbers, respectively.

Clifford introduced dual numbers as

$$d = a + \varepsilon a^*$$

where ε is the dual unit and $\varepsilon^2 = 0, \;\varepsilon \neq 0$.

Now, with the same logic we can define dual Balancing and dual Cobalancing numbers, dual Balancing Quaternions and dual Cobalancing quaternions.

Definition 1 The nth dual Balancing and nth dual Copbalancing numbers are defined by

$$DB_n = B_n + \varepsilon B_{n+1} \tag{6}$$

$$Db_n = b_n + \varepsilon b_{n+1} \tag{7}$$

respectively, where ε is the dual unit and $\varepsilon^2 = 0$, $\varepsilon \neq 0$. Here, B_n is the nth Balancing number and b_n is the nth Cobalancing number.

Definition 2 The nth dual Balancing quaternion DQB_n and the nth dual Cobalancing quaternion DQb_n are defined, respectively, as shown

$$DQB_n = QB_n + \varepsilon QB_{n+1} \tag{8}$$

and

$$DQb_n = Qb_n + \varepsilon Qb_{n+1} \tag{9}$$

where $QB_n = B_n e_0 + B_{n+1} e_1 + B_{n+2} e_2 + B_{n+3} e_3$ is the nth Balancing quaternions and $Qb_n = b_n e_0 + b_{n+1} e_1 + b_{n+2} e_2 + b_{n+3} e_3$ is the nth Cobalancing quaternions.

The dual Balancing quaternions and Cobalancing quaternions has four dual elements and can be shown that

$$\begin{aligned} DQB_n &= QB_n + \varepsilon QB_{n+1} \\ &= (B_n e_0 + B_{n+1} e_1 + B_{n+2} e_2 + B_{n+3} e_3) \\ &\quad + \varepsilon(B_{n+1} e_0 + B_{n+2} e_1 + B_{n+3} e_3 + B_{n+4} e_4) \\ &= (B_n + \varepsilon B_{n+1}) e_0 + (B_{n+1} + \varepsilon B_{n+2}) e_1 \\ &\quad + (B_{n+2} + \varepsilon B_{n+3}) e_2 + (B_{n+3} + \varepsilon B_{n+4}) e_3 \\ &= DB_n e_0 + DB_{n+1} e_1 + DB_{n+2} e_2 + DB_{n+3} e_3 \end{aligned}$$

Now we show the dual Cobalancing quaternions as follows:

$$
\begin{aligned}
DQb_n &= Qb_n + \varepsilon Qb_{n+1} \\
&= (b_n e_0 + b_{n+1}e_1 + b_{n+2}e_2 + b_{n+3}e_3) \\
&\quad + \varepsilon(b_{n+1}e_0 + b_{n+2}e_1 + b_{n+3}e_2 + b_{n+4}e_3) \\
&= (b_n + \varepsilon b_{n+1})e_0 + (b_{n+1} + \varepsilon b_{n+2})e_1 \\
&\quad + (b_{n+2} + \varepsilon b_{n+3})e_2 + (b_{n+3} + \varepsilon b_{n+4})e_3 \\
&= Db_n e_0 + Db_{n+1}e_1 + Db_{n+2}e_2 + Db_{n+3}e_3.
\end{aligned}
$$

Also the scalar part of the dual Balancing quaternion is

$$
\begin{aligned}
S_{DQB_n} &= B_n + \varepsilon B_{n+1} \\
&= DB_n
\end{aligned}
$$

and vectorel part is

$$
\begin{aligned}
\overrightarrow{V_{DQB_n}} &= (B_{n+1} + \varepsilon B_{n+2})e_1 + (B_{n+2} + \varepsilon B_{n+3})e_2 + (B_{n+3} + \varepsilon B_{n+4})e_3 \\
&= DB_{n+1}e_0 + DB_{n+2}e_1 + DB_{n+3}e_3.
\end{aligned}
$$

Let DQB_n and DQM_n be two dual Balancing quaternions such that $DQB_n = QB_n + \varepsilon QB_{n+1}$ and $DQM_n = QM_n + \varepsilon QM_{n+1}$. The addition, subtraction and multiplication of them is shown as

$$
DQB_n \pm DQM_n = (QB_n \pm QM_n) + \varepsilon(QB_{n+1} \pm QM_{n+1})
$$

$$
DQB_n \cdot DQM_n = QB_n \cdot QM_n + \varepsilon(QB_n \cdot QM_{n+1} + QB_{n+1} \cdot QM_n)
$$

where $QB_n = B_n e_0 + B_{n+1}e_1 + B_{n+2}e_2 + B_{n+3}e_3$, $QB_{n+1} = B_{n+1}e_0 + B_{n+2}e_1 + B_{n+3}e_2 + B_{n+4}e_3$, $QM_n = M_n e_0 + M_{n+1}e_1 + M_{n+2}e_2 + M_{n+3}e_3$ and $QM_{n+1} = M_{n+1}e_0 + M_{n+2}e_1 + M_{n+3}e_3 + M_{n+4}e_4$ are Balancing quaternions.

The conjugates of DQB_n is defined by

$$\begin{aligned}
\overline{DQB_n} &= \overline{QB_n + \varepsilon QB_{n+1}} \\
&= \overline{QB_n} + \varepsilon \overline{QB_{n+1}} \\
&= (B_n + \varepsilon B_{n+1})e_0 - (B_{n+1} + \varepsilon B_{n+2})e_1 \\
&\quad - (B_{n+2} + \varepsilon B_{n+3})e_2 - (B_{n+3} + \varepsilon B_{n+4})e_3 \\
&= S_{DQB_n} - \overrightarrow{V_{DQB_n}}
\end{aligned} \tag{10}$$

and the norm of DQB_n can be shown as

$$\|DQB_n\| = N(DQB_n) = DQB_n \cdot \overline{DQB_n} = A^2 + B^2 + C^2 + D^2$$

where $A = B_n + \varepsilon B_{n+1}$, $B = B_{n+1} + \varepsilon B_{n+2}$, $C = B_{n+2} + \varepsilon B_{n+3}$ and $D = B_{n+3} + \varepsilon B_{n+4}$. If $N(DQB_n) = 1$ then DQB_n is a unit dual Balancing quaternion.

The inverse of the dual Balancing quaternions DQB_n is given by

$$DQB_n^{-1} = \frac{\overline{DQB_n}}{N_{DQB_n}} = \frac{DQB_n}{DQB_n \cdot \overline{DQB_n}}$$

Proposition 1 For $n \geq 2$, we have the following properties:

$$DQB_n + \overline{DQB_n} = 2(B_n + \varepsilon B_{n+1}) = 2DB_n \tag{11}$$

$$DQB_n^2 + DQB_n \cdot \overline{DQB_n} = 2DB_n \cdot DQB_n \tag{12}$$

$$DQB_n \cdot \overline{DQB_n} = DB_n^2 + DB_{n+1}^2 + DB_{n+2}^2 + DB_{n+3}^2 \tag{13}$$

Proof. From (8) and (10), we get

$$DQB_n + \overline{DQB_n} = (QB_n + \varepsilon QB_{n+1}) + (\overline{QB_n + \varepsilon QB_{n+1}})$$

$$= \sum_{i=0}^{3}(B_{n+i} + \varepsilon B_{n+i+1})e_i + (B_n + \varepsilon B_{n+1})e_0$$

$$- \sum_{i=1}^{3}(B_{n+i} + \varepsilon B_{n+i+1})e_i$$

$$= 2(B_n + \varepsilon B_{n+1}).$$

Also, from (6), we get

$$DQB_n + \overline{DQB_n} = 2DB_n.$$

From (11), we obtain

$$DQB_n^2 = DQB_n \cdot DQB_n$$

$$= DQB_n \cdot \left(2(B_n + \varepsilon B_{n+1}) - \overline{DQB_n}\right)$$

$$= 2(B_n + \varepsilon B_{n+1}) \cdot DQB_n - DQB_n \cdot \overline{DQB_n}.$$

We get (12)

$$DQB_n^2 + DQB_n \cdot \overline{DQB_n} = 2(B_n + \varepsilon B_{n+1}) \cdot DQB_n$$

$$= 2DB_n \cdot DQB_n.$$

From (8) and (10), we get

$$DQB_n \cdot \overline{DQB_n} = (QB_n + \varepsilon QB_{n+1}) \cdot \overline{(QB_n + \varepsilon QB_{n+1})}$$

$$= \left(\sum_{i=0}^{3} (B_{n+i} + \varepsilon B_{n+i+1}) e_i \right)$$

$$\times \left((B_n + \varepsilon B_{n+1}) e_0 - \sum_{i=1}^{3} (B_{n+i} + \varepsilon B_{n+i+1}) e_i \right)$$

$$= (B_n + \varepsilon B_{n+1})^2 + (B_{n+1} + \varepsilon B_{n+2})^2$$

$$+ (B_{n+2} + \varepsilon B_{n+3})^2 + (B_{n+3} + \varepsilon B_{n+4})^2$$

$$= DB_n^2 + DB_{n+1}^2 + DB_{n+2}^2 + DB_{n+3}^2.$$

□

Proposition 2 For $n \geq 2$, we have the following identities:

$$DQb_n + \overline{DQb_n} = 2(b_n + \varepsilon b_{n+1}) = 2Db_n$$

$$DQb_n^2 + DQb_n \cdot \overline{DQb_n} = 2Db_n \cdot DQb_n$$

$$DQb_n \cdot \overline{DQb_n} = Db_n^2 + Db_{n+1}^2 + Db_{n+2}^2 + Db_{n+3}^2.$$

Proof. The proof is made similar to the above. □

Theorem 1 The dual Balancing and dual Cobalancing quaternions have the second order linear recurrence sequence as for $n \geq 0$,

$$DQB_{n+2} = 6DQB_{n+1} - DQB_n \tag{14}$$

$$DQb_{n+2} = 6DQb_{n+1} - DQb_n + 2(e_0 + e_1 + e_2 + e_3) \cdot (1 + \varepsilon) \tag{15}$$

Proof. From (8), we get

$$6DQB_{n+1} - DQB_n = 6\left(\sum_{i=0}^{3}(B_{n+1+i} + \varepsilon B_{n+2+i})e_i\right)$$

$$- \left(\sum_{i=0}^{3}(B_{n+i} + \varepsilon B_{n+i+1})e_i\right)$$

and since from the recurrence relation of Balancing numbers [1]

$$B_{n+2} = 6B_{n+1} - B_n$$

and from the recurrence of Balancing quaternions [24]

$$QB_{n+2} = 6QB_{n+1} - QB_n.$$

we obtain (14)

$$DQB_{n+2} = 6DQB_{n+1} - DQB_n.$$

Now, we find the recurrence of dual Cobalancing quaternions. From (9), we get

$$6DQb_{n+1} - DQb_n + 2(e_0 + e_1 + e_2 + e_3)(1 + \varepsilon) = 6\left(\sum_{i=0}^{3}(b_{n+1+i} + \varepsilon b_{n+2+i})e_i\right)$$

$$- \left(\sum_{i=0}^{3}(b_{n+i} + \varepsilon b_{n+1+i})e_i\right)$$

$$+ 2(e_0 + e_1 + e_2 + e_3)(1 + \varepsilon)$$

and since from the recurrence relation of Cobalancing numbers [8] and the recurrence relation of Cobalancing quaternions [24]

$$b_{n+2} = 6b_{n+1} - b_n + 2$$

and

$$Qb_{n+2} = 6Qb_{n+1} - Qb_n + 2(e_0 + e_1 + e_2 + e_3)$$

we get (15)

$$DQb_{n+2} = 6DQb_{n+1} - DQb_n + 2(e_0 + e_1 + e_2 + e_3)(1 + \varepsilon).$$

□

Theorem 2 We have the following identities for DQB_n dual Balancing quaternion and DQb_n dual Cobalancing quaternion:

$$DQB_n = \frac{DQb_{n+1} - DQb_n}{2} \tag{16}$$

$$DQB_{n+1} - DQB_n = 2DQb_{n+1} + (e_0 + e_1 + e_2 + e_3)(1 + \varepsilon) \tag{17}$$

$$DQB_n - DQB_{n+1}e_1 - DQB_{n+2}e_2 - DQB_{n+3}e_3 = 204DB_{n+3}. \tag{18}$$

Proof. From (9), we get

$$DQb_{n+1} - DQb_n = (Qb_{n+1} + \varepsilon Qb_{n+2}) - (Qb_n + \varepsilon Qb_{n+1})$$

$$= \left(\sum_{i=0}^{3}(b_{n+1+i} + \varepsilon b_{n+2+i})e_i\right) - \left(\sum_{i=0}^{3}(b_{n+i} + \varepsilon b_{n+1+i})e_i\right).$$

Also from [2]

$$B_n = \frac{b_{n+1} - b_n}{2}$$

and from [24]

$$QB_n = \frac{Qb_{n+1} - Qb_n}{2}$$

we obtain (16)

$$DQB_n = \frac{DQb_{n+1} - DQb_n}{2}.$$

From (8), we have

$$DQB_{n+1} - DQB_n = (QB_{n+1} + \varepsilon QB_{n+2}) - (QB_n + \varepsilon QB_{n+1})$$

$$= \left(\sum_{i=0}^{3} (B_{n+1+i} + \varepsilon B_{n+2+i}) e_i\right) - \left(\sum_{i=0}^{3} (B_{n+i} + \varepsilon B_{n+1+i}) e_i\right)$$

and from [4, 5]

$$B_{n+1} - B_n = 2b_{n+1} + 1$$

we obtain (17)

$$DQB_{n+1} - DQB_n = 2DQb_{n+1} + (e_0 + e_1 + e_2 + e_3)(1 + \varepsilon).$$

Finally, from (8), we have

$$\begin{aligned} DQB_n - DQB_{n+1}e_1 - DQB_{n+2}e_2 - DQB_{n+3}e_3 = &\left(\sum_{i=0}^{3} (B_{n+i} + \varepsilon B_{n+i+1}) e_i\right) \\ &- \left(\sum_{i=0}^{3} (B_{n+1+i} + \varepsilon B_{n+i+2}) e_i\right) e_1 \\ &- \left(\sum_{i=0}^{3} (B_{n+2+i} + \varepsilon B_{n+i+3}) e_i\right) e_2 \\ &- \left(\sum_{i=0}^{3} (B_{n+3+i} + \varepsilon B_{n+i+4}) e_i\right) e_3. \end{aligned}$$

Since from [1]

$$B_{n+2} = 6B_{n+1} - B_n$$

and from [24]

$$\sum_{i=0}^{3} B_{n+2i} = 204 B_{n+3}$$

(18) is obtained by making necessary arrangement. □

Definition 3 Since $B_{-n} = -B_n$ [25, 26], $b_{-n} = (-1)^{n+1} b_{n+1}$ [3, 8] and $QB_{-n} = -QB_n$, $Qb_{-n} = \sum_{i=0}^{3} (-1)^{n+1+i} b_{n+1+i} e_i$ [24], the dual Balancing and the dual Cobalancing quaternions with negative subscripts are defined by

$$\begin{aligned} DQB_{-n} &= QB_{-n} + \varepsilon QB_{-n+1} \\ &= -QB_n - \varepsilon QB_{n+1} \\ &= -DQB_n \end{aligned} \tag{19}$$

and

$$\begin{aligned} DQb_{-n} &= Qb_{-n} + \varepsilon Qb_{-n+1} \\ &= \sum_{i=0}^{3} (-1)^{n+1+i} (b_{n+1+i} - \varepsilon b_{n+2+i}) e_i \\ &= \sum_{i=0}^{3} (-1)^{n+1+i} Db_{n+1+i} \end{aligned} \tag{20}$$

Corollary 1 The following relations are easily seen from the definition of the dual Balancing and the dual Cobalancing quaternions with negative subscripts:

$$DQB_{-n} + \overline{DQB_{-n}} = -2(B_n + \varepsilon B_{n+1}) = -2DB_n$$

$$DQb_{-n} + \overline{DQb_{-n}} = 2(-1)^{n+1} (b_{n+1} - \varepsilon b_{n+2})$$

Theorem 3 [Binet's Formula for the dual Balancing Quaternions] For $n \geq 0$, the Binet's formula for the dual Balancing quaternions is as follows

$$DQB_n = \frac{1}{\alpha - \beta} (\alpha^* \alpha^n - \beta^* \beta^n)$$

where $\alpha' = 1 + \alpha e_1 + \alpha^2 e_2 + \alpha^3 e_3$, $\alpha^* = \alpha'(1 + \varepsilon\alpha)$ and $\beta' = 1 + \beta e_1 + \beta^2 e_2 + \beta^3 e_3$, $\beta^* = \beta'(1 + \varepsilon\beta)$ for taking $\alpha = 3 + \sqrt{8}$ and $\beta = 3 - \sqrt{8}$.

Proof. We can write the Binet's formula for Balancing quaternions QB_n in (8) from [24],

$$
\begin{aligned}
DQB_n &= QB_n + \varepsilon QB_{n+1} \\
&= \left(\frac{1}{\alpha-\beta}(\alpha'\alpha^n - \beta'\beta^n)\right) + \varepsilon\left(\frac{1}{\alpha-\beta}(\alpha'\alpha^{n+1} - \beta'\beta^{n+1})\right) \\
&= \frac{1}{\alpha-\beta}\left(\alpha'\alpha^n - \beta'\beta^n + \varepsilon\alpha'\alpha^{n+1} - \varepsilon\beta'\beta^{n+1}\right) \\
&= \frac{1}{\alpha-\beta}\left(\alpha^n\alpha'(1+\varepsilon\alpha) - \beta^n\beta'(1+\varepsilon\beta)\right) \\
&= \frac{1}{\alpha-\beta}(\alpha^*\alpha^n - \beta^*\beta^n)
\end{aligned}
$$

where $\alpha^* = \alpha'(1+\varepsilon\alpha)$ and $\beta^* = \beta'(1+\varepsilon\beta)$. Finally, the Binet's formula for the dual Balancing quaternions as follows:

$$
DQB_n = \frac{1}{\alpha-\beta}(\alpha^*\alpha^n - \beta^*\beta^n).
$$

□

Theorem 4 [Binet's Formula for the dual Cobalancing Quaternions] For $n \geq 0$, the Binet's formula for the dual Cobalancing quaternions is as follows:

$$
DQb_n = \frac{1}{4\sqrt{2}}\left(\alpha^*\alpha^{2n-1} - \beta^*\beta^{2n-1}\right) - \frac{1}{2}(e_0+e_1+e_2+e_3)(1+\varepsilon)
$$

where $\alpha' = 1+\alpha^2e_1+\alpha^4e_2+\alpha^6e_3$, $\beta' = 1+\beta^2e_1+\beta^4e_2+\beta^6e_3$ and $\alpha^* = \alpha'(1+\varepsilon\alpha^2)$, $\beta^* = \beta'(1+\varepsilon\beta^2)$ for taking $\alpha = 1+\sqrt{2}$ and $\beta = 1-\sqrt{2}$.

Proof. We can write the Binet's formula for Cobalancing quaternions Qb_n in (9) from [24],

$$
\begin{aligned}
DQb_n &= Qb_n + \varepsilon Qb_{n+1} \\
&= \left(\frac{1}{4\sqrt{2}}\left(\alpha'\alpha^{2n-1} - \beta'\beta^{2n-1}\right) - \frac{1}{2}(e_0 + e_1 + e_2 + e_3)\right) \\
&\quad + \varepsilon\left(\frac{1}{4\sqrt{2}}\left(\alpha'\alpha^{2n+1} - \beta'\beta^{2n+1}\right) - \frac{1}{2}(e_0 + e_1 + e_2 + e_3)\right) \\
&= \left(\frac{1}{4\sqrt{2}}\left(\alpha'\alpha^{2n-1} - \beta'\beta^{2n-1} + \varepsilon\alpha'\alpha^{2n+1} - \beta'\beta^{2n+1}\right)\right) \\
&\quad - \frac{1}{2}(e_0 + e_1 + e_2 + e_3) - \varepsilon\frac{1}{2}(e_0 + e_1 + e_2 + e_3) \\
&= \left(\frac{1}{4\sqrt{2}}\left(\alpha'\alpha^{2n-1}(1 + \varepsilon\alpha^2) - \beta'\beta^{2n-1}(1 + \varepsilon\beta^2)\right)\right) \\
&\quad - \frac{1}{2}(e_0 + e_1 + e_2 + e_3)(1 + \varepsilon) \\
&= \frac{1}{4\sqrt{2}}\left(\alpha^*\alpha^{2n-1} - \beta^*\beta^{2n-1}\right) - \frac{1}{2}(e_0 + e_1 + e_2 + e_3)(1 + \varepsilon)
\end{aligned}
$$

where $\alpha^* = \alpha'(1 + \varepsilon\alpha^2)$ and $\beta^* = \beta'(1 + \varepsilon\beta^2)$. Consequently, the Binet's formula for the dual Cobalancing quaternions as follows:

$$
DQb_n = \frac{1}{4\sqrt{2}}\left(\alpha^*\alpha^{2n-1} - \beta^*\beta^{2n-1}\right) - \frac{1}{2}(e_0 + e_1 + e_2 + e_3)(1 + \varepsilon).
$$

□

Theorem 5 The generating function for the dual Balancing quaternions DQB_n is

$$
G(x,\ t) = \frac{(t + \varepsilon)\, e_0 + (1 + \varepsilon(6 - t))\, e_1 + ((6 - t) + \varepsilon(35 - 6t))\, e_2 + ((35 - 6t) + \varepsilon(204 - 35t))\, e_3}{1 - 6t + t^2}.
$$

Proof. Let

$$
G(x,\ t) = \sum_{n=0}^{\infty} DQB_n(x) \cdot t^n
$$

be the generating function of the dual Balancing quaternions.

$$G(x,\ t) = DQB_0 + DQB_1 t + DQB_2 t^2 + \sum_{n=3}^{\infty} DQB_n t^n$$

$$= DQB_0 + DQB_1 t + DQB_2 t^2 + \sum_{n=3}^{\infty} (6DQB_{n-1} - DQB_{n-2})\, t^n$$

$$= DQB_0 + DQB_1 t + DQB_2 t^2 + 6\sum_{n=3}^{\infty} DQB_{n-1} t^n - \sum_{n=3}^{\infty} DQB_{n-2} t^n$$

$$= DQB_0 + DQB_1 t + DQB_2 t^2 + 6t\sum_{n=3}^{\infty} DQB_{n-1} t^{n-1} - t^2\sum_{n=3}^{\infty} DQB_{n-2} t^{n-2}$$

$$= DQB_0 + DQB_1 t + DQB_2 t^2 + 6t\left(\sum_{n=2}^{\infty} DQB_n t^n\right) - t^2\left(\sum_{n=1}^{\infty} DQB_n t^n\right)$$

$$= DQB_0 + DQB_1 t + DQB_2 t^2 + 6t\,(G(x,\ t) - DQB_0 - DQB_1 t) - t^2\,(G(x,\ t) - DQB_0)$$

by making necessary arrangement, the generating function of the dual Balancing quaternion is found as follows:

$$G(x,\ t) = \frac{(t+\varepsilon)\,e_0 + (1+\varepsilon(6-t))\,e_1 + ((6-t)+\varepsilon(35-6t))\,e_2 + ((35-6t)+\varepsilon(204-35t))\,e_3}{1-6t+t^2}.$$

□

Theorem 6 The generating function of dual Cobalancing quaternion DQb_n is

$$G(x,\ t) = \frac{2(e_0+e_1+e_2+e_3)(1+\varepsilon)t^2}{(1-t)(1-6t+t^2)}$$

$$+\frac{2\varepsilon t e_0 + [2\varepsilon(1+t)+2t]\,e_1 + (2+2t+14\varepsilon)\,e_2 + (14+84\varepsilon-12\varepsilon t)e_3}{1-6t+t^2}$$

Proof. The proof easily can be done similarly. □

Now we give the matrix representation of dual Balancing quaternions. Throughout this section, u_0 is a 2×1 matrix defined by $u_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, A is a 2×2 matrix defined by $A = \begin{bmatrix} 0 & 1 \\ -1 & 6 \end{bmatrix}$ and Q is a 2×2 matrix defined by $Q = \begin{bmatrix} -QB_0 & QB_1 \\ -QB_1 & QB_2 \end{bmatrix}$.

In [24], we showed that for $n \geq 1$,

$$\begin{bmatrix} -QB_{n-1} & QB_n \\ -QB_n & QB_{n+1} \end{bmatrix} = \begin{bmatrix} -QB_0 & QB_1 \\ -QB_1 & QB_2 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ -1 & 6 \end{bmatrix}^{n-1}$$

and S is 2×2 matrix defined by $S = Q\cdot A^{n-1}$.

Theorem 7 Let $n \geq 2$ be integer. Then

$$\begin{bmatrix} -DQB_{n-1} & DQB_{n-2} \\ -DQB_n & DQB_{n-1} \end{bmatrix} = \begin{bmatrix} -QB_0 & QB_1 \\ -QB_1 & QB_2 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ -1 & 6 \end{bmatrix}^{n-1} \cdot \begin{bmatrix} 1 & 6-\varepsilon \\ -\varepsilon & 1 \end{bmatrix}.$$

Corollary 2 Let E be 2×2 matrix defined by $E = \begin{bmatrix} 1 & 6-\varepsilon \\ -\varepsilon & 1 \end{bmatrix}$ and D be 2×2 matrix defined by $D = S\cdot E$. Then

$$DQB_{n-1} = u_0^T \cdot D \cdot u_0.$$

Theorem 8 Let $n \geq 2$ be integer. Then

$$DQB_{n-1}^2 - DQB_n \cdot DQB_{n-2} = (2+12\varepsilon) + (-12-72\varepsilon)e_1 + (2+12\varepsilon)e_2 + (-204-1224\varepsilon)e_3.$$

Proof. If $|D|$ is deteminant of matrix D, then from [24],

$$\begin{aligned} |D| &= |S|\cdot|E| \\ &= \left|Q\cdot A^{n-1}\right|\cdot|E| \\ &= |Q|\cdot|A|^{n-1}\cdot|E| \\ &= (-QB_0\cdot QB_2 + QB_1^2)\cdot(1+6\varepsilon-\varepsilon^2) \end{aligned}$$

and $\varepsilon^2 = 0$. So, we get as follows:

$$-DQB_{n-1}^2 + DQB_n \cdot DQB_{n-2} = (-2-12\varepsilon) + (12+72\varepsilon)e_1 + (-2-12\varepsilon)e_2 + (204+1224\varepsilon)e_3.$$

Consequently; we obtain

$$DQB_{n-1}^2 - DQB_n \cdot DQB_{n-2} = (2+12\varepsilon) + (-12-72\varepsilon)e_1 + (2+12\varepsilon)e_2 + (-204-1224\varepsilon)e_3.$$

□

3. Conclusion

This paper presents a study of the dual Balancing and dual Cobalancing numbers, as well as their corresponding quaternions. We provide proofs of Binet's formulas and generating functions, and investigate several interesting results related to these concepts. In addition, we derive E and D matrices for obtaining DQB_n, and establish various properties of the dual Balancing and dual Cobalancing quaternions. Finally, we demonstrate Cassini's identity and its proof using the derived matrix.

Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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