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Research Article

A Study on Approximate Controllability of Ψ-Caputo Fractional Differential Equations with Impulsive Effects

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Abstract: In this article, we studied the approximate controllability of Ψ-Caputo fractional differential systems. We prove the sufficient conditions for an abstract Cauchy problem invloving infinite delay, impulsive and nonlocal conditions. The result is shown by means of the infinitesimal operator, semigroup theory, fractional calculus, and Schauder's fixed point theorem. First, we prove the existence of the mild solution and demonstrate that the Ψ-Caputo fractional system is approximately controllable. Finally, an example is given to analyse the obtained results.

***Keywords*:** Ψ-Caputo fractional derivative, controllability, fixed point theorem, infinitesimal generator, impulsive effects

MSC: 34K37, 34H05, 35A01, 35R12, 47H10

1. Introduction

In recent times, fractional calculus have been studied by many researchers because the fractional differential systems describe many real-world processes related to memory and hereditary properties of various materials more accurately compared to classical differential equations. For more details on fractional calculus and their applications, see [1-6]. The fractional differential system is now receiving a lot of interest because of its amazing implications in displaying the splendours of science and technology. Fractional systems may be used to solve broad spectrum of issues in a number of fields, like elasticity, power systems, electrolysis, fluid circulation, and others. The enlargement of differential equations and inequalities known as differential inclusions, which is sometimes referred to as optimal control theory, has several users and applications. When one is adept at employing differential inclusions, dynamical systems with velocities that aren't solely determined by the system's state are easier to analyse. Numerous studies have been undertaken on boundary value problems. There have been several investigations conducted to find out if there are solutions for fractional differential systems as well as fractional differential inclusions. The following research publications can be referenced to support the concept and the implications discussed in relation to fractional calculus: [7-16].

The concept of controllability, which is important in both pure and applied mathematics, is a central one in

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the field of mathematical control theory. At present, controllability plays a significant role in fractional calculus. Researchers are now working on a novel concept and notion connected to control theory, specifically how to apply control theory to fractional differential systems. The understanding of the exact and approximate controllability of various types of dynamical systems, such as delay or not, has advanced significantly over the past few years thanks to the efforts of numerous researchers. Exact controllability enables to steer the system to arbitrary final state while approximate controllability means that system can be steered to arbitrary small neighbourhood of final state. Moreover, approximate controllable systems are more prevalent and very often approximate controllability is completely adequate in applications. In [17, 18] authors studied the approximate controllability of Caputo fractional differential systems. In [19, 20] studied the approximate coontrollability of stochastic differential systems. Recently, in [21-23] discussed the approximate controllability of fractional stochastic differential systems of order $1 < r < 2$. It is possible to support discussions of theory and applications related to controllability by citing the research papers [24-29].

Impulsive differential systems have an important role in mathematical science and real world. Impulsive effects have a huge impact on how a system behaves. They may create abrupt shifts, breaks, or leaps in the system's variables, causing deviations from the behaviour that was anticipated or forecasted. This modification may have an impact on system dynamics generally, convergence, and stability. The impulsive effect can be purposefully used to affect or control a system. It is possible to influence a system's behaviour, stabilise unstable dynamics, or move it towards desirable states by carefully placing impulses into it. Engineering, physics, and biology are just a few of the disciplines that use impulsive control techniques. The system state can abruptly change in many phenomena and processes, including those in the fields of electronics, telecommunications, economics, mechanics, biology, and medicine, where the impulsive effects can occur, we refer [30-37].

In particular, ones like the fractional derivative with respect to another function are examples of the generic fractional derivative that have recently been established. In order to increase the precision of the objective modelling, Almeida [38] and colleagues presented a novel version of fractional derivative in 2017 by accounting for the Caputo fractional derivative with respect to a second function Ψ, or the Ψ-Caputo fractional derivative. Then, in [39] presented the so-called Ψ-Hilfer derivative, a fractional derivative with respect to an another function. The Ψ-Caputo and Ψ-Hilfer models, which are here stated, have the advantage of allowing the choice of the classical differential operator and the Ψ function, i.e., from the decision of the Ψ function, the conventional differentiation operator may act on the fractional integral operator, or else the fractional integral operator may act on the conventional differentiation operator. These two papers served as inspiration for further study into Ψ-Caputo and Ψ-Hilfer, which led to the creation of novel works. For Caputo fractional differential systems with an infinitesimal generator $\mathscr{A}$, writers [40] investigated the existence, uniqueness, and stability of several sorts of mild solutions. The fixed-point approach was used by the authors [41] to analyse the existence and uniqueness of Ψ-Hilfer neutral equations with indefinite delays. A recent study of the Ψ-Caputo derivative's approximate controllability was published in article [42].

We investigate the approximate controllability of Ψ-Caputo fractional differential equations with impulsive conditions, nonlocal conditions, and indefinite delay since, to our knowledge, no paper has been published on this topic. Inspire by the aforementioned studies, we also investigate the approximate controllability of the impulsive systems provided by:

$$\begin{cases} {}^{C}D_{0^+}^{\eta;\Psi}\mathfrak{u}(t) = \mathscr{A}\mathfrak{u}(t) + \mathscr{B}v(t) + \mathfrak{G}\left(t, \mathfrak{u}_t, \int_0^t e\left(t, s, \mathfrak{u}_s\right) ds\right), t \in [0, b], t \neq t_k, k = 1, 2, \cdots, m, \\ \mathfrak{u}(t_k^+) - \mathfrak{u}(t_k^-) = I(\mathfrak{u}(t_k^-)), \\ \mathfrak{u}(0) = \mathfrak{u}_0 + \xi\left(\mathfrak{u}_{t_1}, \mathfrak{u}_{t_2}, \ldots, \mathfrak{u}_{t_n}\right) \in L^2(D, \mathscr{T}_w), t \in (-\infty, 0], \end{cases} \tag{1}$$

where $\mathscr{A}$ is an infinitesimal generator of the analytic semigroup $\{T(t), t \geq 0\}$ on $\mathfrak{D}$. ${}^{C}D_{0+}^{\eta;\Psi}$ represents the Ψ-Caputo fractional derivative of order η, $0 < \eta < 1$. Let $v(t)$ be the control function in $L^2(\mathscr{W}, \mathscr{U})$, where $\mathscr{U}$ is the Banach space, and $\mathfrak{u}(t)$ be the state in a Banach space $\mathfrak{D}$ with $\|\cdot\|$. The bounded linear operator from $\mathscr{U}$ into $\mathfrak{D}$ is represented here by B. Let $\mathscr{W} = [0, b]$, $\mathfrak{G} : \mathscr{W} \times \mathscr{T}_w \times \mathfrak{D} \to \mathfrak{D}$ be the relevant function, $e : \mathscr{W} \times \mathscr{W} \times \mathscr{T}_w \to \mathfrak{D}$ and $0 < t_1 < t_2 < \cdots < t_n \leq b$, $\xi : \mathscr{T}_w^n \to \mathscr{T}_w$ are the relevant functions, where $\mathscr{T}_w$ is a phase space. The memories of $\mathfrak{u}_t : (-\infty, 0] \to \mathfrak{D}$, such that $\mathfrak{u}_t(s) = \mathfrak{u}(t + s)$ correspond to the phase space $\mathscr{T}_w$, and $I_k : \mathfrak{D} \to \mathfrak{D}$ are the impulsive functions with the jump of t at points of t_k.

The organisation of the work is divided as follows: The Ψ-Caputo fractional, semigroup, and principles of fractional calculus are covered in Section 2. Before expanding to the approximate controllability of systems, we first prove the existence of the mild solution in Section 3. In Section 4, we gave an illustration to highlight our key principles. In the end, a few observations are offered.

2. Preliminaries

The key concepts, theorems, and lemma that are utilised throughout the whole work are discussed here.

Let us assume Ψ is an non-decreasing function with $\Psi'(t) \neq 0$, for every $t \in \mathscr{W}$.

Definition 2.1 [43] The Laplace transform of the functions $\mathfrak{G} : [0, \infty] \to \mathbb{R}$ with respect to Ψ is presented by

$$\mathcal{L}_{\Psi}\{\mathfrak{G}(t)\}(\chi) = \mathfrak{G}(\chi) = \int_{a}^{\infty} \mathfrak{G}(t)e^{-\chi(\Psi(t)-\Psi(a))}\mathfrak{G}(t)\Psi'(t)dt \ \text{ for all } \ \chi \in \mathbb{C}. \tag{2}$$

Definition 2.2 [39] Let $\eta > 0$, $\mathfrak{G}$ be an integrable function defined on $[a, b]$ and $\Psi \in C^1([a, b])$ be an increasing function with $\Psi'(t) \neq 0$ for all $t \in [a, b]$. The Ψ-Riemann-Liouville fractional integral of order η of the function $\mathfrak{G}$ is presented by

$$I_{a^+}^{\eta;\Psi}\mathfrak{G}(\chi) = \frac{1}{\Gamma(\eta)}\int_{a}^{\chi} \Psi'(t)(\Psi(\chi)-\Psi(t))^{\eta-1}\mathfrak{G}(t)dt, \tag{3}$$

where $\eta \in (m - 1, m)$.

Definition 2.3 [39] Let $m - 1 < \eta < m$, $\mathfrak{G}$ be an integrable function defined on $[a, b]$ and $\Psi \in C^1([a, b])$ be an increasing function with $\Psi'(t) \neq 0$ for all $t \in [a, b]$. The Ψ-Riemann-Liouville fractional derivative of order η of the function $\mathfrak{G}$ is presented by

$$D_{a^+}^{\eta;\Psi}\mathfrak{G}(\chi) = \frac{1}{\Gamma(m-\eta)}\left(\frac{1}{\Psi'(\chi)}\frac{d}{d\chi}\right)^m \int_{a}^{\chi} \Psi'(t)(\Psi(\chi)-\Psi(t))^{m-\eta-1}\mathfrak{G}(t)dt. \tag{4}$$

Definition 2.4 [38] Let $m - 1 < \eta < m$, $\mathfrak{G} \in C^n([a, b])$ and $\Psi \in C^m([a, b])$ be an increasing function with $\Psi'(t) \neq 0$ for all $t \in [a, b]$. The Ψ-Caputo fractional derivative of order η is defined by

$${}^{C}D_{a^+}^{\eta;\Psi}\mathfrak{G}(t) = \frac{1}{\Gamma(m-\eta)}\int_{0}^{t} (\Psi(t)-\Psi(\chi))^{m-\eta-1}\mathfrak{G}^{[m]}(\chi)\Psi'(\chi)d\chi, \tag{5}$$

where $m = [\eta] + 1$.

Theorem 2.5 [38] Let $\mathfrak{G} \in C^n(a, b)$ and $\eta > 0$.Then we have

$$I_{a}^{\eta,\Psi}\left({}^{C}D_{a^+}^{\eta;\Psi}F(t)\right) = \mathfrak{G}(t) - \sum_{k=0}^{n-1}\left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^k \left(\mathfrak{G}(a^+)\right)\left(\Psi(t)-\Psi(a)\right)^k. \tag{6}$$

Especially, given $0 < \eta < 1$, we have

$$I_{a}^{\eta,\Psi}\left({}^{C}D_{a^+}^{\eta;\Psi}\ \mathfrak{G}(t)\right) = \mathfrak{G}(t) - \mathfrak{G}(a). \tag{7}$$

By referring [31], we define the abstract phase space $\mathscr{T}_w$. Let $w : (-\infty, 0] \to (0, +\infty)$ be continuous along $\Upsilon = \int_{-\infty}^{0} w(t)dt < +\infty$. Now, for every $n > 0$, we have

$$\mathscr{T} = \{\varpi : [-n, 0] \to \mathfrak{D} \text{ there exists } \varpi(t) \text{ is bounded and measurable}\},$$

and set the space $\mathscr{T}$ with the norm

$$\|\varpi\|_{[-n,0]} = \sup_{\tau\in[-n,0]} \|\varpi(\tau)\|, \forall \varpi \in \mathscr{T}. \tag{8}$$

Now, we define

$$\mathscr{T}_w = \left\{\varpi : (-\infty, 0] \to \mathfrak{D}, \text{ such that } \forall n > 0, \varpi|_{[-n,0]} \in \mathscr{T} \text{ and } \int_{-\infty}^{0} w(\tau)\|\varpi\|_{[\tau,0]} d\tau < +\infty\right\}.$$

If $\mathscr{T}_w$ is endowed with

$$\|\varpi\|_{\Upsilon} = \int_{-\infty}^{0} w(\tau)\|\varpi\|_{[\tau,0]} d\tau, \ \forall \varpi \in \mathscr{T}_w, \tag{9}$$

thus $(\mathscr{T}_w, \|\cdot\|)$ is a Banach space.

Now, we consider the set

$$\mathscr{T}_w' = \{\mathfrak{u} : (-\infty, 0] \to \mathfrak{D}_{\mathfrak{u}_k} \in C(I_k, \mathfrak{D}), \text{ there exists } \mathfrak{u}(t_k^+) \text{ and } \mathfrak{u}(t_k^-) \text{ with } \mathfrak{u}(t_k^+) = \mathfrak{u}(t_k^-),$$

$$\mathfrak{u}(0) \in \mathscr{T}_w, \ k = 0, 1, \cdots, m\},$$

where $I_k = (t_k, t_{k+1})$. Let $\|\cdot\|_{\Upsilon}'$ in $\mathscr{T}_w'$ be the seminorm classified as

$$\|\mathfrak{u}\|_{\Upsilon}' = \|\mathfrak{u}_0\|_{\Upsilon} + \sup\{\|\mathfrak{u}(\tau)\| : \tau \in [0, b]\}, \ \mathfrak{u} \in \mathscr{T}_w'. \tag{10}$$

Lemma 2.6 If $\mathfrak{u} \in \mathscr{T}_w'$, then for $t \in \mathscr{W}$, $\mathfrak{u}_t \in \mathscr{T}_w$. Moreover,

$$\Upsilon |\mathfrak{u}(t)| \le \|\mathfrak{u}_t\|_{\Upsilon} \le \|\mathfrak{u}_0\|_{\Upsilon} + \Upsilon \sup_{r\in[0,t]} |\mathfrak{u}(r)|, \ \Upsilon = \int_{-\infty}^{0} w(t)dt < \infty. \tag{11}$$

Lemma 2.7 [12] Let the linear operator $\mathscr{A}$ be the infinitesimal generator of a C_0 semigroup if and only if

(c_i) $\mathscr{A}$ is closed and $\overline{D(\mathscr{A})} = \mathfrak{D}$.

(c_{ii}) $\rho(\mathscr{A})$ be the resolvent set of $\mathscr{A}$ contains $\mathbb{R}^+$ and, $\forall \lambda > 0$, we write

$$\|R(\lambda, \mathscr{A})\| \le \frac{1}{\lambda},$$

where $R(\lambda, \mathscr{A}) = (\lambda^{\eta} I - \mathscr{A})^{-1} z = \int_0^{\infty} e^{-\lambda^{\alpha} t} T(t) z dt.$

Definition 2.8 Let $0 < \eta < 1$, the Wright type function $W_\eta(t)$ is defined as

$$W_\eta(z) = \sum_{k=0}^{\infty} \frac{(-z)^k}{k!\Gamma(-\eta k + 1 - \eta)}, \; z \in \mathbb{C}. \tag{12}$$

Proposition 2.9 The Wright type function W_η is an entire function with satisfy the succeeding conditions:

1. $W_\eta(\varepsilon) \geq 0$, for $\varepsilon \geq 0$, $\int_0^\infty W_\eta(\varepsilon) d\varepsilon = 1$;

2. $\int_0^\infty W_\eta(\varepsilon)\varepsilon^k d\varepsilon = \frac{\Gamma(1+k)}{\Gamma(1+\eta k)}$, for $k > -1$;

3. $\int_0^\infty W_\eta(\varepsilon) e^{z\varepsilon} d\varepsilon = E_\eta(-z)$, $z \in \mathbb{C}$.

Lemma 2.10 [9] The Ψ-Caputo fractional differential systems (1) is equivalent to the integral equation

$$\mathfrak{u}(t) = \mathfrak{u}_0 + \xi\left(\mathfrak{u}_{t_1}, \mathfrak{u}_{t_2}, \ldots, u_{t_n}\right) + \sum_{0<t_k<t} I_k(\mathfrak{u}(t_k^-)) + \frac{1}{\Gamma(\eta)} \int_0^t \left(\Psi(t) - \Psi(\chi)\right)^{\eta-1}$$

$$\times\left[\mathscr{A}\mathfrak{u}(t) + \mathscr{B}v(t) + \mathfrak{G}\left(\chi, \mathfrak{u}_\chi, \int_0^\chi e\left(\chi, s, \mathfrak{u}_s\right) ds\right)\right]\Psi'(\chi) d\chi, \tag{13}$$

where $t \in [0, b]$.

Proof. Let $0 < t \leq b$, $0 < \eta < 1$, applying the operator $I_{0+}^{\eta;\Psi}$ to left-hand side (LHS) of Equation (1), by using Theorem 2.5 we get

$$I_{0^+}^{\eta;\Psi}\left({}^C D_{0^+} \mathfrak{u}(t)\right) = \mathfrak{u}(t) - \mathfrak{u}_0 - \xi\left(\mathfrak{u}_{t_1}, \mathfrak{u}_{t_2}, \ldots, u_{t_n}\right), \; t \in [0, t_1], \tag{14}$$

$$I_{0^+}^{\eta;\Psi}\left({}^C D_{0^+} \mathfrak{u}(t)\right) = \mathfrak{u}(t) - \mathfrak{u}(t_1^-) - I_1(\mathfrak{u}(t_1^-)), \; t \in (t_1, t_2], \tag{15}$$

$$I_{0^+}^{\eta;\Psi}\left({}^C D_{0^+} \mathfrak{u}(t)\right) = \mathfrak{u}(t) - \mathfrak{u}(t_k^-) - I_1(\mathfrak{u}(t_k^-)), \; t \in (t_k, t_{k+1}]. \tag{16}$$

Thus, the operator $I_{0+}^{\eta;\Psi}$ act on the right hand side of Equation (1),

$$I_{0^+}^{\eta;\Psi}\left(\mathscr{A}\mathfrak{u}(t) + \mathscr{B}v(t) + \mathfrak{G}\left(t, \mathfrak{u}_t, \int_0^t e(t, s, \mathfrak{u}_s) ds\right)\right) = \frac{1}{\Gamma(\eta)} \int_0^t \Psi'(\chi)(\Psi(t) - \Psi(\chi))^{\eta-1} \mathscr{A}\mathfrak{u}(\chi) d\chi$$

$$+ \frac{1}{\Gamma(\eta)} \int_0^t \Psi'(\chi)(\Psi(t) - \Psi(\chi))^{\eta-1} \mathscr{B}v(\chi) d\chi$$

$$+ \frac{1}{\Gamma(\eta)} \int_a^t \Psi'(\chi)(\Psi(t) - \Psi(\chi))^{\eta-1} \mathfrak{G}\left(\chi, \mathfrak{u}_\chi, \int_0^\chi e(\chi, s, \mathfrak{u}_s) ds\right) d\chi. \tag{17}$$

Now, we can deduce the above equations,

$$\mathfrak{u}(t) = \mathfrak{u}_0 + \xi\left(\mathfrak{u}_{t_1}, \mathfrak{u}_{t_2}, \ldots, \mathfrak{u}_{t_n}\right) + \sum_{0<t_k<t} I_k(\mathfrak{u}(t_k^-)) + \frac{1}{\Gamma(\eta)} \int_0^t \left(\Psi(t) - \Psi(\chi)\right)^{\eta-1}$$

$$\times\left[\mathscr{A}\mathrm{u}(t)+\mathscr{B}v(t)+\mathfrak{G}\left(\chi,\mathrm{u}_\chi,\int_0^\chi e(\chi,s,\mathrm{u}_s)ds\right)\right]\Psi'(\chi)d\chi, \tag{18}$$

where $t \in [0, b]$. □

Lemma 2.11 [40] If integral equation (13) holds, then we have

$$\mathrm{u}(t)=\mathcal{S}_\Psi^\eta(t,0)\left[\mathrm{u}_0+\xi\left(\mathrm{u}_{t_1},\mathrm{u}_{t_2},\ldots,u_{t_n}\right)\right]+\sum_{0<t_k<t}\mathcal{S}_\Psi^\eta(t,t_k)I_k(\mathrm{u}(t_k^-))$$

$$+\int_0^t\left(\Psi(t)-\Psi(\chi)\right)^{\eta-1}\mathcal{Q}_\Psi^\eta(t,\chi)\mathscr{B}v(\chi)\Psi'(\chi)d\chi$$

$$+\int_0^t\left(\Psi(t)-\Psi(\chi)\right)^{\eta-1}\mathcal{Q}_\Psi^\eta(t,\chi)\mathfrak{G}\left(\chi,\mathrm{u}_\chi,\int_0^\chi e(\chi,s,\mathrm{u}_s)ds\right)\Psi'(\chi)d\chi,\ \text{for } t\in[0,b]. \tag{19}$$

Proof. Let $\lambda > 0$. Consider the generalized Laplace transform, take

$$\Upsilon_1(\lambda)=\int_0^\infty e^{-\lambda(\Psi(t)-\Psi(0))}\mathrm{u}(\chi)\Psi'(\chi)d\chi, \tag{20}$$

$$\Upsilon_2(\lambda)=\int_0^\infty e^{-\lambda(\Psi(t)-\Psi(0))}v(\chi)\Psi'(\chi)d\chi, \tag{21}$$

$$\Upsilon_3(\lambda)==\int_0^\infty e^{-\lambda(\Psi(t)-\Psi(0))}\mathfrak{G}\left(\chi,\mathrm{u}_\chi,\int_0^\chi e(\chi,s,\mathrm{u}_s)ds\right)\Psi'(\chi)d\chi. \tag{22}$$

Now, apply generalized Laplace transform on Equation (13),

$$\Upsilon_1(\lambda)=\frac{1}{\lambda}\left(\mathrm{u}_0+\xi\left(\mathrm{u}_{t_1},\mathrm{u}_{t_2},\ldots,u_{t_n}\right)+\sum_{0<t_k<t}I_k(\mathrm{u}(t_k^-))\right)+\frac{1}{\lambda^\eta}\left(\mathscr{A}\Upsilon_1(\lambda)+\mathscr{B}\Upsilon_2(\lambda)+\Upsilon_3(\lambda)\right).$$

We can deduce that

$$\Upsilon_1(\lambda)=\lambda^{\eta-1}(\lambda^\eta I-\mathscr{A})^{-1}\left(\mathrm{u}_0+\xi\left(\mathrm{u}_{t_1},\mathrm{u}_{t_2},\ldots,u_{t_n}\right)+\sum_{0<t_k<t}I_k(\mathrm{u}(t_k^-))\right)$$

$$+(\lambda^\eta I-\mathscr{A})^{-1}\mathscr{B}\Upsilon_2(\lambda)+(\lambda^\eta I-\mathscr{A})^{-1}\Upsilon_3(\lambda)$$

$$=I_1+I_2+I_3.$$

Since $(\lambda^\eta I-\mathscr{A})^{-1}z=\int_0^\infty e^{-\lambda^\eta t}T(t)zdt$. By using Lemma 3.1 in [40], we can derive the values of I_1, I_2 and I_3. Then we get,

$$\Upsilon_1(\lambda)=\int_0^\infty e^{-\lambda(\Psi(t)-\Psi(0))}\left(\int_0^\infty\rho_\eta(\theta)T\left(\frac{\left(\Psi(t)-\Psi(0)\right)^\eta}{\theta^\eta}\right)\left(\mathrm{u}_0+\xi\left(\mathrm{u}_{t_1},\mathrm{u}_{t_2},\ldots,u_{t_n}\right)\right)d\theta\right)\Psi'(t)dt$$

$$+\sum_{0<t_k<t}\left(\int_0^\infty e^{-\lambda(\Psi(t)-\Psi(0))}\left(\int_0^\infty \rho_\eta(\theta)T\left(\frac{\left(\Psi(t)-\Psi(t_k)\right)^\eta}{\theta^\eta}\right)I_k(\mathfrak{u}(t_k^-))\right)d\theta\right)\Psi'(t)dt$$

$$+\int_0^\infty e^{-\lambda(\Psi(r)-\Psi(0))}\left(\int_0^r\int_0^\infty \eta\rho_\eta(\theta)\frac{\left(\Psi(r)-\Psi(\chi)\right)^{\eta-1}}{\theta^\eta}T\left(\frac{\left(\Psi(r)-\Psi(\chi)\right)^\eta}{\theta^\eta}\right)v(\chi)\Psi'(\chi)d\theta d\chi\right)\Psi'(r)dr$$

$$+\int_0^\infty e^{-\lambda(\Psi(r)-\Psi(0))}\left(\int_0^r\int_0^\infty \eta\rho_\eta(\theta)\frac{\left(\Psi(r)-\Psi(\chi)\right)^{\eta-1}}{\theta^\eta}T\left(\frac{\left(\Psi(r)-\Psi(\chi)\right)^\eta}{\theta^\eta}\right)\right.$$

$$\left.\times\mathfrak{G}\left(\chi,\mathfrak{u}_\chi,\int_0^\chi e(\chi,s,\mathfrak{u}_s)ds\right)\Psi'(\chi)d\theta d\chi\right)\Psi'(r)dr.$$

Applying Laplace inverse transform, we obtain

$$\mathfrak{u}(t)=\int_0^\infty \rho_\eta(\theta)T\left(\frac{\left(\Psi(t)-\Psi(0)\right)^\eta}{\theta^\eta}\right)\left(\mathfrak{u}_0+\xi\left(\mathfrak{u}_{t_1},\mathfrak{u}_{t_2},\ldots,u_{t_n}\right)\right)d\theta$$

$$+\sum_{0<t_k<t}\left(\int_0^\infty \rho_\eta(\theta)T\left(\frac{\left(\Psi(t)-\Psi(t_k)\right)^\eta}{\theta^\eta}\right)I_k(\mathfrak{u}(t_k^-))\right)d\theta$$

$$+\eta\int_0^t\int_0^\infty \rho_\eta(\theta)\frac{\left(\Psi(t)-\Psi(\chi)\right)^{\eta-1}}{\theta^\eta}T\left(\frac{\left(\Psi(t)-\Psi(\chi)\right)^\eta}{\theta^\eta}\right)v(\chi)\Psi'(\chi)d\chi$$

$$+\eta\int_0^t\int_0^\infty \rho_\eta(\theta)\frac{\left(\Psi(t)-\Psi(\chi)\right)^{\eta-1}}{\theta^\eta}T\left(\frac{\left(\Psi(t)-\Psi(\chi)\right)^\eta}{\theta^\eta}\right)$$

$$\times\mathfrak{G}\left(\chi,\mathfrak{u}_\chi,\int_0^\chi e(\chi,s,\mathfrak{u}_s)\,ds\right)\Psi'(\chi)d\chi$$

$$=\mathcal{S}_\Psi^\eta(t,0)\left[\mathfrak{u}_0+\xi\left(\mathfrak{u}_{t_1},\mathfrak{u}_{t_2},\ldots,u_{t_n}\right)\right]+\sum_{0<t_k<t}\mathcal{S}_\Psi^\eta(t,t_k)I_k(\mathfrak{u}(t_k^-))$$

$$+\int_0^t\left(\Psi(t)-\Psi(\chi)\right)^{\eta-1}\mathcal{Q}_\Psi^\eta(t,\chi)\mathscr{B}v(\chi)\Psi'(\chi)d\chi$$

$$+\int_0^t\left(\Psi(t)-\Psi(\chi)\right)^{\eta-1}\mathcal{Q}_\Psi^\eta(t,\chi)\mathfrak{G}\left(\chi,\mathfrak{u}_\chi,\int_0^\chi e(\chi,s,\mathfrak{u}_s)ds\right)\Psi'(\chi)d\chi,$$

where

$$\mathcal{S}_{\Psi}^{\eta}(t,\chi)\mathrm{u}=\int_{0}^{\infty}\zeta_{\eta}(\theta)T\Big((\Psi(t)-\Psi(\chi))^{\eta}\theta\Big)\mathrm{u}d\theta, \tag{23}$$

and

$$\mathcal{Q}_{\Psi}^{\eta}(t,\chi)\mathrm{u}=\eta\int_{0}^{\infty}\theta\zeta_{\eta}(\theta)T\Big((\Psi(t)-\Psi(\chi))^{\eta}\theta\Big)\mathrm{u}d\theta, \tag{24}$$

for $0\le\chi\le t\le b$ and the probability density function $\zeta_{\eta}(\theta)=\frac{1}{\eta}\theta^{-\frac{1}{\eta}-1}\rho_{\eta}(\theta^{-\frac{1}{\eta}})$ on $(0,\infty)$, i.e., $\zeta_{\eta}(\theta)\ge 0$ for $\theta\in(0,\infty)$ and $\int_{0}^{\infty}\zeta_{\eta}(\theta)d\theta=1$. □

Definition 2.12 A function $\mathrm{u}\in PC([0,b],\mathfrak{D})$ is called mild solution of the system (1) if satisfies

$$\mathrm{u}(t)=\mathcal{S}_{\Psi}^{\eta}(t,0)\Big[\mathrm{u}_0+\xi\Big(\mathrm{u}_{t_1},\mathrm{u}_{t_2},\ldots,u_{t_n}\Big)\Big]+\sum_{0<t_k<t}\mathcal{S}_{\Psi}^{\eta}(t,t_k)I_k(\mathrm{u}(t_k^-))$$

$$+\int_0^t\big(\Psi(t)-\Psi(\chi)\big)^{\eta-1}\mathcal{Q}_{\Psi}^{\eta}(t,\chi)\mathscr{B}v(\chi)\Psi'(\chi)d\chi$$

$$+\int_0^t\big(\Psi(t)-\Psi(\chi)\big)^{\eta-1}\mathcal{Q}_{\Psi}^{\eta}(t,\chi)\mathfrak{G}\Big(\chi,\mathrm{u}_{\chi},\int_0^{\chi}e(\chi,s,\mathrm{u}_s)ds\Big)\Psi'(\chi)d\chi,\ \text{for } t\in[0,b]. \tag{25}$$

Lemma 2.13 [40] The operator $\mathcal{S}_{\Psi}^{\eta}(t,\chi)$ and $\mathcal{Q}_{\Psi}^{\eta}(t,\chi)$ hold the following properties:

(a) For any $0\le\chi\le t$, $\mathcal{S}_{\Psi}^{\eta}(t,\chi)$ and $\mathcal{Q}_{\Psi}^{\eta}(t,\chi)$ are bounded linear operators with

$$\left\|\mathcal{S}_{\Psi}^{\eta}(t,\chi)\mathrm{u}\right\|\le\kappa_{\eta}\left\|\mathrm{u}\right\|\ \text{and}\ \left\|\mathcal{Q}_{\Psi}^{\eta}(t,\chi)\mathrm{u}\right\|\le\frac{\eta\kappa_{\eta}}{\Gamma(1+\eta)}\left\|\mathrm{u}\right\|,$$

for all $\mathrm{u}\in Y$.

(b) The operator $\mathcal{S}_{\Psi}^{\eta}(t,\chi)$ and $\mathcal{Q}_{\Psi}^{\eta}(t,\chi)$ are strongly continuous for all $0\le t_1\le t_2\le b$ we write

$$\left\|\mathcal{S}_{\Psi}^{\eta}(t_2,\chi)\mathrm{u}-\mathcal{S}_{\Psi}^{\eta}(t_2,\chi)\mathrm{u}\right\|\to 0\ \text{and}\ \left\|\mathcal{Q}_{\Psi}^{\eta}(t_2,\chi)-\mathcal{Q}_{\Psi}^{\eta}(t_1,\chi)\right\|\to 0,\ \text{as}\ t_2\to t_1.$$

(c) If $T(t)$ is a compact operator $\forall t>0$, then $\mathcal{S}_{\Psi}^{\eta}(t,\chi)$ and $\mathcal{Q}_{\Psi}^{\eta}(t,\chi)$ are compact for all $t,\chi>0$.

(d) If $\mathcal{S}_{\Psi}^{\eta}(t,\chi)$ and $\mathcal{Q}_{\Psi}^{\eta}(t,\chi)$ are the compact strongly continuous semigroup of bounded linear operator for $t,\chi>0$, then $\mathcal{S}_{\Psi}^{\eta}(t,\chi)$ and $\mathcal{Q}_{\Psi}^{\eta}(t,\chi)$ are continuous in the uniform operator topology.

Lemma 2.14 (Schauder Fixed Point Theorem) [5] If D is a closed, bounded, and convex subset of a Banach space X and $\mathfrak{G}:\mathrm{D}\to\mathrm{D}$ is completely continuous, then $\mathfrak{G}$ has a fixed point in D.

We give the preceding description of an appropriate system, its controllers, and its essential assumptions:

$$^{C}D_{0^+}^{\eta;\Psi}\mathrm{u}(t)=\mathscr{A}\mathrm{u}(t)+\mathscr{B}v(t),\ t\in\mathcal{I}'=(0,b], \tag{26}$$

$$\mathrm{u}(0)=\mathrm{u}_0. \tag{27}$$

The approximate controllability for the linear fractional system (26) is a natural generalization of approximate controllability of linear first order control system. It is convenient at this point to introduce the following controllability and resolvent operators associated with (26)

$$\mathfrak{T}_0^b = \int_0^b \mathcal{Q}_\Psi^\eta(b,\varpi)\mathscr{B}\mathscr{B}^*\mathcal{Q}_\Psi^{\eta^*}(b,\varpi)d\varpi, \tag{28}$$

$$R\left(\gamma,\mathfrak{T}_0^b\right) = \left(\gamma I + \mathfrak{T}_0^b\right)^{-1},\ \gamma > 0, \tag{29}$$

here $\mathscr{B}^*$ and $\mathcal{Q}_\Psi^{\eta^*}$ are the adjoint of $\mathscr{B}$ and $\mathcal{Q}_\eta$ respectively, also $\mathfrak{T}_0^b$ be the linear bounded operator.

Lemma 2.15 The linear fractional control system (26) is approximately controllable on $\mathcal{I}$ if and only if $\gamma R(\gamma, \mathfrak{T}_0^b) \to 0$ as $\gamma \to 0^+$ in the strong operator topology.

Proof. The proof of the Lemma is similar to proof of Theorem 2 in [44]. □

Next, for every $\gamma > 0$, and $\mathfrak{u}_1 \in \mathfrak{D}$, take

$$v(t) = \mathscr{B}^*\mathcal{Q}_\Psi^{\eta^*}(b,t)\mathrm{R}\left(\gamma,\mathfrak{T}_0^b\right)P(\mathfrak{u}(\cdot)), \tag{30}$$

where

$$P(v(\cdot)) = \mathfrak{u}_1 - \mathcal{S}_\Psi^\eta(t,0)\left[\mathfrak{u}_0 + \xi\left(\mathfrak{u}_{t_1},\mathfrak{u}_{t_2},\ldots,u_{t_n}\right)\right] - \sum_{0<\chi_k<t}\mathcal{S}_\Psi^\eta(t,t_k)I_k(\mathfrak{u}(t_k^-))$$

$$-\int_0^b\left(\Psi(b)-\Psi(\varpi)\right)^{\eta-1}\mathcal{Q}\eta(b,\varpi)\mathfrak{G}\left(\omega,\mathfrak{u}_\omega,\int_0^\omega e(\omega,s,\mathfrak{u}_s)ds\right)\Psi'(\varpi)d\varpi. \tag{31}$$

We introducing the succeeding hypotheses:

(H_1) $\{T(t)\}_{t\geq 0}$ is the C_0-semigroup, such that $\sup_{t\in[0,\infty)}\|T(t)\| = M_\eta$ where $M_\eta \geq 1$ and $\left\|\mathrm{R}(\gamma,\mathfrak{T}_0^b)\right\| \leq 1\ \forall \gamma > 0$.

(H_2) For $t \in \mathscr{W}$, $\mathfrak{G}(t,\cdot,\cdot): \mathscr{T}_w \times \mathfrak{D} \to \mathfrak{D}, e(t,s,\cdot): \mathscr{T}_w \to \mathfrak{D}$ are continuous functions and for every $\mathfrak{u} \in \mathcal{X}$, $\mathfrak{G}(\cdot, \mathfrak{u}_t, \int e) : \mathscr{W} \to \mathfrak{D}$ and $e(\cdot, \cdot, \mathfrak{u}_t) : \mathscr{W} \times \mathscr{W} \to \mathfrak{D}$ are strongly measurable.

(H_3) There exists an increasing function $\Lambda : \mathbb{R}^+ \to (0, \infty)$ and $L_{\mathfrak{G},P}(\cdot) \in L^1(\mathscr{W}', \mathbb{R})$, such that $\|\mathfrak{G}(t, \gamma_1, \gamma_2)\| \leq L_{\mathfrak{G},P}(t)\Lambda(\|\gamma_1\|_\Upsilon + \|\gamma_2\|)$ for every $(t, \gamma_1, \gamma_2) \in \mathscr{W} \times \mathscr{T}_w \times \mathfrak{D}$, and there exist a constant $\mathsf{M} > 0$, then

$$\limsup_{P\to\infty}\frac{L_{\mathfrak{G},P}(t)\Lambda\left(\|\gamma_1\|_\Upsilon + \|\gamma_2\|\right)}{P} = \mathsf{M}.$$

(H_4) There exists a constant $E_0 > 0$, such that $\|e(t, s, \gamma)\| \leq E_0(1 + \|\gamma\|_\Upsilon)\ \forall (t, s, \gamma) \in \mathscr{W} \times \mathscr{W} \times \mathscr{T}_w$.

(H_5) The functions $I_k : \mathfrak{D} \to \mathfrak{D}$ are continuous and there exists continuous nondecreasing functions $\mathrm{L}_K : [0, +\infty) \to [0, +\infty]$, such that $\|I_k(\mathfrak{u})\| \leq \mathrm{L}_K(\|\mathfrak{u}\|)$, and

$$\limsup_{P\to+\infty}\frac{\mathrm{L}_k(P)}{P} = \beta_k < \infty,\ k = 1,2,\cdots m.$$

(H_6) The continuous function $\xi : \mathscr{T}_w{}^n \to \mathscr{T}_w$ and $\Xi_n(\xi) > 0$ such that

$$\left\|\xi(a_1,a_2,a_3,\ldots,a_n) - \xi(b_1,b_2,b_3,\ldots,b_n)\right\| \leq \sum_{k=0}^n \Xi_k(\xi)\|a-b\|_\Upsilon,$$

for all $a_n, b_n \in \mathscr{T}_w$ and assume $\mathcal{P}_\xi = \sup\{\|\xi(a_1, a_2, a_3, ..., a_n)\| : a_j \in \mathscr{T}_w\}$.

3. Approximate controllability

Theorem 3.1 If (H_1)-(H_6) satisfies, then the Equation (1) has atleast mild solution on $\mathscr{W}$ with:

$$\left(\frac{\kappa_\eta \mathsf{M}}{\Gamma(\eta+1)}\left(\Psi(b)-\Psi(0)\right)^\eta+\kappa_\eta\beta_k\right)<1.$$

Proof. Let us consider the operator $\Xi : \mathscr{T}_w' \to \mathscr{T}_w'$, classified

$$\Xi(\mathfrak{u}(t))=\begin{cases}\Xi_1(t)+\xi\left(\mathfrak{u}_{t_1},\mathfrak{u}_{t_2},\ldots,\mathfrak{u}_{t_n}\right)(t),\ (-\infty,0],\\ \mathcal{S}_{\eta,\zeta}(t,0)\left[\mathfrak{u}(0)+\xi\left(\mathfrak{u}_{j_1},\mathfrak{u}_{j_2},\ldots,\mathfrak{u}_{j_n}\right)\right]+\int_0^t(\Psi(t)-\Psi(\chi))^{\eta-1}\mathcal{Q}\eta(t,\chi)\Psi'(t)\\ \times\mathfrak{G}\left(\chi,\mathfrak{u}_\chi,\int_0^\chi e(\chi,s,\mathfrak{u}_s)ds\right)d\chi+\sum\limits_{0<t_k<t}\mathcal{S}_\Psi^\eta(t,t_k)I_k(\mathfrak{u}(t_k^-))\\ +\int_0^t(\Psi(t)-\Psi(\chi))^{\eta-1}\mathcal{Q}\eta(t,\chi)\mathscr{B}v(t)\Psi'(\chi)d\chi,\ t\in(0,b].\end{cases}\tag{32}$$

For $\Xi_1 \in \mathscr{T}_w$, we define $\hat{\Omega}$ by

$$\hat{\Omega}(t)=\begin{cases}\Xi_1(t)+\xi(\mathfrak{u}_{t_1},\mathfrak{u}_{t_2},\ldots,\mathfrak{u}_{t_n}),\ t\in(-\infty,0],\\ \mathcal{S}_{\eta,\zeta}(t,0)\left[\mathfrak{u}(0)+\xi\left(\mathfrak{u}_{t_1},\mathfrak{u}_{t_2},\ldots,\mathfrak{u}_{t_n}\right)(0)\right],\ t\in\mathscr{W},\end{cases}\tag{33}$$

then $\hat{\Omega} \in \mathscr{T}_w'$. Let $\mathfrak{u}_t = [y_t + \hat{\Omega}_t]$, $\infty < t \le b$. It is simple to expose that $\mathfrak{u}$ meets from (2.12) if and only if v fulfils y_0. and

$$y(t)=\int_0^t(\Psi(t)-\Psi(\chi))^{\eta-1}\mathcal{Q}\eta(t,\chi)\mathfrak{G}\left(\chi,(y_\chi+\hat{\Omega}_\chi),\int_0^\chi e\left(\chi,s,y_s+\hat{\Omega}_s\right)ds\right)\Psi'(\chi)d\chi$$

$$+\sum_{0<t_k<t}\mathcal{S}_\Psi^\eta(t,t_k)I_k(y_t^k+\hat{\Omega}_t)$$

$$+\int_0^t(\Psi(t)-\Psi(\chi))^{\eta-1}\mathcal{Q}_\eta(t,\chi)\mathscr{B}\mathscr{B}^*\mathcal{Q}_\eta^*(b,\chi)R(\alpha,\mathfrak{T}_0^b)\Big[\mathfrak{u}_1-\mathcal{S}_{\eta,teta}(b,0)\left[\mathfrak{u}(0)+\mathcal{S}(\mathfrak{u}_{t_1},\mathfrak{u}_{t_2},\ldots,\mathfrak{u}_{t_n})\right]$$

$$-\int_0^b(\Psi(b)-\Psi(\varpi))^{\eta-1}\mathcal{Q}_\eta(b,\varpi)\mathfrak{G}\left(\varpi,v_\varpi+\hat{\Omega}_\varpi,\int_0^\varpi e(\varpi,s,y_s+\hat{\Omega}_s)ds\right)\Psi'(\varpi)d\varpi$$

$$-\sum_{0<\varpi_k<b}\mathcal{S}_\Psi^\eta(b,\varpi_k)I_k\left(y(t_k^-)+\hat{\Omega}(t_k^-)\right)\Big]\Psi'(\chi)d\chi.$$

Let $\mathscr{T}_w'' = \{y \in \mathscr{T}_w' : y_0 \in \mathscr{T}_w\}$. For any $y \in \mathscr{T}_w'$,

$$\|y\|_\Upsilon=\|y_0\|_\Upsilon+\sup\{\|y(\omega)\|:0\le\omega\le b\}$$

$$=\sup\{\|y(\omega)\|:0\le\omega\le b\}.$$

Thus, $(\mathscr{T}_w'', \|\cdot\|)$ is a Banach space.

For $P > 0$, choose $\mathscr{T}_P = \{y \in \mathscr{T}_w'' : \| y \|_\Upsilon \le P\}$, then $\mathscr{T}_P \subset \mathscr{T}_w''$ is uniformly bounded, and $\forall y \in \mathscr{T}_P$, from Lemma 2.6,

$$\left\| y_t + \hat{\Omega}_t \right\|_\Upsilon \le \| y_t \|_\Upsilon + \| \hat{\Omega}_t \|_\Upsilon$$

$$\le \Upsilon\left(P + \kappa_\eta \left[\mathfrak{u}_0 + \mathcal{P}_\xi\right]\right) + \|\Omega_1\|_\Upsilon + \left\| \xi(\mathfrak{u}_{t_1}, \mathfrak{u}_{t_2}, \ldots, \mathfrak{u}_{t_n}) \right\|_\Upsilon$$

$$= P'.$$

Consider the operator $\Omega : \mathscr{T}_w'' \to \mathscr{T}_w''$, defined by

$$\Omega y(t) = \begin{cases} 0, \ t \in (-\infty, 0], \\ \int_0^t (\Psi(t) - \Psi(\chi))^{\eta-1} \mathcal{Q}_\eta(t, \chi) \mathfrak{G}\left(\chi, y_\chi + \hat{\Omega}_\chi, \int_0^\chi e\left(\chi, s, y_s + \hat{\Omega}_s\right) ds\right) \Psi'(\chi) d\chi \\ + \sum_{0 < t_k < t} \mathcal{S}_\Psi^\eta(t, t_k) I_k \left(y(t_k^-) + \hat{\Omega}(t_k^-)\right) \\ + \int_0^t (\Psi(t) - \Psi(\chi))^{\eta-1} \mathcal{Q}_\eta(t, \chi) \mathscr{B} v(t) \Psi'(\chi) d\chi, \ t \in \mathscr{W}. \end{cases} \tag{34}$$

Now we expose Ω has a fixed point.

Step 1 We assume that $\Omega(y(t)) \in \mathscr{T}_P$, to expose that $\Omega(\mathscr{T}_P) \subset \mathscr{T}_P$. We assume that for $P > 0$, there exists $t \in [0, b]$, such that

$$\left\| (\Omega y)(t) \right\| > P. \tag{35}$$

Since,

$$\left\| (\Omega y)(t) \right\| \le \left\| \int_0^t \left(\Psi(t) - \Psi(\chi)\right)^{\eta-1} \mathcal{Q}_\Psi^\eta(t, \chi) \mathfrak{G}\left(\chi, y_\chi + \hat{\Psi}_\chi, \int_0^\chi e\left(\chi, s, y_s + \hat{\Psi}_s\right) ds\right) \Psi'(\chi) d\chi \right\|$$

$$+ \left\| \sum_{0 < t_k < t} \mathcal{S}_\Psi^\eta(t, \chi_k) I_k \left(y\left(y(t_k^-) + \hat{\Omega}(t_k^-)\right)\right) \right\|$$

$$+ \left\| \int_0^t \left(\Psi(t) - \Psi(\chi)\right)^{\eta-1} \mathcal{Q}_\Psi^\eta \eta(t, \chi) \mathscr{B} v(\chi) \Psi'(\chi) d\chi \right\|$$

$$\le \frac{\kappa_\eta L_{\mathfrak{G}, P}(b) \Lambda\left(P' + E_0(1 + P')\right)}{\Gamma(\eta)} \times \int_0^t \left(\Psi(t) - \Psi(\chi)\right)^{\eta-1} \Psi'(\chi) d\chi + \kappa_\eta \mathrm{L}_K(P')$$

$$+ \frac{\kappa_\eta \kappa_\mathscr{B}}{\Gamma(\eta)} \int_0^t \left(\Psi(t) - \Psi(\chi)\right)^{\eta-1} \Psi'(\chi) \times \frac{\kappa_\eta \kappa_\mathscr{B}}{\alpha \Gamma(\eta)} \Bigg[\| \mathfrak{u}_b \| - \kappa_\eta \left[\| \mathfrak{u}_0 \| + \mathcal{P}_\xi\right]$$

$$- \frac{\kappa_\eta L_{\mathfrak{G}, P}(b) \Lambda\left(P' + E_0(1 + P')\right)}{\Gamma(\eta)} \times \int_0^b \left(\Psi(b) - \Psi(\varpi)\right)^{\eta-1} \Psi'(\varpi) d\varpi - \kappa_\eta \mathrm{L}_K(P') \Bigg] d\chi$$

$$\leq \frac{\kappa_\eta L_{\mathfrak{G},P}(b)\Lambda\left(P'+E_0(1+P')\right)}{\Gamma(\eta+1)}\left(\Psi(b)-\Psi(0)\right)^\eta+\kappa_\eta \mathrm{L}_K(P')+\left[\frac{\kappa_\eta\kappa_{\mathscr{B}}}{\Gamma(\eta+1)}\right]^2\eta\left(\Psi(b)-\Psi(0)\right)^\eta$$

$$\times\left[\| \mathfrak{u}_b \|-\kappa_\eta\left[\| \mathfrak{u}_0 \|+\mathcal{P}_\xi\right]-\frac{\kappa_\eta L_{\mathfrak{G},P}(b)\Lambda\left(P'+E_0(1+P')\right)}{\Gamma(\eta+1)}\left(\Psi(b)-\Psi(0)\right)^\eta-\kappa_\eta \mathrm{L}_K(P')\right].$$

Dividing to both side by P and taking limit supremum as $P \to \infty$, obtain

$$1\leq\left(\frac{\kappa_\eta \mathsf{M}}{\Gamma(\eta+1)}\left(\Psi(b)-\Psi(0)\right)^\eta+\kappa_\eta\beta_k\right),$$

then we have a contradiction to our assumption (32).

Therefore $\Omega y \in \mathscr{T}_P$.

Step 2 To expose Ω is continuous. Let $\{y^n\} \subset \mathscr{T}_p$, such that $y^n \to y \in \mathscr{T}_p$ as $n \to \infty$. From assumptions (H_2) and (H_3), we can write, for every $t \in \mathscr{W}$,

$$\mathfrak{G}\left(t, y_t^n+\hat{\Omega}_t, \int_0^t e\left(t,s,y_s^n+\hat{\Omega}_s\right)\right)\to\mathfrak{G}\left(t, y_t+\hat{\Omega}_t, \int_0^t e\left(t,s,y_s+\hat{\Omega}_s\right)\right) \text{ as } n\to\infty \quad \forall n\in\mathbb{N}.$$

By Lebesgue dominated convergence theorem, for every $t \in \mathscr{W}$, we write

$$\left\|\left(\Omega y^n\right)(t)-\left(\Omega y\right)(t)\right\|$$

$$\leq\left\|\int_0^t\left(\Psi(t)-\Psi(\chi)\right)^{\eta-1}\mathcal{Q}_\Psi^\eta(t,\chi)\Psi'(\chi)\right.$$

$$\left.\times\left[\mathfrak{G}\left(\chi, y_\chi^n+\hat{\Omega}_\chi, \int_0^t e\left(\chi,s,y_s^n+\hat{\Omega}_s\right)dt\right)-\mathfrak{G}\left(\chi, y_\chi+\hat{\Omega}_\chi, \int_0^t e\left(\chi,s,y_s+\hat{\Omega}_s\right)dt\right)\right]d\chi\right\|$$

$$+\left\|\sum_{0<t_k<t}\left(\mathcal{S}_\Psi^\eta(t,t_k)I_k\left(y^n(t_k^-)+\hat{\Omega}(t_k^-)\right)-\mathcal{S}_\Psi^\eta(t,t_k)I_k\left(y(t_k^-)+\hat{\Omega}(t_k^-)\right)\right)\right\|$$

$$+\left\|\int_0^t\left(\Psi(t)-\Psi(\chi)\right)^{\eta-1}\mathcal{Q}_\Psi^\eta(t,\chi)\Psi'(\chi)\mathscr{B}\mathscr{B}^*\mathcal{Q}_\Psi^{\eta*}(b,t)R(\alpha,\mathfrak{T}_0^b)\int_0^b\left(\Psi(b)-\Psi(\varpi)\right)^{\eta-1}\Psi'(\varpi)\right.$$

$$\left.\times\left[\mathfrak{G}\left(\varpi, y_\varpi^n+\hat{\Omega}_\varpi, \int_0^\varpi e\left(\varpi,s,y_s^n+\hat{\Omega}_s\right)d\varpi\right)-\mathfrak{G}\left(\varpi, y_\varpi+\hat{\Omega}_\varpi, \int_0^\varpi e\left(\varpi,s,y_s+\hat{\Omega}_s\right)d\varpi\right)\right]d\chi\right\|$$

$$\leq\frac{\kappa_\eta}{\Gamma(\eta)}\int_0^t\left(\Psi(t)-\Psi(\chi)\right)^{\eta-1}\Psi'(\chi)$$

$$\times\left\|\left[\mathfrak{G}\left(t, y_t^n+\hat{\Omega}_t, \int_0^t e\left(t,s,y_s^n+\hat{\Omega}_s\right)dt\right)-\mathfrak{G}\left(t, y_t+\hat{\Omega}_t, \int_0^t e\left(t,s,y_s+\hat{\Omega}_s\right)dt\right)\right]\right\|d\chi$$

$$+\kappa_\eta\left\|I_k\left(y^n(t_k^-)+\hat{\Omega}(t_k^-)\right)-\mathcal{S}_\Psi^\eta(t,t_k)I_k\left(y(t_k^-)+\hat{\Omega}(t_k^-)\right)\right\|$$

$$+\left(\frac{\kappa_\eta \kappa_{\mathscr{B}}}{\Gamma(\eta)}\right)^2 \int_0^b \left(\Psi(b)-\Psi(\varpi)\right)^{\eta-1}\Psi'(\varpi)$$

$$\times\left\|\left[\mathfrak{G}\left(\varpi, y_\varpi^n+\hat{\Omega}_\varpi, \int_0^\varpi e\left(\varpi, s, y_s^n+\hat{\Omega}_s\right)d\varpi\right)-\mathfrak{G}\left(\varpi, y_\varpi+\hat{\Omega}_\varpi, \int_0^\varpi e\left(\varpi, s, y_s+\hat{\Omega}_s\right)d\varpi\right)\right]\right\|d\varpi.$$

Apply $n \to \infty$, then $\|(\Omega y^n)(t) - (\Omega y)(t)\| \to 0$. Hence Ω is continuous.

Step 3 Now, we demonstrate that compactness of Ω. For that, first we expose that $\{(\Omega y)(t) : y \in \mathscr{T}_p\}$ is equicontinuous in $\mathfrak{D}$.

For any $y \in \mathscr{T}_p$ and $0 \le t_1 \le t_2 \le b$, we obtain

$$\begin{aligned}
&\|(\Omega y)(t_2)-(\Omega y)(t_1)\| \\
&\le \left\|\int_0^{t_2}\left(\Psi(t_2)-\Psi(\chi)\right)^{\eta-1}\mathcal{Q}_\Psi^\eta(t_2,\chi)\times\mathfrak{G}\left(\chi, y_\chi+\hat{\Omega}_\chi, \int_0^t e\left(\chi, s, y_s+\hat{\Omega}_s\right)\right)\Psi'(\chi)d\chi\right. \\
&\left.-\int_0^{t_1}\left(\Psi(t_1)-\Psi(\chi)\right)^{\eta-1}\mathcal{Q}_\Psi^\eta(t_1,\chi)\times\mathfrak{G}\left(\chi, y_\chi+\hat{\Omega}_\chi, \int_0^\chi e\left(\chi, s, y_s+\hat{\Omega}_s\right)\right)\Psi'(\chi)d\chi\right\| \\
&+\left\|\sum_{0<t_k<t_2}\mathcal{S}_\Psi^\eta(t,t_k)I_k\left(y(t_k^-)+\hat{\Omega}(t_k^-)\right)-\sum_{0<t_k<t_1}\mathcal{S}_\Psi^\eta(t,t_k)I_k\left(y(t_k^-)+\hat{\Omega}(t_k^-)\right)\right\| \\
&+\left\|\int_0^{t_2}\left(\Psi(t_2)-\Psi(\chi)\right)^{\eta-1}\mathcal{Q}_\Psi^\eta(t_2,\chi)\mathscr{B}v(\chi)\Psi'(\chi)d\chi\right. \\
&\left.-\int_0^{t_1}\left(\Psi(t_1)-\Psi(\chi)\right)^{\eta-1}\mathcal{Q}_\Psi^\eta(t_1,\chi)\mathscr{B}v(\chi)\Psi'(\chi)d\chi\right\| \\
&\le \left\|\int_0^{t_2}\left(\Psi(t_2)-\Psi(\chi)\right)^{\eta-1}\mathcal{Q}_\Psi^\eta(t_2,\chi)\times\mathfrak{G}\left(\chi, y_\chi+\hat{\Omega}_\chi, \int_0^t e\left(\chi, s, y_s+\hat{\Omega}_s\right)\right)d\chi\right\| \\
&+\left\|\int_0^{t_1}\left[\left(\Psi(t_2)-\Psi(\chi)\right)^{\eta-1}-\left(\Psi(t_1)-\Psi(\chi)\right)^{\eta-1}\right]\mathcal{Q}_\Psi^\eta(t_2,\chi)\right. \\
&\left.\times\mathfrak{G}\left(\chi, y_\chi+\hat{\Omega}_\chi, \int_0^t e\left(\chi, s, y_s+\hat{\Omega}_s\right)\right)\Psi'(\chi)d\chi\right\| \\
&+\left\|\sum_{0<t_k<t_1}\left[\mathcal{S}_\Psi^\eta(t_2,t_k)-\mathcal{S}_\Psi^\eta(t_1,t_k)\right]I_k\left(y(t_k^-)+\hat{\Omega}(t_k^-)\right)\right\| \\
&+\left\|\sum_{t_1<t_k<t_2}\mathcal{S}_\Psi^\eta(t,t_k)I_k\left(y(t_k^-)+\hat{\Omega}(t_k^-)\right)\right\|
\end{aligned}$$

$$+\left\|\int_0^{t_1}\left(\Psi(t_1)-\Psi(\chi)\right)^{\eta-1}\left[\mathcal{Q}_\Psi^\eta(t_2,\chi)-\mathcal{Q}_\Psi^\eta(t_1,\chi)\right]\mathfrak{G}\left(\chi,y_\chi+\hat{\Omega}_\chi,\int_0^t e\left(\chi,s,y_s+\hat{\Omega}_s\right)\right)\Psi'(\chi)d\chi\right\|$$

$$+\left\|\int_{t_1}^{t_2}\left(\Psi(t_2)-\Psi(t_1)\right)^{\eta-1}\mathcal{Q}_\Psi^\eta(t_2,\chi)\mathscr{B}v(\chi)\Psi'(\chi)d\chi\right\|$$

$$+\left\|\int_0^{t_1}\left[\left(\Psi(t_2)-\Psi(t_1)\right)^{\eta-1}-\left(\Psi(t_1)-\Psi(\chi)\right)^{\eta-1}\right]\mathcal{Q}_\Psi^\eta(t_2,\chi)\mathscr{B}v(\chi)\Psi'(\chi)d\chi\right\|$$

$$+\left\|\int_0^{t_1}\left(\Psi(t_1)-\Psi(\chi)\right)^{\eta-1}\left[\mathcal{Q}_\Psi^\eta(t_2,\chi)-\mathcal{Q}_\Psi^\eta(t_1,\chi)\right]\mathscr{B}v(\chi)\Psi'(\chi)d\chi\right\|$$

$$=\sum_{i=1}^{8}I_i.$$

From Lemma (2.13), we get

$$I_1\le\frac{\kappa_\eta L_{\mathfrak{G},P}(b)\Lambda\left(P'+E_0(1+P')\right)}{\Gamma(\eta+1)}\left(\Psi(t_2)-\Psi(t_1)\right)^\eta,$$

and

$$I_2\le\frac{\kappa_\eta L_{\mathfrak{G},P}(b)\Lambda\left(P'+E_0(1+P')\right)}{\Gamma(\eta+1)}\left[\left(\Psi(t_2)\right)^\eta-\left(\Psi(t_1)\right)^\eta-\left(\Psi(t_2)-\Psi(t_1)\right)^\eta\right].$$

Therefore, $I_1\to 0$, and $I_2\to 0$ as $t_2\to t_1$. Now, we consider

$$I_3\le\sum_{0<t_k<t_1}\mathrm{L}_k\left\|\left[\mathcal{S}_\Psi^\eta(t_2,t_k)-\mathcal{S}_\Psi^\eta(t_1,t_k)\right]\right\|,$$

from strong continuity of $\mathcal{S}_\Psi^\eta(t,z_k)$ we get $I_3\to 0$ as $t_2\to t_1$.

$$I_4=\left\|\sum_{t_1<t_k<t_2}\mathcal{S}_\Psi^\eta(t,t_k)I_k(y\left(y(t_k^-)+\hat{\Omega}(t_k^-)\right)\right\|$$

$$\le\sum_{t_1<t_k<t_2}\kappa_\eta\mathrm{L}_k\left(\|t\|\right),$$

which implies, $I_4\to 0$ as $t_2\to t_1$. Let ϵ be the arbitrary small positive, we write

$$I_5\le\int_0^{t_1-\epsilon}\left(\Psi(t_1)-\Psi(\chi)\right)^{\eta-1}\left[\mathcal{Q}_\Psi^\eta(t_2,\chi)-\mathcal{Q}_\Psi^\eta(t_1,\chi)\right]\mathfrak{G}\left(\chi,y_\chi+\hat{\Omega}_\chi,\int_0^t e\left(\chi,s,y_s+\hat{\Omega}_s\right)\right)\Psi'(\chi)d\chi$$

$$+\int_{t_1-\epsilon}^{t_1}\left(\Psi(t_1)-\Psi(\chi)\right)^{\eta-1}\left[\mathcal{Q}_\Psi^\eta(t_2,\chi)-\mathcal{Q}_\Psi^\eta(t_1,\chi)\right]\mathfrak{G}\left(\chi,y_\chi+\hat{\Omega}_\chi,\int_0^t e\left(\chi,s,y_s+\hat{\Omega}_s\right)\right)\Psi'(\chi)d\chi$$

$$\le L_{\mathfrak{G},P}(b)\Lambda\left(P'+E_0(1+P')\right)\int_0^{t_1-\epsilon}\left(\Psi(t_1)-\Psi(\chi)\right)^{\eta-1}\Psi'(\chi)d\chi \sup_{\chi\in[0,t_1-\epsilon]}\left\|\mathcal{Q}_\Psi^\eta(t_2,\chi)-\mathcal{Q}_\Psi^\eta(t_1,\chi)\right\|$$

$$+\frac{2\kappa_\eta L_{\mathfrak{G},P}(b)\Lambda\left(P'+E_0(1+P')\right)}{\Gamma(\eta)}\int_{t_1-\epsilon}^{t_1}\left(\Psi(t_1)-\Psi(\chi)\right)^{\eta-1}\Psi'(\chi)d\chi.$$

From Lemma (2.13), we obtain $I_5 \to 0$ as $t_2 \to t_1$ and $\epsilon \to 0$. Using the similar procedure, we get I_6, I_7 and I_8 are tend to zero.

Step 4 We need to prove, $\forall t \in [0, b]$, $\Omega(t) = \{(\Omega y)(t) : y \in \mathscr{T}_p\}$ is relatively compact in $\mathfrak{D}$.

Take $0 \le t \le b$ then, $\forall \epsilon > 0$ and $\varpi > 0$, let $y \in \mathscr{T}_p$ and explain the operator $\mathbf{\Omega}^{\epsilon,\varpi}$ on $\mathscr{T}_p$ as

$$\left(\Omega^{\epsilon,\varpi}y\right)(t)=\eta\int_0^{t-\epsilon}\int_\varpi^\infty \varepsilon\zeta_\eta(\varepsilon)\left(\Psi(t)-\Psi(\chi)\right)^{\eta-1}T\left(\left(\Psi(t)-\Psi(0)\right)^\eta\varepsilon\right)$$

$$\times\mathfrak{G}\left(t,y_t+\hat{\Omega}_t,\int_0^t e\left(t,s,y_s+\hat{\Omega}_s\right)\right)\Psi'(\chi)d\varepsilon d\chi$$

$$+\sum_{0<t_k<t}\int_\varpi^\infty\zeta_\eta(\varepsilon)T\left((\Psi(t)-\Psi(t_k))^\eta\varepsilon\right)d\varepsilon I_k\left(y(t_k^-)+\hat{\Omega}(t_k^-)\right)$$

$$+\eta\int_0^{t-\epsilon}\int_\varpi^\infty\varepsilon\zeta_\eta(\varepsilon)\left(\Psi(t)-\Psi(\chi)\right)^{\eta-1}T\left(\left(\Psi(\jmath)-\Psi(0)\right)^\eta\varepsilon\right)\mathscr{B}v(\chi)\Psi'(\chi)d\varepsilon d\chi$$

$$=\eta\int_0^{t-\epsilon}\int_\varpi^\infty\varepsilon\zeta_\eta(\varepsilon)\left(\Psi(t)-\Psi(\chi)\right)^{\eta-1}T\left(\left(\Psi(t)-\Psi(0)\right)^\eta\varepsilon+\epsilon^\eta\varpi-\epsilon^\eta\varpi\right)$$

$$\times\left[\mathfrak{G}\left(t,y_t+\hat{\Omega}_t,\int_0^t e\left(t,s,y_s+\hat{\Omega}_s\right)\right)+\mathscr{B}v(\chi)\right]\Psi'(\chi)d\varepsilon d\chi$$

$$+\sum_{0<t_k<t}\int_\varpi^\infty\zeta_\eta(\varepsilon)T\left((\Psi(t)-\Psi(t_k))^\eta\varepsilon+\epsilon^\eta\varpi-\epsilon^\eta\varpi\right)d\varepsilon I_k\left(y(t_k^-)+\hat{\Omega}(t_k^-)\right)$$

$$=\eta T(\epsilon^\eta\varpi)\int_0^{t-\epsilon}\int_\varpi^\infty\varepsilon\zeta_\eta(\varepsilon)\left(\Psi(t)-\Psi(\chi)\right)^{\eta-1}T\left(\left(\Psi(t)-\Psi(0)\right)^\eta\varepsilon-\epsilon^\eta\varpi\right)$$

$$\times\left[\mathfrak{G}\left(t,y_t+\hat{\Omega}_t,\int_0^t e\left(t,s,y_s+\hat{\Omega}_s\right)\right)+\mathscr{B}v(\chi)\right]\Psi'(\chi)d\varepsilon d\chi$$

$$+T(\epsilon^\eta\varpi)\sum_{0<t_k<t}\int_\varpi^\infty\zeta_\eta(\varepsilon)T\left(\left(\Psi(t)-\Psi(t_k)\right)^\eta\varepsilon-\epsilon^\eta\varpi\right)\mathrm{u}d\varepsilon I_k\left(y(t_k^-)+\hat{\Omega}(t_k^-)\right).$$

Then by compactness of $T(\epsilon^\eta\varpi)$ for $\epsilon^\eta\varpi > 0$, we have $\Omega^{\epsilon,\varpi}(t) = \{(\Omega^{\epsilon,\varpi}y)(t) : y \in \mathscr{T}_p\}$ is relatively compact in $\mathfrak{D}$. Furthermore, for any $\mathrm{u} \in \mathscr{T}_p$ we get

$$\left\|\left(\Omega y\right)(t)-\left(\Omega^{\epsilon,\varpi}y\right)(t)\right\|$$

$$\leq \eta \left\| \int_0^t \int_0^{\varpi} \varepsilon \zeta_\eta(\varepsilon) \left(\Psi(t) - \Psi(\chi)\right)^{\eta-1} T\left(\left(\Psi(t) - \Psi(0)\right)^{\eta} \varepsilon\right) \right.$$

$$\left. \times \left[\mathfrak{G}\left(t, y_t + \hat{\Omega}_t, \int_0^t e\left(t, s, y_s + \hat{\Omega}_s\right)\right) + \mathscr{B}v(\chi) \right] \Psi'(\chi) d\varepsilon d\chi \right\|$$

$$+ \eta \left\| \int_{t-\epsilon}^t \int_{\varpi}^{\infty} \varepsilon \zeta_\eta(\varepsilon) \left(\Psi(t) - \Psi(\chi)\right)^{\eta-1} T\left(\left(\Psi(t) - \Psi(0)\right)^{\eta} \varepsilon\right) \right.$$

$$\left. \times \left[\mathfrak{G}\left(t, y_t + \hat{\Omega}_t, \int_0^t e\left(t, s, y_s + \hat{\Omega}_s\right)\right) + \mathscr{B}v(\chi) \right] \Psi'(\chi) d\varepsilon d\chi \right\|$$

$$+ \left\| \sum_{0<t_k<t} \int_0^{\infty} \zeta_\eta(\varepsilon) T\left(\left(\Psi(t) - \Psi(t_k)\right)^{\eta} \varepsilon\right) d\varepsilon I_k\left(y(t_k^-) + \hat{\Omega}(t_k^-)\right) d\varepsilon \right\|$$

$$+ \left\| \sum_{0<t_k<t} \int_{\varpi}^{\infty} \zeta_\eta(\varepsilon) T\left(\left(\Psi(t) - \Psi(t_k)\right)^{\eta} \varepsilon\right) d\varepsilon I_k\left(y(t_k^-) + \hat{\Omega}(t_k^-)\right) d\varepsilon \right\|$$

$$\leq M_\eta \left[L_{\mathfrak{G},P}(b) \Lambda\left(P' + E_0(1+P')\right) + M_{\mathscr{B}} \| v \| \right] \left(\Psi(t) - \Psi(0)\right)^{\eta} \left(\int_0^{\varpi} \varepsilon \zeta_\eta(\varepsilon) d\varepsilon \right)$$

$$+ M_\eta \left[L_{\mathfrak{G},P}(b) \Lambda\left(P' + E_0(1+P')\right) + M_{\mathscr{B}} \| v \| \right] \left(\Psi(t) - \Psi(t-\epsilon)\right)^{\eta} \left(\int_0^{\infty} \varepsilon \zeta_\eta(\varepsilon) d\varepsilon \right)$$

$$\leq M_\eta \left[L_{\mathscr{B},P}(b) \Lambda\left(P' + E_0(1+P')\right) + M_{\mathscr{B}} \| v \| \right] \left(\Psi(b) - \Psi(0)\right)^{\eta} \left(\int_0^{\varpi} \varepsilon \zeta_\eta(\varepsilon) d\varepsilon \right)$$

$$+ \left[\frac{M_\eta L_{\mathscr{B},P}(b) \Lambda\left(P' + E_0(1+P')\right)}{\Gamma(\eta+1)} + M_{\mathscr{B}} \| v \| \right] \left(\Psi(t) - \Psi(t-\epsilon)\right)^{\eta}$$

$$+ M_\eta \sum_{0<t_k<t} \mathrm{L}_k(P') \int_0^{\infty} \zeta_\eta(\varepsilon) d\varepsilon + + M_\eta \sum_{0<t_k<t} \mathrm{L}_k(P') \int_{\varpi}^{\infty} \zeta_\eta(\varepsilon) d\varepsilon,$$

where $\int_0^{\infty} \varepsilon \zeta_\eta(\varepsilon) d\varepsilon = \dfrac{1}{\Gamma(\eta+1)}$ and $\int_0^{\infty} \zeta_\eta(\varepsilon) d\varepsilon = 1$. According to the absolute continuity of the Lebesgue integral, we have

$$\left\| (\Omega y)(t) - \left(\Omega^{\epsilon,\varpi} y\right)(t) \right\| \to 0 \text{ as } \epsilon, \varpi \to 0.$$

As a result, for $t > 0$ there is an arbitrarily compact set that is near to the set $\Omega(t)$. Therefore, by the Arzela-Ascoli theorem $\Omega(t)$ is relatively compact in $\mathfrak{D}$. Hence, the Schauder's fixed point theorem (2.14) Ω has a fixed point in $\mathscr{T}_p$, which is the mild solution of the system (1). □

We now concentrate on the approximate controllability of Equation (1).

Theorem 3.2 Suppose that (H_1)-(H_5) hold and $\mathfrak{G}$ is uniformly bounded function. Furthermore, the corresponding linear equation (26) is approximate controllable on $\mathscr{W}$, then the system (1) is approximately controllable on $\mathscr{W}$.

Proof. Let $\mathfrak{u}^{\gamma}$ be a fixed point of Ξ in $\mathscr{T}_p$, by Theorem (3.1), any fixed point u^{γ} is a mild solution of the system (1), such that

$$\begin{aligned}
\mathfrak{u}^{\gamma}(t) &= \mathcal{S}_{\Psi}^{\eta}(t,0)\Big[\mathfrak{u}_0+\xi\Big(\mathfrak{u}_{t_1},\mathfrak{u}_{t_2},\ldots,\mathfrak{u}_{t_n}\Big)\Big] \\
&+\int_0^t\big(\Psi(t)-\Psi(\chi)\big)^{\eta-1}\mathcal{Q}_{\Psi}^{\eta}(t,\chi)\mathfrak{G}\Big(t,\mathfrak{u}_t^{\gamma},\int_0^t e\Big(t,s,\mathfrak{u}_s^{\gamma}\Big)ds\Big)\Psi'(\chi)d\chi \\
&+\sum_{0<t_k<t}\mathcal{S}_{\Psi}^{\eta}(t,t_k)I_k\Big(\mathfrak{u}^{\gamma}(t_k^-)\Big) \\
&+\int_0^t\big(\Psi(t)-\Psi(\chi)\big)^{\eta-1}\mathcal{Q}_{\Psi}^{\eta}(t,\chi)\mathscr{B}\mathscr{B}^*\mathcal{Q}_{\Psi}^{\eta*}(b,t)\mathrm{R}\Big(\varpi,\mathfrak{T}_0^b\Big) \\
&\times\Big[\mathfrak{u}_1-\mathcal{S}_{\Psi}^{\eta}(b,0)\Big[\mathfrak{u}_0+\xi\Big(\mathfrak{u}_{t_1},\mathfrak{u}_{t_2},\ldots,\mathfrak{u}_{t_n}\Big)\Big]-\sum_{0<t_k<b}\mathcal{S}_{\Psi}^{\eta}(b,t_k)I_k\Big(\mathfrak{u}^{\gamma}(t_k^-)\Big) \\
&-\int_0^b\big(\Psi(b)-\Psi(\varpi)\big)^{\eta-1}\mathcal{Q}_{\Psi}^{\eta}(b,\varpi)\mathfrak{G}\Big(t,\mathfrak{u}_t^{\gamma},\int_0^t e\Big(t,s,\mathfrak{u}_s^{\gamma}\Big)ds\Big)\Psi'(\varpi)d\varpi\Big]d\chi,\ t\in\mathscr{W}.
\end{aligned}$$

Define

$$\begin{aligned}
P(\mathfrak{u}^{\gamma}) &= \mathfrak{u}_1-\mathcal{S}_{\Psi}^{\eta}(b,0)\Big[\mathfrak{u}_0+\xi\Big(\mathfrak{u}_{t_1},\mathfrak{u}_{t_2},\ldots,u_{t_n}\Big)\Big]-\sum_{0<t_k<b}\mathcal{S}_{\Psi}^{\eta}(b,t_k)I_k\Big(\mathfrak{u}^{\gamma}(t_k^-)\Big) \\
&-\int_0^b\big(\Psi(b)-\Psi(\varpi)\big)^{\eta-1}\mathcal{Q}_{\Psi}^{\eta}(b,\varpi)\mathfrak{G}\Big(t,\mathfrak{u}_t^{\gamma},\int_0^t e\Big(t,s,\mathfrak{u}_s^{\gamma}\Big)ds\Big)\Psi'(\varpi)d\varpi. \qquad (36)
\end{aligned}$$

We have $\Big(I-\mathfrak{T}_0^b\mathrm{R}(\gamma,\mathfrak{T}_0^b)\Big)=\gamma\mathrm{R}(\alpha,\mathfrak{T}_0^b)$, then

$$\begin{aligned}
\mathfrak{u}^{\gamma}(b) &= \mathcal{S}_{\Psi}^{\eta}(b,0)\Big[\mathfrak{u}_0+\xi\Big(\mathfrak{u}_{t_1},\mathfrak{u}_{t_2},\ldots,u_{t_n}\Big)\Big]+\sum_{0<t_k<b}\mathcal{S}_{\Psi}^{\eta}(b,t_k)I_k\Big(\mathfrak{u}^{\gamma}(t_k^-)\Big) \\
&+\int_0^b\big(\Psi(b)-\Psi(\chi)\big)^{\eta-1}\mathcal{Q}_{\Psi}^{\eta}(b,\chi)\mathfrak{G}\Big(t,\mathfrak{u}_t^{\gamma},\int_0^t e\Big(t,s,\mathfrak{u}_s^{\gamma}\Big)ds\Big)\Psi'(\chi)d\chi \\
&+\int_0^b\big(\Psi(b)-\Psi(\chi)\big)^{\eta-1}\mathcal{Q}_{\Psi}^{\eta}(b,\chi)\mathscr{B}\mathscr{B}^*\mathcal{Q}_{\Psi}^{\eta*}(b,t)\mathrm{R}(\gamma,\mathfrak{T}_0^b) \\
&\times\Big[\mathfrak{u}_1-\mathcal{S}_{\Psi}^{\eta}(b,0)\Big[\mathfrak{u}_0+\xi\Big(\mathfrak{u}_{t_1},\mathfrak{u}_{t_2},\ldots,\mathfrak{u}_{t_n}\Big)\Big]-\sum_{0<t_k<b}\mathcal{S}_{\Psi}^{\eta}(b,t_k)I_k\Big(\mathfrak{u}^{\gamma}(t_k^-)\Big) \\
&-\int_0^b\big(\Psi(b)-\Psi(\varpi)\big)^{\eta-1}\mathcal{Q}_{\Psi}^{\eta}(b,\varpi)\mathfrak{G}\Big(t,\mathfrak{u}_t^{\gamma},\int_0^t e\Big(t,s,\mathfrak{u}_s^{\gamma}\Big)ds\Big)\Psi'(\varpi)d\varpi\Big]d\chi
\end{aligned}$$

$$= \mathcal{S}_{\Psi}^{\eta}(b,0)\left[\mathfrak{u}_0 + \xi\left(\mathfrak{u}_{t_1}, \mathfrak{u}_{t_2}, \ldots, u_{t_n}\right)\right] + \sum_{0<t_k<b} \mathcal{S}_{\Psi}^{\eta}(b,t_k) I_k\left(\mathfrak{u}^{\gamma}(t_k^-)\right)$$

$$+ \int_0^b \left(\Psi(b) - \Psi(\chi)\right)^{\eta-1} \mathcal{Q}_{\Psi}^{\eta}(b,\chi) \mathfrak{G}\left(t, \mathfrak{u}_t^{\gamma}, \int_0^t e\left(t,s,\mathfrak{u}_s^{\gamma}\right) ds\right) \Psi'(\chi) d\chi + \mathfrak{T}_0^b \mathrm{R}(\gamma, \mathfrak{T}_0^b) P(\mathfrak{u}^{\alpha})$$

$$= \mathcal{S}_{\Psi}^{\eta}(b,0)\left[\mathfrak{u}_0 + \xi\left(\mathfrak{u}_{t_1}, \mathfrak{u}_{t_2}, \ldots, \mathfrak{u}_{t_n}\right)\right] + \sum_{0<t_k<b} \mathcal{S}_{\Psi}^{\eta}(b,t_k) I_k\left(\mathfrak{u}^{\gamma}(t_k^-)\right)$$

$$+ \int_0^b \left(\Psi(b) - \Psi(\chi)\right)^{\eta-1} \mathcal{Q}_{\Psi}^{\eta}(b,\chi) \mathfrak{G}\left(t, \mathfrak{u}_t^{\gamma}, \int_0^t e\left(t,s,\mathfrak{u}_s^{\gamma}\right) ds\right) \Psi'(\chi) d\chi$$

$$+ P(\mathfrak{u}^{\gamma}) - \gamma \mathrm{R}(\gamma, \mathfrak{T}_0^b) P(\mathfrak{u}^{\alpha})$$

$$= \mathfrak{u}_1 - \alpha \mathrm{R}(\gamma, \mathfrak{T}_0^b) P(\mathfrak{u}^{\gamma}).$$

According the Dunford-Pettis Theorem, there is a subsequence $\left\{\mathfrak{G}\left(t, \mathfrak{u}_t^{\gamma}, \int_0^t e(t,s,\mathfrak{u}_s^{\gamma})\right)\right\}$ that convergent weakly to $\left\{\mathfrak{G}\left(t, \mathfrak{u}_t^{\gamma}, \int_0^t e(t,s,\mathfrak{u}_s^{\gamma}) ds\right)\right\}$ in $L^1(\mathscr{W}, \mathfrak{D})$ and also the functions $I_k(\mathfrak{u})$. Consider,

$$\mathsf{W} = \mathfrak{u}_1 - \mathcal{S}_{\Psi}^{\eta}(b,0)\left[\mathfrak{u}_0 + \xi\left(\mathfrak{u}_{t_1}, \mathfrak{u}_{t_2}, \ldots, \mathfrak{u}_{t_n}\right)\right] - \sum_{0<t_k<b} \mathcal{S}_{\Psi}^{\eta}(b,t_k) I_k\left(\mathfrak{u}^{\gamma}(t_k^-)\right)$$

$$- \int_0^b \left(\Psi(b) - \Psi(\varpi)\right)^{\eta-1} \mathcal{Q}_{\Psi}^{\eta}(b,\varpi) \mathfrak{G}\left(t, \mathfrak{u}_t; \int_0^t e\left(t,s,\mathfrak{u}_s\right) ds\right) \Psi'(\varpi) d\varpi. \tag{37}$$

We get

$$\left\|P(\mathfrak{u}^{\gamma}) - \mathsf{W}\right\| = \left\|\sum_{0<t_k<b}\left[\mathcal{S}_{\Psi}^{\eta}(b,t_k) I_k\left(\mathfrak{u}^{\gamma}(t_k^-)\right) - \mathcal{S}_{\Psi}^{\eta}(b,t_k) I_k\left(\mathfrak{u}(t_k^-)\right)\right]\right\|$$

$$+ \left\|\int_0^b \left(\Psi(b) - \Psi(\varpi)\right)^{\eta-1} \mathcal{Q}_{\Psi}^{\eta}(b,\varpi) \mathfrak{G}\left(t, \mathfrak{u}_t^{\gamma}, \int_0^t e(t,s,\mathfrak{u}_s) ds\right) \Psi'(\varpi) d\varpi\right.$$

$$\left.- \int_0^b \left(\Psi(b) - \Psi(\varpi)\right)^{\eta-1} \mathcal{Q}_{\Psi}^{\eta}(b,\varpi) \mathfrak{G}\left(t, \mathfrak{u}_t, \int_0^t e(t,s,\mathfrak{u}_s) ds\right) \Psi'(\varpi) d\varpi\right\|$$

$$\leq \sum_{0<t_k<b} \left\|\mathcal{S}_{\Psi}^{\eta}(b,t_k)\left[I_k\left(\mathfrak{u}^{\gamma}(t_k^-)\right) - I_k\left(\mathfrak{u}(t_k^-)\right)\right]\right\|$$

$$+ \left\|\int_0^b \left(\Psi(b) - \Psi(\varpi)\right)^{\eta-1} \mathcal{Q}_{\Psi}^{\eta}(b,\varpi) \Psi'(\varpi)\right.$$

$$\times\left[\mathfrak{G}\left(t, \mathrm{u}_t^\gamma, \int_0^t e(t,s,\mathrm{u}_s^\gamma)ds\right) - \mathfrak{G}\left(t, \mathrm{u}_t, \int_0^t e(t,s,\mathrm{u}_s)ds\right)\right]d\varpi\Bigg\|$$

$$\leq \kappa_\eta \sum_{0<t_k<b}\left\|I_k\left(\mathrm{u}^\gamma(t_k) - I_k\left(\mathrm{u}(t_k^-)\right)\right)\right\|$$

$$+\frac{\eta\kappa_\eta}{\Gamma(1+\eta)}\Bigg\|\int_0^b\left(\Psi(b)-\Psi(\varpi)\right)^{\eta-1}\Psi'(\varpi)$$

$$\times\left[\mathfrak{G}\left(t, \mathrm{u}_t^\gamma, \int_0^t e(t,s,\mathrm{u}_s^\gamma)ds\right) - \mathfrak{G}\left(t, \mathrm{u}_t, \int_0^t e(t,s,\mathrm{u}_s)ds\right)\right]d\varpi\Bigg\|.$$

By the uniform boundedness of $\{\mathfrak{G}^\gamma(\varpi)\}$ that $\exists$ some $\mathfrak{G}(\varpi) \in L^1(\mathscr{W}, \mathfrak{D})$ such that,

$$\mathfrak{G}\left(\varpi, \mathrm{u}^\gamma(\varpi)\right) \to \mathfrak{G}\left(\varpi, \mathrm{u}(\varpi)\right) \text{ as } \gamma \to 0.$$

Similarly, $\|I_k(\mathrm{u}^\gamma(t_k) - I_k(\mathrm{u}(t_k)))\| \to 0$ as $\gamma \to 0$. Moreover, approximate controllability of the system (26), we obtain $\gamma\mathrm{R}(\gamma, \mathfrak{T}_0^b) \to 0$ as $\gamma \to 0^+$ in the strong continuous topology. Therefore, we can obtain that as $\gamma \to 0^+$,

$$\left\|\mathrm{u}^\gamma(b) - \mathrm{u}_1\right\| \leq \left\|\gamma\mathrm{R}\left(\gamma, \mathfrak{T}_0^\gamma\right)(\mathrm{W})\right\| + \left\|\gamma\mathrm{R}\left(\gamma, \mathfrak{T}_0^b\right)\left(P(\mathrm{u}^\gamma) - \mathrm{W}\right)\right\|$$

$$\leq \left\|\gamma\mathrm{R}\left(\gamma, \mathfrak{T}_0^b\right)\mathrm{W}\right\| + \left\|(P(\mathrm{u}^\gamma) - \mathrm{W})\right\| \to 0.$$

Hence, the system (1) is approximately controllable on $\mathscr{W}$. □

4. Example

This section looks at an initial value problem based on a Caputo fractional differential equation and shows how fractional derivative with respect to another function may be useful:

$$\begin{cases} {}^C D_\Psi^{\frac{4}{7}} z(t,\epsilon) = \dfrac{\partial^2}{\partial\sigma^2} z(t,\epsilon) + \mathscr{B}\mu(t,\epsilon) \\ \qquad + \mathscr{F}\left(t, \int_{-\infty}^t \mathscr{F}_1(\omega - t)z(\omega,\epsilon)d\omega, \int_0^t\int_{-\infty}^r \mathscr{F}_2\left(t,\epsilon,\lambda - t\right)\chi\left(z(\lambda,\epsilon)\right)d\lambda d\epsilon\right),\ t \neq t_1, \\ z(t_1^+) - z(t_1^-) = \dfrac{1}{1000}\left(\mathrm{u}(t_1^-)\right), \\ z(0,\epsilon) = z_0(\epsilon) + \sum_{i=1}^k Q_i z(t_i + \epsilon),\ 0 < t_1 < t_2 < \cdots < t_k \leq b,\ \epsilon \in [0,\pi], \\ z(t,0) = z(t,\pi) = 0,\ t \in \mathscr{W}, \\ z(t,\sigma) = \Omega(t,\sigma),\ 0 \leq \sigma \leq \pi,\ t \in (-\infty, 0], \end{cases} \tag{38}$$

where ${}^C D_\Psi^{\frac{4}{7}} D$ is the Ψ-Caputo fractional derivative of order $\frac{4}{7}$ and set $\mathfrak{D} = L^2([0, \pi])$, be endowed with the usual $\|\cdot\|_{L^2}$, and $k = 1$.

Let $w(\varepsilon) = e^{4\varepsilon}$, $\varepsilon < 0$ then, $\int_{-\infty}^{0} w(\varepsilon)d\varepsilon = \frac{1}{4}$,

$$\| \varpi \|_{\gamma} = \int_{-\infty}^{0} w(\varepsilon) \sup_{\tau\in[-n,0]} \| \varpi(\varepsilon) \| \, d\varepsilon, \tag{39}$$

since $\varpi(\varepsilon)(\epsilon) = \varpi(\varepsilon, \epsilon)$, $(\varepsilon, \epsilon) \in (-\infty, 0] \times [0, \pi]$. Consider the following:

1. $\mathscr{F}(\cdot, \cdot, \cdot)$ is a continuous function in $\mathscr{W} \times \mathscr{T}_w \times L^2([0, \pi])$ and $\mathscr{F}_1$ also continuous, positive bounded, such tha $\int_{-\infty}^{0} \mathscr{F}_1(t, \epsilon, \lambda)d\lambda < \infty$.

2. $\mathscr{F}_2(t, \epsilon, \lambda)$ is a continuous function in $\mathscr{W} \times [0, \pi] \times \mathscr{T}_w$, such that $\int_{-\infty}^{0} \mathscr{F}_1(t, \epsilon, \lambda)d\lambda < \infty$.

3. The function $\varepsilon(\cdot)$ is continuous and satisfy $0 \leq \varepsilon(z(\lambda)(\epsilon)) \leq \mathscr{R}\left(\int_{-\infty}^{0} e^{4\varepsilon}\|z(\varepsilon, \cdot)\|_{L^2} d\varepsilon\right)$, here $\mathscr{R} : [0, +\infty) \to (0, \infty)$ is an increasing continuous function.

4. $\mathscr{B} : L^2([0, \pi]) \to L^2([0, \pi])$ is the bounded linear operator defined with the control function $\mu(t, \epsilon)$.

5. $I_1(\mathrm{u}(t_1^-))$ is the impulsive functions, and $\|I_1(\mathrm{u}(t_1^-))\| \leq \frac{1}{1000}$.

Now we observe the integro function:

$$\left|\mathfrak{G}(t,\cdot,\cdot)\right|_{L^2} = \left[\int_0^{\pi}\left(\mathcal{F}\left(t, \int_{-\infty}^{t} \mathscr{F}_1(\omega)z(\omega,\epsilon)d\omega, \int_0^t\int_{-\infty}^{r} \mathscr{F}_2(t,\epsilon,\lambda)\chi\left(z(\lambda,\epsilon)\right)d\lambda\right)\right)^2 d\epsilon\right]^{\frac{1}{2}}$$

$$\leq \mathscr{M}\Lambda\left[\left(\int_0^{\pi}\left(\int_{-\infty}^{0} \mathscr{F}_1(\omega)z(\omega,\epsilon)d\omega\right)^2 d\epsilon\right)^{\frac{1}{2}} + \left(\int_0^{\pi}\left(\int_{-\infty}^{0} \mathscr{F}_2(t,\epsilon,\lambda)\chi(z(\lambda,\epsilon))d\lambda\right)^2 d\epsilon\right)^{\frac{1}{2}}\right]$$

$$\leq \mathscr{M}\Lambda[J_1 + J_2].$$

Then,

$$J_1 = \left(\int_0^{\pi}\left(\int_{-\infty}^{0} \mathscr{F}_1(\omega)z(\omega,\epsilon)d\omega\right)^2 d\epsilon\right)^{\frac{1}{2}}$$

$$\leq \left(\int_0^{\pi}\left(\int_{-\infty}^{0} \mathscr{F}_1(\omega)\Omega(t,\epsilon)d\omega\right)^2 d\epsilon\right)^{\frac{1}{2}}$$

$$\leq \left(\int_0^{\pi} \mathscr{L}_2(t,\epsilon)^2 d\epsilon\right)^{\frac{1}{2}}$$

$$\leq \mathscr{L}'(t),$$

and

$$J_2 = \left[\int_0^{\pi}\left(\int_{-\infty}^{0} \mathscr{F}_2(t,\epsilon,\lambda)\chi\left(z(\lambda)(\epsilon)\right)d\lambda\right)^2 d\epsilon\right]^{\frac{1}{2}}$$

$$\leq \left[\int_0^{\pi} \left(\int_{-\infty}^{0} \mathscr{F}_2(t,\epsilon,\lambda) \mathscr{R} \left(\int_{-\infty}^{0} e^{4\varepsilon} \left\| \varpi(\varepsilon)(\cdot) \right\|_{L^2} d\varepsilon \right) d\lambda \right)^2 d\epsilon \right]^{\frac{1}{2}}$$

$$\leq \left[\int_0^{\pi} \left(\int_{-\infty}^{0} \mathscr{F}_2(t,\epsilon,\lambda) \mathscr{R} \left(\int_{-\infty}^{0} e^{4\varepsilon} \sup_{\varepsilon \in [\lambda,0]} \left\| \varpi(\varepsilon) \right\|_{L^2} d\varepsilon \right) d\lambda \right)^2 d\epsilon \right]^{\frac{1}{2}}$$

$$\leq \left[\int_0^{\pi} \left(\int_{-\infty}^{0} \mathscr{F}_2(t,\epsilon,\lambda) d\lambda \right)^2 d\epsilon \right]^{\frac{1}{2}} \mathscr{E}\left(1 + \| \varpi \|_{\Upsilon}\right)$$

$$\leq \left[\int_0^{\pi} \mathscr{L}_1(t,\epsilon)^2 d\epsilon \right]^{\frac{1}{2}} \mathscr{E}\left(1 + \| \varpi \|_{\Upsilon}\right)$$

$$\leq \mathscr{L}''(t) \mathscr{E}\left(1 + \| \varpi \|_{\Upsilon}\right).$$

Therefore,

$$\left| \mathfrak{G}(t,\cdot,\cdot) \right| =_{L^2} \leq \mathscr{M} \Lambda \left[\mathscr{L}'(b) + \mathscr{L}''(b) \mathscr{E}\left(1 + \| \varpi \|_{\Upsilon}\right) \right],$$

where $\mathscr{M}$, Λ are the constants. So the required functions satisfied the hypotheses (H_2) and (H_3).

In [38], the author created graphical representations of fractional derivatives with and without a Ψ-functions when $\Psi(t) = t$, $\ln(t + 1)$, and $\sqrt{t+1}$. For different types of Ψ-function are used in this example, we represent some theoretical differences in the proof. Take $\kappa_\eta = 1$, (32) become:

$$\Psi(t) = t, \quad \Rightarrow \frac{\mathsf{M}}{\Gamma\left(\frac{11}{7}\right)} + \frac{1}{1000} < 1;$$

$$\Psi(t) = \ln(t), \quad \Rightarrow \frac{\mathsf{M}}{\Gamma\left(\frac{11}{7}\right)} \ln(2)^{\frac{4}{11}} + \frac{1}{1000} < 1;$$

$$\Psi(t) = \sqrt{t+1}, \quad \Rightarrow \frac{\mathsf{M}}{\Gamma\left(\frac{11}{7}\right)} \left(\sqrt{2} - 1\right)^{\frac{4}{7}} + \frac{1}{1000} < 1.$$

Also we can verify all hypotheses.

Hence, from Theorem 3.1 the system (38) has a mild solution and which is approximately controllable.

5. Conclusion

In this manuscript, we investigated the approximate controllability of Ψ-Caputo fractional differential equations with infinite delay, impulsive and nonlocal connditions by using fixed-point approach. The primary outcomes are developed by utilising the semigroup concepts, Ψ-Caputo fractional derivative and fixed-point approach. An implication is provided to illustrate the principle. In future, we will focus exact controllability of Ψ-Caputo fractional differential systems with impulsive conditions, and existence Ψ-Hilfer fractional differential system with or without delay via fixed point technique. In future, we will extend our work to higher order fractional derivatives.

Authors' contributions

Conceptualisation, V.B.C.S, R.U, V.M, and S.A.O.; methodology, V.B.C.S.; validation, V.B.C.S, R.U, and V.M.; formal analysis, V.B.C.S.; investigation, R.U, V.M, and S.A.O.; resources, V.B.C.S.; writing original draft preparation, V.B.C.S.; writing review and editing, R.U, V.M, and S.A.O.; visualisation, R.U, V.M, and S.A.O.; supervision, R.U, V.M, and S.A.O.; project administration, R.U, and V.M. The published version of the work has been reviewed and approved by all authors.

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This work does not have any competing interest.

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