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# Topological Indices and Properties of the Prime Ideal Graph of a Commutative Ring and its Line Graph

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**Abstract:** Let $R$ be a commutative ring with identity and $P$ be a prime ideal of $R$. The prime ideal graph, denoted by $\Gamma_P$, is the graph where the set of vertices is $R\backslash\{0\}$ and two vertices are joined by an edge if their product belongs to $P$. This paper will discuss topological indices and some properties of the prime ideal graph of a commutative ring and its line graph. Topological indices, such as the Wiener, first Zagreb, second Zagreb, Harary, Gutman, Schultz, and harmonic indices, are related to the degree of vertices and the diameter of the graphs. In this paper, we also discuss the independence and domination numbers of the line graph.

***Keywords***: prime ideal graph, commutative ring, degree of vertices, diameter, topological indices

**MSC:** 05C25, 05C50

## 1. Introduction

We use graphs in various disciplines, from electrical engineering and chemistry to computer science. In chemistry, one uses graph theory to solve molecular problems. Atoms and the bonds between atoms can be represented as the vertices and the edges of a graph, respectively. There are numerous applications of graph theory and group theory in chemistry, including using topological indices to represent the chemical structure with numerical values. Additionally, topological indices help predict molecular structures' chemical and physical properties.

In addition to using graphs, one also employs them as representations of mathematical systems such as groups and rings. One uses graphs to represent groups, for example, coprime [1, 2], non-coprime [3, 4], relative coprime [5], power [6], commuting [7], cyclic [8], and intersection [9] graphs. Rings are represented by the zero divisor [10], prime [11], and Jacobson [12, 13] graphs. Vertices and edges in each type of graph are determined based on the definitions of the respective graphs [14-16].

In recent years, the representation of rings in graphs has been a topic of increasing research interest. One example is the commutative ring's zero-divisor graph. Given a commutative ring, its zero-divisor graph is a simple graph where each vertex represents an element in the ring, and two vertices $x$ and $y$ are joined by an edge if and only if $xy$ is zero. Beck [17] first introduced this concept in 1988. Furthermore, in 2007, Anderson et al. [18] obtained the zero-divisor


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graph's properties, such as its diameter being greater than or equal to 2 and its girth being greater than or equal to 4. In 2021, Asir et al. [19] proposed one of the topological indices for $\mathbb{Z}_n$, the Wiener index, for its zero-divisor graph. Alali et al. [20] did the same, but on a broader scale, by exploring topological indices and entropy using $M$-polynomials. Readers are encouraged to delve into the study for a deeper comprehension of graphs of algebraic structures and their properties [21-26].

A graph representing a commutative ring focusing on prime ideals was introduced by Salih et al. [27] in 2022. This graph is known as the prime ideal graph. One defines the prime ideal graph as one where the elements of the commutative ring, excluding the zero element, are represented as vertices in the graph. Two vertices, $x$ and $y$ are joined by an edge if and only if $xy$ belongs to a prime ideal. The obtained results include chromatic, clique, independence, and domination numbers [27].

This study will discuss the prime ideal graph of a commutative ring and its line graph's properties, including the vertices' degree, diameter representing graph distance, domination number, independence number, and topological indices. The topological indices that will be investigated, particularly those related to vertex degree and distance, are the Wiener, first Zagreb, second Zagreb, Harary, Gutman, Schultz, and harmonic indices.

## 2. Prime ideal graphs

This section discusses the prime ideal graph's properties.

**Definition 2.1.** [27] For a commutative ring $R$ and its prime ideal $P$, the prime ideal graph $\Gamma_P$ is defined where the vertex set is $R\backslash\{0\}$, and two vertices $r_1$ and $r_2$ are adjacent whenever $r_1 r_2 \in P$.

**Example 2.1.** Let $R = \mathbb{Z}_{10} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Take $P = \{0, 2, 4, 6, 8\}$ as a prime ideal. Then, we have $\Gamma_P$ as shown in Figure 1.

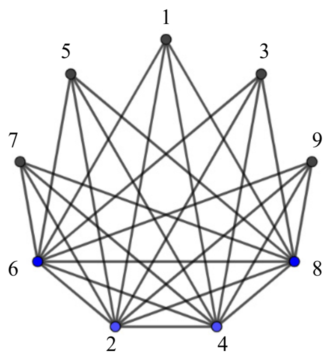


**Figure 1.** $\Gamma_P$ with $P = \{0, 2, 4, 6, 8\}$

**Example 2.2.** Let $R = \dfrac{\mathbb{Z}_2[x]}{\langle x^3+1\rangle} = \{0, 1, x, x+1, x^2, x^2+1, x^2+x, x^2+x+1\}$. Take $P = \langle x+1\rangle = \{x+1, x^2+1, x^2+x\}$. Then, $\Gamma_P$ is as shown in Figure 2.

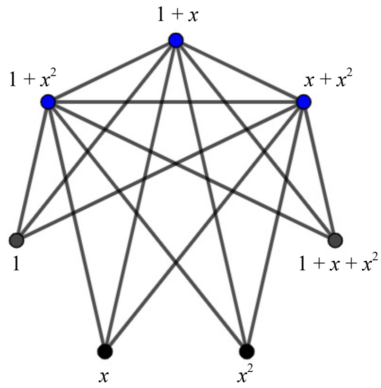


**Figure 2.** $\Gamma_P$ with $P = \{1 + x, 1 + x^2, x + x^2\}$

Any finite commutative ring is in particular Artinian, and commutative Artinian rings $R$ are finite products $\prod Rm$ of Artinian local rings. An example of an Artinian local ring is provided by $\dfrac{\mathbb{Z}_2[x]}{\langle x^3+1\rangle}$, other examples are group algebras of finite Abelian $p$-groups over fields of characteristic, $p$. If $G$ is a finite commutative $p$-groups and $F$ is a field of characteristic $p$, then $FG$ is a local ring which is commutative.

In this paper, we will discuss the results for any finite commutative ring $R$ with a prime ideal $P$.

Let $R$ be a commutative ring with $n$ elements, $P$ a prime ideal with $k$ elements. Denote $R\backslash\{0\}$ as $R^*$ and $P\backslash\{0\}$ as $P^*$, so $|R^*| = n - 1$ and $|P^*| = k - 1$. Let $\Gamma_P$ be the prime ideal graph of $R$. If we write $x, y \in V(\Gamma_P)$, it means that $x \neq y$.

Next, we provide a lemma regarding the degree of a vertex in $\Gamma_P$.

**Lemma 2.1.** In $\Gamma_P$, the degree of a vertex is given by:

$$\deg(x) = \begin{cases} n-2, & \text{for every } x \in P^*, \\ k-1, & \text{for every } x \in R\backslash P. \end{cases}$$

*Proof.* If $x \in P^*$, then, $x$ is joined by an edge to all $y \in R\backslash\{0, x\}$. Consequently, deg $(x) = n - 2$, for every $x \in P^*$. If $x \in R\backslash P$, then $x$ is only adjacent to the vertices in $P$ except 0. Consequently, deg $(x) = k - 1$, for every $x \in R\backslash P$.

Now, let's discuss the distance in graphs. Below is a lemma about distance in the prime ideal graph.

**Lemma 2.2.** The distance between two vertices $x$ and $y$ in $\Gamma_P$ is given by

$$d(x, y) = \begin{cases} 1, & \text{for every } x \in P^*, y \in R, \\ 2, & \text{for every } x, y \in R\backslash P. \end{cases}$$

*Proof.* If $x, y \in R\backslash P$, it means that $x, y \notin P^*$, indicating that $x$ is not joined by an edge to $y$. However, we may find $p \in P^*$ such that $xp \in P^*$ and $yp \in P^*$, resulting in a path between $x$ and $y$ passing through the vertex $p \in P^*$. Consequently, the distance between $x$ and $y$ is 2. If $x \in P^*$, then $x$ is joined by edge to all vertices $y \in R^*$ since $xy \in P^*$. Thus, $d(x, y) = 1$.

Below are several definitions regarding graph properties, namely the clique, domination, independence, and chromatic numbers, as well as the results of previous research on prime ideal graphs.

**Theorem 2.1.** [27] The diameter of $\Gamma_P$ is less than or equal to 2.

**Definition 2.2.** [28] A dominating set is a subset $T$ of $V(\Gamma)$ such that every vertex in $V(\Gamma) \backslash T$ is joined by an edge to some vertex in $W$. The domination number, $\gamma(\Gamma)$, is the order of a minimum dominating set.

**Definition 2.3.** [28] An independent set is a subset $T$ of the vertices of $\Gamma$ which induced subgraph has no edges. The maximum size of an independent set, $\alpha(\Gamma)$, is called the independence number.

**Theorem 2.2.** [27]

1. The domination number of $\Gamma_P$, $\gamma(\Gamma_P) = 1$.
2. The independence number of $\Gamma_P$, $\alpha(\Gamma_P) = n - k$.

Now, we move to the next topic of discussion, namely the topological indices. Some of the topological indices to be discussed include the first Zagreb, second Zagreb, Wiener, Harary, harmonic, Gutman, and Schultz indices. Below, we provide definitions and results of these topological indices.

**Definition 2.4.** [29] For a connected graph Γ, the sum of pairs of vertices in Γ is called the Wiener index of Γ,

$$W(\Gamma) = \sum_{x,y \in V(\Gamma)} d(x, y).$$

**Theorem 2.3.** The Wiener index of the prime ideal graph is

$$W(\Gamma_P) = \binom{k-1}{2} + 2\binom{n-k}{2} + (k-1)(n-k).$$

*Proof.* The distance between vertices is given as follows

1. If $x, y \in P^*$, then the distance between $x$ and $y$ is 1, and there are $\binom{k-1}{2}$ pairs of vertices.
2. If $x, y \in R \backslash P$, then the distance between $x$ and $y$ is 2, and there are $\binom{n-k}{2}$ pairs of vertices.
3. If $x \in P^*$ and $y \in R \backslash P$, then the distance between $x$ and $y$ is 1, and there are $(k-1)(n-k)$ pairs of vertices.

Based on the above cases,

$$\begin{aligned} W(\Gamma_P) &= \sum_{x,y \in V(\Gamma_P)} d(x, y) \\ &= \sum_{x,y \in P^*} d(x, y) + \sum_{x \in P^*, y \in R \backslash P} d(x, y) + \sum_{x,y \in R \backslash P} d(x, y) \\ &= 2\binom{n-k}{2} + \binom{k-1}{2} + (k-1)(n-k). \end{aligned}$$

**Definition 2.5.** [30] For a connected graph Γ, the sum of squares of the degrees of each vertex in Γ is called the first Zagreb index of Γ,

$$M_1(\Gamma) = \sum_{x \in V(\Gamma)} \left(\deg(x)\right)^2.$$

**Theorem 2.4.** The first Zagreb index of the prime ideal graph is given by

$$M_1(\Gamma_P) = (k-1)(n-2)^2 + (n-k)(k-1)^2.$$

*Proof.* Based on Lemma 2.1, we have the following:

$$\begin{aligned} M_1(\Gamma_P) &= \sum_{x \in V(\Gamma_P)} \left(\deg(x)\right)^2 \\ &= \sum_{x \in R \backslash P} \left(\deg(x)\right)^2 + \sum_{x \in P^*} \left(\deg(x)\right)^2 \\ &= (k-1)^2(n-k) + (k-1)(n-2)^2. \end{aligned}$$

**Definition 2.6.** [30] For a connected graph Γ, the sum of the product of degrees of each pair of vertices joined by an edge in Γ is called the second Zagreb index of Γ,

$$M_2(\Gamma) = \sum_{(x,y) \in E(\Gamma)} \deg(x)\deg(y).$$

**Theorem 2.5.** The second Zagreb index of the prime ideal graph is given by

$$M_2(\Gamma_P) = (k-1)^2(n-k)(n-2) + \binom{n-k}{2}(n-2)^2.$$

*Proof.* Based on Lemma 3.1,

1. If $x, y \in P^*$, then deg ($x$) = deg ($y$) = $n - 2$ and there are $\binom{n-k}{2}$ pairs of vertices.
2. If $x \in P^*$ and $y \in R \backslash P$, then deg ($x$) = $n - 2$ and deg ($y$) = $k - 1$, there are $(k - 1)(n - k)$ pairs of vertices.

Hence,

$$\begin{aligned} M_2(\Gamma_P) &= \sum_{(x,y)\in E(\Gamma_P)} \deg(x)\deg(y) \\ &= \sum_{x\in P^*,\ y\in R\backslash P} \deg(x)\deg(y) + \sum_{x,y\in P^*} \deg(x)\deg(y) \\ &= (n-k)(n-2)(k-1)^2 + \binom{n-k}{2}(n-2)^2. \end{aligned}$$

**Definition 2.7.** [31] For a connected graph Γ, the sum of the inverses of the distances between all pairs of vertices in Γ is called the Harary index of Γ,

$$H(\Gamma) = \sum_{x,y\in V(\Gamma)} \frac{1}{d(x,y)}.$$

**Theorem 2.6.** The Harary index of the prime ideal graph is

$$H(\Gamma_P) = \binom{k-1}{2} + \frac{1}{2}\binom{n-k}{2} + (k-1)(n-k).$$

*Proof.* Using the same approach as the Wiener index, we obtain

$$\begin{aligned} H(\Gamma_P) &= \sum_{x,\ y\in V(\Gamma_P)} \frac{1}{d(x,y)} \\ &= \sum_{x,y\in P^*} \frac{1}{d(x,y)} + \sum_{x,\ y\in R\backslash P} \frac{1}{d(x,y)} + \sum_{x\in P^*,\ y\in R\backslash P} \frac{1}{d(x,y)} \\ &= \binom{k-1}{2} + \frac{1}{2}\binom{n-k}{2} + (n-k)(k-1). \end{aligned}$$

**Definition 2.8.** [32] For a connected graph Γ, the sum of the product of the degree of each pair of vertices and the distance between vertices is called the Gutman index of Γ,

$$Gut(\Gamma) = \sum_{x,y\in V(\Gamma)} \deg(x)\deg(y)d(x,y).$$

**Theorem 2.7.** The Gutman index of the prime ideal graph is

$$Gut(\Gamma_P) = (n-2)^2\binom{k-1}{2} + 2(k-1)^2\binom{n-k}{2} + (k-1)^2(n-k).$$

*Proof*. Based on the distance between vertices, and degree of the vertices, we have the following.

$$\begin{aligned} Gut(\Gamma_P) &= \sum_{x,y \in V(\Gamma_P)} \deg(x)\deg(y) d(x, y) \\ &= (n-2)^2 \binom{k-1}{2} + 2(k-1)^2 \binom{n-k}{2} + (k-1)^2 (n-k). \end{aligned}$$

**Definition 2.9.** [33] For a connected graph Γ, the sum of the product between the distance between vertices and the sum of the degrees of each pair of vertices is called the Schultz index of Γ,

$$Sch(\Gamma) = \sum_{x,y \in V(\Gamma)} \big(\deg(x) + \deg(y)\big)\, d(x, y).$$

**Theorem 2.8.** The Schultz index of the prime ideal graph is

$$Sch(\Gamma_P) = 2\binom{k-1}{2}(n-2) + 4\binom{n-k}{2}(k-1) + (k-1)(n+k-3)(n-k).$$

*Proof*. Based on the distance between vertices, and degree of the vertices, as in the Gutman index, we obtain the following.

$$\begin{aligned} Sch(\Gamma_P) &= \sum_{x,y \in V(\Gamma_P)} \big(\deg(x) + \deg(y)\big) d(x, y) \\ &= 2\binom{k-1}{2}(n-2) + 4\binom{n-k}{2}(k-1) + (k-1)(n+k-3)(n-k). \end{aligned}$$

**Definition 2.10.** [34] For a connected graph Γ, the sum of two divided by the sum of the degrees of each pair of vertices joined by an edge is called the harmonic index of Γ

$$Har(\Gamma) = \sum_{x,y \in E(\Gamma)} \frac{2}{\deg(y) + \deg(x)}.$$

**Theorem 2.9.** The harmonic index of the prime ideal graph is

$$Har(\Gamma_P) = (n-k)(k-1)\frac{2}{n+k-3} + \binom{k-1}{2}\frac{1}{n-2}.$$

*Proof*. Based on the degree of vertices, following the same approach as the second Zagreb index, we obtain the following.

$$\begin{aligned} Har(\Gamma_P) &= \sum_{x,y \in E(\Gamma_P)} \frac{2}{\deg(x) + \deg(y)} \\ &= \sum_{x \in P^*,\, y \in R \setminus P} \frac{2}{\deg(x) + \deg(y)} + \sum_{x,\, y \in P^*} \frac{2}{\deg(x) + \deg(y)} \\ &= (n-k)(k-1)\frac{2}{n+k-3} + \binom{k-1}{2}\frac{1}{n-2}. \end{aligned}$$

## 3. The line graph of the prime ideal graph of a commutative ring

This section discusses the line graph of the prime ideal graph of a commutative ring.

**Definition 3.1.** [28] The line graph of a graph $\Gamma$, $L(\Gamma)$, is a graph such that

1. Each vertex of $L(\Gamma)$ is an edge of $\Gamma$.
2. Two vertices of $L(\Gamma)$ are joined by an edge if and only if their corresponding edges in $\Gamma$ have a common endpoint.

**Example 3.1.** Let $R = Z_6$ with $P = \{0, 2, 4\}$. Then, we have $L(\Gamma_P)$ as shown in Figure 3.

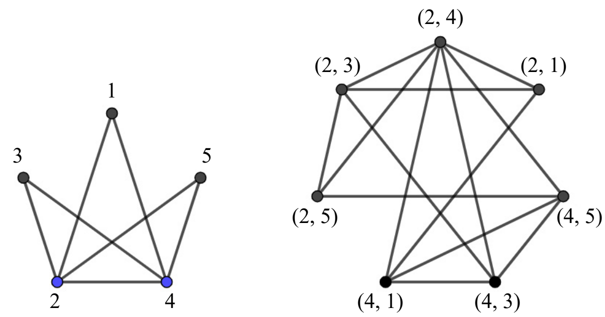


**Figure 3.** $L(\Gamma_P)$ with $P = \{0, 2, 4\}$

Based on the previously mentioned example, which is then generalized, several outcomes regarding the line graph $L(\Gamma_P)$ are obtained. The first result is related to the diameter.

**Theorem 3.1.** If $L(\Gamma_P)$ as the line graph of prime ideal graph of a commutative ring, then $\operatorname{diam}\left(L(\Gamma_P)\right) \le 2$.

*Proof.* If $(u,v),(u,y) \in V\left(L(\Gamma_P)\right)$, such that $(u, v)$ and $(u, y)$ are adjacent, then $d\left((u,v),(u,y)\right) = 1$. Otherwise, let $(u,v) = (p_i, r_k)$ and $(x,y) = (p_j, r_l)$, where $p_i, p_j \in P^*$ and $r_l, r_k \in R \backslash P$, for some $i \ne j$, $k \ne l$ which means $(p_i, r_k)$ is not adjacent to $(p_j, r_l)$. Therefore, $d\left((p_i,r_k),(p_j,r_l)\right) \ne 1$. However, since $p_i \in P^*$ and $r_l \in R \backslash P$, we have $p_i r_l \in P^*$. In other words, there exists $(p_i, r_l) \in V\left(L(\Gamma_P)\right)$ such that $(p_i, r_k)$ is adjacent to $(p_i, r_l)$, and similarly, $(p_i, r_l)$ is adjacent to $(p_j, r_l)$. As a result, $d\left((p_i,r_k),(p_j,r_l)\right) = d\left((u,v),(x,y)\right) = 2$. Then, $\operatorname{diam}\left(L\left(\Gamma_P\right)\right) \le 2$.

Furthermore, we obtain some results concerning the vertices' degree in the line graph of the prime ideal graph.

**Lemma 3.1.** The degree of vertices in $L(\Gamma_P)$ is given by

$$\deg_L(x, y) = \left(\deg(x) - 1\right) + \left(\deg(y) - 1\right)$$

for every $(x, y) \in L(\Gamma_P)$.

*Proof.* Let $L(\Gamma_P)$ be the line graph of the prime ideal graph. Take any adjacent $x, y \in V(\Gamma_P)$, thus $(x, y) \in L(\Gamma_P)$. The pair $(x, y)$ represents the endpoints. Therefore, we have $\deg(x) \ne 0$ and $\deg(y) \ne 0$. The degree of the edge $(x, y)$, $\deg_L(x, y)$ can be interpreted as the sum of the degrees of their same endpoint vertex. However, since $x$ and $y$ are joined by an edge, we subtract 1 from each endpoint's degree. Thus, for every $(x, y) \in V\left(L(\Gamma_P)\right)$, $\deg_L(x, y) = \left(\deg(x) - 1\right) + \left(\deg(y) - 1\right)$.

**Corollary 3.1.** The vertices' degree in the line graph $L(\Gamma_P)$ is given by

1. $\deg(p_i, p_j) = 2(n-3)$, for every $(p_i, p_j) \in V\left(L(\Gamma_P)\right)$, where $p_i, p_j \in P^*$ and $i \ne j$.
2. $\deg(p, r) = n + k - 5$, for every $(p, r) \in V\left(L(\Gamma_P)\right)$, where $p \in P^*$ and $r \in R \backslash P$.

*Proof.* Based on Lemma 3.1, $\deg(x, y) = \left(\deg(x) - 1\right) + \left(\deg(y) - 1\right)$ for every $(x, y) \in L(\Gamma_P)$. Therefore, according to the Lemma 2.1, we obtain:

$$\begin{aligned}\deg(p_i, p_j) &= \left(\deg(p_i) - 1\right) + \left(\deg(p_j) - 1\right) \\ &= 2(n-3),\end{aligned}$$

and

$$\begin{aligned}\deg(p,r) &= \left(\deg(p)-1\right)+\left(\deg(r)-1\right)\\ &= n+k-5.\end{aligned}$$

After obtaining the vertices' degree, the number of edges of the prime ideal graph and its line graph is determined using the handshake lemma.

**Corollary 3.2** The number of edges in the line graph $L(\Gamma_P)$ is

$$\left|E(\Gamma_P)\right| = \frac{1}{2}(k-1)(2n-k-2).$$

*Proof*. Based on Lemma 2.1,

1. $\deg(p) = n-2$, for every $p \in P^*$.
2. $\deg(r) = k-1$, for every $r \in R\backslash P$.

Note that

$$\begin{aligned}2\left|E(\Gamma_P)\right| &= \sum_{x\in V(\Gamma_P)} \deg(x)\\ &= \sum_{p\in P^*}\deg(p) + \sum_{r\in R\backslash P}\deg(r)\\ &= (k-1)(n-2)+(n-k)(k-1)\\ &= (k-1)(2n-k-2).\end{aligned}$$

Thus, we obtain $\left|E(\Gamma_P)\right| = \frac{1}{2}(k-1)(2n-k-2)$.

In the line graph, we have that $\left|V\left(L(\Gamma_P)\right)\right| = \left|E(\Gamma_P)\right|$.

**Corollary 3.3** The number of edges in the line graph $L(\Gamma_P)$ is given by

$$\left|E\left(L(\Gamma_P)\right)\right| = \frac{1}{2}\left(\binom{k-1}{2}(2n-6)+(k-1)(n-k)(n+k-5)\right).$$

*Proof*. Based on Corollary 3.1, we obtain

1. $\deg(p_i, p_j) = 2(n-3)$, for every $(p_i, p_j) \in V\left(L(\Gamma_P)\right)$, where $p_i, p_j \in P^*$ and $i \neq j$.
2. $\deg(p,r) = n+k-5$, for every $(p,r) \in V\left(L(\Gamma_P)\right)$, where $p \in P^*$ and $r \in R\backslash P$.

Note that

$$\begin{aligned}2\left|E\left(L(\Gamma_P)\right)\right| &= \sum_{x\in V\left(L(\Gamma_P)\right)}\deg(x)\\ &= \sum_{(p_i,p_j)\in V\left(L(\Gamma_P)\right)}\deg(p_i,p_j) + \sum_{(p,r)\in V\left(L(\Gamma_P)\right)}\deg(p,r)\\ &= \binom{k-1}{2}(2n-6)+(k-1)(n-k)(n+k-5).\end{aligned}$$

Thus, we obtain

$$\left|E\left(L(\Gamma_P)\right)\right| = \frac{1}{2}\left(\binom{k-1}{2}(2n-6)+(k-1)(n-k)(n+k-5)\right).$$

The following discusses several topological indices of the line graph of the prime ideal graph. We obtain these topological indices by utilizing the degree of vertices, also the number of vertices and edges.

**Theorem 3.2.** The first Zagreb index of $L(\Gamma_P)$ is

$$M_1\left(L(\Gamma_P)\right) = 4\binom{k-1}{2}(n-3)^2 + (n-k)(k-1)(n+k-5)^2.$$

*Proof*. Based on Corollary 3.1, we have $\deg(p_i, p_j) = 2(n-3)$ for every $(p_i, p_j) \in V\left(L(\Gamma_P)\right)$, where $p_i, p_j \in P^*$, $i \neq j$, and $\deg(p, r) = n+k-5$ for every $(p, r) \in V\left(L(\Gamma_P)\right)$, where $p \in P^*$ and $r \in R \backslash P$. The number of such pairs is $\binom{k-1}{2}$ and $(n-k)(k-1)$, respectively. Based on this, we obtain

$$\begin{aligned} M_1\left(L(\Gamma_P)\right) &= \sum_{(x,y)\in V\left(L(\Gamma_P)\right)} \left(\deg(x, y)\right)^2 \\ &= \sum_{(p_i, p_j)\in V\left(L(\Gamma_P)\right)} \left(\deg(p_i, p_j)\right)^2 + \sum_{(p,r)\in V\left(L(\Gamma_P)\right)} \left(\deg(p, r)\right)^2 \\ &= 4\binom{k-1}{2}(n-3)^2 + (n-k)(k-1)(n+k-5)^2. \end{aligned}$$

**Theorem 3.3.** The second Zagreb index of $L(\Gamma_P)$ is

$$M_2\left(L(\Gamma_P)\right) = (n-3)(k-1)\Big(\big(2(k-2)\big(2(k-3)(n-3) + (n+k-5)\big)\big) + (n+k-5)(n-k)\Big)$$

*Proof*. Note that

$$\begin{aligned} M_2\left(L(\Gamma_P)\right) &= \sum_{(x,y)\in E\left(L(\Gamma_P)\right)} \deg(x)\deg(y) \\ &= \sum_{\left((p_i,p_j),(p_i,p_k)\right)\in E\left(L(\Gamma_P)\right)} \deg(p_i, p_j)\deg(p_i, p_k) + \sum_{\left((p_i,p_j),(p_i,r)\right)\in E\left(L(\Gamma_P)\right)} \deg(p_i, p_j)\deg(p_i, r) \\ &+ \sum_{(p_i,r_j),(p_i,r_k)\in E\left(L(\Gamma_P)\right)} \deg(p_i, r_j)\deg(p_i, r_k) + \sum_{(p_i,r_j),(p_k,r_j)\in E\left(L(\Gamma_P)\right)} \deg(p_i, r_j)\deg(p_k, r_j) \\ &= 8\binom{k-1}{2}(k-3)(n-3)^2 + 4\binom{k-1}{2}(n-k)(n-3)(n+k-5) \\ &+ (n-k-1)(k-1)(n-k)(n+k-5)^2 + (k-1)(k-2)(n-k)(n+k-5)^2 \\ &= 4(k-1)(k-2)(k-3)(n-3)^2 + 2(k-1)(k-2)(n-k)(n-3)(n+k-5) \\ &+ (k-1)(n-k)(n-k-1)(n+k-5)^2 + 2(k-1)(n-k)(k-2)(n-3)(n+k-5) \\ &= (n-3)(k-1)\Big(\big(2(k-2)\big(2(k-3)(n-3) + (n+k-5)\big)\big) + (n+k-5)(n-k)\Big). \end{aligned}$$

**Theorem 3.4.** [35] If $\Gamma$ is a simple graph with $\operatorname{diam}(\Gamma) \leq 2$, then

$$W(\Gamma) = |V(\Gamma)|\left(|V(\Gamma)| - 1\right) - |E(\Gamma)|.$$

**Theorem 3.5.** The Wiener index of $L(\Gamma_P)$ is

$$W\left(L(\Gamma_P)\right) = (2n-k-2)(k-1)\big((k-1)(2n-k-2) - 8\big) - (k-1)\big((k-2)(2n-6) + 2(n-k)(n+k-5)\big).$$

*Proof*. Based on Theorem 3.4, Corollary 3.2 and Corollary 3.3, we obtain

$$
\begin{aligned}
W\left(L(\Gamma_P)\right) &= \left|V\left(L(\Gamma_P)\right)\right|\left(\left|V\left(L(\Gamma_P)\right)\right|-1\right)-\left|E\left(L(\Gamma_P)\right)\right| \\
&= \frac{1}{2}(2n-k-2)(k-1)\left(\frac{1}{2}(k-1)(2n-k-2)-1\right)-\frac{1}{2}\left(\binom{k-1}{2}(2n-6)+(k-1)(n-k)(n+k-5)\right) \\
&= \frac{1}{4}(2n-k-2)(k-1)\left((k-1)(2n-k-2)-2\right)-\frac{1}{4}\left((k-1)(k-2)(2n-6)+2(k-1)(n-k)(n+k-5)\right) \\
&= (2n-k-2)(k-1)\left((k-1)(2n-k-2)-8\right)-(k-1)\left((k-2)(2n-6)+2(n-k)(n+k-5)\right) \\
&= (k-1)\left((2n-k-2)\left((k-1)(2n-k-2)-8\right)\right)-\left((k-2)(2n-6)+2(n-k)(n+k-5)\right).
\end{aligned}
$$

**Theorem 3.6.** [35] if $\Gamma$ is a simple graph with $(\Gamma) \leq 2$, then

$$WW(\Gamma) = \frac{3}{2}\left|V(\Gamma)\right|\left(\left|V(\Gamma)\right|-1\right)-2\left|E(\Gamma)\right|.$$

**Theorem 3.7.** The hyper-Wiener index of $L(\Gamma_P)$ is

$$WW\left(L(\Gamma_P)\right) = \left(3(2n-k-2)\right)(k-1)\left((k-1)(2n-k-2)-2\right)-4\left((k-2)(2n-6)+2(n-k)(n+k-5)\right).$$

*Proof*. Based on Theorem 3.6, Corollary 3.2 and Corollary 3.3, we obtain

$$
\begin{aligned}
WW\left(L(\Gamma_P)\right) &= \frac{3}{2}\left|V\left(L(\Gamma_P)\right)\right|\left(\left|V\left(L(\Gamma_P)\right)\right|-1\right)-2\left|E\left(L(\Gamma_P)\right)\right| \\
&= \frac{3}{2}\left(\frac{1}{2}(k-1)(2n-k-2)\right)\left(\frac{1}{2}(k-1)(2n-k-2)-1\right)-2\left(\frac{1}{2}\left(\binom{k-1}{2}(2n-6)+(k-1)(n-k)(n+k-5)\right)\right) \\
&= \frac{3}{8}\left((k-1)(2n-k-2)\right)\left((k-1)(2n-k-2)-2\right)-\frac{1}{2}\left((k-1)(k-2)(2n-6)\right)+2(k-1)(n-k)(n+k-5) \\
&= \left(3(2n-k-2)\right)(k-1)\left((k-1)(2n-k-2)-2\right)-4\left((k-2)(2n-6)+2(n-k)(n+k-5)\right).
\end{aligned}
$$

**Theorem 3.8.** [35] If $\Gamma$ is a simple graph with $(\Gamma) \leq 2$, then

$$H(\Gamma) = \frac{1}{4}\left|V(\Gamma)\right|\left(\left|V(\Gamma)\right|-1\right)+\frac{1}{2}\left|E(\Gamma)\right|.$$

**Theorem 3.9.** The Harary index of $L(\Gamma_P)$ is

$$H\left(L(\Gamma_P)\right) = (2n-k-2)(k-1)\left((k-1)(2n-k-2)-2\right)+(k-2)(2n-6)+4(n-k)(n+k-5).$$

*Proof*. Based on Theorem 3.8, Corollary 3.2 and Corollary 3.3, we obtain

$$
\begin{aligned}
H\left(L(\Gamma_P)\right) &= \frac{1}{4}\left|V\left(L(\Gamma_P)\right)\right|\left(\left|V\left(L(\Gamma_P)\right)\right|-1\right)+\frac{1}{2}\left|E\left(L(\Gamma_P)\right)\right| \\
&= \frac{1}{4}\left(\frac{1}{2}(k-1)(2n-k-2)\right)\left(\frac{1}{2}(k-1)(2n-k-2)-1\right)+\frac{1}{2}\left(\frac{1}{2}\left(\binom{k-1}{2}(2n-6)+(k-1)(n-k)(n+k-5)\right)\right) \\
&= \frac{1}{8}\left((2n-k-2)(k-1)\right)\left((k-1)(2n-k-2)-2\right)+\frac{1}{8}(k-1)(k-2)(2n-6)+\frac{1}{2}(k-1)(n-k)(n+k-5) \\
&= (k-1)\left((2n-k-2)\right)\left((k-1)(2n-k-2)-2\right)+(k-1)\left((k-2)(2n-6)+4(n-k)(n+k-5)\right) \\
&= (2n-k-2)(k-1)\left((k-1)(2n-k-2)-2\right)+(k-2)(2n-6)+4(n-k)(n+k-5).
\end{aligned}
$$

Let $L(\Gamma_P)$ be the line graph of the prime ideal graph. Let $|P^*| = k-1$ with $P^* = \{p_1, p_2, \dots, p_{k-1}\}$. Also, let $|R\backslash P| = n-k$ with $R\backslash P = \{r_1, r_2, \dots, r_{n-k}\}$. In the prime ideal graph, each $p_i$ is adjacent to $p_j$ and $r_l$ resulting in vertices in the line graph, namely $(p_i, p_j)$ and $(p_i, r_l)$ , where $p_i, p_j \in P^*, i \neq j$ and $r_l \in R\backslash P$.

**Theorem 3.10.** The domination number of $L(\Gamma_P)$ is $\binom{k-1}{2}$.

*Proof.* Let $S$ be the dominating set. Define $S = \{(p_i, p_j) : p_i, p_j \in P^*, i \neq j\}$ with $|S| = \binom{k-1}{2}$. We will prove that there is no $S'$ that satisfies $|S'| < |S|$. Suppose there exists $S' = \{(p_u, p_v) : p_u, p_v \in P^*, u, v = 1, 2, \dots, k-2, u \neq v\}$. However, there exist vertices that are not adjacent to $(p_u, p_v)$ namely $(p_{k-1}, r_l)$ for some $p_{k-1} \in P^*$ and $r_l \in R\backslash P$. This gives a contradiction, hence $S'$ is not a dominating set. Therefore, the domination number of $L(\Gamma_P)$ is $\binom{k-1}{2}$.

**Theorem 3.11.** The independence number of $L(\Gamma_P)$ is $|P^*| = k-1$.

*Proof.* Let $X$ be the independence set. Define $X = \{(p_i, r_l), (p_j, r_m) : p_i, p_j \in P^*, r_l, r_m \in R^*, i \neq j \text{ and } l \neq m\}$ with $|X| = |P^*| = k-1$. We will prove that there is no $X'$ that satisfies $|X'| < |X|$. Suppose there exists $X' = X \cup \{(p_j, r_l)\}$ or $X' = X \cup \{(p_i, p_j)\}$. However, there exist vertices $(p_j, p_l)$ or $(p_i, p_j)$ in $X'$ but not in $X$. But $(p_j, p_l)$, $(p_i, p_j)$ adjacent to $(p_i, r_l)$. This leads to a contradiction, and thus, $X'$ is not an independence set. Therefore, the independence number in $L(\Gamma_P)$ is $|P^*| = k-1$.

## 4. Conclusions

In this research, several topological indices related to the vertex degree and the diameter of the prime ideal graph $\Gamma_P$ of a commutative ring, $R$ and its line graph have been computed for any finite commutative ring, $R$ with prime ideal, $P$. We also give the properties of the graphs such as the independence and domination numbers, for both graphs.

## Conflict of interest

There is no conflict of interest for this study.

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