Research Article

Hilfer Fractional Neutral Stochastic Differential Inclusions with Clarke’s Subdifferential Type and fBm: Approximate Boundary Controllability

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Abstract: In this paper, the approximate boundary controllability of Hilfer fractional neutral stochastic differential inclusions with fractional Brownian motion (fBm) and Clarke’s subdifferential in Hilbert space is discussed. The existence of a mild solution of Hilfer fractional neutral stochastic differential inclusions with fractional Brownian motion and Clarke’s subdifferential is proved by using fractional calculus, compact semigroups, the fixed point theorem, stochastic analysis, and multivalued maps. The required conditions for the approximate boundary controllability of this system are defined according to a corresponding linear system that is approximately controllable. To demonstrate how our primary findings may be used, a final example is provided.

Keywords: approximate controllability, Hilfer fractional stochastic differential inclusions, fractional Brownian motion, neutral system

MSC: 93B05, 26A33, 60G22, 34A08

1. Introduction

Recent research has shown that fractional differential equations or inclusions with fractional order are effective modeling tools for a wide range of phenomena in physics, economics, engineering, and so on. In recent years, there has been a significant development in ordinary and partial differential equations involving fractional derivatives. We consult the monographs for further information [1-5]. The Riemann-Liouville fractional derivative and the Caputo fractional derivative are both parts of the Hilfer fractional derivative, which is a general fractional derivative that Hilfer introduced [6]. The subject of fractional differential equations and inclusions has been extensively covered in publications; for example, Gu and Trujillo [7] established the existence of a mild solution for the evolution equation with Hilfer fractional derivative. The approximate controllability of Hilfer fractional differential inclusions with nonlocal conditions was explored by Yang and Wang [8]. Varun Bose and Udhayakumar [9] discussed the existence of Hilfer fractional differential inclusions with almost sectorial operators.

One of the fundamental concepts in the study of mathematical control theory is controllability. It is a characteristic of dynamical systems and has special significance in control theory. Controllability, stability, and stabilizability
of deterministic and stochastic control systems have a number of significant connections. Any control system is considered to be controllable because it is probable to use the set of admissible controls to steer the system from an arbitrarily chosen starting point to an equally chosen ending point, where the starting point and the ending point may differ over the whole space. Ravikumar et al. [10] discussed the null controllability of nonlocal Sobolev-type Hilfer fractional stochastic differential system driven by fractional Brownian motion and Poisson jumps. Sivasankar et al. [11] investigated the nonlocal controllability of Hilfer fractional stochastic differential equations via almost sectorial operators. For further information on this topic, we refer to [12-15] and references therein.

Since they appear in a broad variety of issues in applied mathematics and biological models, including electronics, fluid dynamics, and chemical kinetics, neutral functional differential systems have attracted a lot of attention lately. Over the last decade, a significant number of academics have produced neutral fractional differential systems with or without delays, utilizing a variety of fixed-point procedures, mild solutions, noncompactness measures, and nonlocal conditions. For more details, we may refer to [16-20]. Many researchers have extensively studied the existence of mild solutions for neutral stochastic differential systems in [21-24]. The attention of several authors has been drawn to stochastic differential equations and inclusions driven by fractional Brownian motion (fBm). There have been significant developments about the existence, uniqueness, and controllability of the solution; for example, the existence of neutral stochastic functional differential equations driven by a fBm was examined by Boufoussi and Hajji [25]. Introducing impulsive stochastic functional differential inclusions driven by a fBm with infinite delay was done by Boudaoui et al. [26]. The best way to combine fractional Brownian motion and non-instantaneous impulsive fractional stochastic differential inclusion was explored by Balasubramaniam et al. in their study [27]. Ahmed et al. [28] investigated the approximate controllability of nonlocal Sobolev-type neutral fractional stochastic differential equations with fractional Brownian motion and Clarke subdifferential. Mourad et al. [29] investigated stochastic fractional perturbed control systems with fractional Brownian motion and Sobolev stochastic nonlocal conditions. Mourad [30] established the approximate controllability of fractional neutral stochastic evolution equations in Hilbert spaces with fractional Brownian motion.

Clarke’s subdifferential arises from applied fields such as filtration in porous media and thermo-viscoelasticity and has fascinating applications in non-smooth analysis and optimization [31]. Recent years have seen a rise in research activity in the study of control issues with Clarke’s subdifferential; for example, Ahmed et al. [32] investigated fractional stochastic evolution inclusions with control on the boundary. Kavitha et al. [33] discussed the results on the approximate controllability of Sobolev-type fractional neutral differential inclusions of the Clarke subdifferential type. Sivasankar et al. [34] studied the optimal control problems for Hilfer fractional neutral stochastic evolution hemivariational inequalities.

Only a few authors have investigated the approximate boundary controllability; for example, Wang [35] investigated the approximate boundary controllability for semilinear delay differential equations. Olive [36] introduced the approximate boundary controllability of some linear parabolic systems. Ahmed et al. [37] established the approximate boundary controllability of nonlocal Hilfer fractional stochastic differential systems with fBm and Poisson jumps. Ahmed [38] studied the approximate controllability of neutral fractional stochastic differential systems with control on the boundary. However, to the best of our knowledge, so far, no work has been reported in the literature about the approximate boundary controllability of Hilfer fractional neutral stochastic differential inclusions with fBm and Clarke’s subdifferential.

Inspired by the above-mentioned works, this paper aims to fill this gap. The purpose of this paper is to show the existence of solutions and the approximate boundary controllability of Hilfer fractional neutral stochastic differential inclusions with fBm and Clarke’s subdifferential of the form

\[
\begin{align*}
D_{0+}^{\alpha, \beta}
&\left[m(t) + f\left(t, m(t)\right)\right] \\
&\in F\left[m(t) + f\left(t, m(t)\right)\right] + \sigma(t, m(t)) + \varphi(t, m(t)) \frac{dw(t)}{dt} \\
&+ g\left(t, m(t)\right) dB^H(t), \quad t \in \mathcal{I} = (0, \xi],
\end{align*}
\]

\[
\begin{align*}
\sigma m(t) &= q_1 V(t), \quad t \in \mathcal{I}, \\
I_{0+}^{1-\alpha} m(0) &= m_0,
\end{align*}
\]
where $D_0^\eta$ is the Hilfer fractional derivative (HFD) of order $\ell \in [0, 1]$ and type $\eta \in \left[\frac{1}{2}, 1\right]$, $f$ be bounded linear operator and $\delta : \Lambda \to \mathcal{D}$ be a linear operator, where $\mathcal{D}$ be separable Hilbert space, $q_1 : Y \to \Lambda$ denotes a bounded linear operator, where $Y$ and $\Lambda$ are Hilbert spaces. The state $m(\cdot)$ takes the value in $\Lambda$. Let $A : \Lambda \to \Lambda$ be a linear operator defined by $\text{Dom}(A) = \{m \in \text{Dom}(F) : \delta m = 0\}$, $Am = Fm$, for $m \in \text{Dom}(A)$.

Let $\{w(t)\}_{t \geq 0}$ be a Wiener process that has a finite trace nuclear covariance operator $\Theta \geq 0$ specified on a $(\Omega, S, \{S_t\}_{t \geq 0}, P)$ with values in Hilbert space $\mathcal{D}$. $\{B^H(t)\}_{t \geq 0}$ is a fBm with Hurst parameter $H \in \left[\frac{1}{2}, 1\right]$ defined on a $(\Omega, S, \{S_t\}_{t \geq 0}, P)$ with values in Hilbert space $\Lambda$. Also, $\| \cdot \|$ for $L(\mathcal{D}, \Lambda)$, where $L(\mathcal{D}, \Lambda)$ is the space of all bounded linear operators.

∂$B^H(t, m(t))$ denotes the Clarke’s subdifferential of $B^H(t, m(t))$. $V(\cdot)$ is the control function in $L^2(T, Y)$, the Hilbert space of admissible control functions on $Y$. $\sigma : \Sigma \times \Lambda \to 2^\Lambda$ is a nonempty, bounded, closed, and convex (BCC) multivalued map. $L_\Theta(\mathcal{D}, \Lambda)$ be the space of all $\Theta$-Hilbert Schmidt operators from $\mathcal{D}$ to $\Lambda$. The nonlinear operators $f : T \times \Lambda \to \Lambda$, $\wp : T \times \Lambda \to L_\Theta(\mathcal{D}, \Lambda)$ and $g : T \times \Lambda \to L^2_0(Y, \Lambda)$ are given.

1.1 Novelties of the work

The contributions of this paper exist in the following aspects:

- Hilfer fractional neutral stochastic differential inclusions with fBm and Clarke’s subdifferential are introduced.
- The primary outcomes for the systems (1) are derived by applying fractional calculus, compact semigroups, the fixed point theorem, stochastic analysis and multivalued maps.
- Approximate boundary controllability for Hilfer fractional neutral stochastic differential inclusions with fBm and Clarke’s subdifferential is an unexplored topic in the literature, and this is an additional motivation for writing this manuscript.
- The main findings are demonstrated using an example.

1.2 Structure of the paper

The following describes how the paper is organized: We present some definitions, lemmas, and theorems in Section 2 that are helpful in proving the major results. The sufficient condition to demonstrate approximate boundary controllability for the system (1) is examined in Section 3. In Section 4, we present an example to verify the results of the theoretical work.

2. Preliminaries

The lemmas, theorems, and definitions required to support the primary results are presented in this part of the paper.

Definition 2.1 [2, 6] The HFD of order $0 \leq \ell \leq 1$ and $0 < \eta < 1$ is characterized as

$$D_0^\ell G(t) = \int_0^t \frac{d}{dt} \int_{0^+}^{(t-\eta)s} G(s) \frac{1}{(t-s)^{\ell-\eta}}ds, \quad t > 0, \quad \eta > 0.$$
an operator denoting $\Theta e_k = \lambda_k e_k$ along with finite trace $Tr(\Theta) = \sum_{k=1}^{\infty} \lambda_k < \infty$, $\lambda_k \geq 0$ ($k = 1, 2, \cdots$) and $\{ e_k \}$ ($k = 1, 2, \cdots$) is a complete orthonormal basis in $Y$.

The fBm on $Y$ in infinite dimensions with covariance $\Theta$ is defined as

$$B^H(t) = B^H_0(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k} e_k \beta_k^H(t),$$

where $\beta_k^H(t)$ are real, independent of fBm. In order to define the Wiener integrals with respect to the $\Theta$-fBm, we introduce the space of all $\Theta$-Hilbert Schmidt operators $\varphi : Y \to \Lambda$ as $L^0 \Lambda := L^0(Y, \Lambda)$. We remember that $\varphi \in L(Y, \Lambda)$ is known as $\Theta$-Hilbert Schmidt operators, if

$$\left\| \varphi \right\|_{L^2} := \sum_{k=1}^{\infty} \| \sqrt{\lambda_k} \varphi e_k \|_2 < \infty,$$

and that the space $L^0 \Lambda$ equipped with the inner product $\langle \varphi, \varphi \rangle_{L^2} = \sum_{k=1}^{\infty} \langle \varphi e_k, \varphi e_k \rangle_2$ is a separable Hilbert space.

Let $\varphi(s) ; s \in (0, \xi]$ be a function with values in $L^0 \Lambda$, the Wiener integral of $\varphi$ with respect to $B^H$ is defined by

$$\int_0^t \varphi(s) dB^H(s) = \sum_{k=1}^{\infty} \int_0^t \sqrt{\lambda_k} \varphi(s) e_k d \beta_k^H(s) = \sum_{k=1}^{\infty} \int_0^t K^k(\varphi e_k)(s) d \beta_k(s),$$

(2)

where $\beta_k$ denotes the standard Brownian motion.

**Lemma 2.2** [25] If $\varphi : [0, \xi] \to L^0 \Lambda$ satisfies $\int_0^\xi \| \varphi(s) \|_{L^2}^2 < \infty$, then the above sum in (2) is well defined as a $\Lambda$-valued random variable, and we have

$$E \left\| \int_0^t \varphi(s) dB^H(s) \right\|_2^2 \leq 2H^2 t \int_0^\xi \| \varphi(s) \|_{L^2}^2 ds.$$

**Definition 2.3** [31] Let $\mathcal{B} : \mathcal{O} \to \mathcal{R}$, where $\mathcal{O}$ is a Banach space and $\mathcal{O}^*$ is the dual space of $\mathcal{O}$. The Clarke’s generalized directional derivative of $\mathcal{B}$ at $t \in \mathcal{O}$ in that direction $\zeta \in \mathcal{B}$ is characterized by

$$\mathcal{B}^0(t, \zeta) = \lim_{h \to 0^+} \sup_h \frac{\mathcal{B}(t + h\zeta) - \mathcal{B}(t)}{h},$$

and

$$\partial \mathcal{B}(t) = \{ t^* \in \mathcal{O}^* : \mathcal{B}^0(t, \zeta) \geq \langle t^*, \zeta \rangle, \text{ for all } \zeta \in \mathcal{O} \},$$

where $\partial \mathcal{B}(t)$ is the Clarke’s generalized gradient of $\mathcal{B}$ at $t \in \mathcal{O}$.

Here, $\mathcal{G} : L^2_0(\mathcal{S}, \Lambda) \to 2L^2_0(\mathcal{S}, \Lambda)$ is defined as:

$$\mathcal{G}(m) = \left\{ F \in L^2_0(\mathcal{S}, \Lambda) : F(t) \in \partial \mathcal{B}(t, m(t)) \text{ a.e. } t \in \mathcal{S} \text{ for } m \in L^2_0(\mathcal{S}, \Lambda) \right\}.$$

The Banach space of all continuous functions $m$ from $\mathcal{S}$ into $L^2(\Omega, S, P, \Lambda)$ is denoted by $\lambda := \{ t(\mathcal{S}, L^2(\Omega, S, P, \Lambda))$. 

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within \[ \|m\| = \left( \sup_{t \in \mathbb{T}} E \left[ (t - \langle t, m \rangle)^2 \right] \right)^{1/2} \]. Within this paper, let \( \mathcal{A}_\lambda = \{ m \in \lambda : \|m\|^2 \leq \tau \} \), where \( \tau > 0 \).

We require the following hypotheses to be verified in the findings:

(A1) \( \text{Dom}(F) \subset \text{Dom}(\delta) \) and the restriction of \( \delta \) to \( \text{Dom}(F) \) are continuous with regard to the graph norm of \( \text{Dom}(F) \).

(A2) There exists a linear operator \( \varrho : Y \to V \) such that for all \( V \in Y \) we have \( \varrho V \in \text{Dom}(F) \), \( \delta(\varrho V) = \varrho_1 V \) and \( E\|\varrho V\|^2 \leq C_1 \|\varrho_1 V\|^2 \), where \( C_1 \) is a constant.

(A3) \( A \) is the infinitesimal generator of the bounded operator \( \{ \mathcal{N}(t), t \geq 0 \} \) in \( V \), and there exists a constant \( \Pi > 0 \) such that \( \sup_{t \in \mathbb{T}} \|\mathcal{N}(t)\| \leq \Pi \).

(A4) For all \( t \in (0, \xi) \) and \( V \in Y \), \( \mathcal{N}(t) \varrho V \in \text{Dom}(A) \). Additionally, there exists a constant \( \Pi_1 > 0 \) such that \( \|A \mathcal{N}(t)\| \leq \Pi_1 \).

(A5) If \( f : \mathbb{T} \times V \to V \) is a continuous function, then there exist constants \( C_1, C_2 > 0 \) such that

\[
E \left[ \|f(t, m_1) - f(t, m_2)\|^2 \right] \leq C_1 E \left[ \|m_1 - m_2\|^2 \right],
\]

for \( t \in \mathbb{T}, m_1, m_2 \in V \), and the inequality

\[
E \left[ \|f(t, m)\|^2 \right] \leq C_2 \left( 1 + E \|m\|^2 \right),
\]

for all \( (t, m) \in \mathbb{T} \times V \).

(A6) If \( \sigma : \mathbb{T} \times V \to 2 \) is locally Lipschitz continuous (LLC), for all \( t \in \mathbb{T}, m, m_1, m_2 \in V \), then there exists a constant \( C_3 > 0 \) such that

\[
E \left[ \|\sigma(t, m_1) - \sigma(t, m_2)\|^2 \right] \leq C_3 E \left[ \|m_1 - m_2\|^2 \right], \quad E \left[ \|\sigma(t, m)\|^2 \right] \leq C_3 \left( 1 + E \|m\|^2 \right).
\]

(A7) If \( \varphi : \mathbb{T} \times V \to \mathcal{L}_0(D, V) \) is LLC, for all \( t \in \mathbb{T}, m, m_1, m_2 \in V \), then there exists a constant \( C_4 > 0 \) such that

\[
E \left[ \|\varphi(t, m_1) - \varphi(t, m_2)\|^2 \right] \leq C_4 E \left[ \|m_1 - m_2\|^2 \right], \quad E \left[ \|\varphi(t, m)\|^2 \right] \leq C_4 \left( 1 + E \|m\|^2 \right).
\]

(A8) If \( g : \mathbb{T} \times V \to \mathcal{L}_2(2, V) \) is LLC, for all \( t \in \mathbb{T}, m, m_1, m_2 \in V \), then there exists a constant \( C_5 > 0 \) such that

\[
E \left[ \|g(t, m_1) - g(t, m_2)\|^2 \right] \leq C_5 E \left[ \|m_1 - m_2\|^2 \right], \quad E \left[ \|g(t, m)\|^2 \right] \leq C_5 \left( 1 + E \|m\|^2 \right).
\]

(A9) The following conditions are fulfilled by the function \( \mathcal{B} : \mathbb{T} \times V \to \mathcal{R} \):

(i) \( \mathcal{B}(\cdot, m) : \mathbb{T} \to \mathcal{R} \) be measurable for all \( m \in V \),

(ii) \( \mathcal{B}(t, \cdot) : V \to \mathcal{R} \) be LLC for a.e. \( t \in \mathbb{T} \),

(iii) there exists \( \vartheta \in \mathcal{L}_1(\mathbb{T}, \mathcal{R}^+) \) and a constant \( C_6 > 0 \) that satisfies

\[
E \left[ \|\mathcal{B}(t, m)\|^2 \right] = \sup \left\{ E \left[ \|\mathcal{F}(t)\|^2 : \mathcal{F}(t) \in \partial \mathcal{B}(t, m) \right] : \vartheta(t) + C_6 E \|m\|^2 \right\},
\]

for all \( m \in V \) a.e. \( t \in \mathbb{T} \) and \( m \in V \).

Let \( m(t) \) be the solution of (1). Then, we define \( y(t) = m(t) - \varrho V(t) \). From our hypotheses, it is obvious that \( y(t) \in \text{Dom}(A) \). As a result, (1) may be represented in terms of \( A \) and \( \varrho \) as follows:
\[
\begin{aligned}
&\left\{D_{0^+}^{\alpha,\beta}\left[\tau(t)+f(t,m(t))\right]\right\} \\
&\quad \in A_0^{\beta,\alpha}
+ \frac{1}{(t+\eta-\ell_0)} \int_0^t (t-s)^{\beta-1}\mathbb{B}(s,m(s))ds \\
&\quad + \frac{1}{(t+\eta-\ell_0)} \int_0^t (t-s)^{\beta-1}\mathbb{B}(s,m(s))ds
+ \frac{1}{(t+\eta-\ell_0)} \int_0^t (t-s)^{\beta-1}\varphi(s,m(s))dw(s)
\end{aligned}
\]

Hence, the integral inclusion of (1) is provided by

\[
m(t) + f(t, m(t)) \in \frac{m_0 + f(0, m(0))}{(t+\eta-\ell_0)} + \frac{1}{(t+\eta-\ell_0)} \int_0^t (t-s)^{\beta-1}A_0^{\beta,\alpha}\left[ m(s) + f(s, m(s)) \right]ds
\]

\[
+ \frac{1}{(t+\eta-\ell_0)} \int_0^t (t-s)^{\beta-1}F - A_0^{\beta,\alpha}V(s)ds + \frac{1}{(t+\eta-\ell_0)} \int_0^t (t-s)^{\beta-1}\sigma(s, m(s))ds + \frac{1}{(t+\eta-\ell_0)} \int_0^t (t-s)^{\beta-1}\varphi(s, m(s))dw(s)
\]

\[
+ \frac{1}{(t+\eta-\ell_0)} \int_0^t (t-s)^{\beta-1}g(s, m(s))dB^H(s) + \frac{1}{(t+\eta-\ell_0)} \int_0^t (t-s)^{\beta-1}\partial_0^{\beta,\alpha}2(s, m(s))ds.
\]

Lemma 2.4 \[32\] An \(S_t\)-adapted stochastic process \(m(t) \in L_2^T(\mathcal{A}, \Lambda)\) is said to be a mild solution of the control system (1), and provided that \(\int_0^T (1-\ell_0)^\eta m(0) = m_0\), then there exists \(\mathcal{F} \in L_2^T(\mathcal{A}, \Lambda)\) such that \(\mathcal{F}(t) \in \partial_0^{\beta,\alpha}(t, m(t))\) a.e. \(t \in \mathcal{A}\) and

\[
m(t) = \mathcal{N}_{\ell_0,\eta}(t)m_0 + f(0, m(0)) + \int_0^t \left[ P_0(t-s)F - Ap_0(t-s) \right]\varphi(s)ds
\]

\[
+ \int_0^t P_0(t-s)\sigma(s, m(s))ds + \int_0^t P_0(t-s)\mathcal{F}(s)ds
\]

\[
+ \int_0^t P_0(t-s)\varphi(s, m(s))dw(s) + \int_0^t P_0(t-s)g(s, m(s))dB^H(s), \ t \in \mathcal{A},
\]

where

\[
\mathcal{N}_{\ell_0,\eta}(t) = \int_0^{(1-\eta)\ell_0} P_0(t), \ P_0(t) = t^{\eta-1}T_0(t), \ T_0(t) = \int_0^t \eta^\eta \Psi_0(v)N(v)dv,
\]

with

\[
\Psi_0(v) = \sum_{k=0}^{\infty} \frac{(-v)^{k-1}}{(k-1)!\Gamma(1-k\eta)}, \ 0 < \eta < 1, \ v \in (0, \infty),
\]

the Wright-type function that meets the following inequality

\[
\int_0^\infty v^\rho \Psi_0(v)dv = \frac{\Gamma(1+\rho)}{\Gamma(1+\eta\rho)}, \ v > 0.
\]
Lemma 2.5 [7] The operators $K_{t,u}$ and $P_{t}$ possess the following characteristics:

(i) $\{P_{t}(i) : i > 0\}$ is continuous in the uniform operator topology.

(ii) $K_{t,0}$ and $P_{t}$ are linear bounded operators,
\[
\|P_{t}(i)\| \leq \frac{\|\Pi_{t}^{y-1}\|}{\Gamma(0)} \|m\|, \quad \|K_{t,0}(i)\| \leq \frac{\|\Pi_{t}^{y-1}\|}{\Gamma(t(1-y)+\eta)} \|m\|
\]

(iii) $\{P_{t}(i), K_{t,0(i)}\}_{i>0}$ are strongly continuous.

Lemma 2.6 [37] If the assumption (A4) is satisfied, then
\[
\|AP_{t}(i)\| \leq \frac{\|\Pi_{t}^{y-1}\|}{\Gamma(0)} \|m\|.
\]

Lemma 2.7 [39] If (A9) is satisfied, then the set $S(m)$ has nonempty, convex, and weakly compact values for every $m \in L_{2}^{2}(\mathcal{T}, \Lambda)$.

Lemma 2.8 [39] The operator $S_{t}$ verifies: if $m_{k} \rightarrow m$ in $L_{2}^{2}(\mathcal{T}, \Lambda)$, $\beta_{k} \rightarrow \beta$ weakly in $L_{2}^{2}(\mathcal{T}, \Lambda)$, and $\beta_{k} \in S(m_{k})$, then $\beta \in S(m)$ given that (A9) is fulfilled.

Theorem 2.9 [40] Let $D$ be a locally convex Banach space and $\Psi : D \rightarrow 2^{\mathcal{P}}$ be a compact convex valued (CCV), upper semicontinuous multivalued maps such that there exists a closed neighborhood $L$ of 0 for which $\Psi(L)$ is relatively compact set. If $\Psi = \{m \in D : m_{0} \in \Psi(m), \xi > 1\}$ is bounded, then $\Psi$ has a fixed point.

3. Main results

To investigate the approximate boundary controllability for (1), we consider the linear stochastic system with HFD

\[
D_{0,0}^{\gamma}m(t) = Fm(t) + \sigma(t) + \phi(t) \frac{d\nu(t)}{dt}, \quad t \in \mathcal{T},
\]

\[
\delta m(t) = \theta V(t), \quad t \in \mathcal{T},
\]

\[
\Gamma_{0}^{\gamma}m(0) = m_{0}.
\]

We introduce the operators associated with (5) as
\[
\Gamma_{0}^{\gamma} = \int_{0}^{\gamma} (\xi - s)^{\gamma-1} \left[ T_{\theta}(\xi - s)F - AT_{\theta}(\xi - s) \right] ds,
\]
and
\[
N(\xi, \Gamma_{0}^{\gamma}) = (\xi I + \Gamma_{0}^{\gamma})^{-1}, \quad \xi > 0,
\]
where $\phi$ and $\left[ T_{\theta}(\xi - s)F - AT_{\theta}(\xi - s) \right]$ stands for the adjoint of $\phi$ and $\left[ T_{\theta}(\xi - s)F - AT_{\theta}(\xi - s) \right]$, respectively.

Assume that $m(\xi, m_{0}, V)$ represents the reachable set of the system (1) at the terminal state $\xi$, which corresponds to the control $V$ and the initial value $m_{0}$. Indicate by $N(\xi, m_{0}) = \{m(\xi, m_{0}, V) : V \in L_{2}^{2}(\mathcal{T}, \Lambda)\}$ is the reachable set of the system (1) at terminal state $\xi$ and its closure in $\Lambda$ is marked by $N(\xi, m_{0})$.

Definition 3.1 [41] The control system (1) is approximately controllable on $\mathcal{T}$ if $N(\xi, m_{0}) = L_{2}^{2}(\mathcal{T}, \Lambda)$.

Lemma 3.2 [41] The fractional linear control system (5) is approximately controllable on $\mathcal{T}$ if and only if $N(\xi I + \Gamma_{0}^{\gamma})^{-1} \rightarrow 0$ as $\xi \rightarrow 0$.

Lemma 3.3 For any $\overline{m}_{0} \in L_{2}^{2}(\mathcal{T}, \Lambda)$ there exists $\overline{\xi} \in L_{2}^{2}(\mathcal{T}, L_{2}^{2}(\mathcal{T}, \Lambda))$ and $\overline{\phi} \in L_{2}^{2}(\mathcal{T}, L_{2}^{2}(\mathcal{T}, L_{2}^{2}(\mathcal{T}, \Lambda)))$ such that
\[ \bar{m}_\varepsilon = E \bar{m}_\varepsilon + \int_0^\varepsilon \tilde{z}(s) d\nu(s) + \int_0^\varepsilon \tilde{\phi}(s) d\mathcal{B}^H(s). \]

Now, for any \( m_\varepsilon \in L^2(\Omega, \Lambda) \), as described above, the control function provides

\[ V(t) = \varrho \left[ T_\varepsilon(\xi - s) F - A T_\varepsilon(\xi - s) \right] \left( \xi I + \Gamma_0^\varepsilon \right)^{-1} \left[ E \bar{m}_\varepsilon - \mathcal{N}_{\varepsilon, 0}(\xi) \left[ m_0 + f(0, m(0)) \right] \right] \]

\[ + f(\xi, m(\xi)) - \int_0^\varepsilon P_\varepsilon(\xi - s) \sigma(s, m(s)) ds - \int_0^\varepsilon P_\varepsilon(\xi - s) \mathcal{F}(s) ds \]

\[ - \int_0^\varepsilon P_\varepsilon(\xi - s) \phi(s, m(s)) d\nu(s) - \int_0^\varepsilon P_\varepsilon(\xi - s) g(s, m(s)) d\mathcal{B}^H(s) \]

\[ + \int_0^\varepsilon \tilde{z}(s) d\nu(s) + \int_0^\varepsilon \tilde{\phi}(s) d\mathcal{B}^H(s) \] \( t \in \mathcal{T} \).

**Theorem 3.4** Assume that (A1)-(A9) are satisfied, then the control system (1) provides a mild solution on \( \mathcal{T} \).

\[ \bar{\mathcal{S}}_\varepsilon = \left\{ 49C_2 + \frac{49\Pi^2_{\varepsilon} \xi^{2n-1}}{(2\eta - 1)\Gamma^2(\eta)} \left[ C_3 + Tr(\Theta)C_4 + 2H \xi^{2n-1}C_5 \right] + C_6 \xi \right\} \]

\[ \times \left\{ 1 + \frac{\|F\|_2^2 \|\Pi^2_{\varepsilon} + \Pi^2\|_2^2}{\xi^2(2\eta - 1)\Gamma^4(\eta)} \right\} < 1. \]

**Proof.** Consider the map \( \mathcal{P}_\varepsilon : \lambda \rightarrow 2^I \) as follows

\[ \mathcal{P}_\varepsilon(m) = \left\{ \mathcal{A} \in \lambda : \mathcal{A}(t) = \mathcal{N}_{\varepsilon, 0}(t) \left[ m_0 + f(0, m(0)) \right] - f(t, m(t)) \right\} \]

\[ + \int_0^t \left[ P_\varepsilon(t - s) F - A P_\varepsilon(t - s) \right] \varrho V(s) ds + \int_0^t P_\varepsilon(t - s) \sigma(s, m(s)) ds \]

\[ + \int_0^t P_\varepsilon(t - s) \mathcal{F}(s) ds + \int_0^t P_\varepsilon(t - s) \phi(s, m(s)) d\nu(s) \]

\[ + \int_0^t P_\varepsilon(t - s) g(s, m(s)) d\mathcal{B}^H(s) \].

For \( t \in \mathcal{T} \), and from (A1)-(A9), we have

\[ E \left\| V(t) \right\|^2 \]

\[ \leq 49E \left\| T_\varepsilon(\xi - s) F - A T_\varepsilon(\xi - s) \right\| \left( \xi I + \Gamma_0^\varepsilon \right)^{-1} \left[ E \bar{m}_\varepsilon + \int_0^\varepsilon \tilde{z}(s) d\nu(s) + \int_0^\varepsilon \tilde{\phi}(s) d\mathcal{B}^H(s) \right] \]
In order to verify that $P$, has a fixed point, the proof is divided into six steps.

**Step 1:** For all $m \in \lambda$, $\mathcal{P}(m)$ be nonempty, convex, and weakly compact values.

We use Lemma 2.7 to show that $\mathcal{P}(m)$ is nonempty and has weakly compact values. Moreover, as $\mathcal{S}(m)$ has convex values, if $k_1, k_2 \in \mathcal{S}(m)$, then $l k_1 + (1 - l)k_2 \in \mathcal{S}(m)$ for every $l \in (0, 1)$, which implies clearly that $\mathcal{P}(m)$ is convex.

**Step 2:** $\mathcal{P}$ is bounded on a subset of $\lambda$.

Clearly, $\mathcal{A}$ is a BCC set of $\lambda$. We can prove that $E\|\mathcal{A}(t)\| \leq r, t > 0$, for all $\mathcal{A} \in \mathcal{P}(m), m \in \mathcal{A}$, if $\Phi \in \mathcal{P}(m)$, then there exists a $F \in \mathcal{O}(m)$ in a way that

$$
\Phi(t) = \mathcal{N}_{\alpha}(t)\left[ m_0 + f(0, m(0)) \right] - f(t, m(t)) + \int_0^t P(t - s)F - A P(t - s) \sigma(s, m(s))ds + \int_0^t P(t - s)\mathcal{F}(s)ds.
$$
\[ + \int_0^t P_0(t-s)\varphi(s,m(s))dw(s) + \int_0^t P_0(t-s)g(s,m(s))dB^H(s), \quad t \in \mathbb{T}. \tag{6} \]

Then,

\[
\| \Phi(t) \|_2^2 = 49 \sup_{t \in \mathbb{T}} \left\{ 2^{(1-\eta)(1-\eta)} \left[ E \left\| N_{\epsilon,a}(t) \left[ m_0 + f(0,m(0)) \right] \right\|_2^2 + E \left\| f(t,m(t)) \right\|_2^2 \\
+ E \left\| \int_0^t P_0(t-s)F - AP_0(t-s) \right\|_2^2 \right\} + E \left\| \int_0^t P_0(t-s)\sigma(s,m(s))ds \right\|_2^2 \\
+ E \left\| \int_0^t P_0(t-s)\varphi(s,m(s))dw(s) \right\|_2^2 + E \left\| \int_0^t P_0(t-s)g(s,m(s))dB^H(s) \right\|_2^2 \right\} \\
\leq 49 \left\{ \frac{\Pi^2}{T^2 (1 + \eta)^2} \left[ 2E \left\| m_0 \right\|_2^2 + 2C_2 \right] + 49 \xi^{2(1-\eta)(1-\eta)} C_2 (1 + \tau) \right\} \\
+ 49 \frac{\Pi^2 \xi^{2(1-\eta)^2}}{(2\eta - 1)^2} \left[ \left[ C_3 + Tr(\Theta)C_4 + 2H \xi^{2H-1} C_5 \right] (1 + \tau) + \left\| \mathcal{F} \right\|_{L^2(\mathbb{R}^+)}, C_6 \xi \tau \right] \\
\times \left\{ 1 + \frac{49 \left\| \mathcal{F} \right\|_{L^2(\mathbb{R}^+)^2} \left( \frac{2\eta}{5} \Pi^2 + \Pi^2 \right)^2 }{\xi^2 (2\eta - 1)^4 (\eta)} \right\} \right\} \\
+ \frac{2401 \left\| \mathcal{F} \right\|_{L^2(\mathbb{R}^+)^2} \left[ \left\| \mathcal{F} \right\|_{L^2(\mathbb{R}^+)^2} \left( \frac{2\eta}{5} \Pi^2 + \Pi^2 \right)^2 \left[ E \left\| m_0 \right\|_2^2 + Tr(\Theta) \right\|_{L^2(\mathbb{R}^+)} E \left\| \tilde{z}_0 \right\|_2^2 ds + 2H \xi^{2H-1} \int_0^\tau E \left\| \tilde{z}_0 \right\|_2^2 ds \right] \\
+ \left\| \mathcal{F} \right\|_{L^2(\mathbb{R}^+)^2} \left( \frac{2\eta}{5} \Pi^2 + \Pi^2 \right)^2 \left[ E \left\| m_0 \right\|_2^2 + Tr(\Theta) \right\|_{L^2(\mathbb{R}^+)} E \left\| \tilde{z}_0 \right\|_2^2 ds + 2H \xi^{2H-1} \int_0^\tau E \left\| \tilde{z}_0 \right\|_2^2 ds \right] \\
\right\} \xi^2 (2\eta - 1)^4 (\eta).
\]

As a result, \( \mathbb{S}(\mathbb{A}_\tau) \) is bounded in \( \lambda \).

**Step 3:** \{\( \mathbb{S}_m(m) : m \in \mathbb{A}_\tau \)\} is equiconvergent.

For any \( m \in \mathbb{A}_\tau \), \( \Phi \in \mathbb{S}_m(m) \), there exists a \( \mathcal{F} \in \mathbb{S}(m) \) like that (6) holds for all \( t \in \mathbb{T} \). For \( 0 < t_1 < t_2 < \xi \), we can obtain

\[
E \left\| \Phi(t_2) - \Phi(t_1) \right\|_2^2 \\
\leq 49 E \left\| \left[ N_{\epsilon,a}(t_2) - N_{\epsilon,a}(t_1) \right] \left[ m_0 + f(0,m(0)) \right] \right\|_2^2 + 49 E \left\| f(t_2,m(t_2)) - f(t_1,m(t_1)) \right\|_2^2
\]
\[ +49E \left\| \int_0^t \left[ P_n(t_2-s)F - AP_n(t_2-s) \right] qV(s)ds - \int_0^t \left[ P_n(t_1-s)F - AP_n(t_1-s) \right] qV(s)ds \right\|_2 \]

\[ +49E \left\| \int_0^t P_n(t_2-s)\sigma(s,m(s))ds - \int_0^t P_n(t_1-s)\sigma(s,m(s))ds \right\|_2^2 \]

\[ +49E \left\| \int_0^t P_n(t_2-s)\mathcal{F}(s)ds - \int_0^t P_n(t_1-s)\mathcal{F}(s)ds \right\|_2^2 \]

\[ +49E \left\| \int_0^t P_n(t_2-s)\varphi(s,m(s))dw(s) - \int_0^t P_n(t_1-s)\varphi(s,m(s))dw(s) \right\|_2^2 \]

\[ +49E \left\| \int_0^t P_n(t_2-s)\varphi(s,m(s))dw(s) - \int_0^t P_n(t_1-s)\varphi(s,m(s))dw(s) \right\|_2^2 \]

\[ = 49E \left\| \mathcal{N}_{t_2}(t_2) - \mathcal{N}_{t_1}(t_1) \right\|_2^2 + 49E \left\| f(t_2,m(t_2)) - f(t_1,m(t_1)) \right\|_2^2 \]

\[ +49E \left\| \int_0^t \left[ P_n(t_2-s)F - AP_n(t_2-s) \right] qV(s)ds \right\|_2^2 \]

\[ + \int_0^t \left[ P_n(t_2-s)F - AP_n(t_2-s) - P_n(t_1-s)F + AP_n(t_1-s) \right] qV(s)ds \left\|_2^2 \right. \]

\[ +49E \left\| \int_0^t P_n(t_2-s)\sigma(s,m(s))ds + \int_0^t \left[ P_n(t_2-s) - P_n(t_1-s) \right] \sigma(s,m(s))ds \right\|_2^2 \]

\[ +49E \left\| \int_0^t P_n(t_2-s)\mathcal{F}(s)ds + \int_0^t \left[ P_n(t_2-s) - P_n(t_1-s) \right] \mathcal{F}(s)ds \right\|_2^2 \]

\[ +49E \left\| \int_0^t P_n(t_2-s)\varphi(s,m(s))dw(s) + \int_0^t \left[ P_n(t_2-s) - P_n(t_1-s) \right] \varphi(s,m(s))dw(s) \right\|_2^2 \]

\[ +49E \left\| \int_0^t P_n(t_2-s)g(s,m(s))dB^H(s) + \int_0^t \left[ P_n(t_2-s) - P_n(t_1-s) \right] g(s,m(s))dB^H(s) \right\|_2^2 \]

From the compactness of $\mathcal{N}(t)$ ($t > 0$),

\[ E \left\| \Phi(t_2) - \Phi(t_1) \right\|_2^2 \to 0 \quad \text{as} \quad t_2 \to t_1. \]

Hence, $\mathcal{P}_\epsilon(m)(t)$ is continuous in $\mathcal{S}$. Also, for $t_1 = 0$ and $t_2 \in \mathcal{S}$, we can show that $E \left\| \Phi(t_2) - \Phi(t_1) \right\|_2^2 \to 0$ as $t_1 \to 0$.

As a result, $\{ \mathcal{P}_\epsilon(m) : m \in \mathcal{A}_\epsilon \}$ is equicontinuous.

\textbf{Step 4:} $\mathcal{P}_\epsilon$ is completely continuous.
We show that $\chi(t) = \{\Phi(t) : \Phi \in \mathcal{P}(\mathfrak{A}, t)\}$ is relatively compact in $\Lambda$ for all $t \in T$, $t > 0$. Undoubtedly, $\chi(0)$ is relatively compact in $\mathfrak{A}$. Let $0 < t \leq \xi$ be fixed, $0 < \eta < t$, for $m \in \mathfrak{A}$, we define

$$\Phi^\eta(t) = \frac{\eta}{\Gamma((1-n)/2)} \int_0^\eta \int_s^\infty v(t-s)^{(1-n)/2} \Psi_\eta(v) \left[ \mathcal{N}\left((t-s)^{\eta}v\right) F - AN\left((t-s)^{\eta}v\right) \right] \sigma(s, m(s))dvds$$

$$+ \eta \int_0^\eta \int_s^\infty v(t-s)^{(1-n)/2} \Psi_\eta(v) \left[ \mathcal{N}\left((t-s)^{\eta}v\right) F - AN\left((t-s)^{\eta}v\right) \right] \varphi(s, m(s)) dvds$$

$$+ \eta \int_0^\eta \int_s^\infty v(t-s)^{(1-n)/2} \Psi_\eta(v) \left[ \mathcal{N}\left((t-s)^{\eta}v\right) F - AN\left((t-s)^{\eta}v\right) \right] \varphi(s, m(s)) dvds$$

$$= \frac{\eta \mathcal{N}(\eta^\eta)}{\Gamma((1-n)/2)} \int_0^\eta \int_s^\infty v(t-s)^{(1-n)/2} \Psi_\eta(v) \left[ \mathcal{N}\left((t-s)^{\eta}v\right) F - AN\left((t-s)^{\eta}v\right) \right] \sigma(s, m(s))dvds$$

$$+ \eta \mathcal{N}(\eta^\eta) \int_0^\eta \int_s^\infty v(t-s)^{(1-n)/2} \Psi_\eta(v) \left[ \mathcal{N}\left((t-s)^{\eta}v\right) F - AN\left((t-s)^{\eta}v\right) \right] \varphi(s, m(s)) dvds$$

$$+ \eta \mathcal{N}(\eta^\eta) \int_0^\eta \int_s^\infty v(t-s)^{(1-n)/2} \Psi_\eta(v) \left[ \mathcal{N}\left((t-s)^{\eta}v\right) F - AN\left((t-s)^{\eta}v\right) \right] \varphi(s, m(s)) dvds$$

$$+ \eta \mathcal{N}(\eta^\eta) \int_0^\eta \int_s^\infty v(t-s)^{(1-n)/2} \Psi_\eta(v) \left[ \mathcal{N}\left((t-s)^{\eta}v\right) F - AN\left((t-s)^{\eta}v\right) \right] \varphi(s, m(s)) dvds$$

$$+ \eta \mathcal{N}(\eta^\eta) \int_0^\eta \int_s^\infty v(t-s)^{(1-n)/2} \Psi_\eta(v) \left[ \mathcal{N}\left((t-s)^{\eta}v\right) F - AN\left((t-s)^{\eta}v\right) \right] \varphi(s, m(s)) dvds.$$
\[
\times E \left[ \int_0^t \int_0^{s(t-s)^{\eta-1}} (t-s)^{\eta-1} \Psi_\eta(v) \mathcal{N}(s^\eta v) \left[ m_0 - f(0,m(0)) \right] dv ds \right]^2
\]

+ \frac{49\eta^2}{\Gamma^2(1-n)} \sup_{t \in \mathbb{T}} t^{2(1-n)}

\times E \left[ \int_{t-\eta}^t \int_0^{s(t-s)^{\eta-1}} (t-s)^{\eta-1} \Psi_\eta(v) \mathcal{N}(s^\eta v) \left[ m_0 - f(0,m(0)) \right] dv ds \right]^2

+ 49\eta^2 \sup_{t \in \mathbb{T}} t^{2(1-n)}

\times E \left[ \int_{t-\eta}^t \int_0^{s(t-s)^{\eta-1}} (t-s)^{\eta-1} \Psi_\eta(v) \left[ \mathcal{N}(s^\eta v) F - \mathcal{N}(t-s)^\eta v \right] \varphi \mathcal{F}(s) dv ds \right]^2

+ 49\eta^2 \sup_{t \in \mathbb{T}} t^{2(1-n)} E \left[ \int_0^t \int_0^{s(t-s)^{\eta-1}} (t-s)^{\eta-1} \Psi_\eta(v) \mathcal{N}(s^\eta v) \sigma(s,m(s)) dv ds \right]^2

+ 49\eta^2 \sup_{t \in \mathbb{T}} t^{2(1-n)} E \left[ \int_0^t \int_0^{s(t-s)^{\eta-1}} (t-s)^{\eta-1} \Psi_\eta(v) \mathcal{N}(s^\eta v) \sigma(s,m(s)) dv ds \right]^2

+ 49\eta^2 \sup_{t \in \mathbb{T}} t^{2(1-n)} E \left[ \int_0^t \int_0^{s(t-s)^{\eta-1}} (t-s)^{\eta-1} \Psi_\eta(v) \mathcal{N}(s^\eta v) \mathcal{F}(s) dv ds \right]^2

+ 49\eta^2 \sup_{t \in \mathbb{T}} t^{2(1-n)} E \left[ \int_0^t \int_0^{s(t-s)^{\eta-1}} (t-s)^{\eta-1} \Psi_\eta(v) \mathcal{N}(s^\eta v) \varphi(s,m(s)) dv ds \right]^2

+ 49\eta^2 \sup_{t \in \mathbb{T}} t^{2(1-n)} E \left[ \int_0^t \int_0^{s(t-s)^{\eta-1}} (t-s)^{\eta-1} \Psi_\eta(v) \mathcal{N}(s^\eta v) \varphi(s,m(s)) dv ds \right]^2

+ 49\eta^2 \sup_{t \in \mathbb{T}} t^{2(1-n)} E \left[ \int_0^t \int_0^{s(t-s)^{\eta-1}} (t-s)^{\eta-1} \Psi_\eta(v) \mathcal{N}(s^\eta v) \varphi(s,m(s)) dv ds \right]^2

+ 49\eta^2 \sup_{t \in \mathbb{T}} t^{2(1-n)} E \left[ \int_0^t \int_0^{s(t-s)^{\eta-1}} (t-s)^{\eta-1} \Psi_\eta(v) \mathcal{N}(s^\eta v) \varphi(s,m(s)) dv ds \right]^2

+ 49\eta^2 \sup_{t \in \mathbb{T}} t^{2(1-n)} E \left[ \int_0^t \int_0^{s(t-s)^{\eta-1}} (t-s)^{\eta-1} \Psi_\eta(v) \mathcal{N}(s^\eta v) \varphi(s,m(s)) dv ds \right]^2

+ 49\eta^2 \sup_{t \in \mathbb{T}} t^{2(1-n)} E \left[ \int_0^t \int_0^{s(t-s)^{\eta-1}} (t-s)^{\eta-1} \Psi_\eta(v) \mathcal{N}(s^\eta v) \varphi(s,m(s)) dv ds \right]^2

+ 49\eta^2 \sup_{t \in \mathbb{T}} t^{2(1-n)} E \left[ \int_0^t \int_0^{s(t-s)^{\eta-1}} (t-s)^{\eta-1} \Psi_\eta(v) \mathcal{N}(s^\eta v) \varphi(s,m(s)) dv ds \right]^2

+ 49\eta^2 \sup_{t \in \mathbb{T}} t^{2(1-n)} E \left[ \int_0^t \int_0^{s(t-s)^{\eta-1}} (t-s)^{\eta-1} \Psi_\eta(v) \mathcal{N}(s^\eta v) \varphi(s,m(s)) dv ds \right]^2

+ 49\eta^2 \sup_{t \in \mathbb{T}} t^{2(1-n)} E \left[ \int_0^t \int_0^{s(t-s)^{\eta-1}} (t-s)^{\eta-1} \Psi_\eta(v) \mathcal{N}(s^\eta v) \varphi(s,m(s)) dv ds \right]^2
As a result, the set $\mathcal{X}(t)$ is relatively compact in $\Lambda$. We may infer that $\mathcal{P}_*$ is completely continuous from the Arzela-Ascoli theorem and Step 3.

**Step 5:** The closed graph of $\mathcal{P}_*$.

Let $m_k \to m_*$ in $\lambda$, $\Phi_k \in \mathcal{P}_*(m_k)$ and $\Phi_k \to \Phi_*$. Clearly, $\Phi_* \in \mathcal{P}_*(m_*)$, there exists a $\mathcal{F}_k \in \mathfrak{S}(m_k)$ obtained that

$$
\Phi_k(t) = \mathcal{N}_{c,a}(t)\left[ m_0 + f(0, m_k(0)) \right] - f(t, m_k(t))
$$

$$
+ \int_{0}^{t} \left[ P_0(t-s)F - A P_0(t-s) \right] \varphi(s) ds
$$

$$
+ \int_{0}^{t} P_0(t-s) \sigma(s, m_k(s)) ds + \int_{0}^{t} P_0(t-s) \mathcal{F}_k(s) ds
$$

$$
+ \int_{0}^{t} P_0(t-s) \varphi(s, m_k(s)) dw(s) + \int_{0}^{t} P_0(t-s) g(s, m_k(s)) dB^H (s).
$$

(7)

From (A1)-(A9), we may infer that \((f(\cdot, m_k), \sigma(\cdot, m_k), \mathcal{F}_k, \varphi(\cdot, m_k), g(\cdot, m_k))_{k \geq 1} \subseteq \Lambda \times \Lambda \times L_2^2(\Omega, \lambda) \times L_0 \times L_2^3\) is bounded. Thus, we obtain

$$
(f(\cdot, m_k), \sigma(\cdot, m_k), \mathcal{F}_k, \varphi(\cdot, m_k), g(\cdot, m_k)) \to (f(\cdot, m_*), \sigma(\cdot, m_*), \mathcal{F}_*, \varphi(\cdot, m_*), g(\cdot, m_*)),
$$

(8)

weakly in $\Lambda \times \Lambda \times L_2^2(\Omega, \lambda) \times L_0 \times L_2^3$.

From the compactness of $\mathcal{N}(t)$, (7) and (8), and we get

$$
\Phi_*(t) = \mathcal{N}_{c,a}(t)\left[ m_0 + f(0, m_*(0)) \right] - f(t, m_*(t))
$$

$$
+ \int_{0}^{t} \left[ P_0(t-s)F - A P_0(t-s) \right] \varphi(s) ds
$$

$$
+ \int_{0}^{t} P_0(t-s) \sigma(s, m_*(s)) ds + \int_{0}^{t} P_0(t-s) \mathcal{F}_*(s) ds
$$

$$
+ \int_{0}^{t} P_0(t-s) \varphi(s, m_*(s)) dw(s) + \int_{0}^{t} P_0(t-s) g(s, m_*(s)) dB^H (s).
$$

(9)

Note that $\Phi_k \to \Phi_*$ in $\lambda$ and $\mathcal{F}_k \in \mathfrak{S}(m_k)$. From (9) and Lemma (2.8), we obtain $\mathcal{F}_* \in \mathfrak{S}(m_*)$. Therefore, $\Phi_* \in \mathcal{P}_*(m_*)$, which implies that $\mathcal{P}_*$ has a closed graph and that $\mathcal{P}_*$ is a completely continuous multi-valued map with compact value. Thus, from [32], $\mathcal{P}_*$ is upper semicontinuous.

**Step 6:** A priori estimate.
From Steps 1-5, we found that $\Psi_t$ is CCV. In addition, the upper semicontinuous set $\Psi_t(\mathcal{X}_t)$ is relatively compact. By Theorem (2.9), it remains to demonstrate that $\Psi = \{ m \in \lambda : \zeta m \in \Psi_t, \zeta > 1 \}$ is bounded. For all $m \in \Psi$, there exists a $\mathcal{F} \in \mathcal{G}(m)$ such that

$$m(t) = \zeta^{-1} N_{t, \phi}(t) \left[ m_0 + f(0, m(0)) \right] - \zeta^{-1} f(t, m(t))$$

$$+ \zeta^{-1} \int_0^t \left[ P_0(t-s)F - AP_0(t-s) \right] \varphi(s)ds$$

$$+ \zeta^{-1} \int_0^t P_0(t-s)\sigma(s, (s)) ds + \zeta^{-1} \int_0^t P_0(t-s)\mathcal{F}(s)ds$$

$$+ \zeta^{-1} \int_0^t P_0(t-s)\varphi(s, m(s)) dw(s) + \zeta^{-1} \int_0^t P_0(t-s)g(s, m(s)) dB^H(s). \quad (10)$$

Using the hypotheses (A1)-(A9), we get

$$E \left\| m(t) \right\|^2 \leq 49 \left[ E \left\| N_{t, \phi}(t) \left[ m_0 + f(0, m(0)) \right] \right\|^2 + E \left\| f(t, m(t)) \right\|^2$$

$$+ E \left\| \int_0^t P_0(t-s)F - AP_0(t-s) \right\| \varphi(s)ds \right\|^2 + E \left\| \int_0^t P_0(t-s)\sigma(s, (s)) ds \right\|^2$$

$$+ E \left\| \int_0^t P_0(t-s)\mathcal{F}(s)ds \right\|^2 + E \left\| \int_0^t P_0(t-s)\varphi(s, m(s)) dw(s) \right\|^2$$

$$+ E \left\| \int_0^t P_0(t-s)g(s, m(s)) dB^H(s) \right\|^2 \right\}$$

$$\leq \left\{ \frac{49\Pi_2^{2^{(l-1)(1-n)}}}{\Gamma^2(1-n) + \zeta^2} \left[ 2E \left\| m_0 \right\|^2 + 2C_2 \right] + 49C_2 \left( 1 + E \left\| m(t) \right\|^2 \right)$$

$$+ \frac{49\Pi_2^{2^{2n-1}}}{(2n-1)\Gamma^2(\eta)} \left[ C_5 + Tr(\Theta)C_4 + 2H \zeta^2 + C_5 \right] \left[ 1 + E \left\| m(t) \right\|^2 \right] + \left\| F \right\|_{L^2(\mathcal{H}^+)}$$

$$+ C_6 \zeta E \left\| m(t) \right\|^2 \right\} \times \left\{ 1 + \frac{49 \left\| \phi \right\|^2 \left\| \zeta^{2n-1} \left[ \Gamma^2 + \Pi^2 \right] \right\|^2}{\zeta^2(2n-1)\Gamma^4(\eta)} \right\}$$

$$+ \frac{2401 \left\| \phi \right\|^2 \left\| \zeta^{2n-1} \left[ \Gamma^2 + \Pi^2 \right] \right\|^2 \left[ E \left\| m_0 \right\|^2 + Tr(\Theta) \int_0^\zeta E \left\| \phi \right\|^2 ds + 2H \zeta^2 + \int_0^\zeta E \left\| \phi \right\|^2 ds \right]}{\zeta^2(2n-1)\Gamma^4(\eta)}$$
\[ \leq \delta_1 + \delta_2 E \|n(t)\|_2^2. \quad (11) \]

where

\[
\delta_1 = \left[ \frac{49\Pi^2 \xi^{2(\ell-1)\eta}}{\Gamma^2(\ell(1-\eta)+\eta)} \right] \left( 2E \|n_0\|^2 + 2C_2 \right) + 49C_2
\]

\[ + \frac{49\Pi^2 \xi^{2n-1}}{(2n-1)\Gamma^2(n)(\eta)} \left( C_3 + Tr(\Theta)C_4 + 2H \xi^{2H-1}C_3 \right) + \|F\|^2_{L^2(\mathbb{R},\mathbb{R}^r)} \}
\]

\[ \times \left( 1 + \frac{49\|\|f\|\|^2 \|\varphi\| \xi^{2n-1} \|F\|^2 \Pi^2 + \Pi_1^2 \|^{2}}{\zeta^2(2n-1)\Gamma^4(\eta)} \right) \]

\[ + \frac{2401\|\|f\|\|^2 \|\varphi\| \xi^{2n-1} \|F\|^2 \Pi^2 + \Pi_1^2 \|^{2}}{\zeta^2(2n-1)\Gamma^4(\eta)} \]

and

\[
\delta_2 = \left[ \frac{49\Pi^2 \xi^{2n-1}}{(2n-1)\Gamma^2(n)(\eta)} \right] \left( C_3 + Tr(\Theta)C_4 + 2H \xi^{2H-1}C_3 + C_6 \xi \right)
\]

\[ \times \left( 1 + \frac{49\|\|f\|\|^2 \|\varphi\| \xi^{2n-1} \|F\|^2 \Pi^2 + \Pi_1^2 \|^{2}}{\zeta^2(2n-1)\Gamma^4(\eta)} \right) \]

Since \( \delta_2 < 1 \), from (11), we get

\[ \|m\|^2_{\Xi} = \sup_{t \in \Xi} E \left\| \psi^{(\ell-1)(\eta)} m(t) \right\|^2 \leq \delta_1 + \delta_2 \|m\|^2_{\Xi}. \]

Then, \( \|m\|^2_{\Xi} \leq \frac{\delta_1}{1-\delta_2} \), consequently, \( \Psi \) is bounded. By Theorem (2.9), \( \Psi \) has a fixed point. As a result, the inclusion system (1) is approximately controllable on \( \Xi \).

**Theorem 3.5** Assuming that \((A1)-(A9)\) are fulfilled. Additionally, if the functions \( f, \sigma, \varphi, g \) and \( F \) are all uniformly bounded, then (1) is approximately controllable on \( \Xi \).

**Proof.** Using the stochastic Fubini theorem and \( m^* \) as a fixed point on \( \Xi \), it is evident that

\[ m^*(\xi) = m_0 - \xi(\Pi I + \Gamma_0^{-1})^{-1} \left\{ E\bar{m}_0 - N_{\xi,\eta}(\xi) \left[ m_0 + f \left( 0, m^*(0) \right) \right] \right\} \]

\[ - f \left( \xi, m^*(\xi) \right) - \int_0^\xi P_\eta(\xi-s)\sigma \left( s, m^*(s) \right) ds - \int_0^\xi P_\eta(\xi-s)F^T(s) ds \]
\[-\int_0^\xi P_s(\xi-s)\varphi(s,m^\varepsilon(s))dw(s) - \int_0^\xi P_s(\xi-s)g(s,m^\varepsilon(s))dB^H(s)\]

\[+\int_0^\xi \tilde{X}(s)dw(s) + \int_0^\xi \tilde{\phi}(s)dB^H(s)\}.\]

(12)

From the hypotheses on \(f, \sigma, \varphi, g\) and \(F\) and the proof of the Step 6 in Theorem (3.4), we get

\[E\|f(t,m^\varepsilon(t))\|^2 + E\|\sigma(t,m^\varepsilon(t))\|^2 + E\|\varphi(t,m^\varepsilon(t))\|^2 + E\|g(t,m^\varepsilon(t))\|^2 + E\|F^\varepsilon(t)\|^2 \leq \nu(t).\]

Consequently, the sequence \(\{f(\cdot,m^\varepsilon(\cdot)), \sigma(\cdot,m^\varepsilon(\cdot)), \varphi(\cdot,m^\varepsilon(\cdot)), g(\cdot,m^\varepsilon(\cdot)), F^\varepsilon(\cdot)\}\) be bounded in \(\Lambda \times 2^\Lambda \times L^2(\mathbb{D}, \Lambda) \times L^2(Y, \Lambda) \times L^2(D, \Lambda)\). Thus, there is a subsequent event, indicated by \(\{f(\cdot), \sigma(\cdot), \varphi(\cdot), g(\cdot), F^\varepsilon(\cdot)\}\) that converges weakly to \(\{f(\cdot), \sigma(\cdot), \varphi(\cdot), g(\cdot), F^\varepsilon(\cdot)\}\) in \(\Lambda \times 2^\Lambda \times L^2(\mathbb{D}, \Lambda) \times L^2(Y, \Lambda) \times L^2(D, \Lambda)\). From (12), we have

\[E\left| m^\varepsilon(\xi) - \tilde{m}_0 \right|^2 \leq 49E\left| \zeta I + \Gamma_0^{-1}\right| \left| E\tilde{m}_0 - N_{\tilde{\theta}}(\xi)\right| \left| \left[ m^\varepsilon_0 + f(0,m^\varepsilon(0)) \right] \right|^2\]

\[+49E\left| \zeta I + \Gamma_0^{-1}\right| \left( \zeta I + \Gamma_0^{-1}\right) \left| f(\zeta,m^\varepsilon(\zeta)) \right|^2\]

\[+49E\left| \zeta I + \Gamma_0^{-1}\right| \left( \zeta I + \Gamma_0^{-1}\right) \left| \sigma(s,m^\varepsilon(s)) - \sigma(s) \right| ds\]

\[+49E\left| \zeta I + \Gamma_0^{-1}\right| \left( \zeta I + \Gamma_0^{-1}\right) \left| \varphi(s,m^\varepsilon(s)) - \varphi(s) \right| dw(s)\]

\[+49E\left| \zeta I + \Gamma_0^{-1}\right| \left( \zeta I + \Gamma_0^{-1}\right) \left| g(s,m^\varepsilon(s)) - g(s) \right| dB^H(s)\]

\[+49E\left| \zeta I + \Gamma_0^{-1}\right| \left( \zeta I + \Gamma_0^{-1}\right) \left| \varphi(s,m^\varepsilon(s)) - \varphi(s) \right| dv(s)\]

\[+49E\left| \zeta I + \Gamma_0^{-1}\right| \left( \zeta I + \Gamma_0^{-1}\right) \left| g(s,m^\varepsilon(s)) - g(s) \right| dB^H(s)\]
However, in accordance with the Lemma (3.2), for all $0 \leq s \leq \xi$, $\zeta(s) \rightarrow 0$ strongly as $\zeta \rightarrow 0$, and $\|\zeta(s)\| \leq 1$ in addition to the Lebesgue dominated convergence theorem and the compactness of $P_d(t)$, therefore, $E\|m'(\zeta) - m_d\| \rightarrow 0$ as $\zeta \rightarrow 0$. This demonstrates that (1) is approximately controllable. \hfill \Box

4. Example

The Hilfer fractional neutral stochastic partial differential inclusions with fBm and Clarke’s subdifferential are taken into consideration.

\[
\begin{align*}
\frac{3}{4} e^{\frac{1}{4}} \left[ m(t, \varpi) + f(t, m(t, \varpi)) \right] &+ \frac{2}{7} \left[ m(t, \varpi) + f(t, m(t, \varpi)) \right] + \sigma(t, m(t, \varpi)) + \sigma'(t, m(t, \varpi)) \frac{d\mathcal{B}(t, m(t, \varpi))}{dt} \\
&+ g(t, m(t, \varpi)) \frac{d\mathcal{B}^H(t)}{dt} + \partial \mathcal{B}(t, m(t, \varpi)), \ t \in \mathcal{T} = (0, 1), \ \varpi \in \mathcal{U}, \\
m(t, \varpi) &= \mathcal{V}(t, \varpi), \ t \in \mathcal{T}, \ \varpi \in \mathcal{Y}, \\
\frac{1}{2}, & m(0, \varpi) = m_0(\varpi), \ \varpi \in \mathcal{U},
\end{align*}
\]

where $\frac{3}{4} e^{\frac{1}{4}}$ is the HFD of order $\ell = \frac{3}{4}$, $\eta = \frac{5}{7}$, $0 < t_0 < t_1 < \cdots < t_p < 1$, $\mathcal{U}$ is bounded and open subset of $\mathfrak{H}$, it is smooth enough at the boundary of $\mathcal{Y}$. $w(t)$ is Wiener process and $\mathcal{B}^H$ is a fBm with Hurst parameter $H \in \left(\frac{1}{2}, 1\right]$ and $\mathcal{V} \in L_2(\mathcal{O})$.

The functions

\[f(t, m(t))(\varpi) = f(t, m(t, \varpi)), \ \sigma(t, m(t))(\varpi) = \sigma(t, m(t, \varpi)), \ \varphi(t, m(t))(\varpi) = \varphi(t, m(t, \varpi)),\]

\[g(t, m(t))(\varpi) = g(t, m(t, \varpi)), \text{ and } \partial \mathcal{B}(t, m(t))(\varpi) = \partial \mathcal{B}(t, m(t, \varpi)).\]

Let $\Lambda = \mathfrak{H} = L_2^2(\mathcal{U})$, $\mathcal{Y} = L_2^2(\mathcal{O})$, $\varrho_1 = I$, the identity operator, and $F : \text{Dom}(F) \subset \Lambda \rightarrow \Lambda$ provided by $F = \frac{\partial^2}{\partial \varpi^2}$ with $\text{Dom}(F) = \{m \in \Lambda, \ m, \ \frac{\partial m}{\partial \varpi} \text{ are absolutely continuous}, \ \frac{\partial^2 m}{\partial \varpi^2} \in L_2^2(\mathcal{U})\}$. Then $\Lambda$ can be written as

\[\Delta m = \sum_{k=1}^{\infty} (-k^2)(m, m_k)m_k, \ m \in D(\Delta),\]

where $m_k(\varpi) = \frac{\sqrt{2}}{\sqrt{\pi}} \sin ks$, $k = 1, 2, \cdots$ is the orthogonal base set of eigenvectors of $\Delta$. Furthermore, $\Delta$ is the infinitesimal generator of the bounded linear operator, $\{\mathcal{N}(t)\}_{t \geq 0}$ in $\Lambda$ and provided by

\[\mathcal{N}(t)m = \sum_{k=1}^{\infty} e^{-\frac{t^2}{4}}(m, m_k)m_k,\]
\[\|N(t)\| \leq e^{-t} \leq 1.\] Additionally, the operators \(N_{3,5}(t) \) and \(P_{s}(t)\) may be determined by

\[N_{3,5}(t)m = \frac{5}{7\Gamma\left(\frac{5}{7}\right)} \int_{0}^{\infty} v(t-s)^{\frac{11}{14}} \frac{-2}{7} \mathcal{W}_{v}(v)N(tv)mdvds,\]

\[P_{s}(t)m = \frac{5}{7\Gamma\left(\frac{5}{7}\right)} \int_{0}^{\infty} vs^{\frac{2}{7}} \mathcal{W}_{v}(v)N(tv)mdv.\]

Clearly,

\[\left\|P_{s}(t)\right\| \leq \frac{t^{-\frac{2}{7}}}{\Gamma\left(\frac{5}{7}\right)}, \quad \left\|N_{3,5}(t)\right\| \leq \frac{t^{-\frac{1}{14}}}{\Gamma\left(\frac{13}{14}\right)}.\]

Describe the fBm on \(Y\) using

\[B^{H}(t) = \sum_{k=1}^{\infty} \sqrt{\beta_{k}^{H}(t)e_{k}},\]

where \(H \in \left(\frac{1}{2}, 1\right), \left\{\beta_{k}^{H}\right\}_{k \in \mathbb{N}}\) is a sequence of one-dimensional fBm that are independent of one another.

It is obvious that \(A\) makes the compact semigroup \(\{N(t), t \geq 0\}\) on \(\Lambda\).

Next, we verify that the hypothesis \((A1)-(A9)\) for the above system (13) one by one.

**Verification of A1**

\(\text{Dom}(F) \subset \text{Dom}(\delta)\) and the restriction of \(\delta\) to \(\text{Dom}(F)\) are continuous with regard to the graph norm of \(\text{Dom}(F)\). Therefore, A1 is verified.

**Verification of A2**

There exists a linear operator \(q : Y \rightarrow \Lambda\) such that for all \(V \in Y\) we have \(qV \in \text{Dom}(F)\), \(\delta(qV) = qV\) and \(E||qV||^{2} \leq C_{1}||q_{1}V||\), where \(C_{1}\) is a constant. Hence, A2 is verified.

**Verification of A3**

\(A\) is the infinitesimal generator of bounded operator \(\{N(t), t \geq 0\}\) in \(\Lambda\) and there exists a constant \(\Pi > 0\) such that \(\sup_{t \in T}||N(t)|| \leq \Pi\). Therefore, A3 is verified.

**Verification of A4**

For all \(t \in (0, \xi]\) and \(V \in Y, N(t)qV \in \text{Dom}(A)\). Additionally, there exists a constant \(\Pi_{1} > 0\) such that \(||A\mathcal{N}(t)|| \leq \Pi_{1}\).

Hence, A4 is verified.

**Verification of A5**

Assume that \(f(t, m(t)) = f(t, m(t), m)\).

If \(f : \mathcal{E} \times \Lambda \rightarrow \Lambda\) is continuous function, then there exists a constants \(C_{1}, C_{2} > 0\) such that

\[E\left\|f(t, m_{1}) - f(t, m_{2})\right\|^{2} \leq C_{1}E\left\|m_{1} - m_{2}\right\|^{2},\]

for \(t \in \mathcal{E}, m_{1}, m_{2} \in \Lambda, \sigma \in \mathcal{U}\) and the inequality
\[ E \| f(t, m) \|^2 \leq C_2 \left( 1 + E \| m \|^2 \right), \]

for all \((t, m) \in \mathbb{T} \times \Lambda\).

Therefore the function \(g(t, m(t, \varpi))\) satisfies the condition A5.

**Verification of A6-A8**

Assume that \(\sigma(t, m(t)) = \sigma(t, m(t, \varpi)), \varphi(t, m(t)) = \varphi(t, m(t, \varpi)),\) and \(g(t, m(t)) = g(t, m(t, \varpi))\). The functions \(\sigma(t, m(t, \varpi)), \varphi(t, m(t, \varpi)), g(t, m(t, \varpi))\) are similar to that of the system (1) and there is no more to prove.

**Verification of A9**

Assume that \(\partial B(t, m(t, \varpi)) = \partial B(t, m(t, \varpi))\).

The following conditions are fulfilled by the function \(B : \mathbb{T} \times \Lambda \rightarrow \mathbb{R}\):

(i) \(B(\cdot, m) : \mathbb{T} \rightarrow \mathbb{R}\) be measurable for all \(m \in \Lambda\),

(ii) \(B(t, \cdot) : \Lambda \rightarrow \mathbb{R}\) be LLC for a.e. \(t \in \mathbb{T}\),

(iii) there exists \(\vartheta \in L^1(\mathbb{T}, \mathbb{R}^+)\) and a constant \(C_6 > 0\), that satisfies

\[ E \| \partial B(t, m(t, \varpi)) \|^2 = \sup \left\{ E \| F(t) \|^2 : F(t) \in \partial B(t, m(t, \varpi)) \right\} \leq \vartheta(t) + C_6 E \| m \|^2, \]

for all \(m \in \Lambda\) a.e. \(t \in \mathbb{T}\) and \(\sigma \in \mathcal{U}\).

From the above discussion, we observe that \(\partial B(t, m(t, \varpi))\) satisfies every condition in A9.

It is now possible to write (13) in the form of (1). Clearly, all the assumptions of the Theorem 3.4 and 3.5 are satisfied.

\[
\begin{align*}
49C_2 + \frac{49\Pi^2 \xi^{2n-1}}{(2\eta-1)\Gamma^2(\eta)} \left[ C_3 + Tr(\Theta)C_4 + 2H \xi^{2H-1}C_5 + C_6 \xi \right] \\
\times \left[ \frac{49 \| F \|_{L^2}^2 \xi^{2n-1} \left[ C_3 + Tr(\Theta)C_4 + 2H \xi^{2H-1}C_5 + C_6 \xi \right]^2}{\xi^{(2n-1)\Gamma^2(\eta)}} \right] < 1.
\end{align*}
\]

Thus, (13) is approximately controllable on \((0, 1]\).

5. **Remark**

In this paper, we discuss the approximate boundary controllability of Hilfer fractional neutral stochastic differential inclusions with fBm and Clarke’s subdifferential. The primary results were obtained by using fractional calculus, stochastic analysis theory, and the fixed point theorem. Then, the proposed systems can be extended with impulsive effects and nonlocal conditions.

6. **Conclusion**

In this paper, we investigated the approximate boundary controllability of Hilfer fractional neutral stochastic differential inclusions with fBm and Clarke’s subdifferential. Initially, we worked with stochastic analysis, nonsmooth analysis, semigroup theory, and the fixed point theorem of multivalued mappings to show that there is a mild solution to (1). Then, we offered a sufficient condition for the approximate boundary controllability of Hilfer fractional neutral stochastic differential inclusions with fBm and Clarke’s subdifferential. The primary findings were finally shown using an example. In the future, we will use a fixed point technique to examine the approximate boundary controllability of
Hilfer fractional neutral stochastic integrodifferential equations with fractional Brownian motion and impulses.

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**Data availability**

In order to make their conclusions for this study, the authors did not use any scientific data.

**Conflict of interest**

There is no conflict of interest among the authors.

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