Contemporary Mathematics
http://ojs.wiserpub.com/index.php/CM/

Research Article

Hilfer Fractional Neutral Stochastic Differential Inclusions with Clarke's Subdifferential Type and fBm: Approximate Boundary Controllability

K. Nandhaprasadh, R. Udhayakumar*

Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, Vellore, Tamil Nadu, India
E-mail: udhayakumar.r@vit.ac.in

Received: 28 August 2023; **Revised:** 22 January 2024; **Accepted:** 28 February 2024

Abstract: In this paper, the approximate boundary controllability of Hilfer fractional neutral stochastic differential inclusions with fractional Brownian motion (fBm) and Clarke's subdifferential in Hilbert space is discussed. The existence of a mild solution of Hilfer fractional neutral stochastic differential inclusions with fractional Brownian motion and Clarke's subdifferential is proved by using fractional calculus, compact semigroups, the fixed point theorem, stochastic analysis, and multivalued maps. The required conditions for the approximate boundary controllability of this system are defined according to a corresponding linear system that is approximately controllable. To demonstrate how our primary findings may be used, a final example is provided.

Keywords: approximate controllability, Hilfer fractional stochastic differential inclusions, fractional Brownian motion, neutral system

MSC: 93B05, 26A33, 60G22, 34A08

1. Introduction

Recent research has shown that fractional differential equations or inclusions with fractional order are effective modeling tools for a wide range of phenomena in physics, economics, engineering, and so on. In recent years, there has been a significant development in ordinary and partial differential equations involving fractional derivatives. We consult the monographs for further information [1-5]. The Riemann-Liouville fractional derivative and the Caputo fractional derivative are both parts of the Hilfer fractional derivative, which is a general fractional derivative that Hilfer introduced [6]. The subject of fractional differential equations and inclusions has been extensively covered in publications; for example, Gu and Trujillo [7] established the existence of a mild solution for the evolution equation with Hilfer fractional derivative. The approximate controllability of Hilfer fractional differential inclusions with nonlocal conditions was explored by Yang and Wang [8]. Varun Bose and Udhayakumar [9] discussed the existence of Hilfer fractional differential inclusions with almost sectorial operators.

One of the fundamental concepts in the study of mathematical control theory is controllability. It is a characteristic of dynamical systems and has special significance in control theory. Controllability, stability, and stabilizability

DOI: https://doi.org/10.37256/cm.5120243580

of deterministic and stochastic control systems have a number of significant connections. Any control system is considered to be controllable because it is probable to use the set of admissible controls to steer the system from an arbitrarily chosen starting point to an equally chosen ending point, where the starting point and the ending point may differ over the whole space. Ravikumar et al. [10] discussed the null controllability of nonlocal Sobolev-type Hilfer fractional stochastic differential system driven by fractional Brownian motion and Poisson jumps. Sivasankar et al. [11] investigated the nonlocal controllability of Hilfer fractional stochastic differential equations via almost sectorial operators. For further information on this topic, we refer to [12-15] and references therein.

Since they appear in a broad variety of issues in applied mathematics and biological models, including electronics, fluid dynamics, and chemical kinetics, neutral functional differential systems have attracted a lot of attention lately. Over the last decade, a significant number of academics have produced neutral fractional differential systems with or without delays, utilizing a variety of fixed-point procedures, mild solutions, noncompactness measures, and nonlocal conditions. For more details, we may refer to [16-20]. Many researchers have extensively studied the existence of mild solutions for neutral stochastic differential systems in [21-24]. The attention of several authors has been drawn to stochastic differential equations and inclusions driven by fractional Brownian motion (fBm). There have been significant developments about the existence, uniqueness, and controllability of the solution; for example, the existence of neutral stochastic functional differential equations driven by a fBm was examined by Boufoussi and Hajji [25]. Introducing impulsive stochastic functional differential inclusions driven by a fBm with infinite delay was done by Boudaoui et al. [26]. The best way to combine fractional Brownian motion and non-instantaneous impulsive fractional stochastic differential inclusion was explored by Balasubramaniam et al. in their study [27]. Ahmed et al. [28] investigated the approximate controllability of nonlocal Sobolev-type neutral fractional stochastic differential equations with fractional Brownian motion and Clarke subdifferential. Mourad et al. [29] investigated stochastic fractional perturbed control systems with fractional Brownian motion and Sobolev stochastic nonlocal conditions. Mourad [30] established the approximate controllability of fractional neutral stochastic evolution equations in Hilbert spaces with fractional Brownian motion.

Clarke's subdifferential arises from applied fields such as filtration in porous media and thermo-viscoelasticity and has fascinating applications in non-smooth analysis and optimization [31]. Recent years have seen a rise in research activity in the study of control issues with Clarke's subdifferential; for example, Ahmed et al. [32] investigated fractional stochastic evolution inclusions with control on the boundary. Kavitha et al. [33] discussed the results on the approximate controllability of Sobolev-type fractional neutral differential inclusions of the Clarke subdifferential type. Sivasankar et al. [34] studied the optimal control problems for Hilfer fractional neutral stochastic evolution hemivariational inequalities.

Only a few authors have investigated the approximate boundary controllability; for example, Wang [35] investigated the approximate boundary controllability for semilinear delay differential equations. Olive [36] introduced the approximate boundary controllability of some linear parabolic systems. Ahmed et al. [37] established the approximate boundary controllability of nonlocal Hilfer fractional stochastic differential systems with fBm and Poisson jumps. Ahmed [38] studied the approximate controllability of neutral fractional stochastic differential systems with control on the boundary. However, to the best of our knowledge, so far, no work has been reported in the literature about the approximate boundary controllability of Hilfer fractional neutral stochastic differential inclusions with fBm and Clarke's subdifferential.

Inspired by the above-mentioned works, this paper aims to fill this gap. The purpose of this paper is to show the existence of solutions and the approximate boundary controllability of Hilfer fractional neutral stochastic differential inclusions with fBm and Clarke's subdifferential of the form

$$\begin{cases} D_{0+}^{\ell,\mathfrak{y}}\left[m(t)+f\left(t,m(t)\right)\right] \in F\left[m(t)+f\left(t,m(t)\right)\right]+\sigma\left(t,m(t)\right)+\wp(t,m(t))\dfrac{dw(t)}{dt} \\ \qquad\qquad +g\left(t,m(t)\right)\dfrac{dB^{H}(t)}{dt}+\partial\mathfrak{B}\left(t,m(t)\right),\ t\in\mathfrak{T}=(0,\xi], \\ \delta m(t)=\varrho_1 V(t),\ t\in\mathfrak{T}, \\ I_{0+}^{(1-\ell)(1-\mathfrak{y})}m(0)=m_0, \end{cases} \tag{1}$$

where $D_{0+}^{\ell,\eta}$ is the Hilfer fractional derivative (HFD) of order $\ell \in [0, 1]$ and type $\eta \in \left(\frac{1}{2}, 1\right)$, F be bounded linear operator and $\delta : \Lambda \to \mathfrak{D}$ be a linear operator, where $\mathfrak{D}$ be separable Hilbert space. $\varrho_1 : Y \to \Lambda$ denotes a bounded linear operator, where Y and Λ are Hilbert spaces. The state $m(\cdot)$ takes the value in Λ. Let $A : \Lambda \to \Lambda$ be a linear operator defined by $\mathrm{Dom}(A) = \{m \in \mathrm{Dom}(F) : \delta m = 0\}$, $Am = Fm$, for $m \in \mathrm{Dom}(A)$.

Let $\{w(t)\}_{t \geq 0}$ be a Wiener process that has a finite trace nuclear covariance operator $\Theta \geq 0$ specified on a $(\Omega, S, \{S_t\}_{t\geq 0}, P)$ with values in Hilbert space $\mathfrak{D}$. $\{B^H(t)\}_{t\geq 0}$ is a fBm with Hurst parameter $H \in \left(\frac{1}{2}, 1\right)$ defined on a $(\Omega, S, \{S_t\}_{t\geq 0}, P)$ with values in Hilbert space Λ. Also, $\|\cdot\|$ for $L(\mathcal{D}, \Lambda)$, where $L(\mathcal{D}, \Lambda)$ is the space of all bounded linear operators. $\partial\mathfrak{B}(t, m(t))$ denotes the Clarke's subdifferential of $\mathfrak{B}(t, m(t))$. $V(\cdot)$ is the control function in $L^2(\mathfrak{T}, Y)$, the Hilbert space of admissible control functions on Y. $\sigma : \mathfrak{T} \times \Lambda \to 2^{\Lambda}$ is a nonempty, bounded, closed, and convex (BCC) multivalued map. Let $L_\Theta(\mathcal{D}, \Lambda)$ be the space of all Θ-Hilbert Schmidt operators from $\mathcal{D}$ to Λ. The nonlinear operators $f : \mathfrak{T} \times \Lambda \to \Lambda$, $\wp : \mathfrak{T} \times \Lambda \to L_\Theta(\mathcal{D}, \Lambda)$ and $g : \mathfrak{T} \times \Lambda \to L_2^0(Y, \Lambda)$ are given.

1.1 *Novelties of the work*

The contributions of this paper exist in the following aspects:

• Hilfer fractional neutral stochastic differential inclusions with fBm and Clarke's subdifferential are introduced.

• The primary outcomes for the systems (1) are derived by applying fractional calculus, compact semigroups, the fixed point theorem, stochastic analysis and multivalued maps.

• Approximate boundary controllability for Hilfer fractional neutral stochastic differential inclusions with fBm and Clarke's subdifferential is an unexplored topic in the literature, and this is an additional motivation for writing this manuscript.

• The main findings are demonstrated using an example.

1.2 *Structure of the paper*

The following describes how the paper is organized: We present some definitions, lemmas, and theorems in Section 2 that are helpful in proving the major results. The sufficient condition to demonstrate approximate boundary controllability for the system (1) is examined in Section 3. In Section 4, we present an example to verify the results of the theoretical work.

2. Preliminaries

The lemmas, theorems, and definitions required to support the primary results are presented in this part of the paper.

Definition 2.1 [2, 6] The HFD of order $0 \leq \ell \leq 1$ and $0 < \eta < 1$ is characterized as

$$D_{0+}^{\ell,\eta} G(t) = I_{0+}^{\ell(1-\eta)} \frac{d}{dt} I_{0+}^{(1-\ell)(1-\eta)} G(t),$$

where

$$I^{\eta} G(t) = \frac{1}{\Gamma(\eta)} \int_0^t \frac{G(s)}{(t-s)^{1-\eta}} ds, \ t > 0, \ \eta > 0.$$

Assuming that $(\Omega, S, \{S_t\}_{t\geq 0}, P)$ indicates a complete probability space equipped with a normal filtering $\{S_t\}$, $t \in (0, \xi]$, because S_t corresponds to the σ-algebra produced by $\{w(s), B^H(s), s \in (0, \xi]\}$ and all P-null sets. Let $\Theta \in (Y, Y)$ be an operator denoting $\Theta e_k = \lambda_k e_k$ along with finite trace $Tr(\Theta) = \sum_{k=1}^{\infty} \lambda_k < \infty$, $\lambda_k \geq 0$ ($k = 1, 2, \cdots$) and $\{e_k\}$ ($k = 1, 2, \cdots$) is a complete orthonormal basis in Y.

The fBm on Y in infinite dimensions with covariance Θ is defined as

$$B^H(t) = B_\Theta^H(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k} e_k \beta_k^H(t),$$

where $\beta_k^H(t)$ are real, independent of fBm. In order to define the Wiener integrals with respect to the Θ-fBm, we introduce the space of all Θ-Hilbert Schmidt operators $\varphi : Y \to \Lambda$ as $L_2^0 := L_2^0(Y, \Lambda)$. We remember that $\varphi \in L(Y, \Lambda)$ is known as Θ-Hilbert Schmidt operators, if

$$\|\varphi\|_{L_2^0}^2 := \sum_{k=1}^{\infty} \left\| \sqrt{\lambda}_k \varphi e_k \right\|^2 < \infty,$$

and that the space L_2^0 equipped with the inner product $\langle \vartheta, \varphi \rangle_{L_2^0} = \sum_{k=1}^{\infty} \langle \vartheta e_k, \varphi e_k \rangle$ is a separable Hilbert space.

Let $\varphi(s)$; $s \in (0, \xi]$ be a function with values in $L_0^2(Y, \Lambda)$, the Wiener integral of φ with respect to B^H is defined by

$$\int_0^t \varphi(s) dB^H(s) = \sum_{k=1}^{\infty} \int_0^t \sqrt{\lambda} \varphi(s) e_k d\beta_k^H = \sum_{k=1}^{\infty} \int_0^t \sqrt{\lambda} K^*(\varphi e_k)(s) d\beta_k(s), \tag{2}$$

where β_k denotes the standard Brownian motion.

Lemma 2.2 [25] If $\varphi : [0, \xi] \to L_2^0(Y, \Lambda)$ satisfies $\int_0^\xi \|\varphi(s)\|_{L_2^0}^2 < \infty$, then the above sum in (2) is well defined as a Λ-valued random variable, and we have

$$E \left\| \int_0^t \varphi(s) dB^H(s) \right\|^2 \leq 2Ht^{2H-1} \int_0^t \|\varphi(s)\|_{L_2^0}^2 ds.$$

Definition 2.3 [31] Let $\mathfrak{B} : \mathcal{O} \to \mathfrak{R}$, where $\mathcal{O}$ is a Banach space and $\mathcal{O}^*$ is the dual space of $\mathcal{O}$. The Clarke's generalized directional derivative of $\mathfrak{B}$ at $\iota \in \mathcal{O}$ in that direction $\zeta \in \mathfrak{B}$ is characterized by

$$\mathfrak{B}^0(\iota; \zeta) = \lim_{\hbar \to 0^+} \sup_{\mathfrak{o} \to \iota} \frac{\mathfrak{B}(\mathfrak{o} + \hbar\zeta) - \mathfrak{B}(\mathfrak{o})}{\hbar},$$

and

$$\partial \mathfrak{B}(\iota) = \left\{ \iota^* \in \mathcal{O}^* : \mathfrak{B}^0(\iota; \zeta) \geq \langle \iota^*; \zeta \rangle, \text{ for all } \zeta \in \mathcal{O} \right\},$$

where $\partial\mathfrak{B}(\iota)$ is the Clarke's generalized gradient of $\mathfrak{B}$ at $\iota \in \mathcal{O}$.

Here, $\mathfrak{S} : L_S^2(\mathfrak{T}, \Lambda) \to 2^{L_S^2(\mathfrak{T}, \Lambda)}$ is defined as:

$$\mathfrak{S}(m) = \left\{ \mathcal{F} \in L_S^2(\mathfrak{T}, \Lambda) : \mathcal{F}(\mathrm{t}) \in \partial\mathfrak{B}(t, m(t)) \text{ a.e. } t \in \mathfrak{T} \text{ for } m \in L_S^2(\mathfrak{T}, \Lambda) \right\}.$$

The Banach space of all continuous functions m from $\mathfrak{T}$ into $L^2(\Omega, S, P, \Lambda)$ is denoted by $\lambda := t(\mathfrak{T}, L^2(\Omega, S, P, \Lambda))$

with $\|m\|_{\lambda}=\left(\sup_{t\in\mathfrak{T}} E\left\|t^{(1-\ell)(1-\mathfrak{y})}m(t)\right\|^2\right)^{\left(\frac{1}{2}\right)}$. Within this paper, let $\mathfrak{A}_{\mathfrak{r}}=\{m\in\lambda:\|m\|_{\lambda}^2\leq\mathfrak{r}\}$, where $\mathfrak{r}>0$.

We require the following hypotheses to be verified in the findings:

(**A1**) $\mathrm{Dom}(F)\subset\mathrm{Dom}(\delta)$ and the restriction of δ to $\mathrm{Dom}(F)$ are continuous with regard to the graph norm of $\mathrm{Dom}(F)$.

(**A2**) There exists a linear operator $\varrho: Y\to\Lambda$ such that for all $V\in Y$ we have $\varrho V\in\mathrm{Dom}(F)$, $\delta(\varrho V)=\varrho_1 V$ and $E\|\varrho V\|^2\leq C_1\|\varrho_1 V\|$, where C_1 is a constant.

(**A3**) A is the infinitesimal generator of the bounded operator $\{\aleph(t),\, t\geq 0\}$ in Λ, and there exists a constant $\Pi>0$ such that $\sup_{t\in\mathfrak{T}}\|\aleph(t)\|\leq\Pi$.

(**A4**) For all $t\in(0,\xi]$ and $V\in Y$, $\aleph(t)\varrho V\in\mathrm{Dom}(A)$. Additionally, there exists a constant $\Pi_1>0$ such that $\|A\aleph(t)\|\leq\Pi_1$.

(**A5**) If $f:\mathfrak{T}\times\Lambda\to\Lambda$ is a continuous function, then there exist constants $C_1, C_2>0$ such that

$$E\left\|f(t,m_1)-f(t,m_2)\right\|^2\leq C_1E\left\|m_1-m_2\right\|^2,$$

for $t\in\mathfrak{T}$, $m_1, m_2\in\Lambda$, and the inequality

$$E\left\|f(t,m)\right\|^2\leq C_2\left(1+E\left\|m\right\|^2\right),$$

for all $(t,m)\in\mathfrak{T}\times\Lambda$.

(**A6**) If $\sigma:\mathfrak{T}\times\Lambda\to 2^{\Lambda}$ is locally Lipschitz continuous (LLC), for all $t\in\mathfrak{T}$, m, m_1, $m_2\in\Lambda$, then there exists a constant $C_3>0$ such that

$$E\left\|\sigma(t,m_1)-\sigma(t,m_2)\right\|^2\leq C_3E\left\|m_1-m_2\right\|^2,\;\; E\left\|\sigma(t,m)\right\|^2\leq C_3\left(1+E\left\|m\right\|^2\right).$$

(**A7**) If $\wp:\mathfrak{T}\times\Lambda\to L_{\Theta}(\mathcal{D},\Lambda)$ is LLC, for all $t\in\mathfrak{T}$, m, m_1, $m_2\in\Lambda$, then there exists a constant $C_4>0$ such that

$$E\left\|\wp(t,m_1)-\wp(t,m_2)\right\|^2\leq C_4E\left\|m_1-m_2\right\|^2,\;\; E\left\|\wp(t,m)\right\|^2\leq C_4\left(1+E\left\|m\right\|^2\right).$$

(**A8**) If $g:\mathfrak{T}\times\Lambda\to L_2^0(Y,\Lambda)$ is LLC, for all $t\in\mathfrak{T}$, m, m_1, $m_2\in\Lambda$, then there exists a constant $C_5>0$ such that

$$E\left\|g(t,m_1)-g(t,m_2)\right\|^2\leq C_5E\left\|m_1-m_2\right\|^2,\;\; E\left\|g(t,m)\right\|^2\leq C_5\left(1+E\left\|m\right\|^2\right).$$

(**A9**) The following conditions are fulfilled by the function $\mathfrak{B}:\mathfrak{T}\times\Lambda\to\mathfrak{R}$:

(*i*) $\mathfrak{B}(\cdot,m):\mathfrak{T}\to\mathfrak{R}$ be measurable for all $m\in\Lambda$,

(*ii*) $\mathfrak{B}(t,\cdot):\Lambda\to\mathfrak{R}$ be LLC for a.e. $t\in\mathfrak{T}$,

(*iii*) there exists $\vartheta\in L^1(\mathfrak{T},\mathfrak{R}^+)$ and a constant $C_6>0$ that satisfies

$$E\left\|\partial\mathfrak{B}(t,m)\right\|^2=\sup\left\{E\left\|\mathcal{F}(t)\right\|^2:\mathcal{F}(t)\in\partial\mathfrak{B}(t,m)\}\leq\vartheta(t)+C_6E\left\|m\right\|^2\right\},$$

for all $m\in\Lambda$ a.e. $t\in\mathfrak{T}$ and $m\in\Lambda$.

Let $m(t)$ be the solution of (1). Then, we define $\mathfrak{x}(t)=m(t)-\varrho V(t)$. From our hypotheses, it is obvious that $\mathfrak{x}(t)\in\mathrm{Dom}(A)$. As a result, (1) may be represented in terms of A and ϱ as follows:

$$
\begin{cases}
D_{0+}^{\ell,\mathfrak{y}}\left[\mathfrak{x}(t)+f\left(t,m(t)\right)\right]\in A\left[\mathfrak{x}(t)+f\left(t,m(t)\right)\right]+F\varrho V(t)-\varrho D_{0+}^{\ell,\mathfrak{y}}V(t)+\sigma\left(t,m(t)\right)+\wp\left(t,m(t)\right)\dfrac{dw(t)}{dt}\\
\qquad +g\left(t,m(t)\right)\dfrac{dB^{H}(t)}{dt}+\partial\mathfrak{B}\left(t,m(t)\right),\quad t\in\mathfrak{T}=(0,\xi],\\
I_{0+}^{(1-\ell)(1-\mathfrak{y})}\left[\mathfrak{x}(0)+\varrho V(0)\right]=I_{0+}^{(1-\ell)(1-\mathfrak{y})}m(0)=m_{0}.
\end{cases}
$$

Hence, the integral inclusion of (1) is provided by

$$
m(t)+f\left(t,m(t)\right)\in\frac{m_{0}+f\left(0,m(0)\right)}{\Gamma(\ell+\mathfrak{y}-\ell\mathfrak{y})}t^{(\ell-1)(1-\mathfrak{y})}+\frac{1}{\Gamma(\mathfrak{y})}\int_{0}^{t}(t-s)^{\mathfrak{y}-1}A\left[m(s)+f\left(s,m(s)\right)\right]ds
$$

$$
+\frac{1}{\Gamma(\mathfrak{y})}\int_{0}^{t}(t-s)^{\mathfrak{y}-1}[F-A]\varrho V(s)ds+\frac{1}{\Gamma(\mathfrak{y})}\int_{0}^{t}(t-s)^{\mathfrak{y}-1}\sigma\left(s,m(s)\right)ds+\frac{1}{\Gamma(\mathfrak{y})}\int_{0}^{t}(t-s)^{\mathfrak{y}-1}\wp\left(s,m(s)\right)dw(s)
$$

$$
+\frac{1}{\Gamma(\mathfrak{y})}\int_{0}^{t}(t-s)^{\mathfrak{y}-1}g\left(s,m(s)\right)dB^{H}(s)+\frac{1}{\Gamma(\mathfrak{y})}\int_{0}^{t}(t-s)^{\mathfrak{y}-1}\partial\mathfrak{B}\left(s,m(s)\right)ds. \tag{3}
$$

Lemma 2.4 [32] An S_t-adapted stochastic process $m(\cdot)\in L_S^2(\mathfrak{T},\Lambda)$ is said to be a mild solution of the control system (1), and provided that $I_{0+}^{(1-\ell)(1-\mathfrak{y})}m(0)=m_0$, then there exists $\mathcal{F}\in L_S^2(\mathfrak{T},\Lambda)$ such that $\mathcal{F}(t)\in\partial\mathfrak{B}(t,m(t))$ a.e. $t\in\mathfrak{T}$ and

$$
m(t)=\aleph_{\ell,\mathfrak{y}}(t)\left[m_{0}+f\left(0,m(0)\right)\right]-f\left(t,m(t)\right)+\int_{0}^{t}\left[P_{\mathfrak{y}}(t-s)F-AP_{\mathfrak{y}}(t-s)\right]\varrho V(s)ds
$$

$$
+\int_{0}^{t}P_{\mathfrak{y}}(t-s)\sigma\left(s,m(s)\right)ds+\int_{0}^{t}P_{\mathfrak{y}}(t-s)\mathcal{F}(s)ds
$$

$$
+\int_{0}^{t}P_{\mathfrak{y}}(t-s)\wp\left(s,m(s)\right)dw(s)+\int_{0}^{t}P_{\mathfrak{y}}(t-s)g\left(s,m(s)\right)dB^{H}(s),\ t\in\mathfrak{T}, \tag{4}
$$

where

$$
\aleph_{\ell,\mathfrak{y}}(t)=I_{0+}^{\ell(1-\mathfrak{y})}P_{\mathfrak{y}}(t),\ P_{\mathfrak{y}}(t)=t^{\mathfrak{y}-1}T_{\mathfrak{y}}(t),\ T_{\mathfrak{y}}(t)=\int_{0}^{\infty}\mathfrak{y}v\Psi_{\mathfrak{y}}(v)\aleph(t^{\mathfrak{y}}v)dv,
$$

with

$$
\mathcal{W}_{\mathfrak{y}}(v)=\sum_{k=1}^{\infty}\frac{(-v)^{k-1}}{(k-1)!\Gamma(1-k\mathfrak{y})},\ 0<\mathfrak{y}<1,\ v\in(0,\infty),
$$

the Wright-type function that meets the following inequality

$$
\int_{0}^{\infty}v^{\rho}\mathcal{W}_{\mathfrak{y}}(v)dv=\frac{\Gamma(1+\rho)}{\Gamma(1+\mathfrak{y}\rho)},\ v>0.
$$

Lemma 2.5 [7] The operators $\aleph_{\ell,\eta}$ and P_η possess the following characteristics:
(i) $\{P_\eta(t) : t > 0\}$ is continuous in the uniform operator topology.
(ii) $\aleph_{\ell,\eta}(t)$ and $P_\eta(t)$ are linear bounded operators,

$$\left\|P_\eta(t)m\right\| \le \frac{\Pi t^{\eta-1}}{\Gamma(\eta)}\|m\|, \quad \left\|\aleph_{\ell,\eta}(t)m\right\| \le \frac{\Pi t^{(\ell-1)(1-\eta)}}{\Gamma\left(\ell(1-\eta)+\eta\right)}\|m\|.$$

(iii) $\{P_\eta(t), \aleph_{\ell,\eta}(t)\}_{t>0}$ are strongly continuous.

Lemma 2.6 [37] If the assumption (**A4**) is satisfied, then $\left\|AP_\eta(t)m\right\| \le \dfrac{\Pi_1 t^{\eta-1}}{\Gamma(\eta)}\|m\|$.

Lemma 2.7 [39] If (**A9**) is satisfied, then the set $\mathfrak{S}(m)$ has nonempty, convex, and weakly compact values for every $m \in L_S^2(\mathfrak{T}, \Lambda)$.

Lemma 2.8 [39] The operator $\mathfrak{S}$ verifies: if $m_k \to m$ in $L_S^2(\mathfrak{T}, \Lambda)$, $\beta_k \to \beta$ weakly in $L_S^2(\mathfrak{T}, \Lambda)$, and $\beta_k \in \mathfrak{S}(m_k)$, then $\beta \in \mathfrak{S}(m)$ given that (**A9**) is fulfilled.

Theorem 2.9 [40] Let $\mathcal{D}$ be a locally convex Banach space and $\mathfrak{P}_\varepsilon : \mathcal{D} \to 2^{\mathcal{D}}$ be a compact convex valued (CCV), upper semicontinuous multivalued maps such that there exists a closed neighborhood $\mathfrak{L}$ of 0 for which $\mathfrak{P}_\varepsilon(\mathfrak{L})$ is relatively compact set. If $\Psi = \{m \in \mathcal{D} : \zeta m \in \mathfrak{P}_\varepsilon(m), \zeta > 1\}$ is bounded, then $\mathfrak{P}_\varepsilon$ has a fixed point.

3. Main results

To investigate the approximate boundary controllability for (1), we consider the linear stochastic system with HFD and control on the boundary

$$\begin{cases} D_{0+}^{\ell,\eta} m(t) \in Fm(t) + \sigma(t) + \wp(t)\dfrac{dw(t)}{dt}, \ t \in \mathfrak{T}, \\ \delta m(t) = \varrho_1 V(t), \ t \in \mathfrak{T}, \\ I_{0+}^{(1-\ell)(1-\eta)} m(0) = m_0. \end{cases} \tag{5}$$

We introduce the operators associated with (5) as

$$\Gamma_0^\xi = \int_0^\xi (\xi-s)^{\eta-1}\left[T_\eta(\xi-s)F - AT_\eta(\xi-s)\right]\varrho\varrho^*\left[T_\eta(\xi-s)F - AT_\eta(\xi-s)\right]^* ds,$$

and

$$N(\zeta, \Gamma_0^\xi) = (\zeta I + \Gamma_0^\xi)^{-1}, \ \zeta > 0,$$

where ϱ^* and $[T_\eta(\xi-s)F - AT_\eta(\xi-s)]^*$ stands for the adjoint of ϱ and $[T_\eta(\xi-s)F - AT_\eta(\xi-s)]$, respectively.

Assume that $m(\xi; m_0, V)$ represents the state value of (1) at the terminal state ξ, which corresponds to the control V and the initial value m_0. Indicate by $N(\xi, m_0) = \{m(\xi; m_0, V) : V \in L^2(\mathfrak{T}, Y)\}$ is the reachable set of the system (1) at terminal state ξ and its closure in Λ is marked by $\overline{N(\xi, m_0)}$.

Definition 3.1 [41] The control system (1) is approximately controllable on $\mathfrak{T}$ if $\overline{N(\xi, m_0)} = L^2(\Omega, \Lambda)$.

Lemma 3.2 [41] The fractional linear control system (5) is approximately controllable on $\mathfrak{T}$ if and only if $\zeta(\zeta I + \Gamma_0^\xi)^{-1} \to 0$ as $\zeta \to 0^+$.

Lemma 3.3 For any $\overline{m}_\xi \in L^2(\Omega, \Lambda)$ there exists $\tilde{\chi} \in L^2(\Omega; L^2(\mathfrak{T}, L_\Theta))$ and $\tilde{\varphi} \in L^2(\Omega; L^2(\mathfrak{T}, L_2^0))$ such that

$$\overline{m}_\xi = E\overline{m}_\xi + \int_0^\xi \tilde{\chi}(s)dw(s) + \int_0^\xi \tilde{\varphi}(s)dB^H(s).$$

Now, for any $m_\xi \in L^2(\Omega, \Lambda)$, as described above, the control function provides

$$V(t) = \varrho^*\left[T_\mathfrak{y}(\xi - s)F - AT_\mathfrak{y}(\xi - s)\right]^* (\zeta I + \Gamma_0^\xi)^{-1}\Big\{E\overline{m}_\xi - \aleph_{\ell,\mathfrak{y}}(\xi)\left[m_0 + f\left(0, m(0)\right)\right]$$

$$+ f\left(\xi, m(\xi)\right) - \int_0^\xi P_\mathfrak{y}(\xi - s)\sigma\left(s, m(s)\right)ds - \int_0^\xi P_\mathfrak{y}(\xi - s)\mathcal{F}(s)ds$$

$$-\int_0^\xi P_\mathfrak{y}(\xi - s)\wp\left(s, m(s)\right)dw(s) - \int_0^\xi P_\mathfrak{y}(\xi - s)g\left(s, m(s)\right)dB^H(s)$$

$$+\int_0^\xi \tilde{\chi}(s)dw(s) + \int_0^\xi \tilde{\varphi}(s)dB^H(s)\Big\}, \quad t \in \mathfrak{T}.$$

Theorem 3.4 Assume that (**A1**)-(**A9**) are satisfied, then the control system (1) provides a mild solution on $\mathfrak{T}$.

$$\mathfrak{F}_2 = \left\{49C_2 + \frac{49\Pi^2\xi^{2\mathfrak{y}-1}}{(2\mathfrak{y}-1)\Gamma^2(\mathfrak{y})}\left[\left(C_3 + Tr(\Theta)C_4 + 2H\xi^{2H-1}C_5\right) + C_6\xi\right]\right\}$$

$$\times\left\{1 + \frac{49\|\varrho\|^2\|\varrho^*\|^2\xi^{2\mathfrak{y}-1}\left[\|F\|^2\Pi^2 + \Pi_1^2\right]^2}{\zeta^2(2\mathfrak{y}-1)\Gamma^4(\mathfrak{y})}\right\} < 1.$$

Proof. Consider the map $\mathfrak{P}_\varepsilon : \lambda \to 2^\lambda$ as follows

$$\mathfrak{P}_\varepsilon(m) = \Big\{\mathfrak{A} \in \lambda : \mathfrak{A}(t) = \aleph_{\ell,\mathfrak{y}}(t)\left[m_0 + f\left(0, m(0)\right)\right] - f\left(t, m(t)\right)$$

$$+\int_0^t\left[P_\mathfrak{y}(t-s)F - AP_\varpi(t-s)\right]\varrho V(s)ds + \int_0^t P_\mathfrak{y}(t-s)\sigma\left(s, m(s)\right)ds$$

$$+\int_0^t P_\mathfrak{y}(t-s)\mathcal{F}(s)ds + \int_0^t P_\mathfrak{y}(t-s)\wp\left(s, m(s)\right)dw(s)$$

$$+\int_0^t P_\mathfrak{y}(t-s)g\left(s, m(s)\right)dB^H(s)\Big\}.$$

For $t \in \mathfrak{T}$, and from (**A1**)-(**A9**), we have

$$E\|V(t)\|^2$$

$$\leq 49E\left\|\varrho^*\left[T_\mathfrak{y}(\xi - s)F - AT_\mathfrak{y}(\xi - s)\right]^*(\zeta I + \Gamma_0^\xi)^{-1}\left[E\overline{m}_\xi + \int_0^\xi \tilde{\chi}(s)dw(s) + \int_0^\xi \tilde{\varphi}(s)dB^H(s)\right]\right\|^2$$

$$+49E\left\|\varrho^*\left[T_{\mathfrak{y}}(\xi-s)F-AT_{\mathfrak{y}}(\xi-s)\right]^*(\zeta I+\Gamma_0^{\xi})^{-1}\aleph_{\ell,\mathfrak{y}}(\xi)\left[m_0+f\left(0,m(0)\right)\right]\right\|^2$$

$$+49E\left\|\varrho^*\left[T_{\mathfrak{y}}(\xi-s)F-AT_{\mathfrak{y}}(\xi-s)\right]^*(\zeta I+\Gamma_0^{\xi})^{-1}f\left(\xi,m(\xi)\right)\right\|^2$$

$$+49E\left\|\varrho^*\left[T_{\mathfrak{y}}(\xi-s)F-AT_{\mathfrak{y}}(\xi-s)\right]^*(\zeta I+\Gamma_0^{\xi})^{-1}\int_0^{\xi}P_{\mathfrak{y}}(\xi-s)\sigma\left(s,m(s)\right)ds\right\|^2$$

$$+49E\left\|\varrho^*\left[T_{\mathfrak{y}}(\xi-s)F-AT_{\mathfrak{y}}(\xi-s)\right]^*(\zeta I+\Gamma_0^{\xi})^{-1}\int_0^{\xi}P_{\mathfrak{y}}(\xi-s)\mathcal{F}(s)ds\right\|^2$$

$$+49E\left\|\varrho^*\left[T_{\mathfrak{y}}(\xi-s)F-AT_{\mathfrak{y}}(\xi-s)\right]^*(\zeta I+\Gamma_0^{\xi})^{-1}\int_0^{\xi}P_{\mathfrak{y}}(\xi-s)\wp\left(s,m(s)\right)dw(s)\right\|^2$$

$$+49E\left\|\varrho^*\left[T_{\mathfrak{y}}(\xi-s)F-AT_{\mathfrak{y}}(\xi-s)\right]^*(\zeta I+\Gamma_0^{\xi})^{-1}\int_0^{\xi}P_{\mathfrak{y}}(\xi-s)g\left(s,m(s)\right)dB^H(s)\right\|^2$$

$$\leq\frac{49\left\|\varrho^*\right\|^2\left[\left\|F\right\|^2\Pi^2+\Pi_1^2\right]}{\zeta^2\Gamma^2(\mathfrak{y})}\left\{E\overline{m}_{\xi}+Tr(\Theta)\int_0^{\xi}E\left\|\tilde{\chi}\right\|_{\Theta}^2ds+2H\xi^{2H-1}\int_0^{\xi}E\left\|\tilde{\varphi}\right\|_{L_2^0}^2ds\right.$$

$$+\frac{\Pi^2\xi^{2(\ell-1)(1-\mathfrak{y})}}{\Gamma^2\left(\ell(1-\mathfrak{y})+\mathfrak{y}\right)}\left[2E\left\|m_0\right\|^2+2C_2\right]+C_2(1+\mathfrak{r})$$

$$\left.+\frac{\Pi^2\xi^{2\mathfrak{y}-1}}{(2\mathfrak{y}-1)\Gamma^2(\mathfrak{y})}\left[\left(C_3+Tr(\Theta)C_4+2H\xi^{2H-1}C_5\right)(1+\mathfrak{r})+\left\|\mathcal{F}\right\|_{L^1(\mathfrak{T},\mathfrak{R}^+)}+C_6\xi\mathfrak{r}\right]\right\}.$$

In order to verify that $\mathfrak{P}_{\varepsilon}$ has a fixed point, the proof is divided into six steps.

Step 1: For all $m\in\lambda$, $\mathfrak{P}_{\varepsilon}(m)$ be nonempty, convex, and weakly compact values.

We use Lemma 2.7 to show that $\mathfrak{P}_{\varepsilon}(m)$ is nonempty and has weakly compact values. Moreover, as $\mathfrak{S}(m)$ has convex values, if k_1, $k_2\in\mathfrak{S}(m)$, then $lk_1+(1-l)k_2\in\mathfrak{S}(m)$ for every $l\in(0,1)$, which implies clearly that $\mathfrak{P}_{\varepsilon}(m)$ is convex.

Step 2: $\mathfrak{P}_{\varepsilon}$ is bounded on a subset of λ.

Clearly, $\mathfrak{A}_{\mathfrak{r}}$ is a BCC set of λ. We can prove that $E\|\mathfrak{A}(t)\|^2\leq\tau$, $\tau>0$, for all $\mathfrak{A}\in\mathfrak{P}_{\varepsilon}(m)$, $m\in\mathfrak{A}_{\mathfrak{r}}$. If $\Phi\in\mathfrak{P}_{\varepsilon}(m)$, then there exists a $\mathcal{F}\in\mathfrak{S}(m)$ in a way that

$$\Phi(t)=\aleph_{\ell,\mathfrak{y}}(t)\left[m_0+f\left(0,m(0)\right)\right]-f\left(t,m(t)\right)$$

$$+\int_0^t\left[P_{\mathfrak{y}}(t-s)F-AP_{\varpi}(t-s)\right]\varrho V(s)ds$$

$$+\int_0^tP_{\mathfrak{y}}(t-s)\sigma\left(s,m(s)\right)ds+\int_0^tP_{\mathfrak{y}}(t-s)\mathcal{F}(s)ds$$

$$+\int_0^t P_{\eta}(t-s)\wp\left(s,m(s)\right)dw(s)+\int_0^t P_{\eta}(t-s)g\left(s,m(s)\right)dB^H(s),\ \ t\in\mathfrak{T}. \tag{6}$$

Then,

$$\|\Phi(t)\|_{\lambda}^2 = 49\sup_{t\in\mathfrak{T}} t^{2(1-\ell)(1-\eta)}\Big\{E\left\|\aleph_{\ell,\eta}(t)\left[m_0+f\left(0,m(0)\right)\right]\right\|^2+E\left\|f\left(t,m(t)\right)\right\|^2$$

$$+E\left\|\int_0^t\left[P_{\eta}(t-s)F-AP_{\eta}(t-s)\right]\varrho V(s)ds\right\|^2$$

$$+E\left\|\int_0^t P_{\eta}(t-s)\sigma\left(s,m(s)\right)ds\right\|^2+E\left\|\int_0^t P_{\eta}(t-s)\mathcal{F}(s)ds\right\|^2$$

$$+E\left\|\int_0^t P_{\eta}(t-s)\wp\left(s,m(s)\right)dw(s)\right\|^2+E\left\|\int_0^t P_{\eta}(t-s)g\left(s,m(s)\right)dB^H(s)\right\|^2\Big\}$$

$$\le 49\Bigg\{\frac{\Pi^2}{\Gamma^2\left(\ell(1-\eta)+\eta\right)}\left[2E\|m_0\|^2+2C_2\right]+49\xi^{2(1-\ell)(1-\eta)}C_2(1+\mathfrak{r})$$

$$+49\frac{\Pi^2\xi^{1-2\ell(1-\eta)}}{(2\eta-1)\Gamma^2(\eta)}\left[\left(C_3+Tr(\Theta)C_4+2H\xi^{2H-1}C_5\right)(1+\mathfrak{r})+\|\mathcal{F}\|_{L^1(\mathfrak{T},\mathfrak{R}^+)}+C_6\xi\mathfrak{r}\right]\Bigg\}$$

$$\times\left\{1+\frac{49\|\varrho\|^2\left\|\varrho^*\right\|^2\xi^{2\eta-1}\left[\|F\|^2\Pi^2+\Pi_1^2\right]^2}{\zeta^2(2\eta-1)\Gamma^4(\eta)}\right\}$$

$$+\frac{2401\|\varrho\|^2\left\|\varrho^*\right\|^2\xi^{1-2\ell(1-\eta)}\left[\|F\|^2\Pi^2+\Pi_1^2\right]^2\left[E\|m_{\xi}\|^2+Tr(\Theta)\int_0^{\xi}E\|\tilde{\chi}\|_{\Theta}^2\,ds+2H\xi^{2H-1}\int_0^{\xi}E\|\tilde{\varphi}\|_{L_2^0}^2\,ds\right]}{\zeta^2(2\eta-1)\Gamma^4(\eta)}$$

$$=\tau.$$

As a result, $\mathfrak{P}_{\varepsilon}(\mathfrak{A}_{\mathfrak{r}})$ is bounded in λ.

Step 3: $\{\mathfrak{P}_{\varepsilon}(m) : m\in\mathfrak{A}_{\mathfrak{r}}\}$ is equicontinuous.

For any $m\in\mathfrak{A}_{\mathfrak{r}}$, $\Phi\in\mathfrak{P}_{\varepsilon}(m)$, there exists a $\mathcal{F}\in\mathfrak{S}(m)$ like that (6) holds for all $t\in\mathfrak{T}$. For $0<t_1<t_2<\xi$, we can obtain

$$E\left\|\Phi(t_2)-\Phi(t_1)\right\|_{\lambda}^2$$

$$\le 49E\left\|\left(\aleph_{\ell,\eta}(t_2)-\aleph_{\ell,\eta}(t_1)\right)\left[m_0+f\left(0,m(0)\right)\right]\right\|_{\lambda}^2+49E\left\|f\left(t_2,m(t_2)\right)-f\left(t_1,m(t_1)\right)\right\|_{\lambda}^2$$

$$+49E\left\|\int_0^{t_2}\left[P_\mathfrak{y}(t_2-s)F-AP_\mathfrak{y}(t_2-s)\right]\varrho V(s)ds-\int_0^{t_1}\left[P_\mathfrak{y}(t_1-s)F-AP_\mathfrak{y}(t_1-s)\right]\varrho V(s)ds\right\|_\lambda^2$$

$$+49E\left\|\int_0^{t_2}P_\mathfrak{y}(t_2-s)\sigma\left(s,m(s)\right)ds-\int_0^{t_1}P_\mathfrak{y}(t_1-s)\sigma\left(s,m(s)\right)ds\right\|_\lambda^2$$

$$+49E\left\|\int_0^{t_2}P_\mathfrak{y}(t_2-s)\mathcal{F}(s)ds-\int_0^{t_1}P_\mathfrak{y}(t_1-s)\mathcal{F}(s)ds\right\|_\lambda^2$$

$$+49E\left\|\int_0^{t_2}P_\mathfrak{y}(t_2-s)\wp\left(s,m(s)\right)dw(s)-\int_0^{t_1}P_\mathfrak{y}(t_1-s)\wp\left(s,m(s)\right)dw(s)\right\|_\lambda^2$$

$$+49E\left\|\int_0^{t_2}P_\mathfrak{y}(t_2-s)g\left(s,m(s)\right)dB^H(s)-\int_0^{t_1}P_\mathfrak{y}(t_1-s)g\left(s,m(s)\right)dB^H(s)\right\|_\lambda^2$$

$$=49E\left\|\left(\aleph_{\ell,\mathfrak{y}}(t_2)-\aleph_{\ell,\mathfrak{y}}(t_1)\right)\left[m_0+f\left(0,m(0)\right)\right]\right\|_\lambda^2+49E\left\|f\left(t_2,m(t_2)\right)-f\left(t_1,m(t_1)\right)\right\|_\lambda^2$$

$$+49E\left\|\int_{t_1}^{t_2}\left[P_\mathfrak{y}(t_2-s)F-AP_\mathfrak{y}(t_2-s)\right]\varrho V(s)ds\right.$$

$$\left.+\int_0^{t_1}\left[P_\mathfrak{y}(t_2-s)F-AP_\mathfrak{y}(t_2-s)-P_\mathfrak{y}(t_1-s)F+AP_\mathfrak{y}(t_1-s)\right]\varrho V(s)ds\right\|_\lambda^2$$

$$+49E\left\|\int_{t_1}^{t_2}P_\mathfrak{y}(t_2-s)\sigma\left(s,m(s)\right)ds+\int_0^{t_1}\left[P_\mathfrak{y}(t_2-s)-P_\mathfrak{y}(t_1-s)\right]\sigma\left(s,m(s)\right)ds\right\|_\lambda^2$$

$$+49E\left\|\int_{t_1}^{t_2}P_\mathfrak{y}(t_2-s)\mathcal{F}(s)ds+\int_0^{t_1}\left[P_\mathfrak{y}(t_2-s)-P_\mathfrak{y}(t_1-s)\right]\mathcal{F}(s)ds\right\|_\lambda^2$$

$$+49E\left\|\int_{t_1}^{t_2}P_\mathfrak{y}(t_2-s)\wp\left(s,m(s)\right)dw(s)+\int_0^{t_1}\left[P_\mathfrak{y}(t_2-s)-P_\mathfrak{y}(t_1-s)\right]\wp\left(s,m(s)\right)dw(s)\right\|_\lambda^2$$

$$+49E\left\|\int_{t_1}^{t_2}P_\mathfrak{y}(t_2-s)g\left(s,m(s)\right)dB^H(s)+\int_0^{t_1}\left[P_\mathfrak{y}(t_2-s)-P_\mathfrak{y}(t_1-s)\right]g\left(s,m(s)\right)dB^H(s)\right\|_\lambda^2.$$

From the compactness of $\aleph(t)$ $(t>0)$,

$$E\left\|\Phi(t_2)-\Phi(t_1)\right\|_\lambda^2\to 0\ \text{ as }\ t_2\to t_1.$$

Hence, $\mathfrak{P}_\varepsilon(m)(t)$ is continuous in $\mathfrak{T}$. Also, for $t_1=0$ and $t_2\in\mathfrak{T}$, we can show that $E\|\Phi(t_2)-\Phi(t_1)\|_\lambda^2\to 0$ as $t_1\to 0$. As a result, $\{\mathfrak{P}_\varepsilon(m): m\in\mathfrak{A}_\tau\}$ is equicontinuous.

Step 4: $\mathfrak{P}_\varepsilon$ is completely continuous.

We show that $\chi(t) = \{\Phi(t) : \Phi \in \mathfrak{P}_{\varepsilon}(\mathfrak{A}_r)\}$ is relatively compact in Λ for all $t \in \mathfrak{T}$, $\mathfrak{r} > 0$. Undoubtedly, $\chi(0)$ is relatively compact in $\mathfrak{A}_{\mathfrak{r}}$. Let $0 < t \leq \xi$ be fixed, $0 < \eta < t$, for $m \in \mathfrak{A}_{\mathfrak{r}}$, we define

$$\Phi^{\eta,\iota}(t) = \frac{\mathfrak{y}}{\Gamma\left(\ell(1-\mathfrak{y})\right)}\int_0^{t-\eta}\int_\iota^\infty v(t-s)^{\ell(1-\mathfrak{y})-1}s^{\mathfrak{y}-1}\Psi_{\mathfrak{y}}(v)\aleph(s^{\mathfrak{y}}v)\left[m_0 - f\left(0,m(0)\right)\right]dvds - f\left(t,m(t)\right)$$

$$+\mathfrak{y}\int_0^{t-\eta}\int_\iota^\infty v(t-s)^{\mathfrak{y}-1}\Psi_{\mathfrak{y}}(v)\left[\aleph\left((t-s)^{\mathfrak{y}}v\right)F - A\aleph\left((t-s)^{\mathfrak{y}}v\right)\right]\varrho V(s)dvds$$

$$+\mathfrak{y}\int_0^{t-\eta}\int_\iota^\infty v(t-s)^{\mathfrak{y}-1}\Psi_{\mathfrak{y}}(v)\aleph\left((t-s)^{\mathfrak{y}}v)\sigma(s,m(s)\right)dvds$$

$$+\mathfrak{y}\int_0^{t-\eta}\int_\iota^\infty v(t-s)^{\mathfrak{y}-1}\Psi_{\mathfrak{y}}(v)\aleph\left((t-s)^{\mathfrak{y}}v\right)\mathcal{F}(s)dvds$$

$$+\mathfrak{y}\int_0^{t-\eta}\int_\iota^\infty v(t-s)^{\mathfrak{y}-1}\Psi_{\mathfrak{y}}(v)\aleph\left((t-s)^{\mathfrak{y}}v\right)\wp\left(s,m(s)\right)dvdw(s)$$

$$+\mathfrak{y}\int_0^{t-\eta}\int_\iota^\infty v(t-s)^{\mathfrak{y}-1}\Psi_{\mathfrak{y}}(v)\aleph\left((t-s)^{\mathfrak{y}}v\right)g\left(s,m(s)\right)dvdB^H(s)$$

$$=\frac{\mathfrak{y}\aleph(\eta^{\mathfrak{y}}\iota)}{\Gamma\left(\ell(1-\mathfrak{y})\right)}\int_0^{t-\eta}\int_\iota^\infty v(t-s)^{\ell(1-\mathfrak{y})-1}s^{\mathfrak{y}-1}\Psi_{\mathfrak{y}}(v)\aleph\left(s^{\mathfrak{y}}v-\eta^{\mathfrak{y}}\iota\right)\left[m_0 - f\left(0,m(0)\right)\right]dvds - f\left(t,m(t)\right)$$

$$+\mathfrak{y}\aleph(\eta^{\mathfrak{y}}\iota)\int_0^{t-\eta}\int_\iota^\infty v(t-s)^{\mathfrak{y}-1}\Psi_{\mathfrak{y}}(v)\left[\aleph\left((t-s)^{\mathfrak{y}}v-\eta^{\mathfrak{y}}\iota\right)F - A\aleph\left((t-s)^{\mathfrak{y}}v-\eta^{\mathfrak{y}}\iota\right)\right]\varrho V(s)dvds$$

$$+\mathfrak{y}\aleph(\eta^{\mathfrak{y}}\iota)\int_0^{t-\eta}\int_\iota^\infty v(t-s)^{\mathfrak{y}-1}\Psi_{\mathfrak{y}}(v)\aleph\left((t-s)^{\mathfrak{y}}v-\eta^{\mathfrak{y}}\iota\right)\sigma\left(s,m(s)\right)dvds$$

$$+\mathfrak{y}\aleph(\eta^{\mathfrak{y}}\iota)\int_0^{t-\eta}\int_\iota^\infty v(t-s)^{\mathfrak{y}-1}\Psi_{\mathfrak{y}}(v)\aleph\left((t-s)^{\mathfrak{y}}v-\eta^{\mathfrak{y}}\iota\right)\mathcal{F}(s)dvds$$

$$+\mathfrak{y}\aleph(\eta^{\mathfrak{y}}\iota)\int_0^{t-\eta}\int_\iota^\infty v(t-s)^{\mathfrak{y}-1}\Psi_{\mathfrak{y}}(v)\aleph\left((t-s)^{\mathfrak{y}}v-\eta^{\mathfrak{y}}\iota\right)\wp\left(s,m(s)\right)dvdw(s)$$

$$+\mathfrak{y}\aleph(\eta^{\mathfrak{y}}\iota)\int_0^{t-\eta}\int_\iota^\infty v(t-s)^{\mathfrak{y}-1}\Psi_{\mathfrak{y}}(v)\aleph\left((t-s)^{\mathfrak{y}}v-\eta^{\mathfrak{y}}\iota\right)g\left(s,m(s)\right)dvdB^H(s).$$

Since $\aleph(\eta^{\mathfrak{y}}\iota)$, $\eta^{\mathfrak{y}}\iota > 0$ is a compact operator. Hence, $\chi^{\eta,\iota}(t) = \{\Phi^{\eta,\iota}(t) : \Phi^{\eta,\iota} \in \mathfrak{P}_{\varepsilon}(\mathfrak{A}_{\mathfrak{r}})\}$ is relatively compact in Λ. Additionally, we have

$$E\left\|\Phi(t)-\Phi^{\eta,\iota}(t)\right\|_\lambda^2 = \sup_{t\in\mathfrak{T}} t^{2(1-\ell)(1-\mathfrak{y})}E\left\|\Phi(t)-\Phi^{\eta,\iota}(t)\right\|^2$$

$$\leq \frac{49\mathfrak{y}^2}{\Gamma^2\left(\ell(1-\mathfrak{y})\right)}\sup_{t\in\mathfrak{T}} t^{2(1-\ell)(1-\mathfrak{y})}$$

$$\times E\left\|\int_0^t\int_0^\iota v(t-s)^{\ell(1-\mathfrak{y})-1}s^{\mathfrak{y}-1}\Psi_{\mathfrak{y}}(v)\aleph(s^{\mathfrak{y}}v)\left[m_0-f\left(0,m(0)\right)\right]dvds\right\|^2$$

$$+\frac{49\mathfrak{y}^2}{\Gamma^2\left(\ell(1-\mathfrak{y})\right)}\sup_{t\in\mathfrak{T}}t^{2(1-\ell)(1-\mathfrak{y})}$$

$$\times E\left\|\int_{t-\eta}^t\int_\iota^\infty v(t-s)^{\ell(1-\mathfrak{y})-1}s^{\mathfrak{y}-1}\Psi_{\mathfrak{y}}(v)\aleph(s^{\mathfrak{y}}v)\left[m_0-f\left(0,m(0)\right)\right]dvds\right\|^2$$

$$+49\mathfrak{y}^2\sup_{t\in\mathfrak{T}}t^{2(1-\ell)(1-\mathfrak{y})}$$

$$\times E\left\|\int_0^t\int_0^\iota v(t-s)^{\mathfrak{y}-1}\Psi_{\mathfrak{y}}(v)\left[\aleph\left((t-s)^{\mathfrak{y}}v\right)F-A\aleph\left((t-s)^{\mathfrak{y}}v\right)\right]\varrho V(s)dvds\right\|^2$$

$$+49\mathfrak{y}^2\sup_{t\in\mathfrak{T}}t^{2(1-\ell)(1-\mathfrak{y})}$$

$$\times E\left\|\int_{t-\eta}^t\int_\iota^\infty v(t-s)^{\mathfrak{y}-1}\Psi_{\mathfrak{y}}(v)\left[\aleph\left((t-s)^{\mathfrak{y}}v\right)F-A\aleph\left((t-s)^{\mathfrak{y}}v\right)\right]\varrho V(s)dvds\right\|^2$$

$$+49\mathfrak{y}^2\sup_{t\in\mathfrak{T}}t^{2(1-\ell)(1-\mathfrak{y})}E\left\|\int_0^t\int_0^\iota v(t-s)^{\mathfrak{y}-1}\Psi_{\mathfrak{y}}(v)\aleph\left((t-s)^{\mathfrak{y}}v\right)\sigma\left(s,m(s)\right)dvds\right\|^2$$

$$+49\mathfrak{y}^2\sup_{t\in\mathfrak{T}}t^{2(1-\ell)(1-\mathfrak{y})}E\left\|\int_{t-\eta}^t\int_\iota^\infty v(t-s)^{\mathfrak{y}-1}\Psi_{\mathfrak{y}}(v)\aleph\left((t-s)^{\mathfrak{y}}v\right)\sigma\left(s,m(s)\right)dvds\right\|^2$$

$$+49\mathfrak{y}^2\sup_{t\in\mathfrak{T}}t^{2(1-\ell)(1-\mathfrak{y})}E\left\|\int_0^t\int_0^\iota v(t-s)^{\mathfrak{y}-1}\Psi_{\mathfrak{y}}(v)\aleph\left((t-s)^{\mathfrak{y}}v\right)\mathcal{F}(s)dvds\right\|^2$$

$$+49\mathfrak{y}^2\sup_{t\in\mathfrak{T}}t^{2(1-\ell)(1-\mathfrak{y})}E\left\|\int_{t-\eta}^t\int_\iota^\infty v(t-s)^{\mathfrak{y}-1}\Psi_{\mathfrak{y}}(v)\aleph\left((t-s)^{\mathfrak{y}}v\right)\mathcal{F}(s)dvds\right\|^2$$

$$+49\mathfrak{y}^2\sup_{t\in\mathfrak{T}}t^{2(1-\ell)(1-\mathfrak{y})}E\left\|\int_0^t\int_0^\iota v(t-s)^{\mathfrak{y}-1}\Psi_{\mathfrak{y}}(v)\aleph\left((t-s)^{\mathfrak{y}}v\right)\wp\left(s,m(s)\right)dvdw(s)\right\|^2$$

$$+49\mathfrak{y}^2\sup_{t\in\mathfrak{T}}t^{2(1-\ell)(1-\mathfrak{y})}E\left\|\int_{t-\eta}^t\int_\iota^\infty v(t-s)^{\mathfrak{y}-1}\Psi_{\mathfrak{y}}(v)\aleph\left((t-s)^{\mathfrak{y}}v\right)\wp\left(s,m(s)\right)dvdw(s)\right\|^2$$

$$+49\mathfrak{y}^2\sup_{t\in\mathfrak{T}}t^{2(1-\ell)(1-\mathfrak{y})}E\left\|\int_0^t\int_0^\iota v(t-s)^{\mathfrak{y}-1}\Psi_{\mathfrak{y}}(v)\aleph\left((t-s)^{\mathfrak{y}}v\right)g\left(s,m(s)\right)dvdB^H(s)\right\|^2$$

$$+49\mathfrak{y}^2 \sup_{t\in\mathfrak{T}} t^{2(1-\ell)(1-\mathfrak{y})}$$

$$\times E\left\|\int_{t-\eta}^{t}\int_{\iota}^{\infty} v(t-s)^{\mathfrak{y}-1}\Psi_{\mathfrak{y}}(v)\aleph\left((t-s)^{\mathfrak{y}}v\right)g\left(s,m(s)\right)dvdB^{H}(s)\right\|^2 \to 0,$$

as $\eta \to 0^+$, $\iota \to 0^+$.

As a result, the set $\chi(t)$ is relatively compact in Λ. We may infer that $\mathfrak{P}_\varepsilon$ is completely continuous from the Arzela-Ascoli theorem and Step 3.

Step 5: The closed graph of $\mathfrak{P}_\varepsilon$.

Let $m_k \to m_*$ in λ, $\Phi_k \in \mathfrak{P}_\varepsilon(m_k)$ and $\Phi_k \to \Phi_*$ in λ. We infer that $\Phi_* \in \mathfrak{P}_\varepsilon(m_*)$. Clearly, $\Phi_k \in \mathfrak{P}_\varepsilon(m_k)$, there exists a $\mathcal{F}_k \in \mathfrak{S}(m_k)$ obtained that

$$\Phi_k(t) = \aleph_{\ell,\mathfrak{y}}(t)\left[m_0 + f\left(0, m_k(0)\right)\right] - f\left(t, m_k(t)\right)$$

$$+\int_0^t \left[P_{\mathfrak{y}}(t-s)F - AP_{\varpi}(t-s)\right]\varrho V(s)ds$$

$$+\int_0^t P_{\mathfrak{y}}(t-s)\sigma\left(s, m_k(s)\right)ds + \int_0^t P_{\mathfrak{y}}(t-s)\mathcal{F}_k(s)ds$$

$$+\int_0^t P_{\mathfrak{y}}(t-s)\wp\left(s, m_k(s)\right)dw(s) + \int_0^t P_{\mathfrak{y}}(t-s)g\left(s, m_k(s)\right)dB^H(s). \tag{7}$$

From (**A1**)-(**A9**), we may infer that $\{f(\cdot, m_k), \sigma(\cdot, m_k), \mathcal{F}_k, \wp(\cdot, m_k), g(\cdot, m_k)\}_{k\geq 1} \subseteq \Lambda \times \Lambda \times L_S^2(\mathfrak{T}, \Lambda) \times L_\Theta \times L_2^0$ is bounded. Thus, we obtain

$$\left(f(\cdot,m_k), \sigma(\cdot,m_k), \mathcal{F}_k, \wp(\cdot,m_k), g(\cdot,m_k)\right) \to \left(f(\cdot,m_*), \sigma(\cdot,m_*), \mathcal{F}_*, \wp(\cdot,m_*), g(\cdot,m_*)\right), \tag{8}$$

weakly in $\Lambda \times \Lambda \times L_S^2(\mathfrak{T}, \Lambda) \times L_\Theta \times L_2^0$.

From the compactness of $\aleph(t)$, (7) and (8), and we get

$$\Phi_*(t) = \aleph_{\ell,\mathfrak{y}}(t)\left[m_0 + f\left(0, m_*(0)\right)\right] - f\left(t, m_*(t)\right)$$

$$+\int_0^t \left[P_{\mathfrak{y}}(t-s)F - AP_{\varpi}(t-s)\right]\varrho V(s)ds$$

$$+\int_0^t P_{\mathfrak{y}}(t-s)\sigma\left(s, m_*(s)\right)ds + \int_0^t P_{\mathfrak{y}}(t-s)\mathcal{F}_*(s)ds$$

$$+\int_0^t P_{\mathfrak{y}}(t-s)\wp\left(s, m_*(s)\right)dw(s) + \int_0^t P_{\mathfrak{y}}(t-s)g\left(s, m_*(s)\right)dB^H(s). \tag{9}$$

Note that $\Phi_k \to \Phi_*$ in λ and $\mathcal{F}_k \in \mathfrak{S}(m_k)$. From (9) and Lemma (2.8), we obtain $\mathcal{F}_* \in \mathfrak{S}(m_*)$. Therefore, $\Phi_* \in \mathfrak{P}_\varepsilon(m_*)$, which implies that $\mathfrak{P}_\varepsilon$ has a closed graph and that $\mathfrak{P}_\varepsilon$ is a completely continuous multi-valued map with compact value. Thus, from [32], $\mathfrak{P}_\varepsilon$ is upper semicontinuous.

Step 6: A priori estimate.

From Steps 1-5, we found that $\mathfrak{P}_\varepsilon$ is CCV. In addition, the upper semicontinuous set $\mathfrak{P}_\varepsilon(\mathfrak{A}_\tau)$ is relatively compact. By Theorem (2.9), it remains to demonstrate that $\Psi = \{m \in \lambda : \zeta m \in \mathfrak{P}_\varepsilon, \zeta > 1\}$ is bounded. For all $m \in \Psi$, there exists a $\mathcal{F} \in \mathfrak{S}(m)$ such that

$$m(t) = \zeta^{-1}\aleph_{\ell,\mathfrak{y}}(t)\left[m_0 + f\left(0, m(0)\right)\right] - \zeta^{-1} f\left(t, m(t)\right)$$

$$+\zeta^{-1}\int_0^t \left[P_{\mathfrak{y}}(t-s)F - AP_{\varpi}(t-s)\right]\varrho V(s)ds$$

$$+\zeta^{-1}\int_0^t P_{\mathfrak{y}}(t-s)\sigma\left(s,(s)\right)ds + \zeta^{-1}\int_0^t P_{\mathfrak{y}}(t-s)\mathcal{F}(s)ds$$

$$+\zeta^{-1}\int_0^t P_{\mathfrak{y}}(t-s)\wp\left(s, m(s)\right)dw(s) + \zeta^{-1}\int_0^t P_{\mathfrak{y}}(t-s)g\left(s, m(s)\right)dB^H(s). \tag{10}$$

Using the hypotheses (**A1**)-(**A9**), we get

$$E\|m(t)\|^2 \le 49\left\{E\left\|\aleph_{\ell,\mathfrak{y}}(t)\left[m_0 + f\left(0, m(0)\right)\right]\right\|^2 + E\|f(t, m(t))\|^2\right.$$

$$+E\left\|\int_0^t \left[P_{\mathfrak{y}}(t-s)F - AP_{\varpi}(t-s)\right]\varrho V(s)ds\right\|^2 + E\left\|\int_0^t P_{\mathfrak{y}}(t-s)\sigma(s,(s))ds\right\|^2$$

$$+E\left\|\int_0^t P_{\mathfrak{y}}(t-s)\mathcal{F}(s)ds\right\|^2 + E\left\|\int_0^t P_{\mathfrak{y}}(t-s)\wp\left(s, m(s)\right)dw(s)\right\|^2$$

$$\left.+E\left\|\int_0^t P_{\mathfrak{y}}(t-s)g\left(s, m(s)\right)dB^H(s)\right\|^2\right\}$$

$$\le \left\{\frac{49\Pi^2\xi^{2(\ell-1)(1-\mathfrak{y})}}{\Gamma^2\left(\ell(1-\mathfrak{y})+\mathfrak{y}\right)}\left[2E\|m_0\|^2 + 2C_2\right] + 49C_2\left(1 + E\|m(t)\|^2\right)\right.$$

$$+\frac{49\Pi^2\xi^{2\mathfrak{y}-1}}{(2\mathfrak{y}-1)\Gamma^2(\mathfrak{y})}\left[\left(C_3 + Tr(\Theta)C_4 + 2H\xi^{2H-1}C_5\right)\left(1 + E\|m(t)\|^2\right) + \|\mathcal{F}\|_{L^1(\mathfrak{T},\mathfrak{R}^+)}\right.$$

$$\left.\left.+C_6\xi E\|m(t)\|^2\right]\right\} \times \left\{1 + \frac{49\|\varrho\|^2\|\varrho^*\|^2\xi^{2\mathfrak{y}-1}\left[\|F\|^2\Pi^2 + \Pi_1^2\right]^2}{\zeta^2(2\mathfrak{y}-1)\Gamma^4(\mathfrak{y})}\right\}$$

$$+\frac{2401\|\varrho\|^2\|\varrho^*\|^2\xi^{2\mathfrak{y}-1}\left[\|F\|^2\Pi^2 + \Pi_1^2\right]^2\left[E\|m_\xi\|^2 + Tr(\Theta)\int_0^\xi E\|\tilde{\chi}\|_\Theta^2 ds + 2H\xi^{2H-1}\int_0^\xi E\|\tilde{\varphi}\|_{L_2^0}^2 ds\right]}{\zeta^2(2\mathfrak{y}-1)\Gamma^4(\mathfrak{y})}$$

$$\leq \mathfrak{F}_1 + \mathfrak{F}_2 E\left\|m(t)\right\|^2, \tag{11}$$

where

$$\mathfrak{F}_1 = \left\{ \frac{49\Pi^2 \xi^{2(\ell-1)(1-\mathfrak{y})}}{\Gamma^2\left(\ell(1-\mathfrak{y})+\mathfrak{y}\right)} \left[2E\left\|m_0\right\|^2 + 2C_2 \right] + 49C_2 \right.$$

$$\left. + \frac{49\Pi^2 \xi^{2\mathfrak{y}-1}}{(2\mathfrak{y}-1)\Gamma^2(\mathfrak{y})} \left[\left(C_3 + Tr(\Theta)C_4 + 2H\xi^{2H-1}C_5 \right) + \left\|\mathcal{F}\right\|_{L^1(\mathfrak{T},\mathfrak{R}^+)} \right] \right\}$$

$$\times \left\{ 1 + \frac{49\left\|\varrho\right\|^2 \left\|\varrho^*\right\|^2 \xi^{2\mathfrak{y}-1} \left[\left\|F\right\|^2 \Pi^2 + \Pi_1^2 \right]^2}{\zeta^2 (2\mathfrak{y}-1)\Gamma^4(\mathfrak{y})} \right\}$$

$$+ \frac{2401\left\|\varrho\right\|^2 \left\|\varrho^*\right\|^2 \xi^{2\mathfrak{y}-1} \left[\left\|F\right\|^2 \Pi^2 + \Pi_1^2 \right]^2 \left[E\left\|m_\xi\right\|^2 + Tr(\Theta)\int_0^\xi E\left\|\tilde{\chi}\right\|_\Theta^2 ds + 2H\xi^{2H-1}\int_0^\xi E\left\|\tilde{\varphi}\right\|_{L_2^0}^2 ds \right]}{\zeta^2 (2\mathfrak{y}-1)\Gamma^4(\mathfrak{y})},$$

and

$$\mathfrak{F}_2 = \left\{ 49C_2 + \frac{49\Pi^2 \xi^{2\mathfrak{y}-1}}{(2\mathfrak{y}-1)\Gamma^2(\mathfrak{y})} \left[C_3 + Tr(\Theta)C_4 + 2H\xi^{2H-1}C_5 + C_6\xi \right] \right\}$$

$$\times \left\{ 1 + \frac{49\left\|\varrho\right\|^2 \left\|\varrho^*\right\|^2 \xi^{2\mathfrak{y}-1} \left[\left\|F\right\|^2 \Pi^2 + \Pi_1^2 \right]^2}{\zeta^2 (2\mathfrak{y}-1)\Gamma^4(\mathfrak{y})} \right\}.$$

Since $\mathfrak{F}_2 < 1$, from (11), we get

$$\left\|m\right\|_\lambda^2 = \sup_{t\in\mathfrak{T}} E\left\|t^{(\ell-1)(1-\mathfrak{y})} m(t)\right\|^2 \leq \mathfrak{F}_1 + \mathfrak{F}_2 \left\|m\right\|_\lambda^2.$$

Then, $\left\|m\right\|_\lambda^2 \leq \dfrac{\mathfrak{F}_1}{1-\mathfrak{F}_2}$, consequently, Ψ is bounded. By Theorem (2.9), $\mathfrak{P}_\varepsilon$ has a fixed point. As a result, the inclusion system (1) is approximately controllable on $\mathfrak{T}$. □

Theorem 3.5 Assuming that (**A1**)-(**A9**) are fulfilled. Additionally, if the functions f, σ, $\wp$, g and $\mathcal{F}$ are all uniformly bounded, then (1) is approximately controllable on $\mathfrak{T}$.

Proof. Using the stochastic Fubini theorem and m^ζ as a fixed point on $\mathfrak{P}_\varepsilon$, it is evident that

$$m^\zeta(\xi) = \overline{m}_\xi - \zeta(\zeta I + \Gamma_0^\xi)^{-1} \left\{ E\overline{m}_\xi - \aleph_{\ell,\mathfrak{y}}(\xi)\left[m_0 + f\left(0, m^\zeta(0)\right) \right] \right.$$

$$- f\left(\xi, m^\zeta(\xi)\right) - \int_0^\xi P_\mathfrak{y}(\xi-s)\sigma\left(s, m^\zeta(s)\right)ds - \int_0^\xi P_\mathfrak{y}(\xi-s)\mathcal{F}^\zeta(s)ds$$

$$-\int_0^{\xi} P_{\mathfrak{y}}(\xi-s)\wp\left(s,m^{\zeta}(s)\right)dw(s)-\int_0^{\xi} P_{\mathfrak{y}}(\xi-s)g\left(s,m^{\zeta}(s)\right)dB^{H}(s)$$

$$+\int_0^{\xi}\tilde{\chi}(s)dw(s)+\int_0^{\xi}\tilde{\varphi}(s)dB^{H}(s)\Big\}. \tag{12}$$

From the hypotheses on f, σ, $\wp$, g and $\mathcal{F}$ and the proof of the Step 6 in Theorem (3.4), we get

$$E\left\|f\left(t,m^{\zeta}(t)\right)\right\|^2+E\left\|\sigma\left(t,m^{\zeta}(t)\right)\right\|^2+E\left\|\wp\left(t,m^{\zeta}(t)\right)\right\|^2+E\left\|g\left(t,m^{\zeta}(t)\right)\right\|^2+E\left\|\mathcal{F}^{\zeta}(t)\right\|^2\le\nu(t).$$

Consequently, the sequence $\{f(\cdot, m^{\zeta}(\cdot)), \sigma(\cdot, m^{\zeta}(\cdot)), \wp(\cdot, m^{\zeta}(\cdot)), g(\cdot, m^{\zeta}(\cdot)), \mathcal{F}^{\zeta}(\cdot)\}$ be bounded in $\Lambda \times 2^{\Lambda} \times L_{\Theta}(\mathfrak{D}, \Lambda) \times L_{\Theta}(Y, \Lambda) \times L_S^2(\mathfrak{T}, \Lambda)$. Thus, there is a subsequent event, indicated by $\{f(\cdot, m^{\zeta}(\cdot)), \sigma(\cdot, m^{\zeta}(\cdot)), \wp(\cdot, m^{\zeta}(\cdot)), g(\cdot, m^{\zeta}(\cdot)), \mathcal{F}^{\zeta}(\cdot)\}$ that converges weakly to $\{f(\cdot), \sigma(\cdot), \wp(\cdot), g(\cdot), \mathcal{F}^{\zeta}\}$ in $\Lambda \times 2^{\Lambda} \times L_{\Theta}(\mathfrak{D}, \Lambda) \times L_{\Theta}(Y, \Lambda) \times L_S^2(\mathfrak{T}, \Lambda)$. From (12), we have

$$E\left\|m^{\zeta}(\xi)-\overline{m}_{\xi}\right\|^2\le 49E\left\|\zeta(\zeta I+\Gamma_0^{\xi})^{-1}\left\{E\overline{m}_{\xi}-\aleph_{\ell,\mathfrak{y}}(\xi)\left[m_0^{\zeta}+f\left(0,m^{\zeta}(0)\right)\right]\right\}\right\|^2$$

$$+49E\left\|\zeta(\zeta I+\Gamma_0^{\xi})^{-1}f\left(\xi,m^{\zeta}(\xi)\right)\right\|^2$$

$$+49E\left\|\zeta(\zeta I+\Gamma_0^{\xi})^{-1}\int_0^{\xi}P_{\mathfrak{y}}(\xi-s)\left[\sigma\left(s,m^{\zeta}(s)\right)-\sigma(s)\right]ds\right\|^2$$

$$+49E\left\|\zeta(\zeta I+\Gamma_0^{\xi})^{-1}\int_0^{\xi}P_{\mathfrak{y}}(\xi-s)\sigma(s)ds\right\|^2$$

$$+49E\left\|\zeta(\zeta I+\Gamma_0^{\xi})^{-1}\int_0^{\xi}P_{\mathfrak{y}}(\xi-s)\left[\mathcal{F}^{\zeta}(s)-\mathcal{F}(s)\right]ds\right\|^2$$

$$+49E\left\|\zeta(\zeta I+\Gamma_0^{\xi})^{-1}\int_0^{\xi}P_{\mathfrak{y}}(\xi-s)\mathcal{F}(s)ds\right\|^2$$

$$+49E\left\|\zeta(\zeta I+\Gamma_0^{\xi})^{-1}\int_0^{\xi}P_{\mathfrak{y}}(\xi-s)\left[\wp\left(s,m^{\zeta}(s)\right)-\wp(s)\right]dw(s)\right\|^2$$

$$+49E\left\|\zeta(\zeta I+\Gamma_0^{\xi})^{-1}\int_0^{\xi}P_{\mathfrak{y}}(\xi-s)\wp(s)dw(s)\right\|^2$$

$$+49E\left\|\zeta(\zeta I+\Gamma_0^{\xi})^{-1}\int_0^{\xi}P_{\mathfrak{y}}(\xi-s)\left[g\left(s,m^{\zeta}(s)\right)-g(s)\right]dB^{H}(s)\right\|^2$$

$$+49E\left\|\zeta(\zeta I+\Gamma_0^{\xi})^{-1}\int_0^{\xi}P_{\mathfrak{y}}(\xi-s)g(s)dB^{H}(s)\right\|^2$$

$$+49E\left\|\zeta(\zeta I+\Gamma_0^{\xi})^{-1}\int_0^{\xi}\tilde{\chi}(s)dw(s)\right\|^2+49E\left\|\zeta(\zeta I+\Gamma_0^{\xi})^{-1}\int_0^{\xi}\tilde{\varphi}(s)dB^H(s)\right\|^2.$$

However, in accordance with the Lemma (3.2), for all $0 \le s \le \xi$, $\zeta(\zeta I + \Gamma_0^{\xi})^{-1} \to 0$ strongly as $\zeta \to 0^+$ and $\|\zeta(\zeta I + \Gamma_0^{\xi})^{-1}\| \le 1$ in addition to the Lebesgue dominated convergence theorem and the compactness of $P_{\mathfrak{y}}(t)$, therefore, $E\|m^{\zeta}(\xi) - \overline{m}_{\xi}\|^2 \to 0$ as $\zeta \to 0^+$. This demonstrates that (1) is approximately controllable. □

4. Example

The Hilfer fractional neutral stochastic partial differential inclusions with fBm and Clarke's subdifferential are taken into consideration.

$$\begin{cases} D_{0+}^{\frac{3}{4},\frac{5}{7}}\left[m(t,\varpi)+f\left(t,m(t,\varpi)\right)\right]\in\dfrac{\partial^2}{\partial\varpi^2}\left[m(t,\varpi)+f\left(t,m(t,\varpi)\right)\right]+\sigma\left(t,m(t,\varpi)\right)+\wp\left(t,m(t,\varpi)\right)\dfrac{dw(t)}{dt} \\ \qquad\qquad +g\left(t,m(t,\varpi)\right)\dfrac{dB^H(t)}{dt}+\partial\mathfrak{B}\left(t,m(t,\varpi)\right),\ t\in\mathfrak{T}=(0,1],\ \varpi\in\mho, \\ m(t,\varpi)=V(t,\varpi),\ t\in\mathfrak{T},\ \varpi\in\partial, \\ I_{0+}^{\frac{1}{14}}m(0,\varpi)=m_0(\varpi),\ \varpi\in\mho, \end{cases} \tag{13}$$

where $D_{0+}^{\frac{3}{4},\frac{5}{7}}$ is the HFD of order $\ell=\frac{3}{4}$, $\mathfrak{y}=\frac{5}{7}$, $0 < t_0 < t_1 < \cdots < t_p < 1$, $\mho$ is bounded and open subset of $\mathfrak{R}$, it is smooth enough at the boundary of ∂. $w(t)$ is Wiener process and B^H is a fBm with Hurst parameter $H\in\left(\frac{1}{2},1\right)$ and $V \in L_2(\partial)$. The functions

$$f\left(t,m(t)\right)(\varpi)=f\left(t,m(t,\varpi)\right),\ \sigma\left(t,m(t)\right)(\varpi)=\sigma\left(t,m(t,\varpi)\right),\ \wp\left(t,m(t)\right)(\varpi)=\wp\left(t,m(t,\varpi)\right),$$

$$g\left(t,m(t)\right)(\varpi)=g\left(t,m(t,\varpi)\right),\text{ and }\partial\mathfrak{B}\left(t,m(t)\right)(\varpi)=\partial\mathfrak{B}\left(t,m(t,\varpi)\right).$$

Let $\Lambda = \mathfrak{D} = L^2(\mho)$, $Y = L^2(\partial)$, $\varrho_1 = I$, the identity operator, and $F : \mathrm{Dom}(F) \subset \Lambda \to \Lambda$ provided by $F = \dfrac{\partial^2}{\partial\varpi^2}$ with $\mathrm{Dom}(F) = \{m \in \Lambda, m, \dfrac{\partial}{\partial\varpi}$ are absolutely continuous, $\dfrac{\partial^2 m}{\partial\varpi^2} \in L^2(\mho)\}$.

Then Δ can be written as

$$\Delta m=\sum_{k=1}^{\infty}(-k^2)(m,m_k)m_k,\ m\in D(\Delta),$$

where $m_k(s)=\sqrt{\dfrac{2}{\pi}}\sin ks$, $k = 1, 2, \cdots$ is the orthogonal base set of eigenvectors of Δ. Furthermore, Δ is the infinitesimal generator of the bounded linear operator, $\{\aleph(t)\}_{t\ge 0}$ in Λ and provided by

$$\aleph(t)m=\sum_{k=1}^{\infty}e^{\frac{-k^2}{1+k^2}t}(m,m_k)m_k,$$

$\|\aleph(t)\| \le e^{-t} \le 1$. Additionally, the operators $\aleph_{\frac{3}{4},\frac{5}{7}}(t)$ and $P_{\frac{5}{7}}(t)$ may be determined by

$$\aleph_{\frac{3}{4},\frac{5}{7}}(t)m = \frac{5}{7\Gamma\left(\frac{3}{14}\right)}\int_0^t\int_0^\infty v(t-s)^{\frac{-11}{14}} t^{\frac{-2}{7}} \mathcal{W}_{\frac{5}{7}}(v)\aleph(t^{\frac{5}{7}}v)mdvds,$$

$$P_{\frac{5}{7}}(t)m = \frac{5}{7}\int_0^\infty vs^{\frac{-2}{7}} \mathcal{W}_{\frac{5}{7}}(v)\aleph(t^{\frac{5}{7}}v)mdv.$$

Clearly,

$$\left\|P_{\frac{5}{7}}(t)\right\| \le \frac{t^{\frac{-2}{7}}}{\Gamma\left(\frac{5}{7}\right)}, \quad \left\|\aleph_{\frac{3}{4},\frac{5}{7}}(t)\right\| \le \frac{t^{\frac{-1}{14}}}{\Gamma\left(\frac{13}{14}\right)}.$$

Describe the fBm on Y using

$$B^H(t) = \sum_{k=1}^{\infty}\sqrt{\lambda_k}\beta^H(t)e_k,$$

where $H \in \left(\frac{1}{2},1\right)$, $\{\beta_k^H\}_{k\in N}$ is a sequence of one-dimensional fBm that are independent of one another.

It is obvious that Δ makes the compact semigroup $\{\aleph(t)\}_{t\ge 0}$ on Λ.

Next, we verify that the hypothesis (**A1**)-(**A9**) for the above system (13) one by one.

Verification of A1

$\mathrm{Dom}(F) \subset \mathrm{Dom}(\delta)$ and the restriction of δ to $\mathrm{Dom}(F)$ are continuous with regard to the graph norm of $\mathrm{Dom}(F)$. Therefore, **A1** is verified.

Verification of A2

There exists a linear operator $\varrho : Y \to \Lambda$ such that for all $V \in Y$ we have $\varrho V \in \mathrm{Dom}(F)$, $\delta(\varrho V) = \varrho_1 V$ and $E\|\varrho V\|^2 \le C_1\|\varrho_1 V\|$, where C_1 is a constant. Hence, **A2** is verified.

Verification of A3

A is the infinitesimal generator of bounded operator $\{\aleph(t),\ t \ge 0\}$ in Λ and there exists a constant $\Pi > 0$ such that $\sup_{t\in\mathfrak{T}}\|\aleph(t)\| \le \Pi$. Therefore, **A3** is verified.

Verification of A4

For all $t \in (0, \xi]$ and $V \in Y$, $\aleph(t)\varrho V \in \mathrm{Dom}(A)$. Additionally, there exists a constant $\Pi_1 > 0$ such that $\|A\aleph(t)\| \le \Pi_1$. Hence, **A4** is verified.

Verification of A5

Assume that $f(t, m(t)) = f(t, m(t, \varpi))$.

If $f : \mathfrak{T} \times \Lambda \to \Lambda$ is continuous function, then there exists a constants C_1, $C_2 > 0$ such that

$$E\left\|f(t,m_1) - f(t,m_2)\right\|^2 \le C_1E\left\|m_1 - m_2\right\|^2,$$

for $t \in \mathfrak{T}$, m_1, $m_2 \in \Lambda$, $\varpi \in \mho$ and the inequality

$$E\|f(t,m)\|^2 \le C_2\left(1+E\|m\|^2\right),$$

for all $(t, m) \in \mathfrak{T} \times \Lambda$.

Therefore the function $g(t, m(t, \varpi))$ satisfies the condition **A5**.

Verification of A6-A8

Assume that $\sigma(t, m(t)) = \sigma(t, m(t, \varpi))$, $\wp(t, m(t)) = \wp(t, m(t, \varpi))$, and $g(t, m(t)) = g(t, m(t, \varpi))$. The functions $\sigma(t, m(t, \varpi))$, $\wp(t, m(t, \varpi))$, $g(t, m(t, \varpi))$ are similar to that of the system (1) and there is no more to prove.

Verification of A9

Assume that $\partial\mathfrak{B}(t, m(t)) = \partial\mathfrak{B}(t, m(t, \varpi))$.

The following conditions are fulfilled by the function $\mathfrak{B} : \mathfrak{T} \times \Lambda \to \mathfrak{R}$:

(i) $\mathfrak{B}(\cdot, m) : \mathfrak{T} \to \mathfrak{R}$ be measurable for all $m \in \Lambda$,

(ii) $\mathfrak{B}(t, \cdot) : \Lambda \to \mathfrak{R}$ be LLC for a.e. $t \in \mathfrak{T}$,

(iii) there exists $\vartheta \in L^1(\mathfrak{T}, \mathfrak{R}^+)$ and a constant $C_6 > 0$, that satisfies

$$E\left\|\partial\mathfrak{B}\left(t,m(t,\varpi)\right)\right\|^2 = \sup\left\{E\|\mathcal{F}(t)\|^2 : \mathcal{F}(t) \in \partial\mathfrak{B}\left(t,m(t,\varpi)\right)\right\} \le \vartheta(t) + C_6 E\|m\|^2,$$

for all $m \in \Lambda$ a.e. $t \in \mathfrak{T}$ and $\varpi \in \mho$.

From the above discussion, we observe that $\partial\mathfrak{B}(t, m(t, \varpi))$ satisfies every condition in **A9**.

It is now possible to write (13) in the form of (1). Clearly, all the assumptions of the Theorem 3.4 and 3.5 are satisfied.

$$\left\{49C_2 + \frac{49\Pi^2\xi^{2\mathfrak{y}-1}}{(2\mathfrak{y}-1)\Gamma^2(\mathfrak{y})}\left[\left(C_3 + Tr(\Theta)C_4 + 2H\xi^{2H-1}C_5\right) + C_6\xi\right]\right\}$$

$$\times\left\{1 + \frac{49\|\varrho\|^2\left\|\varrho^*\right\|^2\xi^{2\mathfrak{y}-1}\left[\|F\|^2\Pi^2 + \Pi_1^2\right]^2}{\zeta^2(2\mathfrak{y}-1)\Gamma^4(\mathfrak{y})}\right\} < 1.$$

Thus, (13) is approximately controllable on (0, 1].

5. Remark

In this paper, we discuss the approximate boundary controllability of Hilfer fractional neutral stochastic differential inclusions with fBm and Clarke's subdifferential. The primary results were obtained by using fractional calculus, stochastic analysis theory, and the fixed point theorem. Then, the proposed systems can be extended with impulsive effects and nonlocal conditions.

6. Conclusion

In this paper, we investigated the approximate boundary controllability of Hilfer fractional neutral stochastic differential inclusions with fBm and Clarke's subdifferential. Initially, we worked with stochastic analysis, nonsmooth analysis, semigroup theory, and the fixed point theorem of multivalued mappings to show that there is a mild solution to (1). Then, we offered a sufficient condition for the approximate boundary controllability of Hilfer fractional neutral stochastic differential inclusions with fBm and Clarke's subdifferential. The primary findings were finally shown using an example. In the future, we will use a fixed point technique to examine the approximate boundary controllability of

Hilfer fractional neutral stochastic integrodifferential equations with fractional Brownian motion and impulses.

Acknowledgments

The author thanks the referees and editor valuable remarks and suggestions for enhancing this article.

Data availability

In order to make their conclusions for this study, the authors did not use any scientific data.

Conflict of interest

There is no conflict of interest among the authors.

References

[1] Kilbas AA, Srivastava HM, Trujillo JJ. Theory and applications of fractional differential equations. *North-Holland Mathematics Studies*. Amsterdam: Elsevier; 2006.

[2] Podlubny I. *Fractional Differential Equations*. San Diego: Academic Press; 1999.

[3] Abbas S, Benchohra M, Lazreg JE, Nieto JJ, Zhou Y. *Fractional Differential Equations and Inclusions: Classical and Advanced Topics*. World Scientific; 2023. Available from: doi: 10.1142/12993.

[4] Zhou Y. *Basic Theory of Fractional Differential Equations*. Singapore: World Scientific; 2014. Available from: doi: 10.1142/9069.

[5] Zhou Y. *Fractional Evolution Equations and Inclusions: Analysis and Control*. New York: Elsevier; 2015.

[6] Hilfer R. *Applications of Fractional Calculus in Physics*. Singapore: World Scientific; 2000. Available from: doi: 10.1142/3779.

[7] Gu H, Trujillo JJ. Existence of mild solution for evolution equation with Hilfer fractional derivative. *Applied Mathematics and Computation*. 2015; 257: 344-354. Available from: doi: 10.1016/j.amc.2014.10.083.

[8] Yang M, Wang QR. Approximate controllability of Hilfer fractional differential inclusions with nonlocal conditions. *Mathematical Methods in the Applied Sciences*. 2017; 40(4): 1126-1138. Available from: doi: 10.1002/mma.4040.

[9] Varun Bose CS, Udhayakumar R. A note on the existence of Hilfer fractional differential inclusions with almost sectorial operators. *Mathematical Methods in the Applied Sciences*. 2022; 45(5): 2530-2541. Available from: doi: 10.1002/mma.7938.

[10] Ravikumar K, Ramkumar K, Anguraj A. Null controllability of nonlocal Sobolev-type Hilfer fractional stochastic differential system driven by fractional Brownian motion and Poisson jumps. *Journal of Applied Nonlinear Dynamics*. 2021; 10(4): 617-626. Available from: doi: 10.5890/JAND.2021.12.003.

[11] Sivasankar S, Udhayakumar R, Kishor MH, Alhazmi SE, Al-Omari S. A new result concerning nonlocal controllability of Hilfer fractional stochastic differential equations via almost sectorial operators. *Symmetry*. 2023; 11(1): 159. Available from: doi: 10.3390/math11010159.

[12] Mohan Raja M, Vijayakumar V, Udhayakumar R, Nisar KS. Results on existence and controllability results for fractional evolution inclusions of order $1 < r < 2$ with Clarke's subdifferential type. *Numerical Methods for Partial Differential Equations*. 2020; 40(7): 1-20. Available from: doi: 10.1002/num.22691.

[13] Sakthivel R, Ren Y, Debbouche A, Mahmudov NI. Approximate controllability of fractional stochastic differential inclusions with nonlocal conditions. *Applicable Analysis*. 2016; 95(11): 2361-2382. Available from: doi: 10.1080/00036811.2015.1090562.

[14] Debbouche A, Antonov V. Approximate controllability of semilinear Hilfer fractional differential inclusions with impulsive control inclusion conditions in Banach spaces. *Chaos, Solitons & Fractals*. 2017; 102: 140-148. Available from: doi: 10.1016/j.chaos.2017.03.023.

[15] Tamilalagan P, Balasubramaniam P. Approximate controllability of fractional stochastic differential equations

driven by mixed fractional Brownian motion via resolvent operators. *International Journal of Control*. 2017; 90(8): 1713-1727. Available from: doi: 10.1080/00207179.2016.1219070.
[16] Kavitha K, Vijayakumar V, Udhayakumar R, Nisar KS. Results on the existence of Hilfer fractional neutral evolution equations with infinite delay via measures of noncompactness. *Mathematical Methods in the Applied Sciences*. 2021; 44(2): 1438-1455. Available from: doi: 10.1002/mma.6843.
[17] Varun Bose CS, Udhayakumar R, Elshenhab AM, Sathish Kumar M, Ro JS. Discussion on the approximate controllability of Hilfer fractional neutral integro-differential inclusions via almost sectorial operators. *Fractal Fractional*. 2022; 6(10): 607. Available from: doi: 10.3390/fractalfract6100607.
[18] Varun Bose CS, Udhayakumar R. Existence of mild solutions for Hilfer fractional neutral integro-differential inclusions via almost sectorial operators. *Fractal Fractional*. 2022; 6(9): 532. Available from: doi: 10.3390/fractalfract6090532.
[19] Vijayakumar V, Udhayakumar R, Dineshkumar C. Approximate controllability of second order nonlocal neutral differential evolution inclusions. *IMA Journal of Mathematical Control and Information*. 2021; 38(1): 192-210. Available from: doi: 10.1093/imamci/dnaa001.
[20] Zhou Y, Jiao F. Existence of mild solutions for fractional neutral evolution equations. *Computers and Mathematics with Applications*. 2010; 59(3): 1063-1077. Available from: doi: 10.1016/j.camwa.2009.06.026.
[21] Sivasankar S, Udhayakumar R, Muthukumaran V, Al-Omari S. Approximate controllability outcomes of impulsive second-order stochastic neutral differential evolution systems. *Contemporary Mathematics*. 2023; 4(4): 782-816. Available from: doi: 10.37256/cm.4420233253.
[22] Dineshkumar C, Udhayakumar R. New results concerning to approximate controllability of Hilfer fractional neutral stochastic delay integro-differential systems. *Numerical Methods for Partial Differential Equations*. 2021; 37(2): 1072-1090. Available from: doi: 10.1002/num.22567.
[23] Guendouzi T, Bousmaha L. Approximate controllability of fractional neutral stochastic functional integro-differential inclusions with infinite delay. *Qualitative Theory of Dynamical Systems*. 2014; 13: 89-119. Available from: doi: 10.1007/s12346-014-0107-y.
[24] Sivasankar S, Udhayakumar R. Hilfer fractional neutral stochastic Volterra integrodifferential inclusions via almost sectorial operators. *Mathematics*. 2022; 10(12): 2074. Available from: doi: 10.3390/math10122074.
[25] Boufoussi B, Hajji S. Neutral stochastic functional differential equations driven by a fractional Brownian motion in a Hilbert space. *Statistics and Probability Letters*. 2012; 82(8): 1549-1558. Available from: doi: 10.1016/j.spl.2012.04.013.
[26] Boudaoui A, Caraballo T, Ouahab A. Impulsive stochastic functional differential inclusions driven by a fractional Brownian motion with infinite delay. *Mathematical Methods in the Applied Sciences*. 2016; 39(6): 1435-1451. Available from: doi: 10.1002/mma.3580.
[27] Balasubramaniam P, Sathiyaraj T, Ratnavelu K. Optimality of non-instantaneous impulsive fractional stochastic differential inclusion with fractional Brownian motion. *Bulletin of the Malaysian Mathematical Sciences Society*. 2022; 45: 2787-2819. Available from: doi: 10.1007/s40840-022-01351-8.
[28] Ahmed HM, El-Owaidy HM, Al-Nahhas MA. Neutral fractional stochastic partial differential equations with Clarke subdifferential. *Applicable Analysis*. 2021; 100(15): 3220-3232. Available from: doi: 10.1080/00036811.2020.1714035.
[29] Mourad K, Fateh E, Baleanu D. Stochastic fractional perturbed control systems with fractional Brownian motion and Sobolev stochastic non local conditions. *Collectanea Mathematica*. 2018; 69: 283-296. Available from: doi: 10.1007/s13348-017-0207-5.
[30] Mourad K. Approximate controllability of fractional neutral stochastic evolution equations in Hilbert spaces with fractional Brownian motion. *Stochastic Analysis and Applications*. 2018; 36(2): 209-223. Available from: doi: 10.1080/07362994.2017.1386570.
[31] Clarke FH. *Optimization and Nonsmooth Analysis*. Newyork: Wiley; 1983.
[32] Ahmed HM, El-Borai MM, El-Sayed WG, Elbadrawi AY. Fractional stochastic evolution inclusions with control on the boundary. *Symmetry*. 2023; 15(4): 928. Available from: doi: 10.3390/sym15040928.
[33] Kavitha K, Vijayakumar V, Shukla A, Nisar KS, Udhayakumar R. Results on the approximate controllability of Sobolev-type fractional neutral differential inclusions of Clarke subdifferential type. *Chaos, Solitons and Fractals*. 2021; 151: 111264. Available from: doi: 10.1016/j.chaos.2021.111264.
[34] Sivasankar S, Udhayakumar R, Subramanian V, AlNemer G, Elshenhab AM. Optimal control problems for Hilfer fractional neutral stochastic evolution hemivariational inequalities. *Symmetry*. 2022; 15(1): 18. Available from: doi: 10.3390/sym15010018.
[35] Wang L. Approximate boundary controllability for semilinear delay differential equations. *Journal of Applied*

Mathematics. 2011; 2011: 587890. Available from: doi: 10.1155/2011/587890.
[36] Olive G. Boundary approximate controllability of some linear parabolic systems. *Evolution Equations and Control Theory*. 2014; 3: 167-189. Available from: doi: 10.3934/eect.2014.3.167.
[37] Ahmed HM, El-Borai MM, Ramadan ME. Boundary controllability of nonlocal Hilfer fractional stochastic differential systems with fractional Brownian motion and Poisson jumps. *Advances in Difference Equations*. 2019; 82: 2019. Available from: doi: 10.1186/s13662-019-2028-1.
[38] Ahmed HM. Approximate controllability of neutral fractional stochastic differential systems with control on the boundary. *Numerical Algebra, Control and Optimization*. 2023. Available from: doi: 10.3934/naco.2023013.
[39] Li YX, Lu L. Existence and controllability for stochastic evolution inclusions of Clarke's subdifferential type. *Electronic Journal of Qualitative Theory of Differential Equations*. 2015; 59: 1-16. Available from: doi: 10.14232/ejqtde.2015.1.59.
[40] Dhage BC. Multi-valued mappings and fixed points II. *Tamkang Journal of Mathematics*. 2006; 37: 27-46. Available from: doi: 10.5556/j.tkjm.37.2006.177.
[41] Sakthivel R, Ganesh R, Ren Y, Anthoni SM. Approximate controllability of nonlinear fractional dynamical systems. *Communications in Nonlinear Science and Numerical Simulation*. 2013; 18(12): 3498-3508. Available from: doi: 10.1016/j.cnsns.2013.05.015.