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Research Article

Innovative Method for Computing Approximate Solutions of Non-Homogeneous Wave Equations with Generalized Fractional Derivatives

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Abstract: In this work, a well-known non-homogeneous wave equation with temporal fractional derivative is approximately investigated. A recently defined generalized non-local fractional derivative is utilized as the fractional operator. A novel technique is proposed to approximate the solutions of wave equation with generalized fractional derivative. The proposed method is based on the shifted Chebyshev polynomials and a combination of collocation and residual function methods. Theoretical analysis of the convergence of the proposed method is performed. Approximate solutions are derived in both rectangular and non-rectangular (general) domains.

Keywords: shifted Chebyshev polynomial, wave equation, generalized Caputo fractional derivative, irregular domain

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1. Introduction

It is astounding to learn from the many studies conducted over the past few decades that both ordinary and PDEs with fractional operators may be precisely employed in the modeling of physical processes, such as solid mechanics [1], bioengineering [2], continuum and statistical mechanics [3], and finance [4]. Both local and non-local fractional operators are studied in the literature. The non-local ones, however, are more significant from the perspective of real-applications due to the memory property of fractional derivatives. Riemann-Liouville and Caputo fractional derivatives are the source of the majority of recently defined fractional operators, including the Caputo-Fabrizio [5] and Atangana-Baleanu [6] derivatives.

Numerous scholars focus their research on differential equation solutions. The approximate and exact solutions are the two categories of solutions that are most frequently explored in research. There are just a few methods, like the Lie symmetry method [7-9], and invariant subspace method [10-12], for solving differential equations using FDs

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precisely. Additionally, a variety of approximation techniques are suggested to take into account the numerical solutions of fractional differential equations. In the literature, a number of approaches are suggested to extract the numerical solutions, including [13-15], meshfree methods [16-18], finite element method [19-21], operational matrices [22-24], geometric methods [25-27], and collocation methods [28-29].

The wave equation is a fundamental concept in physics and mathematics that describes the behavior of waves in various mediums, such as sound waves, electromagnetic waves, and water waves. It is a partial differential equation that describes the relationship between the rate of change of a wave and its spatial coordinates. The wave equation has numerous real-world applications, including in the fields of acoustics, seismology, and optics. In acoustics, the wave equation is used to model the behavior of sound waves and their propagation through different materials. In seismology, it is used to study earthquakes and how seismic waves propagate through the Earth's crust. In optics, the wave equation is used to understand the behavior of light waves and their interactions with matter. The wave equation is a powerful tool in modern physics and plays a crucial role in the study of a wide range of phenomena.

The wave equation with fractional derivative is a more generalized form of the traditional wave equation, which incorporates the concept of fractional calculus. In this equation, the time or space derivative is replaced by a fractional derivative. The fractional derivative describes the non-local properties of a wave, which can have significant implications for wave propagation and energy transport. The wave equation with fractional derivative has numerous applications in different fields, including electromagnetic waves, acoustics, and fluid dynamics. It has been used to model wave phenomena in materials with complex structures, such as fractals or porous media. This equation can also be used to study the dynamics of waves in non-linear systems, where traditional wave equations fail to capture the underlying physics. Overall, the wave equation with fractional derivative provides a more accurate description of wave phenomena in complex systems, making it a powerful tool in modern physics and engineering.

The following space-fractional wave equation with a recently developed fractional derivative is examined in this study along with several approximations to its solutions:

$$u_{tt} + \lambda u_t - \mathcal{K}\, {}^{C}\mathcal{D}^{\gamma,\rho}_{0^+,x} u + \mathcal{F}[u] = \mathcal{G}(x, t), x \in (0, x_f), t \in (0, T), \tag{1}$$

with BCs

$$u(0, t) = \psi_0(t), u(x_f, t) = \psi_1(t), t \in (0, T), \tag{2}$$

and ICs

$$u(x, 0) = \phi_0(x), u_t(x, 0) = \phi_1(x), x \in (0, x_f). \tag{3}$$

The following is the summary of this article's structure. Preliminaries that we will need for the follow-up are described in Section 2. The applications of the suggested method are covered in Section 3 and the sections that follow. Section 4 presents error analysis. Section 5 presents a method for resolving a few test issues. Finally, Section 6 presents a brief description of the methodology and the produced outcomes.

2. Preliminaries

In this article, we provide fundamental explanations for fractional derivatives and Chebyshev polynomials. We recommend that the audience familiarize themselves with these definitions and other fractional differentiation concepts in resources such as [7, 30].

2.1 *Fractional operators*

Fractional calculus, which has a three-hundred-year history, is focused on integration and differentiation in any random order, and it has attracted a lot of interest recently. This attractiveness is due to the benefits of using fractional differential equations to simulate the real world. In reality, rather than integer order differentiation, many phenomena across many scientific disciplines are better described by fractional derivatives. The nonlocal aspect of fractional differentiation is what gives fractional derivatives this advantage, because a lot of physical models depend not only on the current time instance but also on the past. As a result, numerous scholars have been inspired during the past 20 years to look for precise or approximative answers to fractional partial differential equations (FPDEs). Various definitions of fractional integration and differentiation that we employ include the following [31-33]:

Definition 1. Assuming its existence, the generalized fractional integral $I_{a+}^{\gamma,\rho}\psi(x)$ of a function ψ of order $\gamma > 0$ with $\rho > 0$ is defined as follows:

$$I_{a^+,x}^{\gamma,\rho}\psi(x) = \frac{\rho^{1-\gamma}}{\Gamma(\gamma)}\int_a^x \sigma^{\rho-1}(x^\rho - \sigma^\rho)^{\gamma-1}\psi(\sigma)d\sigma,\ \gamma,\ \rho > 0.$$

If the interval $[0, +\infty)$ can be point-wise defined for the right hand side of this statement, it is valid.

Definition 2. The definition of the new generalized Caputo-type fractional derivative of order $\gamma > 0$ is as follows:

$${}^{C}\mathcal{D}_{a^+,x}^{\gamma,\rho} f(x) = \frac{\rho^{\gamma-n+1}}{\Gamma(n-\gamma)}\int_a^x \sigma^{\rho-1}(x^\rho - \sigma^\rho)^{n-\gamma-1}\left(\sigma^{1-\rho}\frac{d}{d\sigma}\right)^n f(\sigma)d\sigma,$$

where $n = \lceil\gamma\rceil$, $\rho > 0$, $a \geq 0$, and $f(x) \in C^n[a, b]$.

The computation of the Riemann-Liouville sense of the generalized fractional derivative of power functions is straightforward and can be achieved by applying definition 2:

$${}^{C}\mathcal{D}_{a^+,x}^{\gamma,\rho}(x^\rho - a^\rho)^k = \begin{cases} \rho^\gamma \dfrac{\Gamma(k+1)}{\Gamma(k-\gamma+1)}(x^\rho - a^\rho)^{k-\gamma}, & k \in \mathbb{N}_0,\quad \& \, k \geq \lceil\gamma\rceil \text{ or } k \in \mathbb{N},\quad \& k > \lceil\gamma\rceil \\ 0, & k \in \mathbb{N}_0,\quad \& \, k < \lceil\gamma\rceil. \end{cases} \tag{4}$$

2.2 *Chebyshev polynomials*

The Chebyshev polynomials are a family of orthogonal polynomials with important applications in various areas of mathematics and science, including numerical analysis, signal processing, and approximation theory. These polynomials are named after the Russian mathematician Pafnuty Chebyshev and are defined as solutions to the Chebyshev differential equation. The Chebyshev polynomials have numerous properties, such as recurrence relations, explicit formulas, and orthogonality conditions, which make them useful for a wide range of applications. One notable property of the Chebyshev polynomials is their ability to approximate functions on a given interval with a small error, which is known as the Chebyshev approximation theorem. This theorem is widely used in numerical analysis for approximating functions and solving differential equations. The best approximation property of Chebyshev polynomials is a fundamental theorem in approximation theory that states that the nth-degree Chebyshev polynomial of a function $f(x)$ provides the best possible approximation of $f(x)$ among all polynomials of degree n on a given interval. This means that the difference between $f(x)$ and its nth-degree Chebyshev polynomial is minimized on that interval. The best approximation property of Chebyshev polynomials is particularly useful in numerical analysis for approximating functions, solving differential equations, and optimizing numerical methods. The property is also closely related to the Remez algorithm, which is an iterative procedure for computing the Chebyshev approximation of a function.

It is common knowledge that the first kind of Chebyshev polynomials can be determined by

$$\begin{cases} \mathcal{C}_0(x) = 1, \\ \mathcal{C}_1(x) = x, \\ \mathcal{C}_{n+1}(x) = 2x\mathcal{C}_n(x) - \mathcal{C}_{n-1}(x), \; n \in \mathbb{N}, \text{ and } -1 \le x \le 1. \end{cases}$$

It is worth mentioning that the analytical $\mathcal{C}_n(x)$ can be expressed in the following manner:

$$\mathcal{C}_n(x) = \frac{n}{2} \sum_{\kappa=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^{\kappa} \frac{(n-\kappa-1)!}{\kappa!(n-2\kappa)!} (2x)^{n-2\kappa}, \; n \in \mathbb{N}, \; -1 \le x \le 1. \tag{5}$$

The set of Chebyshev polynomials defined in Eq. (5) are orthogonal w.r.t. the weight function $w(x) = \frac{1}{\sqrt{1-x^2}}$, i.e.

$$\int_{-1}^{1} \mathcal{C}_i(x)\mathcal{C}_j(x)w(x)\, dx = \frac{\pi}{2} \delta_{ij}(1+\delta_{i0}),$$

where δ_{ij} is the Kronecker delta.

If we alter our range to $[0, x_f]$, we need to adapt the Chebyshev polynomials to what are known as the shifted Chebyshev polynomials (SCPs). To do this, let $t = \frac{2x}{x_f} - 1$, which results in $\mathcal{S}_n(x) = \mathcal{C}_n(t)$.

Hence, we do

$$\mathcal{S}_n(x) = \mathcal{C}_n\left(\frac{2x}{x_f} - 1\right) = \mathcal{C}_{2n}\left(\sqrt{\frac{x}{x_f}}\right), \; x \in [0, x_f]. \tag{6}$$

We may also obtain the following analytical form of SCPs by applying (6) in (5):

$$\mathcal{S}_n(x) = n \sum_{\kappa=0}^{n} (-1)^{\kappa} 2^{2n-2\kappa} \frac{(2n-\kappa-1)!}{\kappa!(2n-2\kappa)!} \left(\frac{x}{x_f}\right)^{n-\kappa}, \; n \in \mathbb{N}, \; x \in [0, x_f]. \tag{7}$$

Moreover, from Eq. (7) we have

$$\mathcal{S}_k(0) = (-1)^k, \text{ and } \mathcal{S}_k(x_f) = 1. \tag{8}$$

The SCPs exhibit orthogonality with respect to the weight function $\varpi(x) = \frac{1}{\sqrt{x_f x - x^2}}$ over the interval $[0, x_f]$, meaning that:

$$\int_{0}^{x_f} \mathcal{S}_i(x)\mathcal{S}_j(x)\varpi(x)\, dx = \frac{\pi}{2} \delta_{ij}(1+\delta_{i0}).$$

Think about the subsequent weighted function space

$$L^2_{\varpi} = \{f : [0, x_f] \to \mathbb{R} \mid f \text{ is measurable and } \| f \|_{\varpi} < \infty\},$$

where

$$\| f \|_{\varpi} = \left(\int_0^{x_f} |f(x)|^2 \, \varpi(x) dx \right)^{\frac{1}{2}},$$

is from the following inner product:

$$\langle f, g \rangle_{\varpi} = \int_0^{x_f} f(x) g(x) \varpi(x) dx, f, g \in L^2_{\varpi}.$$

The following theorem can be used to determine which polynomial approximation of f in L^2_{ϖ} is the best:

Theorem 1. [34] Let P_m be the collection of all real polynomials with m as the maximum degree. There exists $f_m \in P_m$ for each $f \in L^2_{\varpi}$ and $m \in \mathbb{N}$, such that

$$\| f - f_m \|_{\varpi} = \inf_{q_m \in P_m} \| f - q_m \|_{\varpi},$$

where $f(x) = \sum_{n=0}^{m} c_n \mathcal{S}_n(x)$, with

$$c_n = \frac{\langle f, \mathcal{S}_n \rangle_{\varpi}}{\| \mathcal{S}_n \|^2_{\varpi}} = \frac{2 - \delta_{n,0}}{\pi} \int_0^{x_f} f(x) \mathcal{S}_n(x) \varpi(x) dx, \; n \in \mathbb{N} \cup \{0\}.$$

Theorem 2. Assuming that $\chi(x) \in C^{m+1}[0, x_f]$ and $\Pi_m \chi$ represents the best approximation to χ from P_m, the maximum limit of the error in the approximation can be determined by:

$$\| \chi - \Pi_m \chi \|_{\varpi} \le \frac{\sqrt[4]{\pi} M_{m+1} x_f^{m+1}}{(m+1)!} \sqrt{\frac{\Gamma(2m + \frac{3}{2})}{\Gamma(2m+3)}},$$

where

$$M_{m+1} = \sup\{ | \chi^{(m+1)}(x) | : x \in [0, x_f] \}.$$

Proof. See Theorem 2 in Ref. [35].

3. Methodology

The steps of our approach utilized to solve the Eq. (1) are covered in this section. Three primary steps are required, and each one has a paragraph that is discussed.

3.1 *Collocating the original problem*

The collocation method is a numerical technique used to solve differential equations by constructing a set of algebraic equations based on the differential equation and evaluating them at specific points, known as collocation points (CPs). Moreover, the Chebyshev collocation method is a numerical technique that employs Chebyshev polynomials as basis functions to construct the algebraic equations needed to solve differential equations using the collocation method. Here, we utilize this method to solve the Eq. (1). Assume that $u(x, t)$ can be approximately represented as

$$\Psi_{\varrho}(x, t) = \sum_{n=0}^{\varrho} \Psi_n(t)\mathcal{S}_n(x). \tag{9}$$

The approximate solution (9) is substituted into (1) to generate

$$\sum_{n=0}^{\varrho} \frac{d^2\Psi_n(t)}{dt^2}\mathcal{S}_n(x) + \lambda\sum_{n=0}^{\varrho} \frac{d\Psi_n(t)}{dt}\mathcal{S}_n(x) - \mathcal{K}\sum_{n=0}^{\varrho} \Psi_n(t)\, {}^{C}\mathcal{D}_{0^+,x}^{\gamma,\rho}\mathcal{S}_n(x) + \mathcal{F}\Big(\sum_{n=0}^{\varrho} \Psi_n(t)\mathcal{S}_n(x)\Big) = \mathcal{G}(x, t). \tag{10}$$

Additionally, the boundary conditions (2) may be considered of as derived from Eqs. (8) and (9).

$$\sum_{n=0}^{\varrho} (-1)^n\Psi_n(t) = \psi_0(t), \text{ and } \sum_{n=0}^{\varrho} \Psi_n(t) = \psi_1(t). \tag{11}$$

From Eqs. (4) and (7), we have

$${}^{C}\mathcal{D}_{0^+,x}^{\gamma,\rho}\mathcal{S}_n(x) = \begin{cases} \displaystyle\sum_{\kappa=0}^{n-\rho\lceil\gamma\rceil} (-1)^{\kappa}\frac{\rho^{\gamma}2^{2n-2\kappa}}{x_f^{n-\kappa}} \frac{n(2n-\kappa-1)!\Gamma(\frac{n-\kappa}{\rho}+1)}{\kappa!(2n-2\kappa)!\Gamma(\frac{n-\kappa}{\rho}+1-\gamma)} x^{n-\kappa-\rho\gamma}, & n = \rho\lceil\gamma\rceil, \cdots, \varrho, \\ 0, & n = 0, 1, \cdots, \rho\lceil\gamma\rceil - 1. \end{cases} \tag{12}$$

Hence from Eq. (12) we can rewrite Eq. (10) equivalently as

$$\sum_{n=0}^{\varrho}\Big[\frac{d^2\Psi_n(t)}{dt^2} + \lambda\frac{d\Psi_n(t)}{dt}\Big]\mathcal{S}_n(x) - \mathcal{K}\sum_{n=\rho\lceil\gamma\rceil}^{\varrho}\sum_{\kappa=0}^{n-\rho\lceil\gamma\rceil} \theta_{n,\kappa}^{\rho,\gamma}\Psi_n(t)x^{n-\kappa-\rho\gamma} + \mathcal{F}\Big(\sum_{n=0}^{\varrho} \Psi_n(t)\mathcal{S}_n(x)\Big) = \mathcal{G}(x, t). \tag{13}$$

where

$$\theta_{n,\kappa}^{\rho,\gamma} = (-1)^{\kappa}\frac{\rho^{\gamma}2^{2n-2\kappa}}{x_f^{n-\kappa}} \frac{n(2n-\kappa-1)!\Gamma(\frac{n-\kappa}{\rho}+1)}{\kappa!(2n-2\kappa)!\Gamma(\frac{n-\kappa}{\rho}+1-\gamma)}.$$

To find $\{\Psi_n(t)\}_{n=0}^{\varrho}$, we must now collocate (13) at $(\varrho - 1)$ points. One can make use of various CPs. If the roots of $\mathcal{S}_{\varrho-1}(x)$, $\left(x_\nu = \frac{x_f}{2}(1+\cos\frac{(2\nu+1)\pi}{2(\varrho-1)}),\ \nu = 0,\ \cdots,\ \varrho-2\right)$ are regarded as the set of CPs, then the following system of ODEs is generated from Eqs. (11) and (13):

$$\begin{cases} \sum_{n=0}^{\varrho}\left[\frac{d^2\Psi_n(t)}{dt^2}+\lambda\frac{d\Psi_n(t)}{dt}\right]\mathcal{S}_n(x_\nu)-\mathcal{K}\sum_{n=\rho\lceil\gamma\rceil}^{\varrho}\sum_{\kappa=0}^{n-\rho\lceil\gamma\rceil}\theta_{n,\kappa}^{\rho,\gamma}\Psi_n(t)x_\nu^{n-\kappa-\rho\gamma}+\mathcal{F}\left(\sum_{n=0}^{\varrho}\Psi_n(t)\mathcal{S}_n(x_\nu)\right) \\ =\mathcal{G}(x_\nu,t),\ \nu=0,\ \cdots,\ \varrho-2, \\ \sum_{n=0}^{\varrho}(-1)^n\Psi_n(t)=\psi_0(t), \\ \sum_{n=0}^{\varrho}\Psi_n(t)=\psi_1(t). \end{cases} \tag{14}$$

$\Psi_{\varrho-1}(t)$ and $\Psi_\varrho(t)$ can be obtained from the last two equations of (14) as

$$\Psi_{\varrho-1}(t)=\frac{(-1)^{\varrho+1}\psi_1(t)+\psi_0(t)+\sum_{n=0}^{\varrho-2}\left(\left[(-1)^\varrho-(-1)^n\right]\Psi_n(t)\right)}{(-1)^{\varrho-1}+(-1)^{\varrho+1}},$$

$$\Psi_\varrho(t)=\frac{(-1)^{\varrho-1}\psi_1(t)-\psi_0(t)+\sum_{n=0}^{\varrho-2}\left(\left[(-1)^\varrho+(-1)^n\right]\Psi_n(t)\right)}{(-1)^{\varrho-1}+(-1)^{\varrho+1}}. \tag{15}$$

Equations in the $\varrho+1$ system and the $\Psi_n(t)$, $n = 0, \ldots, \varrho$ unknown functions make up the (14) system. We provide a semi-analytical approach to solving this situation in the following section.

3.2 *A homogenization for solving (14)*

To derive an approximate solution for (14), it is necessary to solve the following ODEs:

$$\left[\frac{d^2\Psi_\nu(t)}{dt^2}+\lambda\frac{d\Psi_\nu(t)}{dt}\right]\mathcal{S}_\nu(x_\nu)+\sum_{\substack{n=0\\n\neq\nu}}^{\varrho}\left[\frac{d^2\Psi_n(t)}{dt^2}+\lambda\frac{d\Psi_n(t)}{dt}\right]\mathcal{S}_n(x_\nu)-\mathcal{K}\sum_{n=\rho\lceil\gamma\rceil}^{\varrho}\sum_{\kappa=0}^{n-\rho\lceil\gamma\rceil}\theta_{n,\kappa}^{\rho,\gamma}\Psi_n(t)x_\nu^{n-\kappa-\rho\gamma}$$

$$+\mathcal{F}\left(\sum_{n=0}^{\varrho}\Psi_n(t)\mathcal{S}_n(x_\nu)\right)=\mathcal{G}(x_\nu,t),\ \nu=0,\ \cdots,\ \varrho-2. \tag{16}$$

From (15), if we separate the exact values $\Psi_\varrho(t)$ and $\Psi_{\varrho-1}(t)$ from the series sentences, we obtain

$$\left[\Psi''_{\nu}(t)+\lambda\Psi'_{\nu}(t)\right]\mathcal{S}_{\nu}(x_{\nu})+\sum_{\substack{n=0\\n\neq\nu}}^{\varrho-2}\left[\Psi''_{n}(t)+\lambda\Psi'_{n}(t)\right]\mathcal{S}_{n}(x_{\nu})$$

$$+\frac{(-1)^{\varrho+1}\left(\psi''_1(t)+\lambda\psi'_1(t)\right)+\left(\psi''_0(t)+\lambda\psi'_0(t)\right)+\sum_{n=0}^{\varrho-2}\left[(-1)^{\varrho}-(-1)^{n}\right]\left(\Psi''_n(t)+\lambda\Psi'_n(t)\right)}{(-1)^{\varrho-1}+(-1)^{\varrho+1}}\mathcal{S}_{\varrho-1}(x_{\nu})$$

$$+\frac{(-1)^{\varrho-1}\left(\psi''_1(t)+\lambda\psi'_1(t)\right)-\left(\psi''_0(t)+\lambda\psi'_0(t)\right)+\sum_{n=0}^{\varrho-2}\left[(-1)^{\varrho}+(-1)^{n}\right]\left(\Psi''_n(t)+\lambda\Psi'_n(t)\right)}{(-1)^{\varrho-1}+(-1)^{\varrho+1}}\mathcal{S}_{\varrho}(x_{\nu})$$

$$-\mathcal{K}\sum_{n=\rho\lceil\gamma\rceil}^{\varrho}\sum_{\kappa=0}^{n-\rho\lceil\gamma\rceil}\theta_{n,\kappa}^{\rho,\gamma}\Psi_n(t)x_{\nu}^{n-\kappa-\rho\gamma}$$

$$+\mathcal{F}\Big(\frac{(-1)^{\varrho+1}\psi_1(t)+\psi_0(t)}{(-1)^{\varrho-1}+(-1)^{\varrho+1}}\mathcal{S}_{\varrho-1}(x_{\nu})+\frac{(-1)^{\varrho-1}\psi_1(t)-\psi_0(t)}{(-1)^{\varrho-1}+(-1)^{\varrho+1}}\mathcal{S}_{\varrho}(x_{\nu})$$

$$+\sum_{n=0}^{\varrho-2}\left[\frac{(-1)^{\varrho}-(-1)^{n}}{(-1)^{\varrho-1}+(-1)^{\varrho+1}}\mathcal{S}_{\varrho-1}(x_{\nu})+\frac{(-1)^{\varrho}+(-1)^{n}}{(-1)^{\varrho-1}+(-1)^{\varrho+1}}\mathcal{S}_{\varrho}(x_{\nu})+\mathcal{S}_{n}(x_{\nu})\right]\Psi_n(t)\Big)$$

$$=\mathcal{G}(x_{\nu},t),\ \nu=0,\ \cdots,\ \varrho-2. \tag{17}$$

We obtain from Eqs. (3) and (9)

$$\sum_{\nu=0}^{\varrho}\Psi_{\nu}(0)\mathcal{S}_{\nu}(x)=\phi_0(x),\ \sum_{\nu=0}^{\varrho}\Psi'_{\nu}(0)\mathcal{S}_{\nu}(x)=\phi_1(x),\ x\in[0,x_f].$$

If the SCPs can be used to approximate $\phi_0(x)$ and $\phi_1(x)$, then

$$\phi_0(x)=\sum_{\nu=0}^{\varrho}\lambda_{\nu}\mathcal{S}_{\nu}(x),\ \phi_1(x)=\sum_{\nu=0}^{\varrho}\mu_{\nu}\mathcal{S}_{\nu}(x),\ x\in[0,x_f],$$

then we have

$$\sum_{\nu=0}^{\varrho}\Psi_{\nu}(0)\mathcal{S}(x)=\sum_{\nu=0}^{\varrho}\lambda_{\nu}\mathcal{S}_{\nu}(x),\ \sum_{\nu=0}^{\varrho}\Psi'_{\nu}(0)\mathcal{S}(x)=\sum_{\nu=0}^{\varrho}\mu_{\nu}\mathcal{S}_{\nu}(x). \tag{18}$$

Therefore, (16) can be used to determine the initial conditions of ODEs from (18).

$$
\begin{cases}
\Psi_0(0) = \lambda_0 = \dfrac{1}{\pi}\displaystyle\int_0^{x_f} \phi_0(x)\varpi(x)dx, \\
\Psi_\nu(0) = \lambda_\nu = \dfrac{2}{\pi}\displaystyle\int_0^{x_f} \phi_0(x)\varpi(x)dx, \ \nu = 1, \cdots, \varrho - 2,
\end{cases} \tag{19}
$$

and

$$
\begin{cases}
\Psi_0'(0) = \mu_0 = \dfrac{1}{\pi}\displaystyle\int_0^{x_f} \phi_1(x)\varpi(x)dx, \\
\Psi_\nu'(0) = \mu_\nu = \dfrac{2}{\pi}\displaystyle\int_0^{x_f} \phi_1(x)\varpi(x)dx, \ \nu = 1, \cdots, \varrho - 2,
\end{cases} \tag{20}
$$

Let us assume

$$
\hat{\Psi}_{\nu,\Lambda}(t) = v_\nu(t) + \sum_{j=0}^{\Lambda} \epsilon_{\nu j} w_{\nu j}(t), \ \nu = 0, \cdots, \varrho - 2, \tag{21}
$$

where $\{\hat{\Psi}_{\nu,\Lambda}(x)\}_{\nu=0}^{\varrho-2}$ fulfill the initial conditions (19) and (20) and $v_\nu(t)$ and $\{w_{\nu j}(t)\}_{j=0}^{\Lambda}$ are obtained from auxiliary differential equations in this way. Obviously, it occurs when

$$
\begin{cases}
v_\nu(0) + \displaystyle\sum_{j=0}^{\Lambda} \epsilon_{\nu j} w_{\nu j}(0) = \lambda_\nu, \\
v_\nu'(0) + \displaystyle\sum_{j=0}^{\Lambda} \epsilon_{\nu j} w_{\nu j}'(0) = \mu_\nu, \ \nu = 0, \cdots, \varrho - 2,
\end{cases}
$$

or equivalently

$$
v_\nu(0) = \lambda_\nu, \ v_\nu'(0) = \mu_\nu, \ \sum_{j=0}^{\Lambda} \epsilon_{\nu j} w_{\nu j}(0) = \sum_{j=0}^{\Lambda} \epsilon_{\nu j} w_{\nu j}'(0) = 0, \text{ for } \nu = 0, \cdots, \varrho - 2, \ j = 0, \cdots, \Lambda. \tag{22}
$$

Theorem 3. Under non-homogeneous initial conditions, the functions $v_\nu(t)$, $\nu = 0, \ldots, \varrho - 2$ satisfy in the auxiliary differential equations, if

$$
v_\nu(t) = \lambda_\nu + \frac{\mu_\nu}{\lambda} - \frac{\mu_\nu}{\lambda} e^{-\lambda t} + \frac{1}{\lambda T_\nu^*(x_\nu)} \int_0^t \left(1 - e^{\lambda(\tau - t)}\right) \mathcal{G}(x_\nu, \tau) d\tau, \ \nu = 0, \ldots, \varrho - 2. \tag{23}
$$

Proof. To validate the non-homogeneous part of (16), we have

$$
\frac{d^2 v_\nu(t)}{dt^2} + \lambda \frac{dv_\nu(t)}{dt} = \frac{\mathcal{G}(x_\nu, t)}{T_\nu^*(x_\nu)}, \tag{24}
$$

with initial conditions $v_\nu(0) = \lambda_\nu$, $v_\nu'(0) = \mu_\nu$. To solve this equation, we assume $y_\nu(t) = \dfrac{dv_\nu(t)}{dt}$. Then, from Eq. (24) we get

$$\frac{dy_\nu(t)}{dt}+\lambda y_\nu(t)=\frac{\mathcal{G}(x_\nu,t)}{T_\nu^*(x_\nu)}, \tag{25}$$

with initial conditions $y_\nu(0) = \mu_\nu$. This equation is a first order linear ODE with exact solution

$$y_\nu(t)=e^{-\lambda t}\Big[\mu_\nu+\frac{1}{T_\nu^*(x_\nu)}\int_0^t e^{\lambda\tau}\mathcal{G}(x_\nu,\,\tau)d\tau\Big].$$

Equivalently, we have

$$\frac{dv_\nu(t)}{dt}=e^{-\lambda t}\Big[\mu_\nu+\frac{1}{T_\nu^*(x_\nu)}\int_0^t e^{\lambda\tau}\mathcal{G}(x_\nu,\,\tau)d\tau\Big],\ v_\nu(0)=\lambda_\nu.$$

One time integration of this equation yields

$$v_\nu(t)=\lambda_\nu+\mu_\nu\int_0^t e^{-\lambda\sigma}d\sigma+\frac{1}{T_\nu^*(x_\nu)}\int_0^t e^{-\lambda\sigma}\int_0^\sigma e^{\lambda\tau}\mathcal{G}(x_\nu,\,\tau)d\tau d\sigma.$$

Therefore

$$v_\nu(t)=\lambda_\nu+\frac{\mu_\nu}{\lambda}\left(1-e^{-\lambda t}\right)+\frac{1}{T_\nu^*(x_\nu)}\int_0^t\int_0^\sigma e^{\lambda(\tau-\sigma)}\mathcal{G}(x_\nu,\,\tau)d\tau d\sigma. \tag{26}$$

Now, order of integration changing, concludes

$$\int_0^t\int_0^\sigma e^{\lambda(\tau-\sigma)}\mathcal{G}(x_\nu,\,\tau)d\tau d\sigma=\int_0^t\int_\tau^t e^{\lambda(\tau-\sigma)}\mathcal{G}(x_\nu,\,\tau)d\sigma d\tau=\frac{1}{\lambda}\int_0^t\left(1-e^{\lambda(\tau-t)}\right)\mathcal{G}(x_\nu,\,\tau)d\tau. \tag{27}$$

Finally, substituting the Eq. (27) into Eq. (26), completes the proof. □

Theorem 4. Under homogeneous initial conditions, the functions $w_{\nu j}(t)$, $\nu = 0, \ldots, \varrho - 2$, $j = 0, \ldots, \Lambda$, satisfy in the auxiliary differential equations, if

$$w_{\nu j}(t)=\frac{t^{j+1}}{\lambda(j+1)}+\frac{(-1)^{j+2}j!}{\lambda^{j+2}}e^{-\lambda t}+\sum_{\kappa=0}^{j}\frac{(-1)^{\kappa+1}\kappa!\binom{j}{\kappa}t^{j-\kappa}}{\lambda^{\kappa+2}},\ \nu=0,\ \ldots,\ \varrho-2, j=0,\ \ldots,\ \Lambda. \tag{28}$$

Proof. To validate the homogeneous part of Eq. (16), we have

$$\frac{d^2w_{\nu j}(t)}{dt^2}+\lambda\frac{dw_{\nu j}(t)}{dt}=t^j, \tag{29}$$

with initial conditions $w_{\nu j}(0) = w'_{\nu j}(0) = 0$. To solve Eq. (29), let us assume $z_{\nu j} = \frac{dw_{\nu j}(t)}{dt}$. Then, we get

$$\frac{dz_{\nu j}(t)}{dt} + \lambda z_{\nu j}(t) = t^j, \; z_{\nu j}(0) = 0. \tag{30}$$

This equation can be solved exactly with the following form of solution

$$z_{\nu j}(t) = e^{-\lambda t}\int_0^t \tau^j e^{\lambda\tau} d\tau.$$

Using the integration by parts concludes

$$\int_0^t \tau^j e^{\lambda\tau} d\tau = \frac{t^j e^{\lambda t}}{\lambda} - \frac{j}{\lambda}\int_0^t \tau^{j-1} e^{\lambda\tau} d\tau$$

$$= \frac{t^j e^{\lambda t}}{\lambda} - \frac{jt^{j-1}e^{\lambda t}}{\lambda^2} + \frac{j(j-1)}{\lambda^2}\int_0^t \tau^{j-2} e^{\lambda\tau} d\tau = \cdots$$

$$= \frac{t^j e^{\lambda t}}{\lambda} - \frac{jt^{j-1}e^{\lambda t}}{\lambda^2} + \frac{j(j-1)t^{j-2}e^{\lambda t}}{\lambda^3} + \cdots + (-1)^{j-1}\frac{j(j-1)\cdots 2}{\lambda^j} te^{\lambda t} + (-1)^j \frac{j!}{\lambda^j}\int_0^t e^{\lambda\tau} d\tau.$$

Therefore

$$z_{\nu j}(t) = \frac{t^j}{\lambda} - \frac{jt^{j-1}}{\lambda^2} + \frac{j(j-1)t^{j-2}}{\lambda^3} + \cdots + (-1)^{j-1}\frac{j(j-1)\cdots 2}{\lambda^j} t + (-1)^j \frac{j!}{\lambda^{j+1}} + (-1)^{j+1}\frac{j!}{\lambda^{j+1}} e^{-\lambda t}. \tag{31}$$

Hence, from $z_{\nu j} = \frac{dw_{\nu j}(t)}{dt}$, and one time integration of Eq. (31) along with the initial condition $w_{\nu j}(0) = 0$, we obtain

$$w_{\nu j}(t) = \frac{t^{j+1}}{\lambda(j+1)} + \frac{(-1)^{j+2} j!}{\lambda^{j+2}} e^{-\lambda t} + \sum_{i=0}^{j} \frac{(-1)^{i+1} i! \binom{j}{i} t^{j-i}}{\lambda^{i+2}}, \; \nu = 0, \ldots, \varrho - 2, j = 0, \ldots, \Lambda. \; \square$$

Substituting Eqs. (23) and (28) in (21) yields

$$\hat{\Psi}_{\nu,\Lambda}(t) = \lambda_\nu + \frac{\mu_\nu}{\lambda} - \frac{\mu_\nu}{\lambda} e^{-\lambda t} + \frac{1}{\lambda T_\nu^*(x_\nu)}\int_0^t \left(1 - e^{\lambda(\tau - t)}\right)\mathcal{G}(x_\nu, \tau) d\tau$$

$$+ \sum_{j=0}^{\Lambda} \epsilon_{\nu j}\Big[\frac{t^{j+1}}{\lambda(j+1)} + \frac{(-1)^{j+2} j!}{\lambda^{j+2}} e^{-\lambda t} + \sum_{i=0}^{j} \frac{(-1)^{i+1} i! \binom{j}{i} t^{j-i}}{\lambda^{i+2}}\Big]. \tag{32}$$

If $\{\hat{\Psi}_{\nu,\Lambda}(t)\}_{\nu=0}^{\varrho-2}$ was the exact solution of Eq. (17), then

$$\Big[v''_{\nu}(t)+\lambda v'_{\nu}(t)+\sum_{j=0}^{\Lambda}\epsilon_{\nu j}\Big(w''_{\nu j}(t)+\lambda w'_{\nu j}(t)\Big)\Big]\mathcal{S}_{\nu}(x_{\nu})$$

$$+\sum_{\substack{n=0\\ n\neq\nu}}^{\varrho-2}\Big[v''_{n}(t)+\lambda v'_{n}(t)+\sum_{j=0}^{\Lambda}\epsilon_{nj}\Big(w''_{nj}(t)+\lambda w'_{nj}(t)\Big)\Big]\mathcal{S}_{n}(x_{\nu})$$

$$+\frac{(-1)^{\varrho+1}\left(\psi''_1(t)+\lambda\psi'_1(t)\right)+\left(\psi''_0(t)+\lambda\psi'_0(t)\right)}{(-1)^{\varrho-1}+(-1)^{\varrho+1}}\mathcal{S}_{\varrho-1}(x_{\nu})+\frac{(-1)^{\varrho-1}\left(\psi''_1(t)+\lambda\psi'_1(t)\right)-\left(\psi''_0(t)+\lambda\psi'_0(t)\right)}{(-1)^{\varrho-1}+(-1)^{\varrho+1}}\mathcal{S}_{\varrho}(x_{\nu})$$

$$+\sum_{n=0}^{\varrho-2}\left[\frac{(-1)^{\varrho}-(-1)^{n}}{(-1)^{\varrho-1}+(-1)^{\varrho+1}}\mathcal{S}_{\varrho-1}(x_{\nu})+\frac{(-1)^{\varrho}+(-1)^{n}}{(-1)^{\varrho-1}+(-1)^{\varrho+1}}\mathcal{S}_{\varrho}(x_{\nu})\right]\times$$

$$\Big(v''_{n}(t)+\lambda v'_{n}(t)+\sum_{j=0}^{\Lambda}\epsilon_{nj}\Big(w''_{nj}(t)+\lambda w'_{nj}(t)\Big)\Big)$$

$$-\mathcal{K}\sum_{n=\rho\lceil\gamma\rceil}^{\varrho}\sum_{\kappa=0}^{n-\rho\lceil\gamma\rceil}\theta_{n,\kappa}^{\rho,\gamma}\Psi_{n}(t)x_{\nu}^{n-\kappa-\rho\gamma}$$

$$+\mathcal{F}\Big(\frac{(-1)^{\varrho+1}\psi_1(t)+\psi_0(t)}{(-1)^{\varrho-1}+(-1)^{\varrho+1}}\mathcal{S}_{\varrho-1}(x_{\nu})+\frac{(-1)^{\varrho-1}\psi_1(t)-\psi_0(t)}{(-1)^{\varrho-1}+(-1)^{\varrho+1}}\mathcal{S}_{\varrho}(x_{\nu})$$

$$+\sum_{n=0}^{\varrho-2}\left[\frac{(-1)^{\varrho}-(-1)^{n}}{(-1)^{\varrho-1}+(-1)^{\varrho+1}}\mathcal{S}_{\varrho-1}(x_{\nu})+\frac{(-1)^{\varrho}+(-1)^{n}}{(-1)^{\varrho-1}+(-1)^{\varrho+1}}\mathcal{S}_{\varrho}(x_{\nu})+\mathcal{S}_{n}(x_{\nu})\right]\left(v_{n}(t)+\sum_{j=0}^{\Lambda}\epsilon_{nj}w_{nj}(t)\right)\Big)$$

$$=\mathcal{G}(x_{\nu},t),\ \nu=0,\ \cdots,\ \varrho-2. \tag{33}$$

Thus from Theorems 3 and 4, relation (33) can be rewritten as

$$\Big(\sum_{j=0}^{\Lambda}\epsilon_{\nu j}t^{j}\Big)\mathcal{S}_{\nu}(x_{\nu})+\sum_{\substack{n=0\\ n\neq\nu}}^{\varrho-2}\Big(\frac{\mathcal{G}(x_n,t)}{\mathcal{S}_n(x_n)}+\sum_{j=0}^{\Lambda}\epsilon_{nj}t^{j}\Big)\mathcal{S}_{n}(x_{\nu})$$

$$+\frac{(-1)^{\varrho+1}\left(\psi''_1(t)+\lambda\psi'_1(t)\right)+\left(\psi''_0(t)+\lambda\psi'_0(t)\right)}{(-1)^{\varrho-1}+(-1)^{\varrho+1}}\mathcal{S}_{\varrho-1}(x_{\nu})+\frac{(-1)^{\varrho-1}\left(\psi''_1(t)+\lambda\psi'_1(t)\right)-\left(\psi''_0(t)+\lambda\psi'_0(t)\right)}{(-1)^{\varrho-1}+(-1)^{\varrho+1}}\mathcal{S}_{\varrho}(x_{\nu})$$

$$+\sum_{n=0}^{\varrho-2}\left[\frac{(-1)^{\varrho}-(-1)^{n}}{(-1)^{\varrho-1}+(-1)^{\varrho+1}}\mathcal{S}_{\varrho-1}(x_\nu)+\frac{(-1)^{\varrho}+(-1)^{n}}{(-1)^{\varrho-1}+(-1)^{\varrho+1}}\mathcal{S}_{\varrho}(x_\nu)\right]\times\Big(\frac{\mathcal{G}(x_n,t)}{\mathcal{S}_n(x_n)}+\sum_{j=0}^{\Lambda}\epsilon_{nj}t^j\Big)$$

$$-\mathcal{K}\sum_{n=\rho\lceil\gamma\rceil}^{\varrho-2}\sum_{\kappa=0}^{n-\rho\lceil\gamma\rceil}\theta_{n,\kappa}^{\rho,\gamma}\left(v_n(t)+\sum_{j=0}^{\Lambda}\epsilon_{nj}w_{nj}(t)\right)x_\nu^{n-\kappa-\rho\gamma}$$

$$-\mathcal{K}\sum_{\kappa=0}^{\varrho-1-\rho\lceil\gamma\rceil}\theta_{\varrho-1,\kappa}^{\rho,\gamma}\left[\frac{(-1)^{\varrho+1}\psi_1(t)+\psi_0(t)+\sum_{n=0}^{\varrho-2}\Big[(-1)^{\varrho}-(-1)^{n}\Big]\times\left(v_n(t)+\sum_{j=0}^{\Lambda}\epsilon_{nj}w_{nj}(t)\right)}{(-1)^{\varrho-1}+(-1)^{\varrho+1}}\right]x_\nu^{\varrho-1-\kappa-\rho\gamma}$$

$$-\mathcal{K}\sum_{\kappa=0}^{\varrho-\rho\lceil\gamma\rceil}\theta_{\varrho,\kappa}^{\rho,\gamma}\left[\frac{(-1)^{\varrho-1}\psi_1(t)-\psi_0(t)+\sum_{n=0}^{\varrho-2}\Big[(-1)^{\varrho}+(-1)^{n}\Big]\times\left(v_n(t)+\sum_{j=0}^{\Lambda}\epsilon_{nj}w_{nj}(t)\right)}{(-1)^{\varrho-1}+(-1)^{\varrho+1}}\right]x_\nu^{\varrho-1-\kappa-\rho\gamma}$$

$$+\mathcal{F}\Big(\frac{(-1)^{\varrho+1}\psi_1(t)+\psi_0(t)}{(-1)^{\varrho-1}+(-1)^{\varrho+1}}\mathcal{S}_{\varrho-1}(x_\nu)+\frac{(-1)^{\varrho-1}\psi_1(t)-\psi_0(t)}{(-1)^{\varrho-1}+(-1)^{\varrho+1}}\mathcal{S}_{\varrho}(x_\nu)$$

$$+\sum_{n=0}^{\varrho-2}\left[\frac{(-1)^{\varrho}-(-1)^{n}}{(-1)^{\varrho-1}+(-1)^{\varrho+1}}\mathcal{S}_{\varrho-1}(x_\nu)+\frac{(-1)^{\varrho}+(-1)^{n}}{(-1)^{\varrho-1}+(-1)^{\varrho+1}}\mathcal{S}_{\varrho}(x_\nu)+\mathcal{S}_n(x_\nu)\right]\left(v_n(t)+\sum_{j=0}^{\Lambda}\epsilon_{nj}w_{nj}(t)\right)\Big)=0,$$

$$\nu=0,\ \cdots,\ \varrho-2. \tag{34}$$

However, because $\{\hat{\Psi}_{\nu,\Lambda}(x)\}_{\nu=0}^{\varrho-2}$ is not the exact solution of Eq. (17), defining the following residual functions is valid.

$$\mathcal{R}_{\nu,\Lambda}(t;\epsilon)=\Big(\sum_{j=0}^{\Lambda}\epsilon_{\nu j}t^j\Big)\mathcal{S}_\nu(x_\nu)+\sum_{\substack{n=0\\ n\neq\nu}}^{\varrho-2}\Big(\frac{\mathcal{G}(x_n,t)}{\mathcal{S}_n(x_n)}+\sum_{j=0}^{\Lambda}\epsilon_{nj}t^j\Big)\mathcal{S}_n(x_\nu)$$

$$+\frac{(-1)^{\varrho+1}\big(\psi_1''(t)+\lambda\psi_1'(t)\big)+\big(\psi_0''(t)+\lambda\psi_0'(t)\big)}{(-1)^{\varrho-1}+(-1)^{\varrho+1}}\mathcal{S}_{\varrho-1}(x_\nu)+\frac{(-1)^{\varrho-1}\big(\psi_1''(t)+\lambda\psi_1'(t)\big)-\big(\psi_0''(t)+\lambda\psi_0'(t)\big)}{(-1)^{\varrho-1}+(-1)^{\varrho+1}}\mathcal{S}_{\varrho}(x_\nu)$$

$$+\sum_{n=0}^{\varrho-2}\left[\frac{(-1)^{\varrho}-(-1)^{n}}{(-1)^{\varrho-1}+(-1)^{\varrho+1}}\mathcal{S}_{\varrho-1}(x_\nu)+\frac{(-1)^{\varrho}+(-1)^{n}}{(-1)^{\varrho-1}+(-1)^{\varrho+1}}\mathcal{S}_{\varrho}(x_\nu)\right]\times\Big(\frac{\mathcal{G}(x_n,t)}{\mathcal{S}_n(x_n)}+\sum_{j=0}^{\Lambda}\epsilon_{nj}t^j\Big)$$

$$-\mathcal{K}\sum_{n=\rho\lceil\gamma\rceil}^{\varrho-2}\sum_{\kappa=0}^{n-\rho\lceil\gamma\rceil}\theta_{n,\kappa}^{\rho,\gamma}\left(v_n(t)+\sum_{j=0}^{\Lambda}\epsilon_{nj}w_{nj}(t)\right)x_\nu^{n-\kappa-\rho\gamma}$$

$$-\mathcal{K}\sum_{\kappa=0}^{\varrho-1-\rho\lceil\gamma\rceil}\theta_{\varrho-1,\kappa}^{\rho,\gamma}\left[\frac{(-1)^{\varrho+1}\psi_1(t)+\psi_0(t)+\sum_{n=0}^{\varrho-2}\left[(-1)^{\varrho}-(-1)^n\right]\times\left(v_n(t)+\sum_{j=0}^{\Lambda}\epsilon_{nj}w_{nj}(t)\right)}{(-1)^{\varrho-1}+(-1)^{\varrho+1}}\right]x_\nu^{\varrho-1-\kappa-\rho\gamma}$$

$$-\mathcal{K}\sum_{\kappa=0}^{\varrho-\rho\lceil\gamma\rceil}\theta_{\varrho,\kappa}^{\rho,\gamma}\left[\frac{(-1)^{\varrho-1}\psi_1(t)-\psi_0(t)+\sum_{n=0}^{\varrho-2}\left[(-1)^{\varrho}+(-1)^n\right]\times\left(v_n(t)+\sum_{j=0}^{\Lambda}\epsilon_{nj}w_{nj}(t)\right)}{(-1)^{\varrho-1}+(-1)^{\varrho+1}}\right]x_\nu^{\varrho-1-\kappa-\rho\gamma}$$

$$+\mathcal{F}\Big(\frac{(-1)^{\varrho+1}\psi_1(t)+\psi_0(t)}{(-1)^{\varrho-1}+(-1)^{\varrho+1}}\mathcal{S}_{\varrho-1}(x_\nu)+\frac{(-1)^{\varrho-1}\psi_1(t)-\psi_0(t)}{(-1)^{\varrho-1}+(-1)^{\varrho+1}}\mathcal{S}_{\varrho}(x_\nu)$$

$$+\sum_{n=0}^{\varrho-2}\left[\frac{(-1)^{\varrho}-(-1)^n}{(-1)^{\varrho-1}+(-1)^{\varrho+1}}\mathcal{S}_{\varrho-1}(x_\nu)+\frac{(-1)^{\varrho}+(-1)^n}{(-1)^{\varrho-1}+(-1)^{\varrho+1}}\mathcal{S}_{\varrho}(x_\nu)+\mathcal{S}_n(x_\nu)\right]\left(v_n(t)+\sum_{j=0}^{\Lambda}\epsilon_{nj}w_{nj}(t)\right)\Big), \tag{35}$$

where $\epsilon = (\epsilon_{00}, \ldots, \epsilon_{0\Lambda}, \ldots, \epsilon_{\varrho-2,0}, \ldots, \epsilon_{\varrho-2,\Lambda})$. In the subsequent subsection, we demonstrate how to compute $\epsilon_{\nu j}{}_{j=0}^{\Lambda}$ for $\nu = 0, \ldots, \varrho - 2$, with the aim of eliminating or minimizing our residual functions over the interval $[0, T]$ on average.

3.3 *Optimizing the residuals*

Now, we aim to minimize the residual functions (35). For this, we obtain the equivalent reference residual $\{\epsilon_{\nu j}\}_{j=0}^{\Lambda}$ for $\nu = 0, \ldots, \varrho - 2$, such that $E_{\nu j}(\epsilon) = 0$, $(j = 0, \ldots, \Lambda, \nu = 0, \ldots, \varrho - 2)$, where

$$E_{\nu j}(\epsilon)=\int_0^T w_j\,\mathcal{R}_{\nu,j}(t;\ \epsilon)dt,\ \text{for}\ j=0,\ \cdots,\ \Lambda,\ \text{and}\ \nu=0,\ \cdots,\ \varrho-2,$$

and the set $w_j{}_{j=0}^{\Lambda}$ refers to a collection of Dirac delta functions that have been displaced. Specifically,

$$w_j=\delta(t-t_j)=\begin{cases}\infty, & t=t_j,\\ 0, & t\neq t_j.\end{cases}$$

Therefore

$$E_{\nu j}(\epsilon)=\int_0^T w_j\mathcal{R}_{\nu,\Lambda}(t;\ \epsilon)dt=\mathcal{R}_{\nu,\Lambda}(t_j;\ \epsilon)=0, j=0,\ \cdots,\ \Lambda. \tag{36}$$

The residual functions are eliminated at $\Lambda + 1$ Chebyshev collocation points t_j, $j = 0, 1, \ldots, \Lambda$, as given by the relationships in (36). By increasing the value of Λ, we can achieve more accurate approximate solutions, as the residual functions $\mathcal{R}v, \Lambda(t; \epsilon)v = 0^{\varrho-2}$ are then eliminated at more points. To obtain these Chebyshev collocation points, we utilized the formula $t_j = \frac{T}{2}(1+\cos(\frac{j\pi}{\Lambda}))$. To calculate the unknown coefficients $\epsilon_{v\,j}{}_{j=0}^{\Lambda}$ for $v = 0, \ldots, \varrho - 2$, we substituted these collocation points into (36) and solved the resulting system.

4. Convergence analysis

Convergence analysis is a crucial aspect of numerical methods for solving differential equations. It involves the study of how well a numerical solution approximates the exact solution as the step size or grid spacing is decreased. The goal is to establish whether the numerical method is convergent, and if so, to determine the rate of convergence. Convergence can be proven mathematically under certain assumptions, such as Lipschitz continuity of the differential equation or consistency and stability of the numerical method. Additionally, error estimates can be derived to quantify the accuracy of the numerical solution. Overall, convergence analysis is necessary to ensure the reliability and accuracy of numerical methods in practice.

This section discusses the suggested method's convergence analysis for solving the Eq. (1). Eqs. (9), (15) and (32) imply that we have

$$\Psi_v(t) \simeq \hat{\Psi}_{v,\Lambda}(t) = \lambda_v + \frac{\mu_v}{\lambda} - \frac{\mu_v}{\lambda}e^{-\lambda t} + \frac{1}{\lambda T_v^*(x_v)}\int_0^t \left(1 - e^{\lambda(\tau - t)}\right)\mathcal{G}(x_v, \tau)d\tau$$

$$+\sum_{j=0}^{\Lambda}\epsilon_{vj}\left[\frac{t^{j+1}}{\lambda(j+1)} + \frac{(-1)^{j+2}j!}{\lambda^{j+2}}e^{-\lambda t} + \sum_{i=0}^{j}\frac{(-1)^{i+1}i!\binom{j}{i}t^{j-i}}{\lambda^{i+2}}\right],$$

$$\Psi_{\varrho-1}(t) \simeq \hat{\Psi}_{\varrho-1,\Lambda}(t) = \frac{(-1)^{\varrho+1}\psi_1(t) + \psi_0(t) + \sum_{n=0}^{\varrho-2}\left(\left[(-1)^{\varrho} - (-1)^n\right]\hat{\Psi}_{n,\Lambda}(t)\right)}{(-1)^{\varrho-1} + (-1)^{\varrho+1}},$$

$$\Psi_{\varrho}(t) \simeq \hat{\Psi}_{\varrho,\Lambda}(t) = \frac{(-1)^{\varrho-1}\psi_1(t) - \psi_0(t) + \sum_{n=0}^{\varrho-2}\left(\left[(-1)^{\varrho} + (-1)^n\right]\hat{\Psi}_{n,\Lambda}(t)\right)}{(-1)^{\varrho-1} + (-1)^{\varrho+1}}.$$

Therefore, an approximation for $\Psi_{\varrho}(x, t)$ can be expressed as follows:

$$\Psi_{\varrho}(x, t) = \sum_{n=0}^{\varrho}\Psi_n(t)\mathcal{S}_n(x) \simeq \sum_{n=0}^{\varrho}\hat{\Psi}_{n,\Lambda}(t)\mathcal{S}_n(x) = \hat{\Psi}_{\varrho,\Lambda}(x, t).$$

We obtain the following residual function by substituting $\hat{\Psi}_{\varrho,\Lambda}(x, t)$ into the original problem (1)

$$Res^{\Lambda}(x,t,\epsilon)=\frac{\partial^2\hat{\Psi}_{\varrho,\Lambda}(x,t)}{\partial t^2}+\lambda\frac{\partial\hat{\Psi}_{\varrho,\Lambda}(x,t)}{\partial t}-\mathcal{K}\,{}^{C}\mathcal{D}^{\gamma,\rho}_{0^+,x}\hat{\Psi}_{\varrho,\Lambda}(x,t)+\mathcal{F}[\hat{\Psi}_{\varrho,\Lambda}(x,t)]-\mathcal{G}(x,t). \tag{37}$$

The vector ϵ consists of unknown coefficients that need to be determined in order to minimize the residual function $Res^{\Lambda}(x, t, \epsilon)$. To achieve this, we evaluate Eq. (37) at the roots x_ν of the Chebyshev polynomial $\mathcal{S}_{\varrho-1}(x)$, where $\nu = 0, 1, \ldots, \varrho - 2$, and define $Res^{\Lambda}_{\nu}(t, \epsilon) = Res^{\Lambda}(x_\nu, t, \epsilon)$.

It is notable that the generalized fractional term ${}^{C}\mathcal{D}^{\gamma,\rho}_{0^+,x}\hat{\Psi}_{\varrho,\Lambda}(x,t)$ in Eq. (37) is frequently not quite solvable. The values of it should therefore be solved numerically. We use the eight-point Gauss quadrature method in the current study. The integration domain should be moved from $[0, x]$ to $[-1, 1]$ as a result.

The result of the transformation $\sigma=\frac{x}{2}(r+1)$ is

$${}^{C}\mathcal{D}^{\gamma,\rho}_{0^+,x}\hat{\Psi}_{\varrho,\Lambda}(x,t)=\frac{\rho^{\gamma-n+1}}{\Gamma(n-\gamma)}\int_0^x\sigma^{\rho-1}(x^{\rho}-\sigma^{\rho})^{n-\gamma-1}\Big(\sigma^{1-\rho}\frac{\partial}{\partial\sigma}\Big)^n\hat{\Psi}_{\varrho,\Lambda}(\sigma,t)d\sigma$$

$$=\frac{\rho^{\gamma-n+1}}{\Gamma(n-\gamma)}\int_0^x\sigma^{(\rho-1)(1-n)}(x^{\rho}-\sigma^{\rho})^{n-\gamma-1}\frac{\partial^n\hat{\Psi}_{\varrho,\Lambda}(\sigma,t)}{\partial\sigma^n}d\sigma$$

$$=\frac{\rho^{\gamma-n+1}}{\Gamma(n-\gamma)}\int_{-1}^1\left(\frac{x}{2}(r+1)\right)^{(\rho-1)(1-n)}\left[x^{\rho}-\left(\frac{x}{2}(r+1)\right)^{\rho}\right]^{n-\gamma-1}\frac{\partial^n\hat{\Psi}_{\varrho,\Lambda}(\frac{x}{2}(r+1),t)}{\partial r^n}\frac{x}{2}dr$$

$$=\frac{\rho^{\gamma-n+1}}{\Gamma(n-\gamma)}\left(\frac{x}{2}\right)^{n-\rho\gamma}\int_{-1}^1(r+1)^{(\rho-1)(1-n)}\Big(2^{\rho}-(r+1)^{\rho}\Big)^{n-\gamma-1}\frac{\partial^n\hat{\Psi}_{\varrho,\Lambda}(\frac{x}{2}(r+1),t)}{\partial r^n}dr.$$

Now, we can use the Gauss quadrature.

The following theorem shows that we can find μ such that $|Res^{\Lambda}_{\nu}(t, \mu)|$ be as small as possible.

Theorem 5. Assuming that $\mathcal{G}(x, t) \in \mathcal{C}([0, x_f] \times [0, T])$, $\psi_0(t), \psi_1(t) \in \mathcal{C}^1([0, T])$, and $\phi_0(x), \phi_1(x) \in \mathcal{C}^1([0, x_f])$, for any $\varepsilon > 0$, there exists an integer K such that $\Lambda \geq K$ implies $|Res^{\Lambda}_{\nu}(t, \epsilon)| < \varepsilon$.

Proof. Assumptions on $\psi_0(t)$, $\psi_1(t)$, $\phi_0(x)$, $\phi_1(x)$, and $\mathcal{G}(x, t)$ concludes that $|Res^{\Lambda}_{\nu}(t, \epsilon)|$ is a continuous function for any $\nu = 0, 1, \ldots, \varrho - 2$. Therefore, expansion of $|Res^{\Lambda}_{\nu}(t, \epsilon)|$, in terms of the SCPs, implies

$$Res^{\Lambda}_{\nu}(t,\epsilon)=\sum_{\kappa=1}^{\infty}\chi_{\nu,\kappa}\mathcal{S}_{\kappa}(t),\ \nu=0,1,\ldots,\varrho-2,$$

where $\chi_{\nu,\kappa}$, $\nu=0,1,\ldots,\varrho-2$, $\kappa\in\mathbb{N}_0$, and

$$\chi_{\nu,\kappa}=\frac{\langle Res^{\Lambda}_{\nu}(t,\epsilon),\mathcal{S}_{\kappa}(t)\rangle_{\varpi}}{\|\mathcal{S}_{\kappa}(t)\|^2_{\varpi}}.$$

If we take

$$\chi_{\nu,\kappa} = 0, \ \nu = 0, 1, \ldots, \varrho - 2, \ \kappa = 0, 1, \ldots, \Lambda, \tag{38}$$

then we may obtain $\epsilon_{\nu,\kappa}$, $\nu = 0, 1, \ldots, \varrho - 2$, $\kappa = 0, 1, \ldots, m$. From $|\mathcal{S}_\kappa(t)| \le 1$, we get

$$|Res_\nu^\Lambda(t, \epsilon)| = \left|\sum_{\kappa=\Lambda+1}^{\infty} \chi_{\nu,\kappa}\mathcal{S}_\kappa(t)\right| \le \sum_{\kappa=\Lambda+1}^{\infty} |\chi_{\nu,\kappa}|, \ \nu = 0, 1, \ldots, \varrho - 2.$$

Consequently, if the shifted Chebyshev expansion of $Res_\nu^\Lambda(t, \epsilon)$ converges, it follows that for any $\varepsilon > 0$, there exists a natural number K such that $\Lambda \ge K$ yields:

$$|Res_\nu^\Lambda(t, \epsilon)| \le \sum_{\kappa=\Lambda+1}^{\infty} |\chi_{\nu,\kappa}| < \varepsilon, \ \nu = 0, 1, \ldots, \varrho - 2. \ \square$$

5. Example

This section utilizes the proposed method to solve a space-fractional wave equation with a generalized fractional operator. Additionally, the computed approximations and the exact solution of the problem are compared to the results of various numerical approaches discussed in the literature. All numerical calculations were performed using Maple 2018 on an 8 GB memory, Intel Core i5-6400, 2.70 GHz CPU machine running on OS Windows 10 (64 bit). Consider the following equation [36-38]

$$\begin{cases} u_{tt} + 0.5u_t + 0.75\ {}^{C}\mathcal{D}_{0^+,x}^{\gamma,\rho} u + \sin(u) = \mathcal{G}(x, t), & (x, t) \in [0, 1]^2, \\ \phi_0(x) = 0, \ \phi_1(x) = 0, \ \psi_0(t) = 0, \ \psi_1(t) = 0, \end{cases} \tag{39}$$

with the exact solution $u(x, t) = t^2x(1 - x)$. For this example we take the fractional order $\gamma = 1.5$. Moreover for the $\rho = 1$, the non-homogeneous part is $\mathcal{G}(x, t) = (2 + t)x(1 - x) - 3\sqrt{\frac{x}{\pi}}t^2 + \sin(t^2x(1 - x))$, and for the $\rho = 0.5$, the non-homogeneous part is $\mathcal{G}(x, t) = (2 + t)x(1 - x) - \frac{8\sqrt{2}\sqrt[4]{x^5}}{5\sqrt{\pi}}t^2 + \sin(t^2x(1 - x))$.

We solve the test problem on the following domains:

$$\mathcal{D}_1 = \{(x, t) \in \mathbb{R}^2 : 0 \le t, x \le 1\}, \quad (\textit{Rectangular domain}),$$

$$\mathcal{D}_i = \{(x, t) \in \mathbb{R}^2 : x(\theta) = r_i(\theta)\cos(\theta), y(\theta) = r_i(\theta)\sin(\theta), 0 \le \theta \le 2\pi\}, i = 1, 2, \quad (\textit{Polar domains}),$$

where $r_1(\theta) = 1 + 1.2\cos(\theta)\sin^2(\theta)$, and $r_2(\theta) = 1 + \cos(4\theta)\sin^4(4\theta)$.

The assumed fractional order in Figure 1 is $\rho = 1$. Figures 1(a)-(b) display the exact solution and absolute error, respectively, in the rectangular domain $\mathcal{D}_1$. Figure 1(c) presents the irregular (polar) domain $\mathcal{D}_2$, while Figure 1(d) shows the corresponding 3D absolute error for $\varrho = 9$ and $\Lambda = 25$ in this domain. In a similar fashion, Figure 1(e) illustrates the irregular (polar) domain $\mathcal{D}_3$, and Figure 1(f) shows the corresponding 3D absolute error for $\varrho = 9$ and $\Lambda = 25$ in this domain.

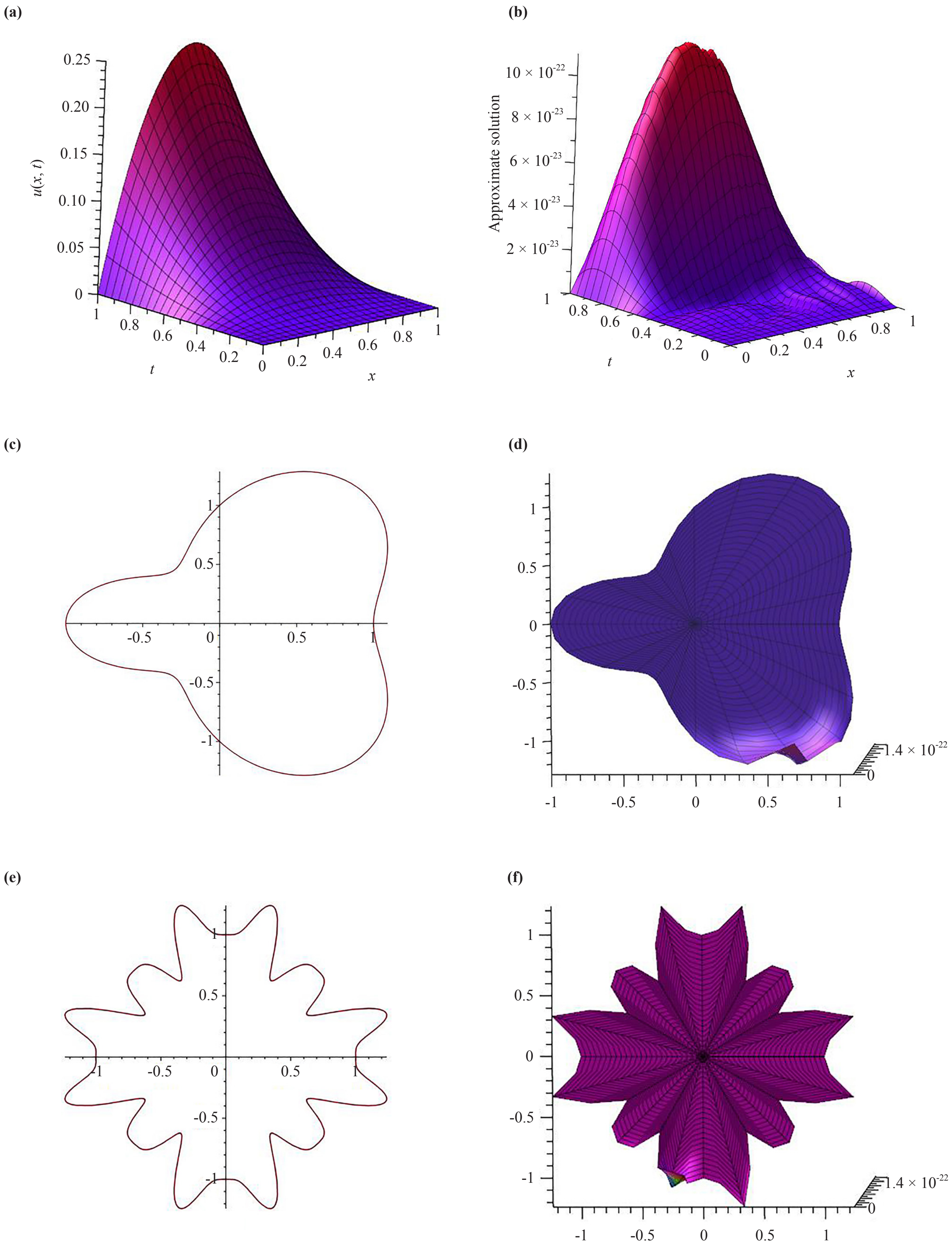

Figure 1. The plots of the model (39) for the fractional order $\rho = 1$

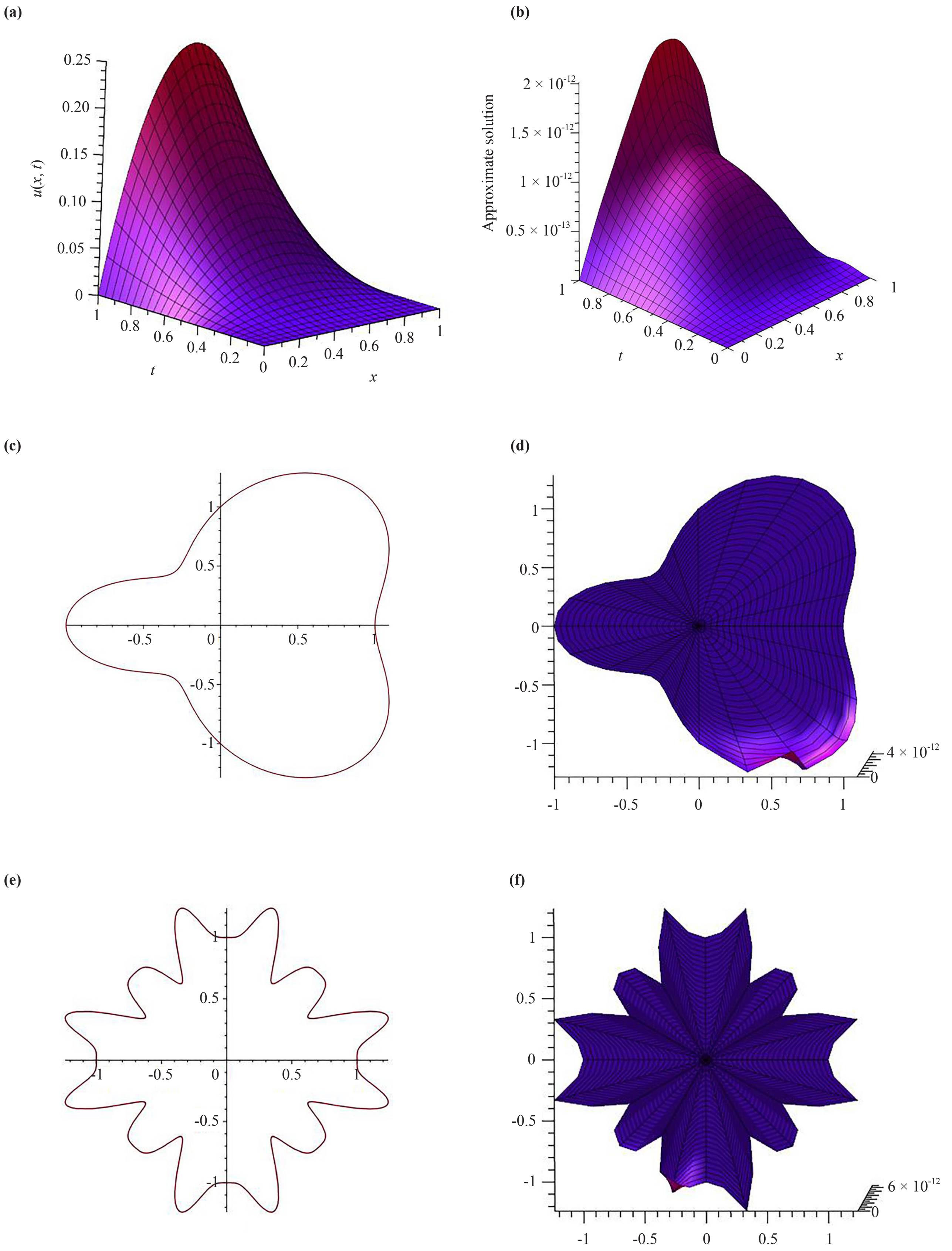

Figure 2. The plots of the model (39) for the fractional order $\rho = 0.5$

The assumption made in Figure 2 is that the fractional order ρ is equal to 0.5. Figures 2(a)-(b) demonstrate the exact solution and absolute error, respectively, in the rectangular region $\mathcal{D}_1$. Figure 2(c) displays the irregular (polar) domain $\mathcal{D}_2$, while Figure 2(d) shows the corresponding 3D absolute error for $\varrho = 9$ and $\Lambda = 25$ in this domain. Similarly, Figure 2(e) presents the irregular (polar) domain $\mathcal{D}_3$, and Figure 2(f) illustrates the corresponding 3D absolute error for $\varrho = 9$ and $\Lambda = 25$ in this domain.

The obtained results by the proposed method for different orders $\rho = 0.5$ and $\rho = 1$ are presented in Table 1, which includes the approximate values and absolute errors.

Table 1. Absolute errors for different values of fractional orders, for the model (39)

	$\rho = 0.5$		$\rho = 1$	
(x, t)	Approximate solution	Absolue error	Approximate solution	Absolue error
(0.1, 0.1)	0.0009	5.46E − 15	0.0009	8.31E − 26
(0.1, 0.5)	0.0225	1.39E − 13	0.0225	3.37E − 24
(0.1, 0.9)	0.0729	3.19E − 13	0.0729	7.76E − 24
(0.5, 0.1)	0.0025	1.10E − 14	0.0025	1.48E − 26
(0.5, 0.5)	0.0625	6.54E − 13	0.0625	1.23E − 23
(0.5, 0.9)	0.2025	1.27E − 12	0.2025	3.62E − 23
(0.9, 0.1)	0.0009	1.64E − 14	0.0009	8.49E − 25
(0.9, 0.5)	0.0225	3.89E − 13	0.0225	1.34E − 24
(0.9, 0.9)	0.0729	6.64E − 13	0.0729	3.72E − 23

6. Conclusions

In conclusion, this paper presented an investigation into a non-homogeneous wave equation with a time fractional derivative using a generalized non-local fractional derivative as the fractional operator. The proposed method for approximating the solutions of the wave equation utilized a novel technique based on the shifted Chebyshev polynomials and a combination of collocation and residual function methods. The method was applied to both rectangular and non-rectangular domains, resulting in approximate solutions. Overall, the results demonstrate the effectiveness and versatility of the proposed method for solving wave equations with generalized fractional derivatives in different types of domains. The authors of the paper identify several contributions that they believe are novel in their study. Firstly, they investigate the non-homogeneous wave equation using a generalized fractional operator, which has not been previously explored. Secondly, they propose a new method that can be applied to other types of equations with general fractional operators. Thirdly, they consider the wave equation with a fractional derivative in non-standard domains, which has not been done before. Finally, they provide a theoretical convergence analysis that can be extended to other equations with generalized fractional derivatives. Overall, the contributions of this study provide new insights into the use of fractional operators in solving differential equations, particularly in non-standard domains. The proposed method has the potential to be applied to other equations beyond the non-homogeneous wave equation, and the convergence analysis can be extended to other types of equations with generalized fractional derivatives. These findings could potentially have implications for various fields of science and engineering where differential equations play a critical role.

Conflict of interest

The authors declare no competing financial interest.

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