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Research Article

Jacobi Rational Operational Approach for Time-Fractional Sub-Diffusion Equation on a Semi-Infinite Domain

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Abstract: In this study, we employ a rational Jacobi collocation technique to effectively address linear time-fractional subdiffusion and reaction sub-diffusion equations. The semi-analytic approximation solution, in this case, represents the spatial and temporal variables as a series of rational Jacobi polynomials. Subsequently, we apply the operational collocation method to convert the target equations into a system of algebraic equations. A comprehensive investigation into the convergence properties of the dual series expansion employed in this approximation is conducted, demonstrating the robustness of the numerical method put forth. To illustrate the method's accuracy and practicality, we present several numerical examples. The advantages of this method are: high accuracy, efficiency, applicability, and high rate of convergence.

***Keywords*:** collocation method, Jacobi Rational (JR) Polynomials, Operational Matrix (OM), sub-diffusion equation

MSC: 35K10, 33C45, 65N30

1. Introduction

A family of orthogonal polynomials known as Jacobi polynomials is commonly defined on a limited interval [-1, 1]. Applying the proper transformations may be used to let Jacobi polynomials suitable in the numerical solution of differential equations on semi-infinite domains. One typical method for applying Jacobi polynomials to semi-infinite domains is to modify the variables to translate the semi-infinite domain into a limited interval where the polynomials are defined. One common approach to applying Jacobi polynomials to semi-infinite domains is to use a change of variables that maps the semi-infinite domain to a bounded interval where the Jacobi polynomials are defined. The most famous mapping, in this case, is $x \to \frac{x-1}{x+1}$, [1-4].

A partial differential equation that captures the behavior of a sub-diffusive process with a fractional derivative in time is the time-fractional sub-diffusion equation (TFSE) on a semi-infinite domain. The equation uses a fractional derivative operator in time and is defined on a semi-infinite domain, generally $[0, \infty)$.

The TFSE on a semi-infinite domain is a partial differential equation that describes the behavior of a sub-diffusive process with a fractiona1 derivative in time. The equation is defined on a semi-infinite domain, typically $[0, \infty)$, and involves a fractional derivative operator in time.

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The general form of the TFSE on a semi-infinite domain is:

$$\partial_t^{\alpha} u(x, t) = D\partial_{xx}^2 u(x, t),$$

where $u(x, t)$ is the unknown function representing the quantity of interest, D is the diffusion coefficient, ∂_t^{α} denotes the fractional derivative of order α in time, and ∂_{xx} represents the second derivative in space. The fractional derivative operator ∂_t^{α} is defined using fractional calculus and captures the nonlocal memory effects inherent in sub-diffusive processes. It generalizes the standard integer-order derivatives to include fractional orders. It is significant to note that proper boundary conditions at $x = 0$ may be necessary to solve the TFSE on a semi-infinite domain and guarantee well-posedness. Depending on the physical or mathematical context of the issue, these boundary conditions could be reflecting, absorbing, or of other sorts. Recently, the TFSE was treated by many iterative and numerical methods, see for instance, [5-9].

A mathematical model that integrates fractional calculus, reaction kinetics, and sub-diffusion to describe the development of a quantity in a system is known as the Time Fractional Reaction Sub-diffusion Equation (TFRSE). It is a development of the standard reaction subdiffusion equation that uses fractional time derivatives to represent memory effects and long-range dependencies found in some physical and biological systems.

The general form of the TFRSE can be written as:

$$\partial_t^{\alpha} u(x, t) + u(x, t) = D\partial_{xx}^2 u(x, t),$$

where $u(x, t)$ represents the quantity of interest, which depends on the spatial variable x, time t, and D is the diffusion coefficient, which can be a constant or a function of space. The term $u(x, t)$ in the LHS represents the reaction term, which describes the local interaction or transformation of the quantity u. To take into consideration non-local and memory-dependent behaviors in the system, the TFRSE mixes the effects of diffusion and reaction with fractional derivatives. The degree of memory and long-range dependency in the system is determined by the order of the fractional derivative, where $\alpha = 1$ corresponds to the traditional reaction sub-diffusion equation. It's important to note that research into analysis and solution methods for fractional differential equations is currently ongoing and that the time-fractional response sub-diffusion equation's specific approaches might change depending on the particular situation and its characteristics. Numerous iterative and finite difference techniques were used to solve the TFRSE in one and two spatial dimensions; for examples, see [10-13]. Fore more studies that used fractional differential equations. please see [14-20].

Collocation method is one of the most important spectral methods [21-24] for solving partial differential equations (PDEs). It is based on the concept of utilizing a series expansion to represent the unknown function in the PDE and then using the collocation approach [25-26] to the numerical approximation of the derivatives in the PDE. When solving nonlinear or linear PDEs with boundary or initia conditions, the approach is especially helpful. The main idea of the OM collocation approach is to create an OM linked to the PDE's derivative operators. The OM is a matrix that connects the derivatives of the unknown function at a set of collocation points to the coefficients of the series expansion of the unknown function. The PDE may be converted into a set of algebraic equations using this matrix, which can then be solved to get the unknown coefficients. A versatile and effective method for numerically solving PDEs is the OM collocation method. It may be expanded to handle systems of PDEs and can deal with issues with a variety of boundary conditions. The number of basis functions utilized, the number of collocation points employed, and the smoothness of the solution all affect the method's accuracy and convergence. Therefore, while using the approach to solve particular issues, great thought should be given to these elements, for recent research on collocation operational method for handling PDEs, see [27-31].

The main aims of this paper can be summarized in the following four-fold:

• Presenting a new technique for solving the TFSE and TFRSE via basis functions based on JR Polynomials by applying the spectral collocation method.

• Reducing the solution of the equation with its conditions into a system of algebraic equations, that can be solved using a suitable solver.

• Discussion of the error bound of the proposed method in detail.

• Presenting some comparisons to show the efficiency of our methods.

In accordance with the aforementioned aspects, the advantage of the proposed methods is: By choosing JR Polynomials as basis functions, and taking a few terms of retained modes, we get an approximate solution with excellent precision and less calculations.

The structure of the sequel of the work is as follows: In Sec 2, we cover some essential formulae and relations of fractional calculus and Jacobi polynomials and recall the OM of integer order derivatives. In Sec 3 and Sec 4, we build and implement two spectral collocation operational algorithms for handling the TFSE and TFRSE, respectively. The Error bound was studied in Sec 5. Some numerical results with comparisons were exhibited in Sec 6, and finally, some concluding remarks were reported in Sec 7.

2. Preliminaries and notations

2.1 *The fractional derivative in the Caputo sense*

In order to familiarise ourselves with the necessary fractional calculus theory, we first go throught he fundamental definitions and characteristics of the fractional integral and derivative in this subsection. Over the past 200 years, numerous definitions and analyzes of fractional calculus have been put forth. Fractional operators like as Riemann-Liouville, Reize, Caputo, and Grünwald-Letnikov are included in these definitions. The Riemann-Liouville operator and the Caputo operator are the two definitionst hat are used the most frequently. We provide some definitions as well as certain fractional calculus characteristics.

Definition 2.1 [32] The Riemann-Liouville fractional integral operator of order μ ($\mu \geq 0$) is defined as

$$\begin{aligned} &J^{\mu} f(\tau) = \frac{1}{\Gamma(\mu)} \int_0^{\tau} (\tau - t)^{\mu-1} f(t)dt, \quad \mu > 0, \quad \tau > 0, \\ &J^0 f(\tau) = f(\tau). \end{aligned} \tag{1}$$

Definition 2.2 [32] The Caputo fractional derivatives of order μ is defined as

$$D^{\mu} f(\tau) = J^{m-\mu} D^m f(\tau) = \frac{1}{\Gamma(m-\mu)} \int_0^{\tau} (\tau - t)^{m-\mu-1} \frac{d^m}{dt^m} f(t)dt, \quad m-1 < \mu \leq m, \quad \tau > 0 \tag{2}$$

where D^m is the classical differential operator of order m. For the Caputo derivative we have

$$D^{\mu}\tau^{\beta} = \begin{cases} 0, & \text{for } \beta < \mu, \\ \dfrac{\Gamma(\beta+1)}{\Gamma(\beta+1-\mu)}\tau^{\beta-\mu}, & \text{for } \beta > \mu. \end{cases} \tag{3}$$

Remember that the Caputo differential operator and the standard differential operator of an integer order are equivalent for $\mu \in N$.

The Caputo's fractional differentiation is a linear operation, much like the integer-order differentiation; i.e.

$$D^{\mu}(\lambda f(\tau) + \upsilon g(\tau)) = \lambda D^{\mu} f(\tau) + \upsilon D^{\mu} g(\tau), \tag{4}$$

where λ and υ are constants.

2.2 *Shifted jacobi OM*

The following recurrence relation can be used to derive the well-known Jacobi polynomials, which are defined on the interval [-1, 1]

$$\mathfrak{J}_i^{(\rho,\varrho)}(t) = \frac{(\rho+\varrho+2i-1)\{\rho^2-\varrho^2+t(\rho+\varrho+2i)(\rho+\varrho+2i-2)\}}{2i(\rho+\varrho+i)(\rho+\varrho+2i-2)}\mathfrak{J}_{i-1}^{(\rho,\varrho)}(t)$$

$$-\frac{(\rho+i-1)(\varrho+i-1)(\rho+\varrho+2i)}{i(\rho+\varrho+i)(\rho+\varrho+2i-2)}\mathfrak{J}_{i-2}^{(\rho,\varrho)}(t), \qquad i = 2, 3, \cdots,$$

where

$$\mathfrak{J}_0^{(\rho,\varrho)}(t) = 1 \quad \text{and} \quad \mathfrak{J}_1^{(\rho,\varrho)}(t) = \frac{\rho+\varrho+2}{2}(t) + \frac{\rho-\varrho}{2}.$$

We defined the so-called shifted Jacobi polynomials by include the change of variable $t = \frac{2\tau}{\mathcal{L}} - 1$ in order to apply these polynomials to the range $\tau \in [0, \mathcal{L}]$. Let the shifted Jacobi polynomials $\mathfrak{J}_i^{(\rho,\varrho)}(\frac{2\tau}{\mathcal{L}} - 1)$ be denoted by $\mathfrak{J}_{\mathcal{L},i}^{(\rho,\varrho)}(\tau)$.

Then $\mathfrak{J}_{\mathcal{L},i}^{(\rho,\varrho)}(\tau)$ can be obtained as follows:

$$\mathfrak{J}_{\mathcal{L},i}^{(\rho,\varrho)}(\tau) = \frac{(\rho+\varrho+2i-1)\{\rho^2-\varrho^2+(\frac{2\tau}{\mathcal{L}}-1)(\rho+\varrho+2i)(\rho+\varrho+2i-2)\}}{2i(\rho+\varrho+i)(\rho+\varrho+2i-2)}\mathfrak{J}_{\mathcal{L},i-1}^{(\rho,\varrho)}(\tau)$$

$$-\frac{(\rho+i-1)(\varrho+i-1)(\rho+\varrho+2i)}{i(\rho+\varrho+i)(\rho+\varrho+2i-2)}\mathfrak{J}_{\mathcal{L},i-2}^{(\rho,\varrho)}(\tau), \qquad i = 2, 3, \cdots, \tag{5}$$

where

$$\mathfrak{J}_{\mathcal{L},0}^{(\rho,\varrho)}(\tau) = 1 \quad \text{and} \quad \mathfrak{J}_{\mathcal{L},1}^{(\rho,\varrho)}(\tau) = \frac{\rho+\varrho+2}{2}\left(\frac{2\tau}{\mathcal{L}}-1\right) + \frac{\rho-\varrho}{2}.$$

The analytic form of the shifted Jacobi polynomials $\mathfrak{J}_{\mathcal{L},i}^{(\rho,\varrho)}(\tau)$ of degree i is given by

$$\mathfrak{J}_{\mathcal{L},i}^{(\rho,\varrho)}(\tau) = \sum_{k=0}^{i}(-1)^{i-k}\frac{\Gamma(i+\varrho+1)\Gamma(i+k+\rho+\varrho+1)}{\Gamma(k+\varrho+1)\Gamma(i+\rho+\varrho+1)(i-k)!\,k!\,\mathcal{L}^k}\tau^k, \tag{6}$$

where

$$\mathfrak{J}_{\mathcal{L},i}^{(\rho,\varrho)}(0) = (-1)^i\frac{\Gamma(i+\varrho+1)}{\Gamma(\varrho+1)\,i!}, \qquad \mathfrak{J}_{\mathcal{L},i}^{(\rho,\varrho)}(\mathcal{L}) = \frac{\Gamma(i+\rho+1)}{\Gamma(\rho+1)\,i!}.$$

The most frequently used of these polynomials are

$$C_{\mathcal{L},i}^{\rho}(\tau) = \frac{i!\,\Gamma(\rho+\frac{1}{2})}{\Gamma(i+\rho+\frac{1}{2})}\mathfrak{J}_{\mathcal{L},i}^{(\rho-\frac{1}{2},\varrho-\frac{1}{2})}(\tau), \qquad T_{\mathcal{L},i}(\tau) = \frac{i!\,\Gamma(\frac{1}{2})}{\Gamma(i+\frac{1}{2})}\mathfrak{J}_{\mathcal{L},i}^{(-\frac{1}{2},-\frac{1}{2})}(\tau),$$

$$P_{\mathcal{L},i}(\tau) = \mathfrak{J}_{\mathcal{L},i}^{(0,0)}(\tau), \qquad U_{\mathcal{L},i}(\tau) = \frac{(i+1)!\,\Gamma(\frac{1}{2})}{\Gamma(i+\frac{3}{2})}\mathfrak{J}_{\mathcal{L},i}^{(\frac{1}{2},\frac{1}{2})}(\tau), \tag{7}$$

$$V_{\mathcal{L},i}(\tau) = \frac{(2i)!!}{(2i-1)!!}\mathfrak{J}_{\mathcal{L},i}^{(\frac{1}{2},-\frac{1}{2})}(\tau), \qquad W_{\mathcal{L},i}(\tau) = \frac{(2i)!!}{(2i-1)!!}\mathfrak{J}_{L,i}^{(-\frac{1}{2},\frac{1}{2})}(\tau),$$

where $C^{\rho}_{\mathcal{L},i}(\tau)$ and $P_{\mathcal{L},i}(\tau)$ are the shifted Gegenbauer (ultraspherical) polynomials and the shifted Legendre polynomials. Also, $T_{\mathcal{L},i}(\tau)$, $U_{\mathcal{L},i}(\tau)$, $V_{\mathcal{L},i}(\tau)$ and $W_{\mathcal{L},i}(\tau)$ are the shifted Chebyshev polynomials of first, second, third and fourth kinds respectively.

The orthogonality condition is

$$\int_0^{\mathcal{L}} \mathfrak{J}^{(\rho,\varrho)}_{\mathcal{L},j}(\tau)\mathfrak{J}^{(\rho,\varrho)}_{\mathcal{L},k}(\tau)w^{(\rho,\varrho)}_{\mathcal{L}}(\tau)d\tau = h_k\delta_{jk}, \tag{8}$$

where $w^{(\rho,\varrho)}_{\mathcal{L}}(\tau) = \tau^{\varrho}(\mathcal{L}-\tau)^{\rho}$, $h_k = \dfrac{\mathcal{L}^{\rho+\varrho+1}\Gamma(k+\rho+1)\Gamma(k+\varrho+1)}{(2k+\rho+\varrho+1)k!\Gamma(k+\rho+\varrho+1)}$ and δ_{jk} is the Kronecker delta function. If $f(\tau)$ is a polynomial of degree n, then the following shifted Jacobi polynomials can be used to describe it as:

$$u(\tau) = \sum_{j=0}^{N} c_j \mathfrak{J}^{(\rho,\varrho)}_{\mathcal{L},j}(\tau) = \mathbf{C}^T\Psi_N(\tau), \tag{9}$$

where the coefficients c_j are given by

$$c_j = \frac{1}{h_j}\int_0^{\mathcal{L}} w^{(\rho,\varrho)}_{\mathcal{L}}(\tau)u(\tau)\mathfrak{J}^{(\rho,\varrho)}_{\mathcal{L},j}(\tau)d\tau \qquad j = 0,1,\cdots. \tag{10}$$

If the shifted Jacobi coefficient vector **C** and the shifted Jacobi vector $\Psi_N(\tau)$ are given by

$$\mathbf{C}^T = [c_0,\ c_1,\ \cdots,\ c_N],$$

$$\Psi_N(\tau) = [\mathfrak{J}^{(\rho,\varrho)}_{\mathcal{L},0}(\tau),\ \mathfrak{J}^{(\rho,\varrho)}_{\mathcal{L},1}(\tau),\ \cdots,\ \mathfrak{J}^{(\rho,\varrho)}_{\mathcal{L},N}(\tau)]^T. \tag{11}$$

Theorem 1 Let $\Psi N(\tau)$ be shifted Jacobi vector defined in Eq. (11) and also suppose $\mu > 0$ then

$$D^{\mu}\Psi_N(\tau) \simeq {}_{\tau}\mathbf{D}_{\mu}\Psi_N(\tau), \tag{12}$$

where ${}_{\tau}\boldsymbol{D}_{\mu}$ is the $(N+1)\times(N+1)$ OM of derivatives of order μ in the Caputo sense and is defined as follows:

$${}_{\tau}\mathbf{D}_{\mu} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 0 \\ \Upsilon_{\mu}(\lceil\mu\rceil,0) & \Upsilon_{\mu}(\lceil\mu\rceil,1) & \Upsilon_{\mu}(\lceil\mu\rceil,2) & \dots & \Upsilon_{\mu}(\lceil\mu\rceil,N) \\ \vdots & \vdots & \vdots & \dots & \vdots \\ \Upsilon_{\mu}(i,0) & \Upsilon_{\mu}(i,1) & \Upsilon_{\mu}(i,2) & \dots & \Upsilon_{\mu}(i,N) \\ \vdots & \vdots & \vdots & \dots & \vdots \\ \Upsilon_{\mu}(N,0) & \Upsilon_{\mu}(N,1) & \Upsilon_{\mu}(N,2) & \dots & \Upsilon_{\mu}(N,N) \end{pmatrix}, \tag{13}$$

where

$$\Upsilon_{\mu}(i,j) = \sum_{k=\lceil\mu\rceil}^{i}\Omega_{ijk},$$

and $\mathbf{\Omega}_{ijk}$ is given by

$$\Omega_{ijk} = \frac{(-1)^{i-k}\ \mathcal{L}^{\rho+\varrho-\mu+1}\Gamma(j+\varrho+1)\Gamma(i+\varrho+1)\ \Gamma(i+k+\rho+\varrho+1)}{h_j\ \Gamma(j+\rho+\varrho+1)\Gamma(k+\varrho+1)\Gamma(i+\rho+\varrho+1)\Gamma(k-\mu+1)(i-k)!}$$

$$\times\sum_{l=0}^{j}\frac{(-1)^{j-l}\Gamma(j+l+\rho+\varrho+1)\Gamma(\rho+1)\Gamma(l+k+\varrho-\mu+1)}{\Gamma(l+\varrho+1)\Gamma(l+k+\rho+\varrho-\mu+2)(j-l)!l!}. \tag{14}$$

Note that in ${}_{\tau}\boldsymbol{D}_{\mu}$, the first $[\mu]$ rows, are all zero.

Proof. For the proof, see [33].

2.3 *JR functions*

Consider the classical Jacobi polynomials $\mathfrak{J}_k^{(\rho,\varrho)}(z)$ on the interval $[-1, 1]$ with the weight function $\omega^{(\rho,\varrho)}(z) = (1-z)^{\rho}(1+z)^{\varrho}$, $\rho, \varrho > -1$,

$$\mathfrak{J}_0^{(\rho,\varrho)}(z)=1,\ \ \mathfrak{J}_1^{(\rho,\varrho)}(z)=\frac{1}{2}(\rho-\varrho+z(\rho+\varrho+2)),$$

the set $\{\mathfrak{J}_k^{(\rho,\varrho)}(z): k = 0, 1, \ldots\}$ forms a complete orthogonal system in the weighted Hilbert space $L^2_{\omega^{\rho,\varrho}(z)}[-1, 1]$ with inner product

$$(u,v)_{\omega^{(\rho,\varrho)}(z)} = \int_{-1}^{1} u(z)v(z)\omega^{(\rho,\varrho)}(z)dz,$$

and norm

$$\|u\|_{\omega^{(\rho,\varrho)}(z)} = (u,u)^{\frac{1}{2}}_{\omega^{(\rho,\varrho)}(z)}.$$

Let the JR functions $\mathfrak{J}_i^{(\rho,\varrho)}\left(\frac{\xi-1}{\xi+1}\right)$ be denoted by $\mathfrak{R}_i^{(\rho,\varrho)}(\xi)$, $\xi\in[0,\infty)$. Then $\mathfrak{R}_i^{(\rho,\varrho)}(\xi)$ can be generated with the aid of the following recurrence formula:

$$\mathfrak{R}_{k+1}^{(\rho,\varrho)}(\xi)=\frac{(2k+\rho+\varrho+1)(2k+\rho+\varrho+2)}{(k+1)(k+\rho+\varrho+1)}\Big[\Big(\frac{((\rho+1)(\rho+\varrho)+2k^2+2k(\rho+\varrho+1))}{(2k+\rho+\varrho)(2k+\rho+\varrho+2)}-\frac{1}{x+1}\Big)\mathfrak{R}_k^{(\rho,\varrho)}(\xi)$$

$$-\frac{(k+\rho)(k+\varrho)}{(2k+\rho+\varrho)(2k+\rho+\varrho+1)}R_{k-1}^{(\rho,\varrho)}(\xi)\Big],\quad k\geq 1, \tag{15}$$

where

$$\mathfrak{R}_0^{(\rho,\varrho)}(\xi)=1,\quad \mathfrak{R}_1^{(\rho,\varrho)}(\xi)=\frac{\xi(\rho+1)-\varrho-1}{\xi+1},$$

and

$$(k+\rho+\varrho)\mathfrak{R}_i^{(\rho,\varrho)}(\xi)=(k+\varrho)\mathfrak{R}_i^{(\rho,\varrho-1)}(\xi)+(k+\rho)\mathfrak{R}_i^{(\rho-1,\varrho)}(\xi).$$

The JR functions $\mathfrak{R}_i^{(\rho,\varrho)}(\xi)$ can be expressed as

$$\mathfrak{R}_i^{(\rho,\varrho)}(\xi)=\sum_{k=0}^{i}(-1)^k\frac{\Gamma(i+\rho+1)\Gamma(i+k+\rho+\varrho+1)}{\Gamma(\rho+k+1)\Gamma(i+\rho+\varrho+1)(i-k)!k!}\frac{1}{(\xi+1)^k},$$

where

$$\Re_i^{(\rho,\varrho)}(0) = (-1)^i \frac{\Gamma(i+\varrho+1)}{i!\Gamma(\varrho+1)}, \tag{16}$$

$$D\Re_i^{(\rho,\varrho)}(0) = \frac{(-1)^{i-1}\Gamma(i+\varrho+1)(i+\rho+\varrho+1)}{(i-1)!\Gamma(\varrho+2)}. \tag{17}$$

Let $\chi_{\Re}^{(\rho,\varrho)}(\xi) = \xi^{\varrho}(\xi+1)^{-\rho-\varrho-2}$, ρ, $\varrho > -1$.
The orthogonality relation of JR functions is

$$\int_0^\infty \Re_i^{(\rho,\varrho)}(\xi)\Re_j^{(\rho,\varrho)}(\xi)\chi_{\Re}^{(\rho,\varrho)}(\xi)d\xi = \begin{cases} \kappa_i^{(\rho,\varrho)}, & i = j, \\ 0, & i \neq j, \end{cases} \tag{18}$$

where

$$\kappa_i^{(\rho,\varrho)} = \frac{\Gamma(i+\rho+1)\Gamma(i+\varrho+1)}{(2i+\rho+\varrho+1)\Gamma(i+1)\Gamma(i+\rho+\varrho+1)}.$$

Any function $u(\xi) \in L^2_{\chi_{\Re}^{(\rho,\varrho)}(\xi)}[0, \infty)$ may be written in terms of JR fun ctions as

$$u(\xi) = \sum_{j=0}^{\infty} c_j \Re_j^{(\rho,\varrho)}(\xi),$$

where the coefficients c_j are given by

$$c_j = \frac{1}{\kappa_j^{(\rho,\varrho)}} \int_0^\infty u(\xi)\Re_j^{(\rho,\varrho)}(\xi)\chi_{\Re}^{(\rho,\varrho)}(\xi)d\xi,\ j = 0,\ 1,\ \ldots.$$

Now, approximate $u(\xi)$ by $(N + 1)$ terms of JR functions, one has

$$u(\xi) \simeq \sum_{j=0}^{N} c_j \Re_j^{(\rho,\varrho)}(\xi) = \mathbf{C}^T \Phi_N(\xi), \tag{19}$$

where $\mathbf{C}$ and $\Phi(\xi)$ are the unknown coefficients vector and the JR function vector respectively and are given by:

$$\mathbf{C} = [c_0,\ c_1,\ \ldots,\ c_N]^T, \tag{20}$$

$$\Phi_N(\xi) = [\Re_0^{(\rho,\varrho)}(\xi),\ \Re_1^{(\rho,\varrho)}(\xi),\ \ldots,\ \Re_N^{(\rho,\varrho)}(\xi)]^T. \tag{21}$$

2.4 *The derivative OM of JR function*

Theorem 2 Assume $\Phi_N(\xi)$ is the JR vector defined in (21). Then the derivative of $\Phi_N(\xi)$ is given by

$$\Phi'_N(\xi) = \frac{d\Phi_N(\xi)}{d\xi} \simeq \boldsymbol{D}\Phi_N(\xi), \tag{22}$$

where $\boldsymbol{D}$ is the $(N + 1)^2$ OM of the derivative. And the nonzero elements D_{rs} for $0 \leq r, s \leq N$ are given as follows:

$$\mathcal{D}_{r+1,r} = (\rho+r+1)\Big(\frac{r}{\rho+\varrho+r+1} - \frac{r(\rho+r+1)}{\rho+\varrho+2r+2} + \frac{(r+1)(\rho+r+2)}{\rho+\varrho+2r+3}\Big),$$

$$\mathcal{D}_{rr} = \frac{(r-1)r(\rho+r)}{\rho+\varrho+2r} - \frac{r(r+1)(\rho+r+1)}{\rho+\varrho+2r+2},$$

$$\mathcal{D}_{r,r+1} = \frac{r(r+1)(\rho+\varrho+r+1)}{(\rho+\varrho+2r+1)(\rho+\varrho+2r+2)},$$

$$\mathcal{D}_{rs} = (-1)^{r+s+1}(2s+\rho+\varrho+1)\prod_{k=1}^{r-s}\frac{(\rho+r-k+1)}{(\rho+\varrho+r-k+1)}, \quad s < r-1,$$

The general form of the matrix ***D*** is a lower-Heisenberg matrix.

Proof. For the proof, see [2].

Now, based on the class of JR functions, we can deduce the following corollaries:

Corollary 1 (Legendre Case) If $\rho = \varrho = 0$, we get the rational Legendre functions. Moreover, the nonzero elements D_{rs} for $0 \le r, s \le N$ can be expressed as:

$$\mathcal{D}_{r+1,r} = \frac{r(7r+13)+4}{4r+6}, \quad \mathcal{D}_{rr} = -r, \quad \mathcal{D}_{r,r+1} = \frac{r(r+1)}{4r+2},$$

$$\mathcal{D}_{rs} = (-1)^{r+s+1}(2s+1), \quad s < r-1,$$

Corollary 2 (ChebyshevT Case) If $\rho = \varrho = -\frac{1}{2}$, we get the first kind rational Chebyshev functions. Moreover, the nonzero elements D_{rs} for $0 \le r, s \le N$ can be expressed as:

$$\mathcal{D}_{r+1,r} = \frac{7}{8}(2r+1), \quad \mathcal{D}_{rr} = -r, \quad \mathcal{D}_{r,r+1} = \frac{r(r+1)}{4r+2},$$

$$\mathcal{D}_{rs} = \frac{2(-1)^{r+s+1}s(r-\frac{1}{2})_{r-s}}{(1-r)_{r-s}}, \quad s < r-1,$$

Corollary 3 (ChebyshevU Case) If $\rho = \varrho = \frac{1}{2}$, we get the second kind rational Chebyshev functions. Moreover, the nonzero elements D_{rs} for $0 \le r, s \le N$ can be expressed as:

$$\mathcal{D}_{r+1,r} = \frac{1}{8}\left(14r + \frac{9}{r+2} + 3\right), \quad \mathcal{D}_{rr} = -r, \quad \mathcal{D}_{r,r+1} = \frac{r(r+2)}{4r+6},$$

$$\mathcal{D}_{rs} = \frac{2(-1)^{r+s+1}(s+1)(r-\frac{1}{2})_{r-s}}{(-r-1)_{r-s}}, \quad s < r-1,$$

Corollary 4 (ChebyshevV Case) If $\rho = -\frac{1}{2}$, $\varrho = \frac{1}{2}$, we get the third kind rational Chebyshev functions. Moreover, the nonzero elements D_{rs} for $0 \le r, s \le N$ can be expressed as:

$$\mathcal{D}_{r+1,r} = \frac{(2r+1)(7r+2)}{8(r+1)}, \quad \mathcal{D}_{rr} = \frac{1}{4} - r, \quad \mathcal{D}_{r,r+1} = \frac{r(r+1)}{4r+2},$$

$$\mathcal{D}_{rs} = \frac{(2s+1)(-1)^{r+s+1}\Gamma(-r)\left(\frac{1}{2}-r\right)_{r-s}}{\Gamma(-s)}, \quad s < r-1.$$

Corollary 5 (ChebyshevW Case) If $\rho = \frac{1}{2}$, $\varrho = -\frac{1}{2}$, we get the fourth kind rational Chebyshev functions. Moreover, the nonzero elements D_{rs} for $0 \le r, s \le N$ can be expressed as:

$$\mathcal{D}_{r+1,r}=\frac{(2r+1)(7r+10)}{8(r+1)},\quad \mathcal{D}_{rr}=-r-\frac{1}{4},\quad \mathcal{D}_{r,r+1}=\frac{r(r+1)}{4r+2},$$

$$\mathcal{D}_{rs}=\frac{2(-1)^{r+s}\Gamma(-r)\Gamma\left(\frac{1}{2}-s\right)}{\Gamma\left(-r-\frac{1}{2}\right)\Gamma(-s)},\quad s<r-1,$$

Remark 1 The OM for n-th derivative can be derived as

$$\frac{d^n\Phi_N(\xi)}{dx^n}=(\boldsymbol{D}^{(1)})^n\Phi_N(\xi), \tag{23}$$

where $n\in N$ and the superscript in $\boldsymbol{D}^{(1)}$, denotes matrix powers. Thus

$$\boldsymbol{D}^{(n)}=(\boldsymbol{D}^{(1)})^n,\ n=1,2,\ldots. \tag{24}$$

3. Collocation approach for the TFSE

This section's goal is to develop a method for the Jacobi spectral collocation equation, in conjunction with the OMs of the Caputo fractional derivative for shifted Jacobi polynomials and derivative for JR functions to numerically solve the TFSE on a semi-infinite domain. Let us consider the TFSE of the form [34]

$$\frac{\partial^\mu\mathcal{W}(\xi,\tau)}{\partial\tau^\mu}-\delta\frac{\partial^2\mathcal{W}(\xi,\tau)}{\partial\xi^2}=\mathcal{H}(\xi,\tau),\ (\xi,\tau)\in[0,\infty)\times[0,\mathcal{L}], \tag{25}$$

subject to the initial condition

$$\mathcal{W}(\xi,0)=\mathcal{G}_1(\xi),\quad \xi\in[0,\infty), \tag{26}$$

and the boundary conditions

$$\mathcal{W}(0,\tau)=\mathcal{G}_2(\tau),\quad \frac{\partial\mathcal{W}(0,\tau)}{\partial\xi}=\mathcal{G}_3(\tau),\quad \tau\in[0,\mathcal{L}], \tag{27}$$

where $0<\mu\le 1$, δ is constant, while $\mathcal{H}(\xi,\tau)$, $\mathcal{G}_1(\xi)$, $\mathcal{G}_2(\tau)$ and $\mathcal{G}_3(\tau)$ are given functions. We approximate $\mathcal{W}(\xi,\tau)$ $\frac{\partial^\mu\mathcal{W}(\xi,\tau)}{\partial\tau^\mu}$ and $\frac{\partial^2\mathcal{W}(\xi,\tau)}{\partial\xi^2}$ by shifted Jacobi polynomials and JR functions as

$$\mathcal{W}(\xi,\tau)\approx\mathcal{W}_N(\xi,\tau)=\sum_{i=0}^{N}\sum_{j=0}^{N}w_{ij}\Re_i^{(\rho,\varrho)}(\xi)\Im_j^{(\rho,\varrho)}(\tau)=\Psi_N(\tau)\mathbf{W}\Phi_N(\xi), \tag{28}$$

$$\frac{\partial^\mu\mathcal{W}_N(\xi,\tau)}{\partial\tau^\mu}=\sum_{i=0}^{N}\sum_{j=0}^{N}w_{ij}\Re_i^{(\rho,\varrho)}(\xi)\frac{\partial^\mu\Im_j^{(\rho,\varrho)}(\tau)}{\partial\tau^\mu}={}_\tau\mathbf{D}_\mu\Psi_N(\tau)\mathbf{W}\Phi_N(\xi), \tag{29}$$

$$\frac{\partial^2\mathcal{W}_N(\xi,\tau)}{\partial\xi^2}=\sum_{i=0}^{N}\sum_{j=0}^{N}w_{ij}\frac{\partial^2\Re_i^{(\rho,\varrho)}(\xi)}{\partial\xi^2}\Im_j^{(\rho,\varrho)}(\tau)=\Psi_N(\tau)\mathbf{W}\mathbf{D}^{(2)}\Phi_N(\xi), \tag{30}$$

where the matrix $\mathbf{W}$ is given by

$$\mathbf{W} = \begin{pmatrix} w_{00} & w_{01} & \cdots & w_{0N} \\ w_{10} & w_{11} & \cdots & w_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ w_{N0} & w_{N1} & \cdots & w_{NN} \end{pmatrix}.$$

Now, using Eqs. (28), (29) and (30), then it is easy to write

$$ {}_{\tau}\mathbf{D}_{\mu}\Psi_{N}(\tau)\mathbf{W}\Phi_{N}(\xi) - \delta\Psi_{N}(\tau)\mathbf{W}\mathbf{D}^{(2)}\Phi_{N}(\xi) = \mathcal{H}(\xi,\tau), \tag{31}$$

$$\Psi_{N}(0)\mathbf{W}\Phi_{N}(\xi) = \mathcal{G}_{1}(\xi), \tag{32}$$

$$\Psi_{N}(\tau)\mathbf{W}\Phi_{N}(0) = \mathcal{G}_{2}(\tau), \tag{33}$$

$$\Psi_{N}(\tau)\mathbf{W}\mathbf{D}\Phi_{N}(0) = \mathcal{G}_{3}(\tau), \tag{34}$$

Now, we take the collocation procedure for solving Eqs. (31)-(34). Suppose $\tau_{\mathcal{L},j}^{(\rho,\varrho)}$, $j = 0, 1, \cdots, N-1$ are the roots of $\mathfrak{J}_{N}^{(\rho,\varrho)}(\tau)$, while $\xi_{i}^{(\rho,\varrho)}$, $i = 0, 1, \cdots, N$ are the roots of $\mathfrak{R}_{N+1}^{(\rho,\varrho)}(\xi)$. Then the application of collocation scheme based on these collocation points enables us to write (31)-(34) as:

$$ {}_{\tau}\mathbf{D}_{\mu}\Psi_{N}(\tau_{\mathcal{L},j}^{(\rho,\varrho)})\mathbf{W}\Phi_{N}(\xi_{i}^{(\rho,\varrho)}) - \delta\Psi_{N}(\tau_{\mathcal{L},j}^{(\rho,\varrho)})\mathbf{W}\mathbf{D}^{(2)}\Phi_{N}(\xi_{i}^{(\rho,\varrho)}) = \mathcal{H}(\xi_{i}^{(\rho,\varrho)},\tau_{\mathcal{L},j}^{(\rho,\varrho)}),\ 1 \le i \le N-1,\quad 0 \le j \le N-1, \tag{35}$$

$$\Psi_{N}(0)\mathbf{W}\Phi_{N}(\xi_{i}^{(\rho,\varrho)}) = \mathcal{G}_{1}(\xi_{i}^{(\rho,\varrho)}),\quad 0 \le i \le N, \tag{36}$$

$$\Psi_{N}(\tau_{\mathcal{L},j}^{(\rho,\varrho)})\mathbf{W}\Phi_{N}(0) = \mathcal{G}_{2}(\tau_{\mathcal{L},j}^{(\rho,\varrho)}),\quad 0 \le j \le N-1, \tag{37}$$

$$\Psi_{N}(\tau_{\mathcal{L},j}^{(\rho,\varrho)})\mathbf{W}\mathbf{D}\Phi_{N}(0) = \mathcal{G}_{3}(\tau_{\mathcal{L},j}^{(\rho,\varrho)}),\quad 0 \le j \le N-1. \tag{38}$$

This yield an algebraic system of $(N + 1)^2$ equations for obtaining w_{ij}, it can be resolved using an appropriate numerical solution, such as Newton's iterative method.

Algorithm 1 Coding algorithm for the TFSE.

Input μ, δ, $\mathcal{L}$, $\mathcal{G}_1(\xi)$, $\mathcal{G}_2(\tau)$, $\mathcal{G}_3(\tau)$ and $\mathcal{H}(\xi, \tau)$.

Step 1 Assume an approximate solution $\mathcal{W}_N(\xi,\tau) = \Psi_N(\tau)\mathbf{W}\Phi_N(\xi)$ as in (28).

Step 2 Using Eqs. (28), (29) and (30) to get the matrix form of Eqs. (31)-(34).

Step 3 Apply the collocation method to obtain a system of equations as in (35)-(38).

Step 4 Use FindRoot command with initial guess $\{w_{ij} = 10^{-i-j}, i, j : 0, 1, \ldots, N\}$, to solve the system (35)-(38) to get w_{ij}.

Output $\mathcal{W}_N(\xi, \tau)$.

4. Collocation approach for the TFRSE

The numerical method introduced in the part before is used in this one to solve the TFRSE in the form [35]

$$\frac{\partial^{\nu}\mathcal{W}(\xi,\tau)}{\partial\tau^{\nu}} - \frac{\partial^{2}\mathcal{W}(\xi,\tau)}{\partial\xi^{2}} + \mathcal{W}(\xi,\tau) = \mathcal{K}(\xi,\tau),\quad (\xi,\tau) \in [0,\infty)\times[0,\mathcal{L}], \tag{39}$$

subject to initial boundary value conditions as follows

$$\mathcal{W}(\xi,0)=\mathcal{G}(\xi), \quad \xi\in[0,\infty), \tag{40}$$

$$\mathcal{W}(0,\tau)=0, \quad \lim_{\xi\to\infty}\mathcal{W}(\xi,\tau)=0, \quad \tau\in[0,\mathcal{L}], \tag{41}$$

where $0 < v \le 1$, while $\mathcal{K}(\xi, \tau)$ and $\mathcal{G}(\xi)$ are given functions. After approximating $\mathcal{W}(\xi, \tau)$ by the shifted Jacobi polynomials and JR functions as in (28) and making use of (12), we can write

$$\frac{\partial^v \mathcal{W}_N(\xi,\tau)}{\partial\tau^v}=\sum_{i=0}^{N}\sum_{j=0}^{N} w_{ij}\mathfrak{R}_i^{(\rho,\varrho)}(\xi)\frac{\partial^v \mathfrak{J}_j^{(\rho,\varrho)}(\tau)}{\partial\tau^v}=_\tau \mathbf{D}_v\Psi_N(\tau)\mathbf{W}\Phi_N(\xi). \tag{42}$$

By substituting (28), (42) and (30) in (39), (40) and (41) we get

$$_\tau\mathbf{D}_v\Psi_N(\tau)\mathbf{W}\Phi_N(\xi)-\Psi_N(\tau)\mathbf{W}\mathbf{D}^{(2)}\Phi_N(\xi)+\Psi_N(\tau)\mathbf{W}\Phi_N(\xi)=\mathcal{K}(\xi,\tau), \tag{43}$$

$$\Psi_N(0)\mathbf{W}\Phi_N(\xi)=\mathcal{G}(\xi), \tag{44}$$

$$\Psi_N(\tau)\mathbf{W}\Phi_N(0)=0, \tag{45}$$

$$\Psi_N(\tau)\mathbf{W}\mathbf{D}\lim_{\xi\to\infty}\Phi_N(\xi)=0. \tag{46}$$

Similarly, as in the pervious section, the collocation method enables us to write (43)-(46) after using the collocation points as:

$$_\tau\mathbf{D}_v\Psi_N(\tau_{\mathcal{L},j}^{(\rho,\varrho)})\mathbf{W}\Phi_N(\xi_i^{(\rho,\varrho)})-\Psi_N(\tau_{\mathcal{L},j}^{(\rho,\varrho)})\mathbf{W}\mathbf{D}^{(2)}\Phi_N(\xi_i^{(\rho,\varrho)})+\Psi_N(\tau_{\mathcal{L},j}^{(\rho,\varrho)})\mathbf{W}\Phi_N(\xi)=\mathcal{K}(\xi_i^{(\rho,\varrho)},\tau_{\mathcal{L},j}^{(\rho,\varrho)}), \tag{47}$$

$$\Psi_N(0)\mathbf{W}\Phi_N(\xi_i^{(\rho,\varrho)})=\mathcal{G}(\xi_i^{(\rho,\varrho)}), \tag{48}$$

$$\Psi_N(\tau_{\mathcal{L},j}^{(\rho,\varrho)})\mathbf{W}\Phi_N(0)=0, \tag{49}$$

$$\Psi_N(\tau_{\mathcal{L},j}^{(\rho,\varrho)})\mathbf{W}\mathbf{D}\lim_{\xi\to\infty}\Phi_N(\xi)=0. \tag{50}$$

This creates algebraic equations of the form $(N + 1)^2$, which can be resolved using Newton's iterative approach. Consequently $\mathcal{W}_N(\xi, \tau)$ given in (28) can be calculated.

Algorithm 2 Coding algorithm for the TFRSE.

Input v, $\mathcal{L}$, $\mathcal{G}(\xi)$ and $\mathcal{K}(\xi, \tau)$

Step 1 Assume an approximate solution $\mathcal{W}_N(\xi,\tau)=\Psi_N(\tau)\mathbf{W}\Phi_N(\xi)$ as in (28).

Step 2 Using Eqs. (28), (30) and (42) to get the matrix form of Eqs. (43)-(46).

Step 3 Apply the collocation method to obtain a system of equations as in (47)-(50).

Step 4 Use FindRoot command with initial guess $\{w_{ij}=10^{-i-j}, i, j: 0, 1, \dots, N\}$, to solve the system (47)-(50) to get w_{ij}.

Output $\mathcal{W}_N(\xi, \tau)$.

5. Error bound

Lemma 1 [36] For $n \ge 1$, $n + a > 1$ and $n + b > 1$, where a, b, are any constants, we have

$$\frac{\Gamma(\mathrm{n}+\mathrm{a})}{\Gamma(\mathrm{n}+\mathrm{b})} \leq \mathbf{o}_{\mathrm{n}}^{\mathrm{a,b}} \mathrm{n}^{\mathrm{a-b}}, \tag{51}$$

where

$$\mathbf{o}_{\mathrm{n}}^{\mathrm{a,b}} = exp\left(\frac{\mathrm{a}-\mathrm{b}}{2(\mathrm{n}+\mathrm{b}-1)} + \frac{1}{12(\mathrm{n}+\mathrm{a}-1)} + \frac{(\mathrm{a}-\mathrm{b})^2}{n} \right) = 1 + O(\mathrm{n}^{-1}). \tag{52}$$

Theorem 3 Assume that $\frac{\partial^{i+j}\,\mathrm{u}(\xi,\,\tau)}{\partial \xi^i\,\partial \tau^j} \in \mathbf{C}([0,\,\infty)\times[0,\,\mathcal{L}))$, $i, j = 0, 1, 2, ..., N$, $\mathcal{W}(\xi,\,\tau)$ is the exact solution in L^2_{ω}, where $\omega = \xi^{\varrho}\,\tau^{\varrho}\,(\xi+1)^{-\rho-\varrho-2}\,(\mathcal{L}-\tau)^{\rho}$ and $\mathcal{W}_N(\xi,\,\tau)$ is the approximate solution obtained from the method belonging to $\Delta = \mathrm{span}\{\Re_i^{(\rho,\varrho)}(\xi)\Im_j^{(\rho,\varrho)}(\tau) : i,\, j = 0,\ 1,\ \ldots,\ N\}$, also set

$$\mathcal{Y}_N = \sup_{(\xi,\tau)\in[0,\infty)\times[0,\mathcal{L})} \left| \frac{\partial^{2(N+1)}\mathcal{W}(\xi,\tau)}{\partial \xi^{N+1}\,\partial \tau^{N+1}} \right|, \tag{53}$$

Then, the following estimation holds:

$$\left\|\mathcal{W}(\xi,\tau) - \mathcal{W}_N(\xi,\tau)\right\|_{L^2_{\omega}} \lesssim \frac{(-1)^N\,\mathcal{Y}_N\,\mathcal{L}^{\frac{2N+\varrho+\rho+3}{2}}}{((N+1)!)^2\,(N)^{\frac{\varrho-2\rho}{2}}}, \tag{54}$$

where $r_1 \lesssim r_2$ means that there exist positive integer n such that $r_1 \leq n r_2$.

Proof. Assume that

$$\mathcal{Q}_N(\xi,\tau) = \sum_{i=0}^{N}\sum_{j=0}^{N-i} \left(\frac{\partial^{i+j}\,\mathcal{W}(\xi,\tau)}{\partial \xi^i\,\partial \tau^j} \right)_{(0,0)} \frac{\xi^i\,\tau^j}{i!\,j!}, \tag{55}$$

is the Taylor expansion of $\mathcal{W}(\xi,\,\tau)$ about the point (0, 0), and [37]

$$\mathcal{W}(\xi,\tau) - \mathcal{Q}_N(\xi,\tau) = \frac{\xi^{N+1}\,\tau^{N+1}\,\partial^{2(N+1)}\,\mathcal{W}(n_1,n_2)}{((N+1)!)^2\,\partial \xi^{N+1}\,\partial \tau^{N+1}}, \quad (n_1,n_2) \in [0,\infty)\times[0,\mathcal{L}). \tag{56}$$

Since $\mathcal{W}_N(\xi,\,\tau)$ is the best approximate solution of $\mathcal{W}(\xi,\,\tau)$ then according to the definition of the best approximation, we get

$$\begin{aligned} \left\|\mathcal{W}(\xi,\tau) - \mathcal{W}_N(\xi,\tau)\right\|^2_{L^2_{\omega}} &\leq \left\|\mathcal{W}(\xi,\tau) - \mathcal{Q}_{N,N}(\xi,\tau)\right\|^2_{L^2_{\omega}} \int_0^{\infty}\int_0^{\mathcal{L}} \frac{\mathcal{Y}_N{}^2\,\xi^{2(N+1)}\,\tau^{2(N+1)}}{((N+1)!)^4}\,\omega\, d\xi\, d\tau \\ &= \frac{\mathcal{Y}_N{}^2}{((N+1)!)^4} \int_0^{\infty} \xi^{2N+\varrho+2}\,(\xi+1)^{-\rho-\varrho-2}\, d\xi \times \int_0^{\mathcal{L}} \tau^{2N+\varrho+2}\,(\mathcal{L}-\tau)^{\rho}\, d\tau. \end{aligned} \tag{57}$$

Based on the definition of beta function, the right hand side of the previous relation can be computed to give the following result

$$\begin{aligned} \left\|\mathcal{W}(\xi,\tau) - \mathcal{W}_{N,N}(\xi,\tau)\right\|^2_{L^2_{\omega}} &\leq \frac{\mathcal{Y}_N{}^2\,\mathcal{L}^2\,N+\varrho+\rho+3}{((N+1)!)^4}\,\beta(2N+\varrho+3, \rho-2N-1)\,\beta(2N+\varrho+3, \rho+1) \\ &= \frac{\mathcal{Y}_N{}^2\,\mathcal{L}^{2N+\varrho+\rho+3}}{((N+1)!)^4} \left(\frac{\Gamma(2N+\varrho+3)\Gamma(\rho-2N-1)}{\Gamma(\varrho+\rho+2)} \right) \left(\frac{\Gamma(2N+\varrho+3)\Gamma(\rho+1)}{\Gamma(2N+\varrho+\rho+4)} \right). \end{aligned} \tag{58}$$

Therefore, the application of Lemma 1 and the following relation

$$\Gamma(a-n)=(-1)^{n-1}\frac{\Gamma(-a)\Gamma(1+a)}{\Gamma(n+1-a)},\quad n\in\mathbb{Z},$$

enable us to write the following estimation

$$\|\mathcal{W}(\xi,\tau)-\mathcal{W}_N(\xi,\tau)\|_{L^2_\omega}\lesssim\frac{(-1)^N\mathcal{Y}_N\mathcal{L}^{\frac{2N+\varrho+\rho+3}{2}}}{((N+1)!)^2(N)^{\frac{\varrho-2\rho}{2}}}. \tag{59}$$

□

Theorem 4 Suppose that $\mathcal{W}(\xi,\tau)$, $\mathcal{W}_N(\xi,\tau)$ and $\dfrac{\partial^{i+j}\mathcal{W}(\xi,\tau)}{\partial\xi^i\partial\tau^j}$ satisfy the condition of Theorem 3 and set

$$\mathcal{X}_{N,m}=\sup_{(\xi,\tau)\in[0,\infty)\times[0,\mathcal{L})}\left|\frac{\partial^{2N-m+2}\mathcal{W}(\xi,\tau)}{\partial\xi^{N-m+1}\partial\tau^{N+1}}\right|,\quad m=1,2. \tag{60}$$

Then, the following estimation holds:

$$\left\|\frac{\partial^m(\mathcal{W}(\xi,\tau)-\mathcal{W}_N(\xi,\tau))}{\partial\xi^m}\right\|_{L^2_\omega}\lesssim\frac{(-1)^{N-m}\mathcal{X}_{N,m}\mathcal{L}^{\frac{2N+\varrho+\rho+3}{2}}}{(N-m+1)!(N+1)!(N-m)^{\frac{\varrho-\rho+1}{2}}(N)^{-\frac{\rho+1}{2}}}. \tag{61}$$

Proof. Assume that $\dfrac{\partial^m\mathcal{Q}_N(\xi,\tau)}{\partial\xi^m}$ is the Taylor expansion of $\dfrac{\partial^m\mathcal{W}(\xi,\tau)}{\partial\xi^m}$ about the point (0, 0), then the residual between $\dfrac{\partial^m\mathcal{W}(\xi,\tau)}{\partial\xi^m}$ and $\dfrac{\partial^m\mathcal{Q}_N(\xi,\tau)}{\partial\xi^m}$ can be written as [37]

$$\frac{\partial^m(\mathcal{W}(\xi,\tau)-\mathcal{Q}_N(\xi,\tau))}{\partial\xi^m}=\frac{\xi^{N-m+1}\tau^{N+1}\partial^{2N-m+2}\mathcal{W}(\bar{n}_1,\bar{n}_2)}{(N+1)!(N-m+1)!\partial\xi^{N-m+1}\partial\tau^{N+1}},\quad(\bar{n}_1,\bar{n}_2)\in[0,\infty)\times[0,\mathcal{L}). \tag{62}$$

Since $\dfrac{\partial^m\mathcal{W}_N(\xi,\tau)}{\partial\xi^m}$ is the best approximate solution of $\dfrac{\partial^m\mathcal{W}(\xi,\tau)}{\partial\xi^m}$, then according to the definition of the best approximation, we get

$$\left\|\frac{\partial^m(\mathcal{W}(\xi,\tau)-\mathcal{W}_N(\xi,\tau))}{\partial\xi^m}\right\|_{L^2_\omega}\le\left\|\frac{\partial^m(\mathcal{W}(\xi,\tau)-\mathcal{Q}_N(\xi,\tau))}{\partial\xi^m}\right\|_{L^2_\omega}. \tag{63}$$

Now, imitating similar steps as in Theorem 3, we get the desired result. □

Theorem 5 Suppose that $\dfrac{\partial^\nu\mathcal{W}(\xi,\tau)}{\partial\tau^\nu}\in\mathbf{C}([0,\infty)\times[0,\mathcal{L}))$ satisfy the conditions of Theorem 3 and set

$$\mathcal{B}_{N,\nu}=\sup_{(\xi,\tau)\in[0,\infty)\times[0,\mathcal{L})}\left|\frac{\partial^{2N-\nu+2}\mathcal{W}(\xi,\tau)}{\partial\xi^{N+1}\partial\tau^{N-\nu+1}}\right|,\quad 0<\nu\le 1. \tag{64}$$

Then, the following estimation holds:

$$\left\|\frac{\partial^\nu(\mathcal{W}(\xi,\tau)-\mathcal{W}_N(\xi,\tau))}{\partial\tau^\nu}\right\|_{L^2_\omega}\lesssim\frac{(-1)^N\mathcal{B}_{N,\nu}\mathcal{L}^{\frac{2N-2\nu+\varrho+\rho+3}{2}}}{(N-\nu+1)!(N+1)!(N)^{\frac{\varrho-\rho+1}{2}}(N-\nu)^{-\frac{\rho+1}{2}}}. \tag{65}$$

Proof. The proof of this theorem can be easily obtained after using the properties of the Caputo operator (3) and imitating similar steps as in Theorems 3 and 4. □

Theorem 6 Assume that $\mathcal{R}_N^1(\xi, \tau)$ be the residual of Eq. (25), then $\|\mathcal{R}_N^1(\xi, \tau)\|_{L^2_\omega}$ will be sufficiently small for the sufficiently large values of *N*.

Proof. $\mathcal{R}_N^1(\xi, \tau)$ of Eq. (25) can be written as

$$\mathcal{R}_N^1(\xi,\tau) = \mathcal{H}(\xi,\tau) - \frac{\partial^\mu \mathcal{W}_N(\xi,\tau)}{\partial\tau^\mu} + \delta\frac{\partial^2\mathcal{W}_N(\xi,\tau)}{\partial\xi^2}$$

$$= \frac{\partial^\mu(\mathcal{W}(\xi,\tau)-\mathcal{W}_N(\xi,\tau))}{\partial\tau^\mu} - \delta\frac{\partial^2(\mathcal{W}(\xi,\tau)-\mathcal{W}_N(\xi,\tau))(\xi,\tau)}{\partial\xi^2}. \tag{66}$$

Taking L^2-norm and using Theorems 4 and 5, we get

$$\|\mathcal{R}_N^1(\xi,\tau)\|_{L^2_\omega} \lesssim \frac{(-1)^N\mathcal{B}_{N,\nu}\mathcal{L}^{\frac{2N-2\nu+\varrho+\rho+3}{2}}}{(N-\nu+1)!(N+1)!(N)^{\frac{\varrho-\rho+1}{2}}(N-\nu)^{-\frac{\rho+1}{2}}} + \frac{(-1)^{N-2}\mathcal{X}_{N,2}\mathcal{L}^{\frac{2N+\delta\varrho+\rho+3}{2}}}{(N-1)!(N+1)!(N-2)^{\frac{\varrho-\rho+1}{2}}(N)^{-\frac{\rho+1}{2}}}. \tag{67}$$

Finally, it is clear from Eq. (67) that $\|\mathcal{R}_N^1(\xi, \tau)\|_{L^2_\omega}$ will be sufficiently small for the sufficiently large values of *N*. This completes the proof of this theorem. □

Theorem 7 Assume that $\mathcal{R}_N^2(\xi, \tau)$ be the residual of Eq. (39), then $\|\mathcal{R}_N^2(\xi, \tau)\|_{L^2_\omega}$ will be sufficiently small for the sufficiently large values of *N*.

Proof. $\mathcal{R}_N^2(\xi, \tau)$ of Eq. (39) can be written as

$$\mathcal{R}_N^2(\xi,\tau) = \mathcal{K}(\xi,\tau) - \frac{\partial^\nu \mathcal{W}_N(\xi,\tau)}{\partial\tau^\nu} + \frac{\partial^2\mathcal{W}_N(\xi,\tau)}{\partial\xi^2} - \mathcal{W}_N(\xi,\tau)$$

$$= \frac{\partial^\nu(\mathcal{W}(\xi,\tau)-\mathcal{W}_N(\xi,\tau))}{\partial\tau^\nu} - \frac{\partial^2(\mathcal{W}(\xi,\tau)-\mathcal{W}_N(\xi,\tau))}{\partial\xi^2} + (\mathcal{W}(\xi,\tau)-\mathcal{W}_N(\xi,\tau)). \tag{68}$$

Taking L^2-norm and using Theorems 3, 4 and 5, we get

$$\|\mathcal{R}_N^2(\xi,\tau)\|_{L^2_\omega} \lesssim \frac{(-1)^N\mathcal{B}_{N,\nu}\mathcal{L}^{\frac{2N-2\nu+\varrho+\rho+3}{2}}}{(N-\nu+1)!(N+1)!(N)^{\frac{\varrho-\rho+1}{2}}(N-\nu)^{-\frac{\rho+1}{2}}} + \frac{(-1)^{N-2}\mathcal{X}_{N,2}\mathcal{L}^{\frac{2N+\varrho+\rho+3}{2}}}{(N-1)!(N+1)!(N-2)^{\frac{\varrho-\rho+1}{2}}(N)^{-\frac{\rho+1}{2}}} + \frac{(-1)^N\mathcal{Y}_N\mathcal{L}^{\frac{2N+\varrho+\rho+3}{2}}}{((N+1)!)^2(N)^{\frac{\varrho-2\rho}{2}}}. \tag{69}$$

At the end, it is clear from Eq. (69) that $\|\mathcal{R}_N^2(\xi, \tau)\|_{L^2_\omega}$ will be sufficiently small for the sufficiently large values of *N*. This completes the proof of this theorem. □

6. Numerical results and comparisons

This section is devoted to testing the performance of our proposed collocation technique. Some test problems are solved and some comparisons are presented to check the applicability and accuracy of our proposed scheme and all of them were performed on the computer using a program written in MATHEMATICA 11.3.

The absolute errors in the given tables are

$$E(\xi,\tau) = |\mathcal{W}(\xi,\tau) - \mathcal{W}_N(\xi,\tau)|, \tag{70}$$

where $\mathcal{W}(\xi, \tau)$ and $\mathcal{W}_N(\xi, \tau)$ are the exact solution and the numerical solution, respectively, at the point (ξ, τ), respectively. Moreover, the maximum absolute errors are given by

$$L^{\infty} = \text{Max}\{E(\xi, \tau) : \forall(\xi, \tau) \in [0, \infty) \times [0, \mathcal{L})\}. \tag{71}$$

Also we can denote to L^2 by

$$L^2 = \sqrt{\frac{\sum_{i=0}^{N}\sum_{j=0}^{N}\left(\mathcal{W}(\xi_i^{(\rho,\varrho)}, \tau_{\mathcal{L},j}^{(\rho,\varrho)}) - \mathcal{W}_N(\xi_i^{(\rho,\varrho)}, \tau_{\mathcal{L},j}^{(\rho,\varrho)})\right)^2}{(N+1)^2}}. \tag{72}$$

Example 1 We consider the TFSE [34]:

$$\frac{\partial^{\mu}\mathcal{W}(\xi, \tau)}{\partial \tau^{\mu}} - \frac{\partial^2 \mathcal{W}(\xi, \tau)}{\partial \xi^2} = \mathcal{H}(\xi, \tau), \quad (\xi, \tau) \in [0, \infty) \times [0, \mathcal{L}], \quad \mu \in (0, 1), \tag{73}$$

with the initial condition,

$$\mathcal{W}(\xi, 0) = 0, \quad \xi \in [0, \infty),$$

and the boundary conditions

$$\mathcal{W}(0, \tau) = \tau^3, \quad \frac{\partial \mathcal{W}(0, \tau)}{\partial \xi} = -\tau^3, \quad \tau \in [0, \mathcal{L}],$$

where

$$\mathcal{H}(\xi, \tau) = \tau^3 e^{-\xi}\left(\frac{6\tau^{-\mu}}{\Gamma(4-\mu)} - 1\right),$$

and the exact solution given by

$$\mathcal{W}(\xi, \tau) = \tau^{\mu} e^{-\xi}, \ (\xi, \tau) \in [0, \infty) \times [0, \mathcal{L}]. \tag{74}$$

For the numerical implementations, we consider the domain [0, 20] × [0, 1].

In Table 1, we have presented that the absolute errors obtained by using collocation method with $\rho = \varrho = \frac{1}{2}$ (second kind shifted Chebyshev collocation method), $\rho = \varrho = 0$ (shifted Legendre collocation method) and $\rho = \varrho = -\frac{1}{2}$ (first kind shifted Chebyshev collocation method) at $N = 12$ and $N = 16$. Our results compare favorably with those obtained by Chebyshev-Laguerre-Gauss-Radau collocation scheme [34], but increasing the number of approximation terms in our algorithm definitely give more accurate results. Meanwhile, we list absolute errors at $N = 12$, $\rho = -\varrho = \frac{1}{2}$ and various choices of μ in Table 2. In Figure 1, we plot the log scale L^{∞} error versus $\rho = \varrho$, N and $\mu = 0.9$. While, in Figure 2, we plotted the τ-direction (a) and ξ-direction (b) curves of exact and numerical solutions with the value of the parameter listed in their captions, respectively. Figure 3 shows the space-time of the approximate solution (left) and its absolute error function (right) for Example 1 with $\rho = -\varrho = \frac{1}{2}$, $\mu = 1$ and $N = 12$. In the case of $\rho = -\varrho = \frac{1}{2}$, $\mu = 1$ and $N = 12$, the absolute error curves in ξ-direction and τ-direction for Example 1 is shown in Figure 4. For various values of μ, the space-time graphs of the absolute error function with $\rho = -\varrho = \frac{1}{2}$ are displayed in Figure 5.

Table 1. Maximum absolute errors for Example 1 at $\mu = 0.5$

$\rho = \varrho$	Our method at $N =$ 12	Our method at $N =$ 16	Bhrawy et al. [34] ($N = 24$, $M = 16$)
$\frac{1}{2}$	2.90×10^{-4}	7.88×10^{-5}	2.49×10^{-4}
0	3.61×10^{-4}	3.17×10^{-5}	1.04×10^{-4}
$-\frac{1}{2}$	7.57×10^{-4}	2.12×10^{-5}	4.61×10^{-5}

Table 2. Absolute errors at $N = 12$, $\rho = -\varrho = \frac{1}{2}$ and various choices of μ for Example 1

ξ	τ	$\mu = 0.1$	$\mu = 0.4$	$\mu = 0.7$	$\mu = 1$	CPU time (s)
2	0.1	2.75×10^{-8}	3.78×10^{-8}	8.40×10^{-9}	1.06×10^{-8}	63.985
	0.5	3.74×10^{-6}	3.26×10^{-6}	2.51×10^{-6}	2.67×10^{-6}	64.017
	0.9	2.20×10^{-5}	2.05×10^{-5}	1.72×10^{-5}	2.12×10^{-5}	64.064
4	0.1	5.03×10^{-8}	1.61×10^{-8}	6.50×10^{-9}	5.18×10^{-8}	64.095
	0.5	7.01×10^{-6}	4.28×10^{-6}	2.24×10^{-6}	1.82×10^{-6}	64.142
	0.9	4.26×10^{-5}	3.08×10^{-5}	2.05×10^{-5}	1.30×10^{-6}	64.189
6	0.1	1.08×10^{-7}	1.62×10^{-7}	1.01×10^{-7}	4.82×10^{-8}	64.220
	0.5	1.37×10^{-5}	1.46×10^{-5}	1.58×10^{-5}	3.29×10^{-6}	64.267
	0.9	3.27×10^{-5}	8.27×10^{-5}	8.61×10^{-5}	5.46×10^{-5}	64.329
8	0.1	2.75×10^{-8}	4.68×10^{-8}	3.76×10^{-8}	3.11×10^{-8}	64.376
	0.5	5.05×10^{-6}	5.67×10^{-6}	4.00×10^{-6}	5.22×10^{-6}	64.407
	0.9	3.35×10^{-5}	7.60×10^{-6}	1.08×10^{-5}	5.90×10^{-5}	64.454
10	0.1	1.18×10^{-7}	3.08×10^{-7}	1.98×10^{-7}	6.21×10^{-10}	64.501
	0.5	1.25×10^{-5}	2.24×10^{-5}	2.92×10^{-5}	1.00×10^{-7}	64.547
	0.9	6.63×10^{-5}	1.10×10^{-4}	1.43×10^{-4}	2.73×10^{-6}	64.594
12	0.1	1.77×10^{-5}	4.05×10^{-7}	2.41×10^{-7}	4.20×10^{-8}	64.641
	0.5	1.99×10^{-5}	3.04×10^{-5}	3.79×10^{-5}	1.47×10^{-5}	64.688
	0.9	1.09×10^{-4}	1.56×10^{-4}	1.89×10^{-4}	1.21×10^{-4}	64.735
14	0.1	1.55×10^{-7}	3.46×10^{-7}	1.89×10^{-7}	1.27×10^{-7}	64.782
	0.5	1.80×10^{-5}	2.57×10^{-5}	3.15×10^{-5}	3.25×10^{-5}	64.829
	0.9	1.00×10^{-4}	1.34×10^{-4}	1.59×10^{-4}	2.47×10^{-4}	64.876
16	0.1	9.60×10^{-8}	1.90×10^{-7}	9.41×10^{-8}	2.13×10^{-7}	64.923
	0.5	1.17×10^{-5}	1.45×10^{-5}	1.70×10^{-5}	4.73×10^{-5}	64.970
	0.9	6.69×10^{-5}	7.87×10^{-5}	8.79×10^{-5}	3.45×10^{-4}	65.001
18	0.1	3.20×10^{-8}	9.63×10^{-9}	6.30×10^{-9}	2.73×10^{-7}	65.048
	0.5	5.06×10^{-6}	1.58×10^{-6}	3.43×10^{-7}	5.61×10^{-5}	65.095
	0.9	3.22×10^{-5}	1.57×10^{-5}	4.78×10^{-6}	3.99×10^{-4}	65.142
20	0.1	1.95×10^{-8}	2.22×10^{-7}	9.26×10^{-8}	2.98×10^{-7}	65.204
	0.5	4.17×10^{-8}	1.05×10^{-5}	1.75×10^{-5}	5.82×10^{-5}	65.235
	0.9	7.49×10^{-6}	4.07×10^{-5}	7.43×10^{-5}	4.08×10^{-4}	65.298

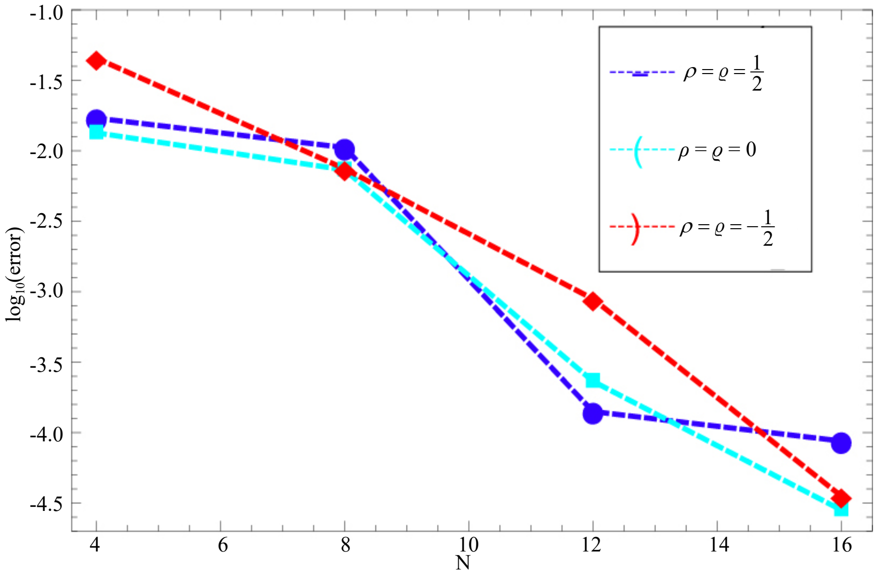

Figure 1. The log scale L^∞ error for Example 1 versus $\rho = \varrho$, N and $\mu = 0.9$

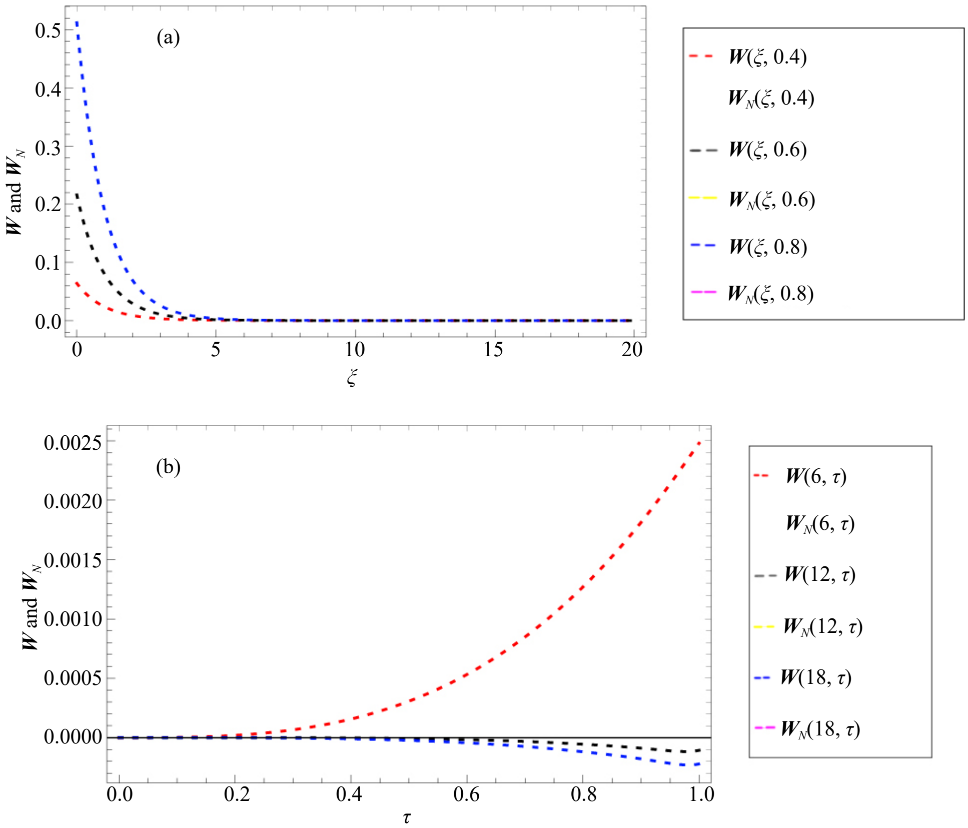

Figure 2. τ-direction curves of exact and numerical solutions (a) and ξ-direction curves of exact and numerical solutions (b) for Example 1 with $\rho = \varrho = 0$, $\mu = 0.9$ and $N = 12$

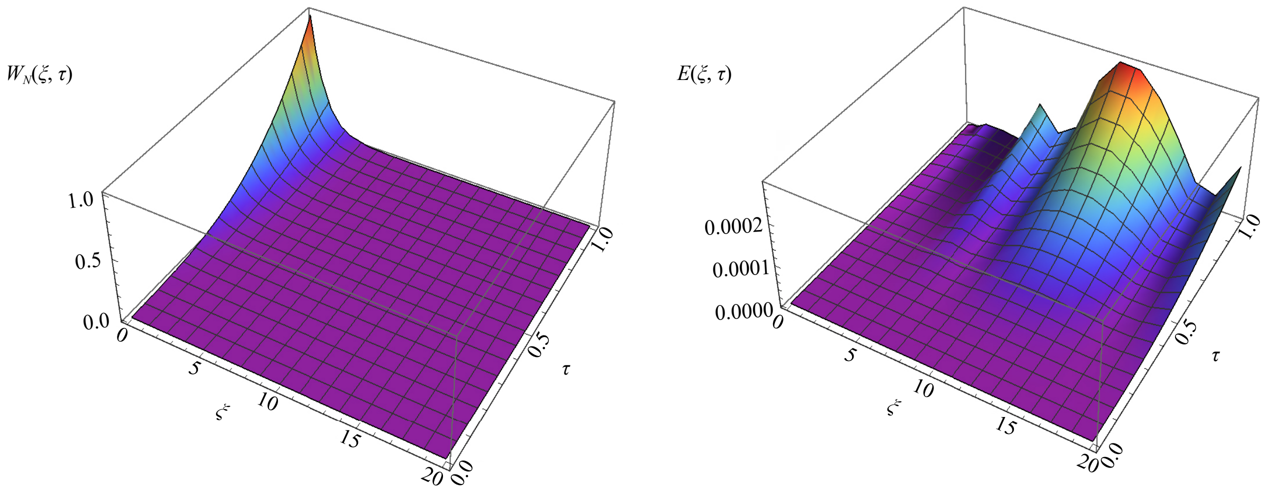

Figure 3. The space-time graphs of the approximate solution (left) and its absolute error function (right) for Example 1 with $\rho = -\varrho = \frac{1}{2}$, $\mu = 1$ and $N = 12$

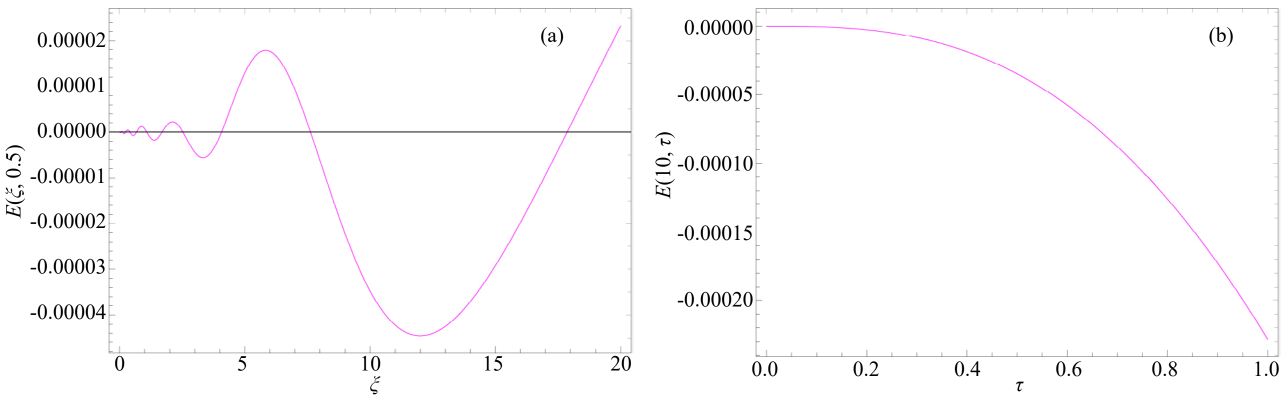

Figure 4. ξ-direction of absolute error (a) and τ-direction of absolute error (b) for Example 1 with $\rho = -\varrho = \frac{1}{2}$, $\mu = 1$ and $N = 12$

Example 2 Consider the following TFRSE [35]:

$$\frac{\partial^\nu \mathcal{W}(\xi, \tau)}{\partial \tau^\nu} - \frac{\partial^2 \mathcal{W}(\xi, \tau)}{\partial \xi^2} + \mathcal{W}(\xi, \tau) = \mathcal{K}(\xi, \tau), \quad (\xi, \tau) \in [0, \infty) \times [0, \mathcal{L}], \quad \nu \in (0, 1), \tag{75}$$

subject to the initial condition

$$\mathcal{W}(\xi, 0) = 0, \quad \xi \in [0, \infty), \tag{76}$$

together with boundary conditions

$$W(0, \tau) = 0, \quad \lim_{\xi \to \infty} \mathcal{W}(\xi, \tau) = 0, \quad \tau \in [0, \mathcal{L}], \tag{77}$$

where

$$\mathcal{K}(\xi, \tau) = \frac{6}{\Gamma(4 - \nu)} \tau^{3-\nu} e^{-\xi} \sin(\xi) + 2\tau^3 e^{-\xi} \cos(\xi) + \tau^3 e^{-\xi} \sin(\xi).$$

The exact solution is given by

$$\mathcal{W}(\xi, \tau) = \tau^{\nu} e^{-\xi} \sin(\xi).$$

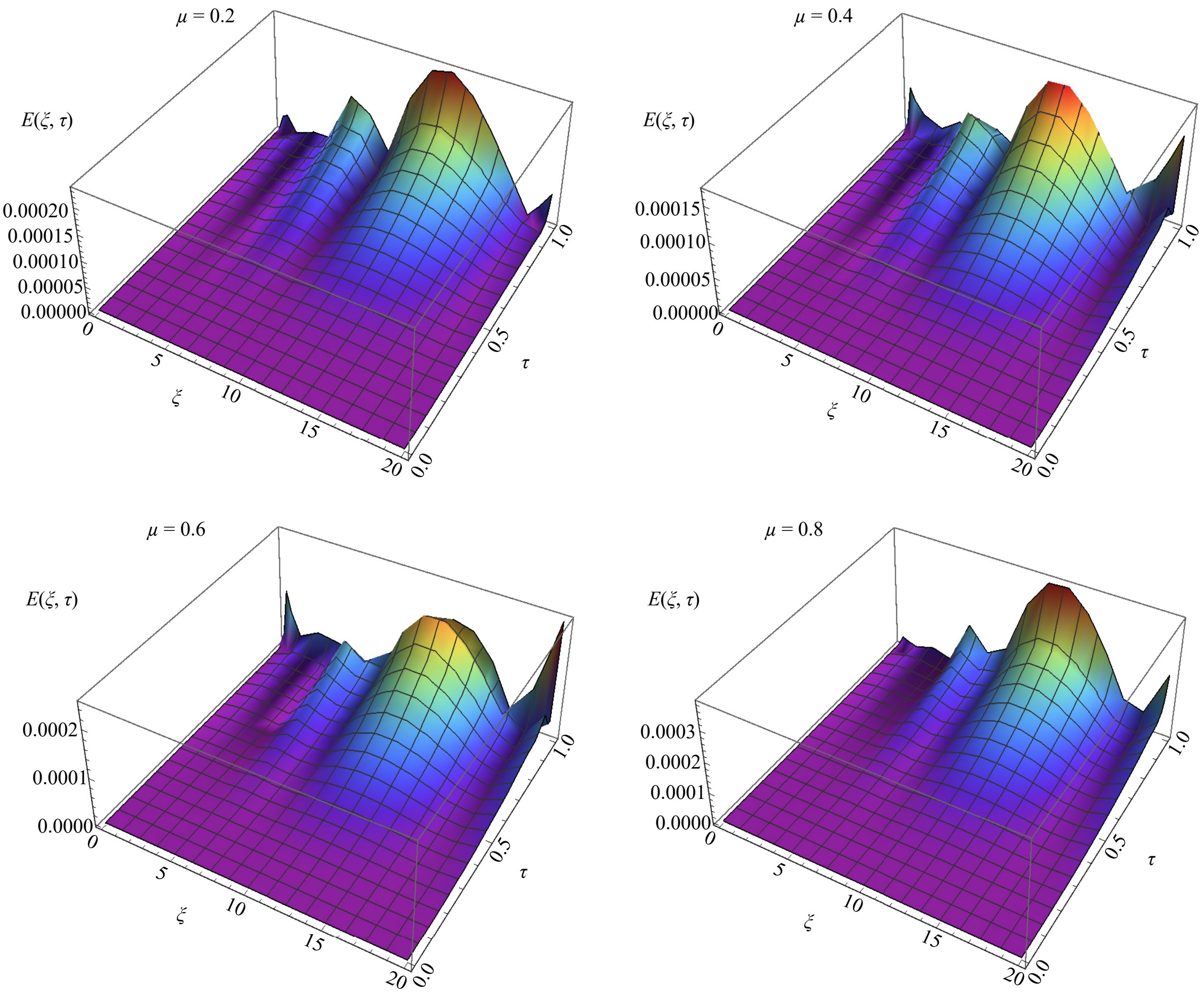

Figure 5. The space-time graphs of the absolute error functions for Example 1 at various choices of μ with $\rho = -\varrho = \frac{1}{2}$ and $N = 12$

The tables show the convergence results of the present method for different ρ, ϱ, ν and $N = 16$. Tables 3 and 4, we have presented that the L^{∞} and L^2-errors. Figure 6 plots the space-time graphs of the approximate solution (left) and its absolute error function (right) with $\rho = -\varrho = \frac{1}{2}$, $\nu = \frac{1}{2}$ and $N = 12$, while Figure 7 presents the ξ-direction of absolute error (a) and τ-direction of absolute error (b) for Example 2 with $\rho = -\varrho = \frac{1}{2}$, $\nu = 0.5$ and $N = 12$.

Table 3. The L^{∞}-errors for different ρ, ϱ, v and $N = 16$ for Example 2

v	$\rho = \varrho = -\frac{1}{2}$	CPU time (s)	$\rho = \varrho = 0$	CPU time (s)	$\rho = \varrho = \frac{1}{2}$	CPU time (s)
0.1	3.70×10^{-4}	444.857	6.88×10^{-4}	374.611	1.73×10^{-3}	265.955
0.2	3.68×10^{-4}	449.393	6.95×10^{-4}	407.827	1.74×10^{-3}	288.392
0.3	3.68×10^{-4}	454.720	7.00×10^{-4}	375.532	1.75×10^{-3}	324.720
0.4	3.46×10^{-4}	477.514	7.06×10^{-4}	468.141	1.77×10^{-3}	293.891
0.5	3.66×10^{-4}	454.999	7.12×10^{-4}	421.565	1.78×10^{-3}	320.032
0.6	3.65×10^{-4}	394.688	7.18×10^{-4}	386.532	1.79×10^{-3}	314.281
0.7	3.64×10^{-4}	477.829	7.24×10^{-4}	360.313	1.81×10^{-3}	321.515
0.8	3.62×10^{-4}	400.033	7.30×10^{-4}	333.984	1.82×10^{-3}	348.109
0.9	3.61×10^{-4}	517.766	7.36×10^{-4}	406.328	1.84×10^{-3}	280.810

Table 4. The L^{2}-errors for different ρ, ϱ, v and $N = 16$ for Example 2

v	$\rho = \varrho = -\frac{1}{2}$	CPU time (s)	$\rho = \varrho = 0$	CPU time (s)	$\rho = \varrho = \frac{1}{2}$	CPU time (s)
0.1	1.99×10^{-4}	400.907	2.73×10^{-4}	349.001	5.09×10^{-4}	253.405
0.2	1.94×10^{-4}	419.545	2.73×10^{-4}	329.203	5.09×10^{-4}	269.703
0.3	1.89×10^{-4}	399.138	2.73×10^{-4}	326.314	5.10×10^{-4}	269.626
0.4	1.85×10^{-4}	434.829	2.74×10^{-4}	322.861	5.12×10^{-4}	263.093
0.5	1.81×10^{-4}	397.719	2.75×10^{-4}	324.358	5.14×10^{-4}	255.937
0.6	1.77×10^{-4}	438.862	2.76×10^{-4}	306.971	5.16×10^{-4}	276.030
0.7	1.74×10^{-4}	378.170	2.77×10^{-4}	356.562	5.19×10^{-4}	239.668
0.8	1.71×10^{-4}	389.769	2.79×10^{-4}	394.607	5.22×10^{-4}	250.954
0.9	1.68×10^{-4}	412.984	2.80×10^{-4}	398.187	5.25×10^{-4}	295.346

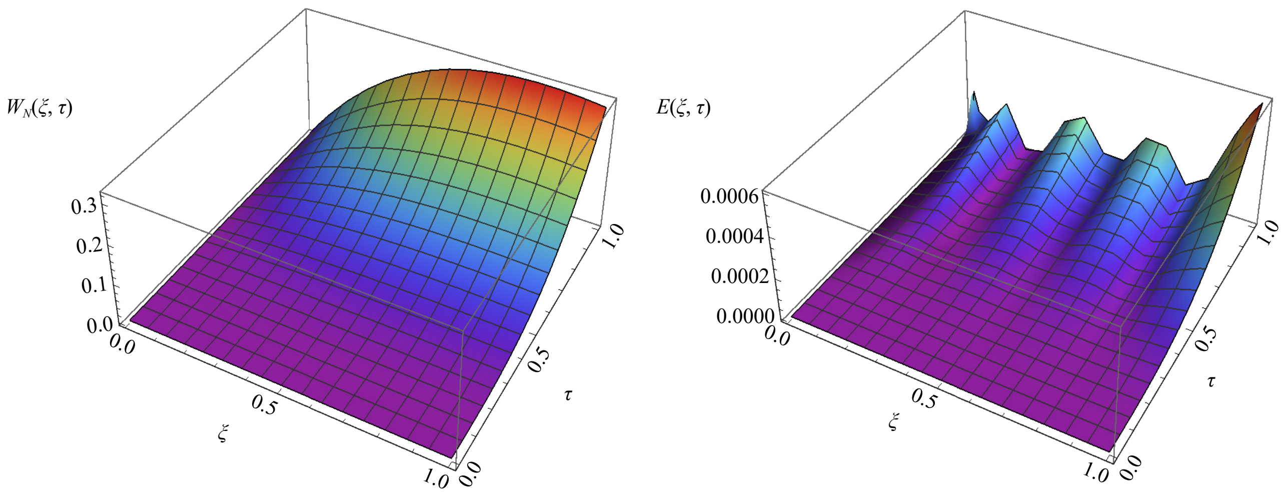

Figure 6. The space-time graphs of the approximate solution (left) and its absolute error function (right) for Example 2 with $\rho = -\varrho = \frac{1}{2}$, $v = 12$ and $N = 12$

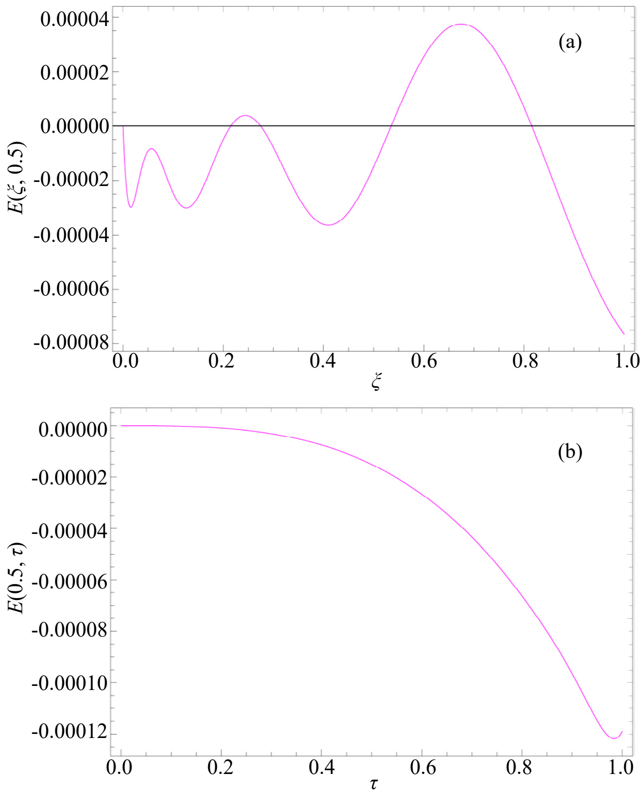

Figure 7. ξ-direction of absolute error (a) and τ -direction of absolute error (b) for Example 2 with $\rho = -\varrho = \frac{1}{2}$, $\nu = 0.5$ and $N = 12$

Example 3 At the end of this section, we consider the fractional diffusion equation (75) with the analytical solution which is given by $\mathcal{W}(\xi, \tau) = \tau^{\nu+2} e^{-\xi} \sin(\xi)$ and $\mathcal{K}(\xi, \tau) = e^{-\xi} \tau^{\nu+2} \sin(\xi) + 2e^{-\xi} \tau^{\nu+2} \cos(\xi) - \frac{\pi \tau^2 e^{-\xi} \csc(\pi\nu) \sin(\xi)}{2\Gamma(-\nu-2)}$. The numerical results are shown in Tables 5, 6 and Figure 8.

The convergence results of the present method with various choices of Jacobi parameters ρ, ϱ and various choices of the fractional derivative ν are given in Tables 5 and 6, while Figure 8 presents the ξ-direction of absolute error (a) and τ-direction of absolute error (b) for Example 3 with $\rho = \varrho = 0$, $\nu = 0.5$ and $N = 12$.

Table 5. The L^∞-errors for different ρ, ϱ, ν and $N = 12$ for Example 3

ν	$\rho = \varrho = -\frac{1}{2}$	CPU time (s)	$\rho = \varrho = 0$	CPU time (s)	$\rho = \varrho = \frac{1}{2}$	CPU time (s)
0.1	1.46×10^{-3}	80.158	2.39×10^{-3}	60.658	4.78×10^{-3}	65.198
0.2	2.41×10^{-3}	79.828	2.31×10^{-3}	62.437	4.22×10^{-3}	65.967
0.3	1.21×10^{-3}	75.33	2.30×10^{-3}	61.456	4.53×10^{-3}	65.703
0.4	1.34×10^{-3}	76.073	2.39×10^{-3}	61.482	4.71×10^{-3}	63.702
0.5	1.31×10^{-3}	79.250	2.32×10^{-3}	59.799	4.72×10^{-3}	65.280
0.6	1.28×10^{-3}	78.873	2.33×10^{-3}	58.657	4.66×10^{-3}	65.142
0.7	1.30×10^{-3}	75.001	2.29×10^{-3}	61.987	4.65×10^{-3}	65.406
0.8	1.30×10^{-3}	75.889	2.30×10^{-3}	60.734	4.71×10^{-3}	64.765
0.9	1.30×10^{-3}	72.86	2.30×10^{-3}	58.124	4.66×10^{-3}	60.141

Table 6. The L^2-errors for different ρ, ϱ, v and $N = 12$ for Example 3

v	$\rho = \varrho = -\frac{1}{2}$	CPU time (s)	$\rho = \varrho = 0$	CPU time (s)	$\rho = \varrho = \frac{1}{2}$	CPU time (s)
0.1	5.27×10^{-4}	71.423	7.78×10^{-4}	60.79	1.43×10^{-3}	62.390
0.2	5.52×10^{-4}	72.000	7.63×10^{-4}	61.38	1.39×10^{-3}	62.139
0.3	5.11×10^{-4}	72.267	7.52×10^{-4}	61.612	1.37×10^{-3}	62.203
0.4	5.03×10^{-4}	73.076	7.45×10^{-4}	61.592	1.36×10^{-3}	60.889
0.5	4.92×10^{-4}	76.25	7.33×10^{-4}	60.908	1.34×10^{-3}	62.249
0.6	4.80×10^{-4}	75.873	7.24×10^{-4}	59.828	1.32×10^{-3}	62.017
0.7	4.69×10^{-4}	72.801	7.14×10^{-4}	62.096	1.30×10^{-3}	62.797
0.8	4.58×10^{-4}	72.420	7.06×10^{-4}	61.875	1.29×10^{-3}	61.218
0.9	4.47×10^{-4}	72.235	6.99×10^{-4}	59.515	1.27×10^{-3}	57.657

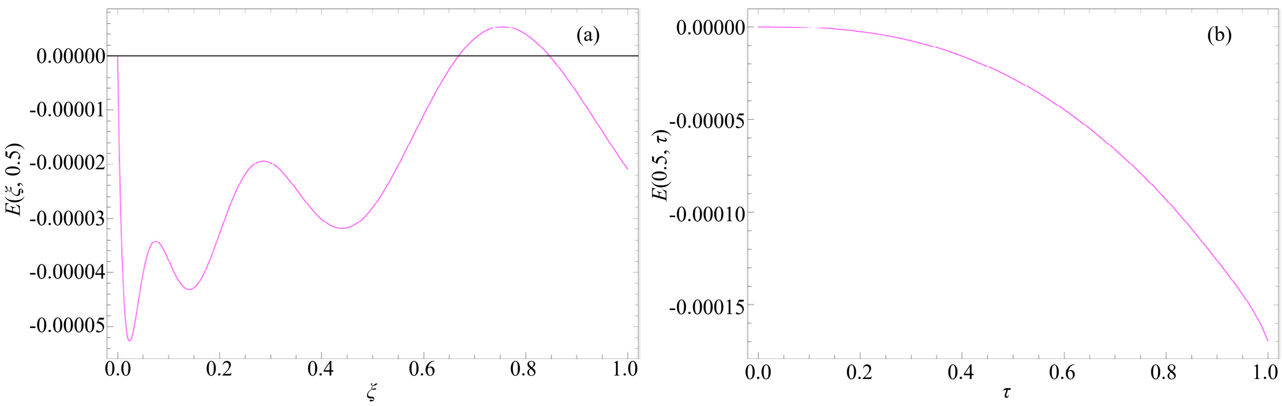

Figure 8. ξ-direction of absolute error (a) and τ-direction of absolute error (b) for Example 3 with $\rho = \varrho = 0$, $v = 0.5$ and $N = 12$

7. Concluding remarks

In this exploratory study, we employed the operational rational Jacobi approach to analyze two TFSE and TFRSE with their conditions. The OMs were constructed and used with the help of the collocation method to transform the whole problem into a simple system of algebraic equations. The outcomes are encouraging. In the near future, we want to expand the existing methods to handle the same PDEs in two and three dimensions, for instance, [38-39]. Also, we plan to generalize the method to handle nonlinear fractional PDEs.

Conflict of interest

The authors declare no competing financial interest.

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