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Research Article

# Quiescent Optical Solitons for the Concatenation Model Having Nonlinear Chromatic Dispersion with Differential Group Delay

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**Abstract:** This study recovers quiescent optical solitons within a complex concatenation model featuring nonlinear chromatic dispersion and differential group delay. It employs the sine-Gordon equation approach and the projective Riccati equation scheme. The results include the successful recovery of soliton solutions and the identification of parameter restrictions for their existence. The novelty lies in the application of these methods to this specific problem, offering valuable insights for optical communication system design.

***Keywords*:** sine-Gordon equation, birefringence, Kudryashov, Riccati

**MSC:** 78A60

## 1. Introduction

One of the most innovative models that has been ever proposed for the transmission of optical solitons across transcontinental and transoceanic distances through optical fibers, about a decade ago, is the concatenation model [1, 2]. This is made possible with the conjunction of the nonlinear Schrödinger's equation (NLSE), the Lakshmanan-Porsezian-Daniel (LPD) model and the Sasa-Satsuma equation (SSE). There are several aspects of this model that has been studied in the past. A few notable ones are the Painleve analysis, retrieval of optical solitons by the method of undetermined coefficients and extracting the conservation laws [1, 2]. Later, the model was also studied for bifurcation analysis [1] and the numerical schemes were also developed [2]. The model with power-law of self-phase modulation was also studied [3]. Subsequently, the quiescent optical solitons were also recovered with nonlinear chromatic dispersion (CD) [4]. The model was also later studied with the inclusion of spatio-temporal dispersion in addition to the pre-existing CD [5]. This yielded a mechanism to control the Internet bottleneck effect. It was also extended to birefringent fibers where


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the 1-soliton solutions were retrieved by the aid of undermined coefficients [6]. All of these developments occurred during 2023.

It is now time to extend these developments further along. This paper addresses the model with differential group delay with nonlinear CD along the two components. Thus the quiescent optical solitons along the two components are retrieved. Two integration schemes have made this retrieval possible. They are the generalized sine-Gordon equation method and the projective Riccati's equation approach. A full spectrum of quiescent optical solitons in birefringent fibers is retrieved with the implementation of the two approaches. The details are listed and the results are exhibited in the rest of the paper after a quick and succinct introduction to the model along with a rapid recapitulation of the integration algorithms.

The study has several limitations. They include a specific concatenation model with certain assumptions, heavy reliance on theoretical mathematical methods, parameter restrictions, and specificity to ertain types of birefringent fibers.

This study's significance lies in its exploration of optical solitons within a complex concatenation model, considering factors like nonlinear CD and differential group delay. Using established mathematical techniques, the study successfully recovers soliton solutions and identifies crucial parameter restrictions. These findings are valuable for designing robust optical communication systems, especially in complex scenarios. The study's novelty comes from applying these methods to this specific problem, advancing our understanding of optical solitons and their role in optical communication.

The study's motivation stems from the need for efficient long-distance optical communication. About a decade ago, the concatenation model was introduced for transmitting optical solitons through optical fibers, combining various mathematical models. This model has been extensively explored, addressing aspects like soliton retrieval, conservation laws, and extending to consider dispersion effects. Notably, it was used to address the Internet bottleneck issue. This study takes a step forward by considering differential group delay with nonlinear CD, recovering quiescent optical solitons using two integration schemes. The findings expand our understanding of optical communication and offer practical insights.

The study's findings of bright, dark, and singular soliton solutions in birefringent fibers offer a rich understanding of the optical behaviors achievable in complex fiber optic systems. Bright solitons are optical wave solutions that are characterized by their localized, peak-like intensity profiles. They are crucial for minimizing signal distortion in optical communication. Dark solitons are optical wave solutions characterized by localized intensity depressions within a broader background. They find applications in optical switching and pulse shaping. Singular solitons are unique in that they have special properties, often involving singular behavior at the center. This singularity represents an extreme point in the soliton's intensity profile. They are valuable for creating highly focused optical beams in technologies like super-resolution imaging and precision laser microfabrication.

## 2. Governing model

The concatenation model with nonlinear chromatic dispersion reads as [1-6]:

$$iu_t + a\left(|u|^n u\right)_{xx} + b|u|^2 u + c_1\left[\sigma_1 u_{xxxx} + \sigma_2 (u_x)^2 u^* + \sigma_3 |u_x|^2 u + \sigma_4 |u|^2 u_{xx} + \sigma_5 u^2 u_{xx}^* + \sigma_6 |u|^4 u\right]$$

$$+ic_2\left[\sigma_7 u_{xxx} + \sigma_8 |u|^2 u_x + \sigma_9 u^2 u_x^*\right] = 0. \tag{1}$$

Equation (1) is the concatenation model that serves as the conjunction of three well-known equations in nonlinear optics. The first three terms for $n = 0$ is the NLSE with Kerr law nonlinearity. The coefficient of $c_1$ is from LPD model while the coefficient of $c_2$ comes from SSE. Thus, equation (1) as it stands is the concatenation model that is used to model the propagation of pulses through optical fibers across inter-continental distances. The power-law parameter $n$ makes the CD nonlinear. For $n = 0$, the CD is rendered to be linear. Here $u(x, t)$ is the wave variable that is dependent on the spatial and temporal co-ordinates given by $x$ and $t$ respectively. For the NLSE component of the model, the first term is the linear temporal evolution while $a$ and $b$ are the coefficients of CD and self-phase modulation, respectively with $i = \sqrt{-1}$.

For birefringent fibers, equation (1) would split into two components as:

$$
\begin{aligned}
& iq_t + a_1\left(|q|^n q\right)_{xx} + \left(b_1|q|^2 + c_1|r|^2\right)q \\
& +c_{11}\left[\sigma_{11}q_{xxxx} + \left\{\alpha_1\left(q_x\right)^2 + \beta_1\left(r_x\right)^2\right\}q^* + \left(\gamma_1|q_x|^2 + \lambda_1|r_x|^2\right)q + \left(\delta_1|q|^2 + \zeta_1|r|^2\right)q_{xx}\right. \\
& \left. +\left(\mu_1 q^2 + \rho_1 r^2\right)q_{xx}^* + \left(f_1|q|^4 + g_1|q|^2|r|^2 + h_1|r|^4\right)q\right] \\
& +ic_{21}\left[\sigma_{71}q_{xxx} + \left(\eta_1|q|^2 + \theta_1|r|^2\right)q_x + \left(\varepsilon_1 q^2 + \tau_1 r^2\right)q_x^*\right] = 0, \qquad (2)
\end{aligned}
$$

and

$$
\begin{aligned}
& ir_t + a_2\left(|r|^n r\right)_{xx} + \left(b_2|r|^2 + c_2|q|^2\right)r \\
& +c_{12}\left[\sigma_{12}r_{xxxx} + \left\{\alpha_2\left(r_x\right)^2 + \beta_2\left(q_x\right)^2\right\}r^* + \left(\gamma_2|r_x|^2 + \lambda_2|q_x|^2\right)r + \left(\delta_2|r|^2 + \zeta_2|q|^2\right)r_{xx}\right. \\
& \left. +\left(\mu_2 r^2 + \rho_2 q^2\right)r_{xx}^* + \left(f_2|r|^4 + g_2|r|^2|q|^2 + h_2|q|^4\right)r\right] \\
& +ic_{22}\left[\sigma_{72}r_{xxx} + \left(\eta_2|r|^2 + \theta_2|q|^2\right)r_x + \left(\varepsilon_2 r^2 + \tau_2 q^2\right)r_x^*\right] = 0. \qquad (3)
\end{aligned}
$$

In order to solve the coupled system (2) and (3), we consider the solution structure:

$$q(x,t) = U_1(kx)e^{i(\omega t + \theta_0)}, \qquad (4)$$

and

$$r(x,t) = U_2(kx)e^{i(\omega t + \theta_0)}. \qquad (5)$$

Here, $U_j(kx)(\, j = 1, 2)$ represents the amplitude component of the soliton solution, while $\omega$ represents the wave number, and $\theta_0$ is the phase constant. Substituting (4) and (5) into (2) and (3) and then decomposing into real and imaginary parts, the real parts give

$$
\begin{aligned}
& a_1k^2n(n+1)U_1^nU_1'^2 + a_1k^2(n+1)U_1^{n+1}U_1'' + U_1^4\left(b_1 + c_{11}\gamma_1\kappa^2\right) + c_{11}f_1U_1^6 + c_{11}g_1U_2^2U_1^4 \\
& +c_{11}h_1U_2^4U_1^2 + c_{11}k^4\sigma_{11}U_1^{(4)}U_1 + c_{11}k^2\left(\alpha_1 + \gamma_1\right)U_1^2U_1'^2 + c_{11}k^2\left(\beta_1 + \lambda_1\right)U_1^2U_2'^2
\end{aligned}
$$

$$+c_{11}k^2\left(\delta_1+\mu_1\right)U_1^3U_1''+c_{11}k^2\left(\zeta_1+\rho_1\right)U_2^2U_1U_1''+U_2^2U_1^2\left(c_{11}\kappa^2\lambda_1+c_1\right)-\omega U_1^2=0, \tag{6}$$

and

$$a_2k^2n(n+1)U_2^nU_2'^2+a_2k^2(n+1)U_2^{n+1}U_2''+U_2^4\left(b_2+c_{12}\gamma_2\kappa^2\right)+c_{12}f_2U_2^6+c_{12}g_2U_1^2U_2^4$$

$$+c_{12}h_2U_1^4U_2^2+c_{12}k^4\sigma_{12}U_2^{(4)}U_2+c_{12}k^2\left(\alpha_2+\gamma_2\right)U_2^2U_2'^2+c_{12}k^2\left(\beta_2+\lambda_2\right)U_2^2U_1'^2$$

$$+c_{12}k^2\left(\delta_2+\mu_2\right)U_2^3U_2''+c_{12}k^2\left(\zeta_2+\rho_2\right)U_1^2U_2U_2''+U_1^2U_2^2\left(c_{12}\kappa^2\lambda_2+c_2\right)-\omega U_2^2=0, \tag{7}$$

while the imaginary parts yield

$$\alpha_{71}c_{21}k^3U_1U_1^{(3)}+c_{21}k\left(\varepsilon_1+\eta_1\right)U_1^3U_1'+c_{21}k\left(\theta_1+\tau_1\right)U_1U_2^2U_1'=0, \tag{8}$$

and

$$\alpha_{72}c_{22}k^3U_2U_2^{(3)}+c_{22}k\left(\varepsilon_2+\eta_2\right)U_2^3U_2'+c_{22}k\left(\theta_2+\tau_2\right)U_1^2U_2U_2'=0. \tag{9}$$

Using the balancing principle leads to $U_2=\varrho U_1$, $\varrho\neq 0$, then Eqs. (6) and (7) become

$$a_1k^2n(n+1)U_1^nU_1'^2+a_1k^2(n+1)U_1^{n+1}U_1''+U_1^4\left(b_1+c_{11}\kappa^2\left(\gamma_1+\lambda_1\varrho^2\right)+c_1\varrho^2\right)$$

$$+c_{11}U_1^6\left(f_1+g_1\varrho^2+h_1\varrho^4\right)+c_{11}k^4\sigma_{11}U_1^{(4)}U_1+c_{11}k^2U_1^2U_1'^2\left(\alpha_1+\varrho^2\left(\beta_1+\lambda_1\right)+\gamma_1\right)$$

$$+c_{11}k^2U_1^3U_1''\left(\delta_1+\varrho^2\left(\zeta_1+\rho_1\right)+\mu_1\right)-\omega U_1^2=0, \tag{10}$$

and

$$a_2k^2n(n+1)\varrho^{n+2}U_1^nU_1'^2+a_2k^2(n+1)\varrho^{n+2}U_1^{n+1}U_1''+\varrho^2U_1^4\left(b_2\varrho^2+c_{12}\kappa^2\left(\gamma_2\varrho^2+\lambda_2\right)+c_2\right)$$

$$+c_{12}\varrho^2U_1^6\left(f_2\varrho^4+g_2\varrho^2+h_2\right)+c_{12}k^4\sigma_{12}\varrho^2U_1^{(4)}U_1+c_{12}k^2\varrho^2U_1^2U_1'^2\left(\varrho^2\left(\alpha_2+\gamma_2\right)+\beta_2+\lambda_2\right)$$

$$+c_{12}k^2\varrho^2U_1^3U_1''\left(\varrho^2\left(\delta_2+\mu_2\right)+\zeta_2+\rho_2\right)-\omega\varrho^2U_1^2=0, \tag{11}$$

while Eqs. (8) and (10) come out as

$$\alpha_{71}c_{21}k^3U_1U_1^{(3)}+c_{21}k\left(\varepsilon_1+\eta_1+\varrho^2\left(\theta_1+\tau_1\right)\right)U_1^3U_1'=0, \tag{12}$$

and

$$\alpha_{72}c_{22}k^3\varrho^2U_1U_1^{(3)}+c_{22}k\varrho^2\left(\varrho^2\left(\varepsilon_2+\eta_2\right)+\theta_2+\tau_2\right)U_1^3U_1'=0. \tag{13}$$

By comparing Eqs. (11) with (12) and reducing them to a single equation with setting $n = 2$ for integrability, we are able to derive the following parametric restrictions:

$$b_1 + c_{11}\kappa^2\left(\gamma_1 + \lambda_1\varrho^2\right) + c_1\varrho^2 = b_2\varrho^2 + c_{12}\kappa^2\left(\gamma_2\varrho^2 + \lambda_2\right) + c_2, \tag{14}$$

$$c_{11}\left(f_1 + g_1\varrho^2 + h_1\varrho^4\right) = c_{12}\left(f_2\varrho^4 + g_2\varrho^2 + h_2\right), \tag{15}$$

$$c_{11}\left(\alpha_1 + \varrho^2\left(\beta_1 + \lambda_1\right) + \gamma_1\right) = c_{12}\left(\varrho^2\left(\alpha_2 + \gamma_2\right) + \beta_2 + \lambda_2\right), \tag{16}$$

$$a_1 = a_2\varrho^2, \tag{17}$$

$$c_{11}\left(\delta_1 + \varrho^2\left(\zeta_1 + \rho_1\right) + \mu_1\right) = c_{12}\left(\varrho^2\left(\delta_2 + \mu_2\right) + \zeta_2 + \rho_2\right), \tag{18}$$

and

$$c_{11}\sigma_{11} = c_{12}\sigma_{12}. \tag{19}$$

From the imaginary parts (13) and (14), we can equate the coefficients of the linearly independent functions with being zero then we get the following parametric restrictions:

$$\alpha_{71} = \alpha_{72} = 0, \tag{20}$$

and

$$\varepsilon_1 + \eta_1 + \varrho^2\left(\theta_1 + \tau_1\right) = \varrho^2\left(\varepsilon_2 + \eta_2\right) + \theta_2 + \tau_2 = 0. \tag{21}$$

As a consequence, Eqs. (1) and (2) become

$$\begin{aligned}
&iq_t + a_1\left(|q|^2 q\right)_{xx} + \left(b_1|q|^2 + c_1|r|^2\right)q \\
&+c_{11}\Big[\sigma_{11}q_{xxxx} + \left\{\alpha_1\left(q_x\right)^2 + \beta_1\left(r_x\right)^2\right\}q^* + \left(\gamma_1|q_x|^2 + \lambda_1|r_x|^2\right)q + \left(\delta_1|q|^2 + \zeta_1|r|^2\right)q_{xx} \\
&+\left(\mu_1 q^2 + \rho_1 r^2\right)q_{xx}^* + \left(f_1|q|^4 + g_1|q|^2|r|^2 + h_1|r|^4\right)q\Big] \\
&+ic_{21}\left[\left(\eta_1|q|^2 + \theta_1|r|^2\right)q_x + \left(\varepsilon_1 q^2 + \tau_1 r^2\right)q_x^*\right] = 0,
\end{aligned} \tag{22}$$

and

$$ir_t + a_2\left(|r|^2 r\right)_{xx} + \left(b_2|r|^2 + c_2|q|^2\right)r$$

$$+c_{12}\left[\sigma_{12}r_{xxxx}+\left\{\alpha_2\left(r_x\right)^2+\beta_2\left(q_x\right)^2\right\}r^*+\left(\gamma_2\left|r_x\right|^2+\lambda_2\left|q_x\right|^2\right)r+\left(\delta_2|r|^2+\zeta_2|q|^2\right)r_{xx}\right.$$

$$\left.+\left(\mu_2 r^2+\rho_2 q^2\right)r_{xx}^*+\left(f_2|r|^4+g_2|r|^2|q|^2+h_2|q|^4\right)r\right]$$

$$+ic_{22}\left[\left(\eta_2|r|^2+\theta_2|q|^2\right)r_x+\left(\varepsilon_2 r^2+\tau_2 q^2\right)r_x^*\right]=0, \tag{23}$$

and Eq. (11) shapes up as

$$k^2\left(6a_1+c_{11}\left(\alpha_1+\varrho^2\left(\beta_1+\lambda_1\right)+\gamma_1\right)\right)U_1^2U_1'^2+k^2\left(3a_1+c_{11}\left(\delta_1+\varrho^2\left(\zeta_1+\rho_1\right)+\mu_1\right)\right)U_1^3U_1''$$

$$+\left(b_1+c_{11}\kappa^2\left(\gamma_1+\lambda_1\varrho^2\right)+c_1\varrho^2\right)U_1^4+c_{11}\left(f_1+g_1\varrho^2+h_1\varrho^4\right)U_1^6+c_{11}k^4\sigma_{11}U_1^{(4)}U_1-\omega U_1^2=0, \tag{24}$$

which can be set as

$$k^2U_1^{(4)}+Q_4U_1U_1'^2+Q_5U_1^2U_1''+Q_3U_1^5+Q_2U_1^3+Q_1U_1=0, \tag{25}$$

with

$$\begin{cases} Q_1=-\dfrac{\omega}{c_{11}k^2\sigma_{11}}, \\ Q_2=\dfrac{b_1+c_{11}\kappa^2\left(\gamma_1+\lambda_1\varrho^2\right)+c_1\varrho^2}{c_{11}k^2\sigma_{11}}, \\ Q_3=\dfrac{f_1+g_1\varrho^2+h_1\varrho^4}{k^2\sigma_{11}}, \\ Q_4=\dfrac{6a_1+c_{11}\left(\alpha_1+\beta_1\varrho^2+\gamma_1+\lambda_1\varrho^2\right)}{c_{11}\sigma_{11}}, \\ Q_5=\dfrac{3a_1+c_{11}\left(\delta_1+\zeta_1\varrho^2+\mu_1+\rho_1\varrho^2\right)}{c_{11}\sigma_{11}}. \end{cases} \tag{26}$$

## 3. The integration algorithms: a rapid recapitulation

Consider a governing model [7-25]:

$$F(u,u_x,u_t,u_{xt},u_{xx},\ldots)=0, \tag{27}$$

where $u=u(x,t)$ denotes a wave profile, while $t$ and $x$ depict the time and space variables in sequence.

The relations

$$u(x,t)=U(\xi),\ \ \xi=k(x-\upsilon t), \tag{28}$$

condenses (27) to

$$P(U, -k\upsilon U', kU', k^2U'', ...) = 0, \tag{29}$$

where $k$ is the wave width, $\xi$ is the wave variable, and $\upsilon$ is the wave velocity.

### 3.1 *The generalized sine-Gordon equation method*

The basic steps of the generalized sine-Gordon equation method are as follows:

**Step 1** Assume Eq. (29) has the formal solution

$$U(\xi) = A_0 + \sum_{j=1}^{N} \cos[G(\xi)]^{j-1} \left( A_j \sin[G(\xi)] + B_j \cos[G(\xi)] \right), \tag{30}$$

along with the general sine-Gordon travelling wave reduction equation

$$G'(\xi) = \sqrt{\lambda + \mu \sin[G(\xi)]^2}. \tag{31}$$

**Step 2** Eq. (31) possesses the following cases:

**Case 1** $\lambda = 0$, $\mu = 1$

$$G'(\xi) = \mp \sin[G(\xi)], \tag{32}$$

which gives the hyperbolic function solutions

$$\sin[G(\xi)] = \text{sech}[\xi], \text{ and } \cos[G(\xi)] = \pm \tanh[\xi], \tag{33}$$

or

$$\sin[G(\xi)] = \pm i \,\text{csch}[\xi], \text{ and } \cos[G(\xi)] = \pm \coth[\xi]. \tag{34}$$

**Case 2** $\lambda = 1$, $\mu = -m^2$

$$G'(\xi) = \sqrt{1 - m^2 \sin[G(\xi)]^2}, \tag{35}$$

which allows us the Jacobi'e elliptic function (JEF) solutions

$$\sin[G(\xi)] = \text{sn}[\xi; m], \text{ and } \cos[G(\xi)] = \text{cn}[\xi; m], \tag{36}$$

or

$$\sin[G(\xi)] = \frac{1}{m} \text{ns}[\xi; m], \text{ and } \cos[G(\xi)] = -\frac{i}{m} \text{ds}[\xi; m]. \tag{37}$$

**Case 3** $\lambda = m^2$, $\mu = -1$

$$G'(\xi) = \sqrt{m^2 - \sin[G(\xi)]^2}, \tag{38}$$

which leaves us with the JEF solutions

$$\sin[G(\xi)] = m\,\mathrm{sn}[\xi; m], \text{ and } \cos[G(\xi)] = \mathrm{dn}[\xi; m], \tag{39}$$

or

$$\sin[G(\xi)] = \mathrm{ns}[\xi; m], \text{ and } \cos[G(\xi)] = -i\,\mathrm{cs}[\xi; m], \tag{40}$$

where $i = \sqrt{-1}$ and $0 < m < 1$.

**Step 3** Substituting (30) along with (31) into Eq. (29), we get a polynomial in $\sin[G(\xi)]$ and $\cos[G(\xi)]$ which equal to zero. The obtained coefficients of this polynomial give the needed parameters in (28) and (30).

**Remark** The approach of the JEFs toward their hyperbolic function limits as m tends to unity is listed as

$$\lim_{m\to 1} \mathrm{cn}[\xi; m] = \lim_{m\to 1} \mathrm{dn}[\xi; m] = \mathrm{sech}[\xi],$$

$$\lim_{m\to 1} \mathrm{sn}[\xi; m] = \tanh[\xi],$$

$$\lim_{m\to 1} \mathrm{cs}[\xi; m] = \lim_{m\to 1} \mathrm{ds}[\xi; m] = \mathrm{csch}[\xi],$$

$$\lim_{m\to 1} \mathrm{ns}[\xi; m] = \coth[\xi]. \tag{41}$$

## 3.2 *The projective Riccati's equation method*

The basic algorithms for the projective Riccati's equation method are:

**Step 1** Assume Eq. (28) has the formal solution

$$U(\xi) = a_0 + \sum_{i=1}^{N} \psi^{i-1}(\xi)\left(a_i \psi(\xi) + b_i \phi(\xi)\right), \tag{42}$$

where $\psi(\xi)$ and $\phi(\xi)$ satisfy the following ODEs:

$$\psi'(\xi) = -\psi(\xi)\phi(\xi),$$

$$\phi'(\xi) = 1 - \phi^2(\xi) - \tau\psi(\xi), \tag{43}$$

with

$$\phi(\xi)^2 = 1 - 2\tau\psi(\xi) + R(\tau)\psi(\xi)^2. \tag{44}$$

Here $r$ is constant and $N$ is a positive integer that comes from the balancing principle in Eq. (14), where $a_0$, $a_i$ and $b_i$ ($i = 0, 1, ..., N$) are constants.

**Step 2** The solutions of (43) are listed as follows:

**Case 1** $R(\tau) = 0$

$$\psi(\xi)=\frac{1}{2\tau}\operatorname{sech}^2\left[\frac{\xi}{2}\right], \text{ and } \phi(\xi)=\tanh\left[\frac{\xi}{2}\right], \tag{45}$$

or

$$\psi(\xi)=-\frac{1}{2\tau}\operatorname{csch}^2\left[\frac{\xi}{2}\right], \text{ and } \phi(\xi)=\coth\left[\frac{\xi}{2}\right]. \tag{46}$$

**Case 2** $R(\tau)=\frac{24}{25}\tau^2$

$$\psi(\xi)=\frac{1}{\tau}\frac{5\operatorname{sech}[\xi]}{5\operatorname{sech}[\xi]\pm 1}, \text{ and } \phi(\xi)=\frac{\tanh[\xi]}{1\pm 5\operatorname{sech}[\xi]}. \tag{47}$$

**Case 3** $R(\tau)=\frac{5}{9}\tau^2$

$$\psi(\xi)=\frac{1}{\tau}\frac{3\operatorname{sech}[\xi]}{3\operatorname{sech}[\xi]\pm 2}, \text{ and } \phi(\xi)=\frac{2}{2\coth[\xi]\pm 3\operatorname{csch}[\xi]}. \tag{48}$$

**Case 4** $R(\tau)=\tau^2-1$

$$\psi(\xi)=\frac{4\operatorname{sech}[\xi]}{3\tanh[\xi]+4\tau\operatorname{sech}[\xi]+5}, \text{ and } \phi(\xi)=\frac{5\tanh[\xi]+3}{3\tanh[\xi]+4\tau\operatorname{sech}[\xi]+5}, \tag{49}$$

or

$$\psi(\xi)=\frac{\operatorname{sech}[\xi]}{\tau\operatorname{sech}[\xi]+1}, \text{ and } \phi(\xi)=\frac{\tanh[\xi]}{\tau\operatorname{sech}[\xi]+1}. \tag{50}$$

**Case 5** $R(\tau)=\tau^2+1$

$$\psi(\xi)=\frac{\operatorname{csch}[\xi]}{\tau\operatorname{csch}[\xi]+1}, \text{ and } \phi(\xi)=\frac{\coth[\xi]}{\tau\operatorname{csch}[\xi]+1}. \tag{51}$$

**Step 2** nserting (42) along with (43) and (44) into Eq. (29), we get a polynomial of $\psi(\xi)$ and $\phi(\xi)$ which equal to zero. The obtained coefficients of this polynomial results in the needed parameters in (28) and (42).

# 4. Quiescent optical solitons

## 4.1 *Generalized sine-Gordon's equation approach*

Balancing $U''''$ with $U^5$ in Eq. (25) gives $N=1$. Thus the solution takes the form

$$U(\xi)=A_0+A_1\sin[G(\xi)]+B_1\cos[G(\xi)]. \tag{52}$$

Inserting (52) together with (31) into Eq. (25), we get a system of algebraic equations:

$$A_1\left(-A_1^2\left(10B_1^2Q_3+\mu\left(Q_4+2Q_5\right)\right)+A_1^4Q_3+3B_1^2\mu\left(Q_4+2Q_5\right)+5B_1^4Q_3+24k^2\mu^2\right)=0, \tag{53}$$

$$2A_0A_1B_1\left(10Q_3\left(B_1^2-A_1^2\right)+\mu\left(Q_4+4Q_5\right)\right)=0, \tag{54}$$

$$A_1\left(2A_0^2\mu Q_5-2A_1^4Q_3-4k^2\mu(5\lambda+7\mu)+B_1^2\left(30A_0^2Q_3-Q_4(3\lambda+4\mu)-Q_5(3\lambda+5\mu)+3Q_2\right)\right.$$

$$\left.+A_1^2\left(10Q_3\left(B_1^2-A_0^2\right)+Q_4(\lambda+\mu)+Q_5(\lambda+3\mu)-Q_2\right)\right)=0, \tag{55}$$

$$2A_0A_1B_1\left(10\left(A_0^2+A_1^2\right)Q_3-Q_4(\lambda+\mu)-Q_5(2\lambda+3\mu)+3Q_2\right)=0, \tag{56}$$

$$A_1\left(A_0^2\left(10A_1^2Q_3-Q_5(\lambda+\mu)+3Q_2\right)+A_1^2\left(A_1^2Q_3-Q_5(\lambda+\mu)+Q_2\right)+5A_0^4Q_3\right.$$

$$\left.+(\lambda+\mu)\left(B_1^2Q_4+k^2(\lambda+5\mu)\right)+Q_1\right)=0, \tag{57}$$

$$B_1\left(-A_1^2\left(10B_1^2Q_3+3\mu\left(Q_4+2Q_5\right)\right)+5A_1^4Q_3+B_1^2\mu\left(Q_4+2Q_5\right)+B_1^4Q_3+24k^2\mu^2\right)=0, \tag{58}$$

$$A_0\left(-A_1^2\left(30B_1^2Q_3+\mu\left(Q_4+4Q_5\right)\right)+5A_1^4Q_3+B_1^2\mu\left(Q_4+4Q_5\right)+5B_1^4Q_3\right)=0, \tag{59}$$

$$B_1^2\left(-10A_0^2Q_3+\left(Q_4+Q_5\right)(\lambda+2\mu)-Q_2\right)-2A_0^2\mu Q_5+10A_1^4Q_3+20k^2\mu(\lambda+2\mu)\Big)$$

$$-B_1\left(A_1^2\left(10Q_3\left(3A_0^2-B_1^2\right)-Q_4(3\lambda+5\mu)-Q_5(3\lambda+10\mu)+3Q_2\right)=0, \tag{60}$$

$$A_0\left(A_1^2\left(-10Q_3\left(A_0^2-3B_1^2\right)+Q_4(\lambda+\mu)+2Q_5(\lambda+3\mu)-3Q_2\right)\right.$$

$$\left.+B_1^2\left(10A_0^2Q_3-\left(Q_4+2Q_5\right)(\lambda+2\mu)+3Q_2\right)-10A_1^4Q_3\right)=0, \tag{61}$$

$$B_1\left(A_0^2\left(30A_1^2Q_3-Q_5(\lambda+2\mu)+3Q_2\right)+A_1^2\left(-2Q_4(\lambda+\mu)-Q_5(3\lambda+4\mu)+3Q_2\right)\right.$$

$$\left.+5A_0^4Q_3+5A_1^4Q_3+B_1^2Q_4(\lambda+\mu)+k^2\left(\lambda^2+16\lambda\mu+16\mu^2\right)+Q_1\right)=0, \tag{62}$$

$$A_0\left(A_1^2\left(5A_1^2Q_3-2Q_5(\lambda+\mu)+3Q_2\right)+A_0^4Q_3+A_0^2\left(10A_1^2Q_3+Q_2\right)+B_1^2Q_4(\lambda+\mu)+Q_1\right)=0. \tag{63}$$

Solving these equations together yields the following results:

**Case 1**

**Result 1**

$$A_0 = 0,\ A_1 = \sqrt{-\frac{5Q_1}{Q_2 + Q_4 + Q_5}} \pm 2,\ B_1 = 0,\ k = \sqrt{-Q_1},$$

$$Q_3 = \frac{(6Q_2 + Q_4 - 4Q_5)(Q_2 + Q_4 + Q_5)}{100Q_1}. \tag{64}$$

Plugging (64) with (33) or (34) into (52), we respectively retrieve bright and singular soliton solutions:

$$q(x,t) = \pm 2\sqrt{-\frac{5Q_1}{Q_2 + Q_4 + Q_5}} \operatorname{sech}\left[\sqrt{-Q_1}x\right] e^{i(\omega t + \theta_0)}, \tag{65}$$

$$r(x,t) = \varrho q(x,t), \tag{66}$$

where

$$Q_1 < 0,\ Q_2 + Q_4 + Q_5 > 0,$$

$$q(x,t) = \pm 2\sqrt{\frac{5Q_1}{Q_2 + Q_4 + Q_5}} \operatorname{csch}\left[\sqrt{-Q_1}x\right] e^{i(\omega t + \theta_0)}, \tag{67}$$

$$r(x,t) = \varrho q(x,t), \tag{68}$$

where

$$Q_1 < 0,\ Q_2 + Q_4 + Q_5 < 0.$$

Figure 1 represents the surface plots of quiescent bright optical soliton solutions (65) and (66) in birefringent fibers with the concatenation model. The parameter values that have been chosen are: $\omega = 1$, $k = 1$, $c_{11} = 1$, $\sigma_{11} = 1$, $b_1 = 1$, $\kappa = 1$, $\gamma_1 = 1$, $\lambda_1 = 1$, $c_1 = 1$, $\varrho = 2$, $a_1 = 1$, $\alpha_1 = 1$, $\beta_1 = 1$, $\delta_1 = 1$, $\zeta_1 = 1$, $\mu_1 = 1$, and $\rho_1 = 1$.

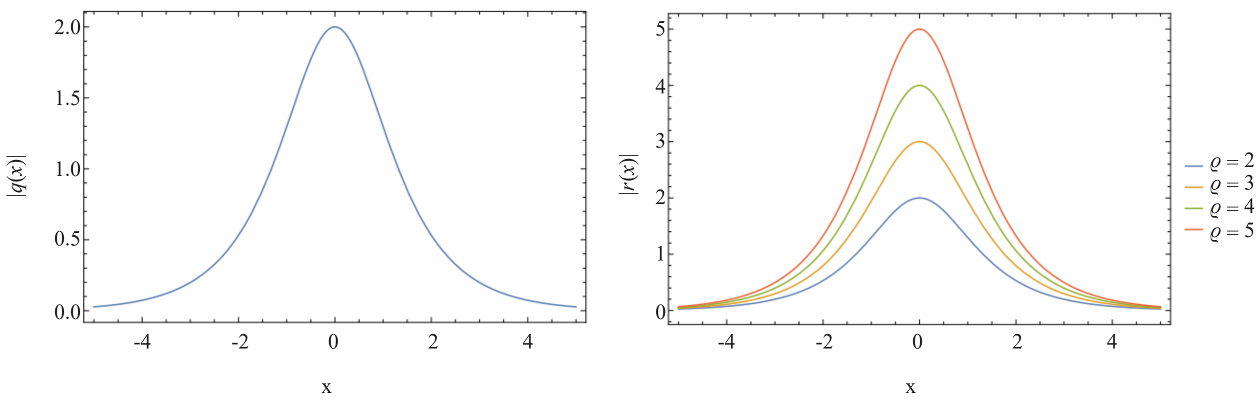


**Figure 1.** Profile of quiescent bright optical soliton solutions

**Result 2**

$$A_0 = A_1 = 0,\ B_1 = \pm\sqrt{-\frac{5Q_1}{2Q_2 + Q_4 - 4Q_5}},\ k = \frac{1}{2}\sqrt{-\frac{Q_1\left(Q_2 - 2\left(Q_4 + Q_5\right)\right)}{2\left(2Q_2 + Q_4 - 4Q_5\right)}},$$

$$Q_3 = \frac{\left(2Q_2 + Q_4 - 4Q_5\right)\left(3Q_2 - Q_4 + 4Q_5\right)}{25Q_1}. \tag{69}$$

Inserting (69) with (33) or (34) into (52), we respectively obtain dark and singular soliton solutions:

$$q(x,t) = \pm\sqrt{-\frac{5Q_1}{2Q_2 + Q_4 - 4Q_5}}\tanh\left[\frac{1}{2}\sqrt{-\frac{Q_1\left(Q_2 - 2\left(Q_4 + Q_5\right)\right)}{2\left(2Q_2 + Q_4 - 4Q_5\right)}}x\right]e^{i(\omega t + \theta_0)}, \tag{70}$$

$$r(x,t) = \varrho q(x,t), \tag{71}$$

$$q(x,t) = \pm\sqrt{-\frac{5Q_1}{2Q_2 + Q_4 - 4Q_5}}\coth\left[\frac{1}{2}\sqrt{-\frac{Q_1\left(Q_2 - 2\left(Q_4 + Q_5\right)\right)}{2\left(2Q_2 + Q_4 - 4Q_5\right)}}x\right]e^{i(\omega t + \theta_0)}, \tag{72}$$

$$r(x,t) = \varrho q(x,t), \tag{73}$$

where

$$Q_1\left(2Q_2 + Q_4 - 4Q_5\right) < 0,\ Q_2 - 2\left(Q_4 + Q_5\right) > 0.$$

In Figure 2, we observe a numerical simulation for a quiescent dark optical soliton pair (70) and (71) in birefringent fibers with the concatenation model. In the context of these plots, the parameter values employed are the following: $\omega = 1$, $k = 1$, $c_{11} = 1$, $\sigma_{11} = 1$, $b_1 = 1$, $\kappa = 1$, $\gamma_1 = 1$, $\lambda_1 = 1$, $c_1 = 1$, $\varrho = 2$, $a_1 = 1$, $\alpha_1 = 1$, $\beta_1 = 1$, $\delta_1 = 1$, $\zeta_1 = 1$, $\mu_1 = 1$, and $\rho_1 = 1$.

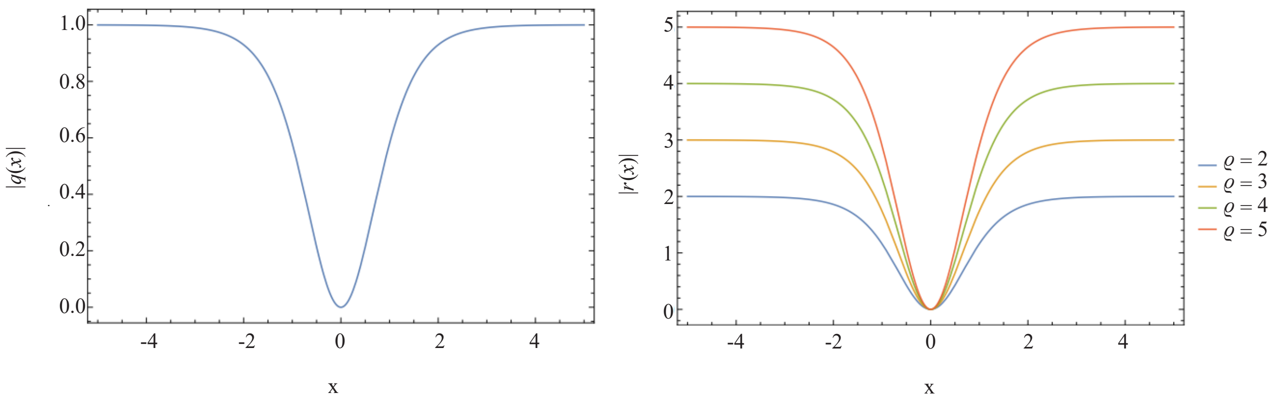


**Figure 2.** Profile of quiescent dark optical soliton solutions

**Result 3**

$$A_0 = 0,\ A_1 = \sqrt{\frac{5Q_1}{8Q_2 + Q_4 - 4Q_5}} \pm 2,\ B_1 = \pm iA_1,\ k = 2\sqrt{\frac{Q_1\left(-2Q_2 + Q_4 + Q_5\right)}{8Q_2 + Q_4 - 4Q_5}},$$

$$Q_3 = \frac{\left(8Q_2 + Q_4 - 4Q_5\right)\left(12Q_2 - Q_4 + 4Q_5\right)}{400Q_1}. \tag{74}$$

Substituting (74) with (33) or (34) into (52), we respectively arrive at complexion, dark and singular soliton solutions:

$$q(x,t) = \pm 2\sqrt{\frac{5Q_1}{8Q_2 + Q_4 - 4Q_5}}\left\{\operatorname{sech}\left[2\sqrt{\frac{Q_1\left(-2Q_2 + Q_4 + Q_5\right)}{8Q_2 + Q_4 - 4Q_5}}x\right]\right.$$

$$\left.\pm i\tanh\left[2\sqrt{\frac{Q_1\left(-2Q_2 + Q_4 + Q_5\right)}{8Q_2 + Q_4 - 4Q_5}}x\right]\right\}e^{i(\omega t + \theta_0)}, \tag{75}$$

$$r(x,t) = \varrho q(x,t), \tag{76}$$

where

$$Q_1\left(8Q_2 + Q_4 - 4Q_5\right) > 0,\ -2Q_2 + Q_4 + Q_5 > 0,$$

$$q(x,t) = \pm\sqrt{-\frac{5Q_1}{2Q_2 + Q_4 - 4Q_5}}\tanh\left[\frac{1}{2}\sqrt{-\frac{Q_1\left(Q_2 - 2\left(Q_4 + Q_5\right)\right)}{2\left(2Q_2 + Q_4 - 4Q_5\right)}}x\right]e^{i(\omega t + \theta_0)}, \tag{77}$$

$$r(x,t) = \varrho q(x,t), \tag{78}$$

$$q(x,t) = \pm\sqrt{-\frac{5Q_1}{2Q_2 + Q_4 - 4Q_5}}\coth\left[\frac{1}{2}\sqrt{-\frac{Q_1\left(Q_2 - 2\left(Q_4 + Q_5\right)\right)}{2\left(2Q_2 + Q_4 - 4Q_5\right)}}x\right]e^{i(\omega t + \theta_0)}, \tag{79}$$

$$r(x,t) = \varrho q(x,t), \tag{80}$$

where

$$Q_1\left(2Q_2 + Q_4 - 4Q_5\right) < 0,\ Q_2 - 2\left(Q_4 + Q_5\right) > 0.$$

**Case 2**
**Result 1**

$$A_0 = 0,\ A_1 = \pm 2\sqrt{\frac{5m^2\left(m^2+1\right)Q_1}{\left(m^2+1\right)\left(\left(m^4-6m^2+1\right)Q_4+\left(m^4+14m^2+1\right)Q_5\right)-\left(m^4+14m^2+1\right)Q_2}},$$

$$B_1 = 0,\ k = \sqrt{-\frac{Q_1\left(Q_2-\left(m^2+1\right)\left(Q_4+Q_5\right)\right)}{\left(m^4+14m^2+1\right)Q_2-\left(m^2+1\right)\left(\left(m^4-6m^2+1\right)Q_4+\left(m^4+14m^2+1\right)Q_5\right)}},$$

$$Q_3 = \frac{\left(\left(m^2+1\right)\left(Q_4-4Q_5\right)-6Q_2\right)\left(\left(m^2+1\right)\left(\left(m^4-6m^2+1\right)Q_4+\left(m^4+14m^2+1\right)Q_5\right)-\left(m^4+14m^2+1\right)Q_2\right)}{100\left(m^2+1\right)^2 Q_1}. \tag{81}$$

Inserting (81) with (36) or (37) into (52), we secure JEF solutions:

$$q(x,t) = \pm 2\sqrt{\frac{5m^2\left(m^2+1\right)Q_1}{\left(m^2+1\right)\left(\left(m^4-6m^2+1\right)Q_4+\left(m^4+14m^2+1\right)Q_5\right)-\left(m^4+14m^2+1\right)Q_2}}$$

$$\times \mathrm{sn}\left[\sqrt{-\frac{Q_1\left(Q_2-\left(m^2+1\right)\left(Q_4+Q_5\right)\right)}{\left(m^4+14m^2+1\right)Q_2-\left(m^2+1\right)\left(\left(m^4-6m^2+1\right)Q_4+\left(m^4+14m^2+1\right)Q_5\right)}}x;m\right]$$

$$\times e^{i(\omega t+\theta_0)}, \tag{82}$$

$$r(x,t) = \varrho q(x,t), \tag{83}$$

$$q(x,t) = \pm 2\sqrt{\frac{5\left(m^2+1\right)Q_1}{\left(m^2+1\right)\left(\left(m^4-6m^2+1\right)Q_4+\left(m^4+14m^2+1\right)Q_5\right)-\left(m^4+14m^2+1\right)Q_2}}$$

$$\times \mathrm{ns}\left[\sqrt{-\frac{Q_1\left(Q_2-\left(m^2+1\right)\left(Q_4+Q_5\right)\right)}{\left(m^4+14m^2+1\right)Q_2-\left(m^2+1\right)\left(\left(m^4-6m^2+1\right)Q_4+\left(m^4+14m^2+1\right)Q_5\right)}}x;m\right]$$

$$\times e^{i(\omega t+\theta_0)}, \tag{84}$$

$$r(x,t) = \varrho q(x,t), \tag{85}$$

where

$$5\left(m^2+1\right)Q_1\left(\left(m^2+1\right)\left(\left(m^4-6m^2+1\right)Q_4+\left(m^4+14m^2+1\right)Q_5\right)-\left(m^4+14m^2+1\right)Q_2\right) > 0,$$

$$Q_1\left(Q_2-\left(m^2+1\right)\left(Q_4+Q_5\right)\right)\left(\left(m^4+14m^2+1\right)Q_2-\left(m^2+1\right)\left(\left(m^4-6m^2+1\right)Q_4+\left(m^4+14m^2+1\right)Q_5\right)\right)<0.$$

**Result 2**

$$B_1=\pm 2\sqrt{\frac{5m^2\left(2m^2-1\right)Q_1}{\left(1-2m^2\left(3-4m^2\right)^2\right)Q_5+\left(-16m^4+16m^2-1\right)Q_2+\left(8m^6-12m^4+2m^2+1\right)Q_4}},$$

$$k=\sqrt{-\frac{Q_1\left(\left(2m^2-1\right)\left(Q_4+Q_5\right)+Q_2\right)}{\left(2m^2\left(3-4m^2\right)^2-1\right)Q_5+\left(16m^4-16m^2+1\right)Q_2-\left(8m^6-12m^4+2m^2+1\right)Q_4}},$$

$$A_0=A_1=0, Q_3=\frac{\left(\begin{array}{c}\left(\left(2m^2-1\right)\left(Q_4-4Q_5\right)+6Q_2\right)\\ \left(\left(2m^2\left(3-4m^2\right)^2-1\right)Q_5+\left(16m^4-16m^2+1\right)Q_2-\left(8m^6-12m^4+2m^2+1\right)Q_4\right)\end{array}\right)}{100\left(1-2m^2\right)^2Q_1}. \tag{86}$$

Plugging (86) with (36) or (37) into (52), we acquire JEF solutions:

$$q(x,t)=\pm 2\sqrt{\frac{5m^2\left(2m^2-1\right)Q_1}{\left(1-2m^2\left(3-4m^2\right)^2\right)Q_5+\left(-16m^4+16m^2-1\right)Q_2+\left(8m^6-12m^4+2m^2+1\right)Q_4}}$$

$$\times\text{cn}\left[\sqrt{-\frac{Q_1\left(\left(2m^2-1\right)\left(Q_4+Q_5\right)+Q_2\right)}{\left(2m^2\left(3-4m^2\right)^2-1\right)Q_5+\left(16m^4-16m^2+1\right)Q_2-\left(8m^6-12m^4+2m^2+1\right)Q_4}}x;m\right]$$

$$\times e^{i(\omega t+\theta_0)}, \tag{87}$$

$$r(x,t)=\varrho q(x,t), \tag{88}$$

where

$$5m^2\left(2m^2-1\right)Q_1\left(\begin{array}{c}\left(1-2m^2\left(3-4m^2\right)^2\right)Q_5\\ +\left(-16m^4+16m^2-1\right)Q_2+\left(8m^6-12m^4+2m^2+1\right)Q_4\end{array}\right)>0,$$

$$Q_1\left(\left(2m^2-1\right)\left(Q_4+Q_5\right)+Q_2\right)\left(\begin{array}{c}\left(2m^2\left(3-4m^2\right)^2-1\right)Q_5\\ +\left(16m^4-16m^2+1\right)Q_2-\left(8m^6-12m^4+2m^2+1\right)Q_4\end{array}\right)<0,$$

$$q(x,t)=\pm 2\sqrt{\frac{5\left(1-2m^2\right)Q_1}{\left(1-2m^2\left(3-4m^2\right)^2\right)Q_5+\left(-16m^4+16m^2-1\right)Q_2+\left(8m^6-12m^4+2m^2+1\right)Q_4}}$$

$$\times \text{ds}\left[\sqrt{-\frac{Q_1\left(\left(2m^2-1\right)\left(Q_4+Q_5\right)+Q_2\right)}{\left(2m^2\left(3-4m^2\right)^2-1\right)Q_5+\left(16m^4-16m^2+1\right)Q_2-\left(8m^6-12m^4+2m^2+1\right)Q_4}}x;m\right]$$

$$\times e^{i(\omega t+\theta_0)}, \tag{89}$$

$$r(x,t)=\varrho q(x,t), \tag{90}$$

where

$$5\left(1-2m^2\right)Q_1\left(\begin{array}{c}\left(1-2m^2\left(3-4m^2\right)^2\right)Q_5\\ +\left(-16m^4+16m^2-1\right)Q_2+\left(8m^6-12m^4+2m^2+1\right)Q_4\end{array}\right)>0,$$

$$Q_1\left(\left(2m^2-1\right)\left(Q_4+Q_5\right)+Q_2\right)\left(\begin{array}{c}\left(2m^2\left(3-4m^2\right)^2-1\right)Q_5\\ +\left(16m^4-16m^2+1\right)Q_2-\left(8m^6-12m^4+2m^2+1\right)Q_4\end{array}\right)<0.$$

**Result 3**

$$B_1=\pm 2\sqrt{\frac{5m^2\left(m^2-2\right)Q_1}{\left(m^2-2\right)\left(\left(m^4+4m^2-4\right)Q_4-4\left(m^4-m^2+1\right)Q_5\right)-8\left(m^4-m^2+1\right)Q_2}},$$

$$k=2\sqrt{-\frac{Q_1\left(\left(m^2-2\right)\left(Q_4+Q_5\right)+2Q_2\right)}{8\left(m^4-m^2+1\right)Q_2-\left(m^2-2\right)\left(\left(m^4+4m^2-4\right)Q_4-4\left(m^4-m^2+1\right)Q_5\right)}},$$

$$A_0=0,\ A_1=\pm iB_1,\ Q_3=\frac{\left(\begin{array}{c}\left(\left(m^2-2\right)\left(Q_4-4Q_5\right)+12Q_2\right)\\ \left(8\left(m^4-m^2+1\right)Q_2-\left(m^6+2m^4-12m^2+8\right)Q_4+4\left(m^6-3m^4+3m^2-2\right)Q_5\right)\end{array}\right)}{400\left(m^2-2\right)^2Q_1}. \tag{91}$$

Substituting (91) with (36) or (37) into (52), we derive JEF solutions:

$$q(x,t)=\pm 2\sqrt{\frac{5m^2\left(m^2-2\right)Q_1}{\left(m^2-2\right)\left(\left(m^4+4m^2-4\right)Q_4-4\left(m^4-m^2+1\right)Q_5\right)-8\left(m^4-m^2+1\right)Q_2}}$$

$$\times\left\{\operatorname{cn}\left[2\sqrt{-\frac{Q_1\left(\left(m^2-2\right)\left(Q_4+Q_5\right)+2Q_2\right)}{8\left(m^4-m^2+1\right)Q_2-\left(m^2-2\right)\left(\left(m^4+4m^2-4\right)Q_4-4\left(m^4-m^2+1\right)Q_5\right)}}x;m\right]\right.$$

$$\left.\pm i\operatorname{sn}\left[2\sqrt{-\frac{Q_1\left(\left(m^2-2\right)\left(Q_4+Q_5\right)+2Q_2\right)}{8\left(m^4-m^2+1\right)Q_2-\left(m^2-2\right)\left(\left(m^4+4m^2-4\right)Q_4-4\left(m^4-m^2+1\right)Q_5\right)}}x;m\right]\right\}$$

$$\times e^{i(\omega t+\theta_0)}, \tag{92}$$

$$r(x,t)=\varrho q(x,t), \tag{93}$$

where

$$5m^2\left(m^2-2\right)Q_1\left(\left(m^2-2\right)\left(\left(m^4+4m^2-4\right)Q_4-4\left(m^4-m^2+1\right)Q_5\right)-8\left(m^4-m^2+1\right)Q_2\right)>0,$$

$$Q_1\left(\left(m^2-2\right)\left(Q_4+Q_5\right)+2Q_2\right)\begin{pmatrix}8\left(m^4-m^2+1\right)Q_2\\-\left(m^2-2\right)\left(\left(m^4+4m^2-4\right)Q_4-4\left(m^4-m^2+1\right)Q_5\right)\end{pmatrix}<0,$$

$$q(x,t)=\pm 2\sqrt{\frac{5\left(2-m^2\right)Q_1}{\left(m^2-2\right)\left(\left(m^4+4m^2-4\right)Q_4-4\left(m^4-m^2+1\right)Q_5\right)-8\left(m^4-m^2+1\right)Q_2}}$$

$$\times\left\{\pm\operatorname{ns}\left[2\sqrt{-\frac{Q_1\left(\left(m^2-2\right)\left(Q_4+Q_5\right)+2Q_2\right)}{8\left(m^4-m^2+1\right)Q_2-\left(m^2-2\right)\left(\left(m^4+4m^2-4\right)Q_4-4\left(m^4-m^2+1\right)Q_5\right)}}x;m\right]\right.$$

$$\left.-\operatorname{ds}\left[2\sqrt{-\frac{Q_1\left(\left(m^2-2\right)\left(Q_4+Q_5\right)+2Q_2\right)}{8\left(m^4-m^2+1\right)Q_2-\left(m^2-2\right)\left(\left(m^4+4m^2-4\right)Q_4-4\left(m^4-m^2+1\right)Q_5\right)}}x;m\right]\right\}$$

$$\times e^{i(\omega t+\theta_0)}, \tag{94}$$

$$r(x,t)=\varrho q(x,t), \tag{95}$$

where

$$5\left(2-m^2\right)Q_1\left(\left(m^2-2\right)\left(\left(m^4+4m^2-4\right)Q_4-4\left(m^4-m^2+1\right)Q_5\right)-8\left(m^4-m^2+1\right)Q_2\right)>0,$$

$$Q_1\left(\left(m^2-2\right)\left(Q_4+Q_5\right)+2Q_2\right)\begin{pmatrix} 8\left(m^4-m^2+1\right)Q_2 \\ -\left(m^2-2\right)\left(\left(m^4+4m^2-4\right)Q_4-4\left(m^4-m^2+1\right)Q_5\right)\end{pmatrix}<0.$$

**Case 3**
**Result 1**

$$A_0=B_1=0,\ A_1=\pm 2\sqrt{\frac{5\left(m^2+1\right)Q_1}{\left(m^2+1\right)\left(\left(m^4-6m^2+1\right)Q_4+\left(m^4+14m^2+1\right)Q_5\right)-\left(m^4+14m^2+1\right)Q_2}},$$

$$k=\sqrt{-\frac{Q_1\left(Q_2-\left(m^2+1\right)\left(Q_4+Q_5\right)\right)}{\left(m^4+14m^2+1\right)Q_2-\left(m^2+1\right)\left(\left(m^4-6m^2+1\right)Q_4+\left(m^4+14m^2+1\right)Q_5\right)}},$$

$$Q_3=\frac{\begin{pmatrix}\left(6Q_2-\left(m^2+1\right)\left(Q_4-4Q_5\right)\right)\left(\left(m^4+14m^2+1\right)Q_2\right. \\ \left.-\left(m^2+1\right)\left(\left(m^4-6m^2+1\right)Q_4+\left(m^4+14m^2+1\right)Q_5\right)\right)\end{pmatrix}}{100\left(m^2+1\right)^2 Q_1}. \tag{96}$$

Inserting (96) with (39) or (40) into (52), we recover JEF solutions:

$$q(x,t)=\pm 2\sqrt{\frac{5\left(m^2+1\right)Q_1}{\left(m^2+1\right)\left(\left(m^4-6m^2+1\right)Q_4+\left(m^4+14m^2+1\right)Q_5\right)-\left(m^4+14m^2+1\right)Q_2}}$$

$$\times\operatorname{sn}\left[\sqrt{-\frac{Q_1\left(Q_2-\left(m^2+1\right)\left(Q_4+Q_5\right)\right)}{\left(m^4+14m^2+1\right)Q_2-\left(m^2+1\right)\left(\left(m^4-6m^2+1\right)Q_4+\left(m^4+14m^2+1\right)Q_5\right)}}x;m\right]$$

$$\times e^{i(\omega t+\theta_0)}, \tag{97}$$

$$r(x,t)=\varrho q(x,t), \tag{98}$$

$$q(x,t)=\pm 2\sqrt{\frac{5\left(m^2+1\right)Q_1}{\left(m^2+1\right)\left(\left(m^4-6m^2+1\right)Q_4+\left(m^4+14m^2+1\right)Q_5\right)-\left(m^4+14m^2+1\right)Q_2}}$$

$$\times\operatorname{ns}\left[\sqrt{-\frac{Q_1\left(Q_2-\left(m^2+1\right)\left(Q_4+Q_5\right)\right)}{\left(m^4+14m^2+1\right)Q_2-\left(m^2+1\right)\left(\left(m^4-6m^2+1\right)Q_4+\left(m^4+14m^2+1\right)Q_5\right)}}x;m\right]$$

$$\times e^{i(\omega t+\theta_0)}, \tag{99}$$

$$r(x,t)=\varrho q(x,t), \tag{100}$$

where

$$5\left(m^2+1\right)Q_1\left(\left(m^2+1\right)\left(\left(m^4-6m^2+1\right)Q_4+\left(m^4+14m^2+1\right)Q_5\right)-\left(m^4+14m^2+1\right)Q_2\right)>0,$$

$$Q_1\left(Q_2-\left(m^2+1\right)(Q_4+Q_5)\right)\begin{pmatrix}\left(m^4+14m^2+1\right)Q_2\\ -\left(m^2+1\right)\left(\left(m^4-6m^2+1\right)Q_4+\left(m^4+14m^2+1\right)Q_5\right)\end{pmatrix}<0.$$

**Result 2**

$$A_0=0, B_1=\pm 2\sqrt{\frac{5\left(2-m^2\right)Q_1}{\left(m^2-2\right)\left(\left(m^4+4m^2-4\right)Q_4+\left(m^4-16m^2+16\right)Q_5\right)-\left(m^4-16m^2+16\right)Q_2}},$$

$$A_1=0, k=\sqrt{-\frac{Q_1\left(Q_2-\left(m^2-2\right)(Q_4+Q_5)\right)}{\left(m^4-16m^2+16\right)Q_2-\left(m^2-2\right)\left(\left(m^4+4m^2-4\right)Q_4+\left(m^4-16m^2+16\right)Q_5\right)}},$$

$$Q_3=\frac{\begin{pmatrix}\left(\left(m^2-2\right)(Q_4-4Q_5)-6Q_2\right)\\ \left(\left(m^2-2\right)\left(\left(m^4+4m^2-4\right)Q_4+\left(m^4-16m^2+16\right)Q_5\right)-\left(m^4-16m^2+16\right)Q_2\right)\end{pmatrix}}{100\left(m^2-2\right)^2Q_1}. \tag{101}$$

Plugging (101) with (39) or (40) into (52), we retrieve JEF solutions:

$$q(x,t)=\pm\sqrt{\frac{5\left(2-m^2\right)Q_1}{\left(m^2-2\right)\left(\left(m^4+4m^2-4\right)Q_4+\left(m^4-16m^2+16\right)Q_5\right)-\left(m^4-16m^2+16\right)Q_2}}$$

$$\times\text{dn}\left[\sqrt{-\frac{Q_1\left(Q_2-\left(m^2-2\right)(Q_4+Q_5)\right)}{\left(m^4-16m^2+16\right)Q_2-\left(m^2-2\right)\left(\left(m^4+4m^2-4\right)Q_4+\left(m^4-16m^2+16\right)Q_5\right)}}x;m\right]$$

$$\times e^{i(\omega t+\theta_0)}, \tag{102}$$

$$r(x,t)=\varrho q(x,t), \tag{103}$$

where

$$5\left(2-m^2\right)Q_1\left(\left(m^2-2\right)\left(\left(m^4+4m^2-4\right)Q_4+\left(m^4-16m^2+16\right)Q_5\right)-\left(m^4-16m^2+16\right)Q_2\right)>0,$$

$$Q_1\left(Q_2-\left(m^2-2\right)(Q_4+Q_5)\right)\begin{pmatrix}\left(m^4-16m^2+16\right)Q_2\\ -\left(m^2-2\right)\left(\left(m^4+4m^2-4\right)Q_4+\left(m^4-16m^2+16\right)Q_5\right)\end{pmatrix}<0,$$

$$q(x,t)=\pm\sqrt{\frac{5\left(m^2-2\right)Q_1}{\left(m^2-2\right)\left(\left(m^4+4m^2-4\right)Q_4+\left(m^4-16m^2+16\right)Q_5\right)-\left(m^4-16m^2+16\right)Q_2}}$$

$$\times\text{cs}\left[\sqrt{-\frac{Q_1\left(Q_2-\left(m^2-2\right)(Q_4+Q_5)\right)}{\left(m^4-16m^2+16\right)Q_2-\left(m^2-2\right)\left(\left(m^4+4m^2-4\right)Q_4+\left(m^4-16m^2+16\right)Q_5\right)}}x;m\right]$$

$$\times e^{i(\omega t+\theta_0)}, \tag{104}$$

$$r(x,t)=\varrho q(x,t), \tag{105}$$

where

$$5\left(m^2-2\right)Q_1\left(\left(m^2-2\right)\left(\left(m^4+4m^2-4\right)Q_4+\left(m^4-16m^2+16\right)Q_5\right)-\left(m^4-16m^2+16\right)Q_2\right)>0,$$

$$Q_1\left(Q_2-\left(m^2-2\right)(Q_4+Q_5)\right)\begin{pmatrix}\left(m^4-16m^2+16\right)Q_2\\ -\left(m^2-2\right)\left(\left(m^4+4m^2-4\right)Q_4+\left(m^4-16m^2+16\right)Q_5\right)\end{pmatrix}<0.$$

**Result 3**

$$A_0=0,\ A_1=\pm iB_1,\ B_1=\pm2\sqrt{\frac{5\left(1-2m^2\right)Q_1}{\left(2m^2-1\right)\left(\left(4m^4-4m^2-1\right)Q_4+4\left(m^4-m^2+1\right)Q_5\right)-8\left(m^4-m^2+1\right)Q_2}},$$

$$k=2\sqrt{-\frac{Q_1\left(2Q_2-\left(2m^2-1\right)(Q_4+Q_5)\right)}{8\left(m^4-m^2+1\right)Q_2-\left(2m^2-1\right)\left(\left(4m^4-4m^2-1\right)Q_4+4\left(m^4-m^2+1\right)Q_5\right)}},$$

$$Q_3=\frac{\begin{pmatrix}\left(12Q_2-\left(2m^2-1\right)(Q_4-4Q_5)\right)\\ \left(8\left(m^4-m^2+1\right)Q_2-\left(8m^6-12m^4+2m^2+1\right)Q_4+4\left(-2m^6+3m^4-3m^2+1\right)Q_5\right)\end{pmatrix}}{400\left(1-2m^2\right)^2Q_1}. \tag{106}$$

Substituting (106) with (39) or (40) into (52), we arrive at JEF solutions:

$$q(x,t) = \pm 2\sqrt{\frac{5\left(1-2m^2\right)Q_1}{\left(2m^2-1\right)\left(\left(4m^4-4m^2-1\right)Q_4+4\left(m^4-m^2+1\right)Q_5\right)-8\left(m^4-m^2+1\right)Q_2}}$$

$$\times\left\{\mathrm{dn}\left[2\sqrt{-\frac{Q_1\left(2Q_2-\left(2m^2-1\right)(Q_4+Q_5)\right)}{8\left(m^4-m^2+1\right)Q_2-\left(2m^2-1\right)\left(\left(4m^4-4m^2-1\right)Q_4+4\left(m^4-m^2+1\right)Q_5\right)}}x;m\right]\right.$$

$$\left.\pm im\,\mathrm{sn}\left[2\sqrt{-\frac{Q_1\left(2Q_2-\left(2m^2-1\right)(Q_4+Q_5)\right)}{8\left(m^4-m^2+1\right)Q_2-\left(2m^2-1\right)\left(\left(4m^4-4m^2-1\right)Q_4+4\left(m^4-m^2+1\right)Q_5\right)}}x;m\right]\right\}$$

$$\times e^{i(\omega t+\theta_0)}, \tag{107}$$

$$r(x,t) = \varrho q(x,t), \tag{108}$$

where

$$5\left(1-2m^2\right)Q_1\left(\left(2m^2-1\right)\left(\left(4m^4-4m^2-1\right)Q_4+4\left(m^4-m^2+1\right)Q_5\right)-8\left(m^4-m^2+1\right)Q_2\right)>0,$$

$$Q_1\left(2Q_2-\left(2m^2-1\right)(Q_4+Q_5)\right)\left(\begin{array}{c}8\left(m^4-m^2+1\right)Q_2\\ -\left(2m^2-1\right)\left(\left(4m^4-4m^2-1\right)Q_4+4\left(m^4-m^2+1\right)Q_5\right)\end{array}\right)<0,$$

$$q(x,t) = \pm 2\sqrt{\frac{5\left(2m^2-1\right)Q_1}{\left(2m^2-1\right)\left(\left(4m^4-4m^2-1\right)Q_4+4\left(m^4-m^2+1\right)Q_5\right)-8\left(m^4-m^2+1\right)Q_2}}$$

$$\times\left\{\pm\mathrm{ns}\left[2\sqrt{-\frac{Q_1\left(2Q_2-\left(2m^2-1\right)(Q_4+Q_5)\right)}{8\left(m^4-m^2+1\right)Q_2-\left(2m^2-1\right)\left(\left(4m^4-4m^2-1\right)Q_4+4\left(m^4-m^2+1\right)Q_5\right)}}x;m\right]\right.$$

$$\left.-\mathrm{cs}\left[2\sqrt{-\frac{Q_1\left(2Q_2-\left(2m^2-1\right)(Q_4+Q_5)\right)}{8\left(m^4-m^2+1\right)Q_2-\left(2m^2-1\right)\left(\left(4m^4-4m^2-1\right)Q_4+4\left(m^4-m^2+1\right)Q_5\right)}}x;m\right]\right\}$$

$$\times e^{i(\omega t+\theta_0)}, \tag{109}$$

$$r(x,t) = \varrho q(x,t), \tag{110}$$

where

$$5\left(2m^2-1\right)Q_1\left(\left(2m^2-1\right)\begin{pmatrix}\left(4m^4-4m^2-1\right)Q_4\\+4\left(m^4-m^2+1\right)Q_5\end{pmatrix}-8\left(m^4-m^2+1\right)Q_2\right)>0,$$

$$Q_1\left(2Q_2-\left(2m^2-1\right)(Q_4+Q_5)\right)\begin{pmatrix}8\left(m^4-m^2+1\right)Q_2\\-\left(2m^2-1\right)\left(\left(4m^4-4m^2-1\right)Q_4+4\left(m^4-m^2+1\right)Q_5\right)\end{pmatrix}<0.$$

## 4.2 *Projective Riccati's equation procedure*

Balancing $U''''$ with $U^5$ in Eq. (25) causes to $N = 1$. Therefore the solution simplifies to the form

$$U(\xi)=\sigma_0+\sigma_1\psi(\xi)+\varrho_1\phi(\xi). \tag{111}$$

Inserting (111) along with (43) and (44) into Eq. (25), we secure a system of algebraic equations:

$$\sigma_1 R(\tau)^2\left(24k^2+5Q_3\varrho_1^4+3Q_4\varrho_1^2+6Q_5\varrho_1^2\right)+\sigma_1^3\left(2\left(5Q_3\varrho_1^2+Q_5\right)+Q_4\right)R(\tau)+Q_3\sigma_1^5=0, \tag{112}$$

$$\varrho_1\left(R(\tau)^2\left(24k^2+Q_3\varrho_1^4+Q_4\varrho_1^2+2Q_5\varrho_1^2\right)+\sigma_1^2\left(10Q_3\varrho_1^2+3Q_4+6Q_5\right)R(\tau)+5Q_3\sigma_1^4\right)=0, \tag{113}$$

$$\sigma_1 R(\tau)\left(-10\left(6k^2\tau+Q_3\left(2\tau\varrho_1^4-3\sigma_0\sigma_1\varrho_1^2\right)\right)+Q_4\left(\sigma_0\sigma_1-8\tau\varrho_1^2\right)+Q_5\left(4\sigma_0\sigma_1-17\tau\varrho_1^2\right)\right)$$

$$+\sigma_0\varrho_1^2\left(5Q_3\varrho_1^2+Q_4+4Q_5\right)R(\tau)^2+\sigma_1^3\left(-2Q_4\tau-3Q_5\tau+5Q_3\left(\sigma_0\sigma_1-4\tau\varrho_1^2\right)\right)=0, \tag{114}$$

$$-\varrho_1\left(R(\tau)\left(36k^2\tau+2Q_4\left(\tau\varrho_1^2-\sigma_0\sigma_1\right)+Q_5\left(5\tau\varrho_1^2-8\sigma_0\sigma_1\right)+4Q_3\left(\tau\varrho_1^4-5\sigma_0\sigma_1\varrho_1^2\right)\right)\right.$$

$$\left.+\sigma_1^2\left(4Q_4\tau+7Q_5\tau+20Q_3\left(\tau\varrho_1^2-\sigma_0\sigma_1\right)\right)\right)=0, \tag{115}$$

$$R(\tau)\left(20k^2\sigma_1-20Q_3\sigma_0\tau\varrho_1^4+2Q_4\varrho_1^2\left(\sigma_1-\sigma_0\tau\right)+Q_5\left(2\sigma_0^2\sigma_1+\varrho_1^2\left(7\sigma_1-10\sigma_0\tau\right)\right)+10Q_3\sigma_1\varrho_1^4\right.$$

$$\left.+30Q_3\sigma_0^2\sigma_1\varrho_1^2+3Q_2\sigma_1\varrho_1^2\right)+\sigma_1\left(30k^2\tau^2+10Q_3\sigma_0^2\sigma_1^2+Q_2\sigma_1^2+Q_5\left(-6\sigma_0\sigma_1\tau+\sigma_1^2+10\tau^2\varrho_1^2\right)\right.$$

$$\left.+Q_4\left(\sigma_1\left(\sigma_1-2\sigma_0\tau\right)+5\tau^2\varrho_1^2\right)-60Q_3\sigma_0\sigma_1\tau\varrho_1^2+10Q_3\sigma_1^2\varrho_1^2+20Q_3\tau^2\varrho_1^4\right)=0, \tag{116}$$

$$\varrho_1\left(R(\tau)\left(8k^2+2Q_5\left(\sigma_0^2+\varrho_1^2\right)+2Q_3\left(5\sigma_0^2\varrho_1^2+\varrho_1^4\right)+Q_2\varrho_1^2\right)+6k^2\tau^2-8Q_5\sigma_0\sigma_1\tau\right.$$

$$+30Q_3\sigma_0^2\sigma_1^2+3Q_2\sigma_1^2+2Q_5\sigma_1^2+Q_4\left(\sigma_1\left(\sigma_1-2\sigma_0\tau\right)+\tau^2\varrho_1^2\right)-40Q_3\sigma_0\sigma_1\tau\varrho_1^2$$

$$\left.+10Q_3\sigma_1^2\varrho_1^2+4Q_3\tau^2\varrho_1^4+2Q_5\tau^2\varrho_1^2\right)=0, \tag{117}$$

$$-15k^2\sigma_1\tau+\sigma_0\varrho_1^2 R(\tau)\left(10Q_3\left(\sigma_0^2+\varrho_1^2\right)+3Q_2+4Q_5\right)-3Q_5\sigma_0^2\sigma_1\tau+10Q_3\sigma_0^3\sigma_1^2+3Q_2\sigma_0\sigma_1^2+2Q_5\sigma_0\sigma_1^2$$

$$+20Q_3\sigma_0\tau^2\varrho_1^4+4Q_5\sigma_0\tau^2\varrho_1^2-20Q_3\sigma_1\tau\varrho_1^4-60Q_3\sigma_0^2\sigma_1\tau\varrho_1^2-6Q_2\sigma_1\tau\varrho_1^2-7Q_5\sigma_1\tau\varrho_1^2$$

$$+Q_4\left(\sigma_0\sigma_1^2+\tau\varrho_1^2\left(\sigma_0\tau-2\sigma_1\right)\right)+30Q_3\sigma_0\sigma_1^2\varrho_1^2=0, \tag{118}$$

$$-\varrho_1\left(k^2\tau-20Q_3\sigma_0^3\sigma_1+20Q_3\sigma_0^2\tau\varrho_1^2+Q_5\left(\sigma_0^2\tau-2\sigma_0\sigma_1+\tau\varrho_1^2\right)+2Q_2\left(\tau\varrho_1^2-3\sigma_0\sigma_1\right)\right.$$

$$\left.-20Q_3\sigma_0\sigma_1\varrho_1^2+4Q_3\tau\varrho_1^4\right)=0, \tag{119}$$

$$k^2\sigma_1+5Q_3\sigma_0^4\sigma_1+Q_1\sigma_1-20Q_3\sigma_0\tau\varrho_1^4-20Q_3\sigma_0^3\tau\varrho_1^2+Q_5\left(\sigma_0^2\sigma_1+\varrho_1^2\left(\sigma_1-2\sigma_0\tau\right)\right)$$

$$+Q_2\left(3\sigma_0^2\sigma_1+\varrho_1^2\left(3\sigma_1-6\sigma_0\tau\right)\right)+5Q_3\sigma_1\varrho_1^4+30Q_3\sigma_0^2\sigma_1\varrho_1^2=0, \tag{120}$$

$$\varrho_1\left(Q_2\left(3\sigma_0^2+\varrho_1^2\right)+Q_3\left(5\sigma_0^4+10\sigma_0^2\varrho_1^2+\varrho_1^4\right)+Q_1\right)=0, \tag{121}$$

$$\sigma_0\left(Q_2\left(\sigma_0^2+3\varrho_1^2\right)+Q_3\left(\sigma_0^4+10\sigma_0^2\varrho_1^2+5\varrho_1^4\right)+Q_1\right)=0. \tag{122}$$

Solving these equations together leaves us with the following results:
**Case 1** $R(\tau)=0$

$$\sigma_0=\sigma_1=0,\ \varrho_1=\sqrt{-\frac{5Q_1}{8Q_2+Q_4-4Q_5}}\pm 2,\ k=2\sqrt{\frac{Q_1\left(-2Q_2+Q_4+Q_5\right)}{8Q_2+Q_4-4Q_5}},$$

$$Q_3=\frac{\left(8Q_2+Q_4-4Q_5\right)\left(12Q_2-Q_4+4Q_5\right)}{400Q_1}. \tag{123}$$

Plugging (123) with (45) or (46) into (111), we respectively derive dark and singular soliton solutions:

$$q(x,t)=\pm 2\sqrt{-\frac{5Q_1}{8Q_2+Q_4-4Q_5}}\tanh\left[\sqrt{\frac{Q_1\left(-2Q_2+Q_4+Q_5\right)}{8Q_2+Q_4-4Q_5}}x\right]e^{i(\omega t+\theta_0)}, \tag{124}$$

$$r(x,t)=\varrho q(x,t), \tag{125}$$

$$q(x,t)=\pm 2\sqrt{-\frac{5Q_1}{8Q_2+Q_4-4Q_5}}\coth\left[\sqrt{\frac{Q_1\left(-2Q_2+Q_4+Q_5\right)}{8Q_2+Q_4-4Q_5}}x\right]e^{i(\omega t+\theta_0)}, \tag{126}$$

$$r(x,t)=\varrho q(x,t), \tag{127}$$

where

$$Q_1\left(8Q_2+Q_4-4Q_5\right)<0,\ \ -2Q_2+Q_4+Q_5<0.$$

**Case 2** $R(\tau)=\dfrac{24}{25}\tau^2$

$$\sigma_0=0,\ \varrho_1=\sqrt{-\frac{5Q_1}{8Q_2+Q_4-4Q_5}}\pm 2,\ k=2\sqrt{\frac{Q_1\left(-2Q_2+Q_4+Q_5\right)}{8Q_2+Q_4-4Q_5}},$$

$$\sigma_1=\pm\varrho_1\frac{2\sqrt{6}\tau}{5},\ Q_3=\frac{\left(8Q_2+Q_4-4Q_5\right)\left(12Q_2-Q_4+4Q_5\right)}{400Q_1}. \tag{128}$$

Substituting (128) with (47) into (111), we catch out combo dark-bright soliton solutions:

$$q(x,t)=\pm\frac{2\sqrt{-\frac{5Q_1}{8Q_2+Q_4-4Q_5}}\left(\tanh\left[2\sqrt{\frac{Q_1\left(-2Q_2+Q_4+Q_5\right)}{8Q_2+Q_4-4Q_5}}x\right]\pm 2\sqrt{6}\operatorname{sech}\left[2\sqrt{\frac{Q_1\left(-2Q_2+Q_4+Q_5\right)}{8Q_2+Q_4-4Q_5}}x\right]\right)}{1\pm 5\operatorname{sech}\left[2\sqrt{\frac{Q_1\left(-2Q_2+Q_4+Q_5\right)}{8Q_2+Q_4-4Q_5}}x\right]}$$

$$\times e^{i(\omega t+\theta_0)}, \tag{129}$$

$$r(x,t)=\varrho q(x,t), \tag{130}$$

where

$$Q_1\left(8Q_2+Q_4-4Q_5\right)<0,\ \ -2Q_2+Q_4+Q_5<0.$$

**Case 3** $R(\tau)=\dfrac{5}{9}\tau^2$

$$\sigma_0=0,\ \sigma_1=\pm\varrho_1\frac{\sqrt{5}\tau}{3},\ \varrho_1=\pm 2\sqrt{-\frac{5Q_1}{8Q_2+Q_4-4Q_5}},\ k=2\sqrt{-\frac{Q_1\left(2Q_2-Q_4-Q_5\right)}{8Q_2+Q_4-4Q_5}},$$

$$Q_3=\frac{\left(8Q_2+Q_4-4Q_5\right)\left(12Q_2-Q_4+4Q_5\right)}{400Q_1}. \tag{131}$$

Inserting (131) with (48) into (111), we retrieve combo bright-singular soliton solutions:

$$q(x,t)=\pm 2\sqrt{-\frac{5Q_1}{8Q_2+Q_4-4Q_5}}\left\{\frac{\sqrt{5}\pm 3\operatorname{sech}\left[2\sqrt{-\frac{Q_1\left(2Q_2-Q_4-Q_5\right)}{8Q_2+Q_4-4Q_5}}x\right]}{39\operatorname{sech}\left[2\sqrt{-\frac{Q_1\left(2Q_2-Q_4-Q_5\right)}{8Q_2+Q_4-4Q_5}}x\right]\pm 6}\right.$$

$$+\frac{2}{2\coth\left[2\sqrt{-\frac{Q_1\left(2Q_2-Q_4-Q_5\right)}{8Q_2+Q_4-4Q_5}}x\right]\pm 3\operatorname{csch}\left[2\sqrt{-\frac{Q_1\left(2Q_2-Q_4-Q_5\right)}{8Q_2+Q_4-4Q_5}}x\right]}\Bigg\}e^{i(\omega t+\theta_0)}, \tag{132}$$

$$r(x,t)=\varrho q(x,t), \tag{133}$$

where

$$Q_1\left(8Q_2+Q_4-4Q_5\right)<0,\ 2Q_2-Q_4-Q_5>0.$$

**Case 4** $R(\tau)=\tau^2-1$

$$\sigma_0=0,\ \varrho_1=\sqrt{\frac{5Q_1}{-8Q_2-Q_4+4Q_5}}\pm 2,\ k=2\sqrt{\frac{Q_1\left(-2Q_2+Q_4+Q_5\right)}{8Q_2+Q_4-4Q_5}},$$

$$\sigma_1=\pm\varrho_1\sqrt{\tau^2-1},\ Q_3=\frac{\left(8Q_2+Q_4-4Q_5\right)\left(12Q_2-Q_4+4Q_5\right)}{400Q_1}. \tag{134}$$

Plugging (134) with (49) or (50) into (111), we acquire combo bright-dark soliton solutions:

$$q(x,t)=\pm 2\sqrt{\frac{5Q_1}{-8Q_2-Q_4+4Q_5}}$$

$$\times\left\{\frac{\pm 4\sqrt{\tau^2-1}\operatorname{sech}\left[2\sqrt{\frac{Q_1\left(-2Q_2+Q_4+Q_5\right)}{8Q_2+Q_4-4Q_5}}x\right]+5\operatorname{sech}\left[2\sqrt{\frac{Q_1\left(-2Q_2+Q_4+Q_5\right)}{8Q_2+Q_4-4Q_5}}x\right]+3}{4\tau\operatorname{sech}\left[2\sqrt{\frac{Q_1\left(-2Q_2+Q_4+Q_5\right)}{8Q_2+Q_4-4Q_5}}x\right]+3\tanh\left[2\sqrt{\frac{Q_1\left(-2Q_2+Q_4+Q_5\right)}{8Q_2+Q_4-4Q_5}}x\right]+5}\right\}$$

$$\times e^{i(\omega t+\theta_0)}, \tag{135}$$

$$r(x,t)=\varrho q(x,t), \tag{136}$$

or

$$q(x,t)=\pm 2\sqrt{\frac{5Q_1}{-8Q_2-Q_4+4Q_5}}$$

$$\times\left\{\frac{\tanh\left[2\sqrt{\frac{Q_1\left(-2Q_2+Q_4+Q_5\right)}{8Q_2+Q_4-4Q_5}}x\right]\pm\sqrt{\tau^2-1}\,\mathrm{sech}\left[2\sqrt{\frac{Q_1\left(-2Q_2+Q_4+Q_5\right)}{8Q_2+Q_4-4Q_5}}x\right]}{\tau\mathrm{sech}\left[2\sqrt{\frac{Q_1\left(-2Q_2+Q_4+Q_5\right)}{8Q_2+Q_4-4Q_5}}x\right]+1}\right\}$$

$$\times e^{i(\omega t+\theta_0)}, \tag{137}$$

$$r(x,t)=\varrho q(x,t), \tag{138}$$

where

$$Q_1\left(8Q_2+Q_4-4Q_5\right)<0,\ -2Q_2+Q_4+Q_5<0.$$

**Case 5** $R(\tau)=\tau^2+1$

$$\sigma_0=0,\ \varrho_1=\pm2\sqrt{\frac{5Q_1}{-8Q_2-Q_4+4Q_5}},\ k=2\sqrt{\frac{Q_1\left(-2Q_2+Q_4+Q_5\right)}{8Q_2+Q_4-4Q_5}},$$

$$\sigma_1=\pm\varrho_1\sqrt{\tau^2+1},\ Q_3=\frac{\left(8Q_2+Q_4-4Q_5\right)\left(12Q_2-Q_4+4Q_5\right)}{400Q_1}. \tag{139}$$

Substituting (139) with (51) into (111), we arrive at combo singular soliton solutions:

$$q(x,t)=\pm2\sqrt{\frac{5Q_1}{-8Q_2-Q_4+4Q_5}}$$

$$\times\left\{\frac{\coth\left[2\sqrt{\frac{Q_1\left(-2Q_2+Q_4+Q_5\right)}{8Q_2+Q_4-4Q_5}}x\right]\pm\sqrt{\tau^2+1}\,\mathrm{csch}\left[2\sqrt{\frac{Q_1\left(-2Q_2+Q_4+Q_5\right)}{8Q_2+Q_4-4Q_5}}x\right]}{\tau\mathrm{csch}\left[2\sqrt{\frac{Q_1\left(-2Q_2+Q_4+Q_5\right)}{8Q_2+Q_4-4Q_5}}x\right]+1}\right\}$$

$$\times e^{i(\omega t+\theta_0)}, \tag{140}$$

$$r(x,t)=\varrho q(x,t), \tag{141}$$

where

$$Q_1\left(8Q_2+Q_4-4Q_5\right)<0,\ -2Q_2+Q_4+Q_5<0.$$

## 5. Conclusions

This paper retrieved quiescent optical soliton for the concatenation model in birefringent fibers having nonlinear CD. Two integration schemes made it possible to recover a complete spectrum of such solitons along the two components. The results are nice and interesting and therefore carries a stern message. The ground handling staff of the telecommunication engineers must exercise extreme caution while laying the optical fibers underground or undersea. Any disruption with the CD would lead to the soliton transmission being stalled. This would lead to catastrophic consequences. Later, the model would be naturally extended to DWDM systems where the possibility of such quiescent optical solitons would be investigated. Such studies are under way. The results would be later disseminated upon their availability after aligning them with some of the pre-existing results [26-31].

## Conflict of interest

The authors claim there is no conflict of interest in this work.

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