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Research Article

The Order Gcd Graph

Pinku Sarkar, Kuntala Patra*

Department of Mathematics, Gauhati University, Guwahati, Assam, India
E-mail: kpatra@gauhati.ac.in

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Abstract: In this paper we define a simple undirected graph $O_g(Z_n)$ whose vertices are all the elements of Z_n and two distinct vertices a, b are adjacent if and only if $\gcd(o(a), o(b)) = o(a \cdot b)$. We introduced that the graph $O_g(Z_n)$ is complete graph for $n = pq$. where p, q are distinct prime. The graph $O_g(Z_n)$ is not planar for $n = p^2$ and $n = 2^k \cdot p$. where, p is any odd prime and $k > 1$. Also, we have discussed the Eulerian property of the graph and find the degree of vertices and clique number of the graph.

***Keywords*:** clique number, planar graph, complete graph, the order gcd graph

MSC: 05C25, 05C10, 05C45

1. Introduction

In 1889 Cayley [1] introduced a graph represents a finite group. There are many other graphs associated with finite group [2–4]. Different types of graphs associated with a commutative and noncommutative group were defined by many authors [5–9]. Zero divisor graph is defined by Beck [10] in 1988, the graph is an undirected graph and elements of a commutative ring are R considered as vertices of the graph and two distinct vertices a, b are adjacent if and only if $ab = 0$. Erdos and Sarkozy [11] was defined co-prime graph in 1997 , If G is finite group the co-prime graph of G is denoted by Γ_G, elements of G are considered as vertices and two distinct vertices a, b are adjacent if and only if $\gcd(o(a), o(b)) = 1$. After two years in 1999 [12] M. S. Lucido defined prime graph $\Gamma(G)$ associated with a finite group G whose vertices are the primes dividing the order of G and two vertices p, q are adjacent if there is an element in G of order pq. This works motivated us to construct a new graph associated with a finite group Z_n and this graph is called Order gcd graph whose vertices are all the elements of Z_n and two distinct vertices a, b are adjacent if and only if $\gcd(o(a), o(b)) = o(ab)$. In this paper we introduce planarity, Eulerian property of the graph for different values of n. Here we also find the clique number of the graph and degree of vertices of the graph for different values of n.

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2. Main results

Definition 1 A simple undirected graph $O_g(Z_n)$ whose vertices are all the elements of Z_n and two distinct vertices a, b are adjacent if and only if $\gcd(o(a), o(b)) = o(a \cdot b)$.

For $n = 2, 3, 4, 5$ the graphs it becomes complete graph K_2, K_3, K_4, K_5 respectively. For $n = 8$.

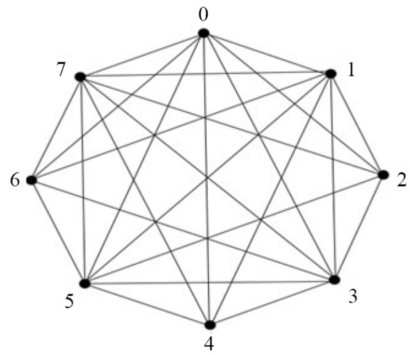

Figure 1. $O_g(Z_8)$

Theorem 1 The graph $O_g\left(Z_{p^2}\right)$ is $K_{p^2} - E$. Where; $E = \{\overline{ab} : a, b$ are non-zero zero divisor $\}$.

Proof. Every element of Z_{p^2} are taken as vertices of the graph $O_g\left(Z_{p^2}\right)$. Let u be the unit element of Z_{p^2}. For any vertex $a(\neq u)$,

$\gcd(o(u), o(a)) = \gcd\left(p^2, o(a)\right) = o(a) = o(au)$ [Since, $\gcd(a, u) = 1$ then $o(a) = o(au)$].

So, u is adjacent with a. Therefore, any unit element is adjacent with every vertex of the graph $O_g\left(Z_{p^2}\right)$.

Zero element of the graph is adjacent with every element. Because, $\gcd(o(a), o(0)) = \gcd(o(a), 1) = 1 = o(0) = o(a \cdot 0)$.

Let, z be non-zero zero-divisor of Z_{p^2}, which is adjacent to every unit element (Since, any unit element is adjacent with every vertex of the graph $O_g\left(Z_{p^2}\right)$). Also, z is adjacent with 0.

z is not adjacent with any non-zero zero-divisor of Z_{p^2}; because, if possible z is adjacent with any nonzero zero-divisor v, then $\gcd(o(z), o(v)) = o(zv)$, but $\gcd(o(z), o(v)) = \gcd(p, p) = p$ and $o(zv) = o(0) = 1$, which contradict that $\gcd(o(z), o(v)) = o(zv)$. Therefore, any two non-zero zero-divisors are not adjacent.

We get unit elements and zero vertex are adjacent with every vertex of the graph except itself. Nonzero zero-divisor are adjacent with unit element and zero vertex but not adjacent with any non-zero zero-divisor. Hence, the graph $O_g\left(Z_{p^2}\right)$ is $K_{p^2} - E$. Where; $E = \{\overline{ab} : a, b$ are non-zero zero divisor$\}$. □

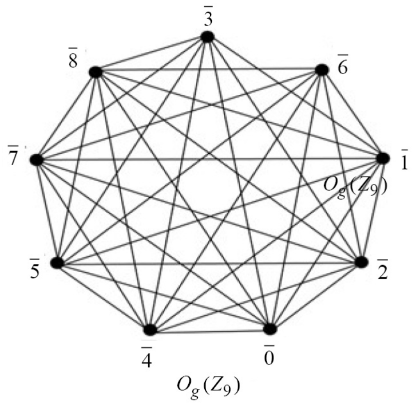

Figure 2. $O_g(Z_9)$

Theorem 2 Degree of zero element and any unit element in $O_g(Z_n)$ is $n-1$.

Proof. let u be unit element and a be any element of $Z_n \cdot \gcd(o(u),\ o(a)) = \gcd(n,\ o(a)) = o(a) = o(au)$ [Since, $\gcd(a,\ u) = 1$ then $o(a) = o(au)$]. Therefore, u is adjacent to any vertex except itself (since graph is simple). Hence, degree of any unit element is $n-1$. Also, $\gcd(o(0),\ o(a)) = \gcd(1,\ o(a)) = 1 = o(0)$. So, zero vertex is adjacent with every vertex except itself. Hence, degree of zero vertex is $n-1$. □

Theorem 3 Degree of any zero-divisor in $O_g\left(Z_{p^2}\right)$ is $\varphi\left(p^2\right)+1$.

Proof. let a and b be two non-zero zero-divisor, where $a \neq b$. So, $a,\ b$ both are multiples of p.

Then $a \cdot b \equiv 0 \left(\bmod p^2\right)$. So, $o(ab) = o(0) = 1$. But, $\gcd(o(a),\ o(b)) = \gcd(p,\ p) = p$.

Therefore, a is not adjacent to b. In Z_{p^2} any two non-zero zero-divisors are not adjacent. And unit elements are adjacent with every element except itself. So, any non-zero zero-divisors are adjacent with every unit element and zero elements. Since, there are $\varphi\left(p^2\right)$ unit elements in Z_{p^2}. Hence, degree of any zero divisor in $O_g\left(Z_{p^2}\right)$ is $\varphi\left(p^2\right)+1$. □

Theorem 4 The graph $O_g\left(Z_{p^2}\right)$ is not planar for any odd prime p.

Proof. Number of unit elements in $O_g\left(Z_{p^2}\right)$ is $\varphi\left(p^2\right) = p(p-1)$. For any odd prime $p,\ \varphi\left(p^2\right) > 5$ and unit elements are adjacent with each other. Therefore, the graph $O_g\left(Z_{p^2}\right)$ always has a complete subgraph K_5 for any odd prime p. Hence, the graph $O_g\left(Z_{p^2}\right)$ is not planar for any odd prime p. □

Theorem 5 The graph $O_g\left(Z_{p^2}\right)$ is not Eulerian for any odd prime p.

Proof. In the graph $O_g\left(Z_{p^2}\right)$ degree of any zero-divisor is $\varphi\left(p^2\right)+1 = p(p-1)$ which is odd for any odd prime p. Therefore, $O_g\left(Z_{p^2}\right)$ is not Eulerian for any odd prime p. □

Theorem 6 Clique number of the graph $O_g\left(Z_{p^2}\right)$ is p^2-p+2.

Proof. In the graph $O_g\left(Z_{p^2}\right)$ there are $(p-1)$ non-zero zero-divisors. If we delate $(p-2)$ number of non-zero zero-divisor, the graph will be complete graph K_{p^2-p+2}, which is maximal complete subgraph. Therefore, Clique number of the graph $O_g\left(Z_{p^2}\right)$ is p^2-p+2. □

Theorem 7 The graph $O_g(Z_n)$ is complete graph if n is a prime and $n = pq$ where $p,\ q$ are distinct prime.

Proof. Let a be any vertex of $O_g(Z_n)$. Then order of any vertex ' a ' is divisor of n.

If $n = p$, then any non-zero vertex of the graph $O_g(Z_n)$ is unit element of Z_n. Degree of zero element and any unit element in $O_g(Z_p)$ is $p-1$. So, the graph $O_g(Z_p)$ is complete graph. Where; p is a prime.

If $n = pq$ where $p,\ q$ are distinct prime, then possible values of order of ' a ' are $p,\ q,\ 1$.

For any vertex $a,\ b$ of the graph $O_g(Z_n)$

Case 1 If $o(a) = p$ and $o(b) = p$

$$\gcd(o(a),\ o(b)) = \gcd(p,\ p) = p$$

Since, $o(a) = p$ and $n = pq$, so, a is multiple of q but not multiple of p. And, since $o(b) = p$ and $n = pq$, so, b is multiple of q but not multiple of p.

So, $o(ab) = p$

Therefore, $\gcd(o(a),\ o(b)) = o(ab)$. So, a is adjacent to b.

Case 2 If $o(a) = p$ and $o(b) = q$

$$\gcd(o(a),\ o(b)) = \gcd(p,\ q) = 1$$

since, $o(a) = p$ and $n = pq$, so, a is multiple of q but not multiple of p. And, since $o(b) = q$ and $n = pq$, so, b is multiple of p but not multiple of q.

So, ab is multiple of pq

$$\therefore ab \equiv 0 \bmod (pq) \Rightarrow o(ab) = o(0) = 1$$

Therefore, $\gcd(o(a),\ o(b)) = 1 = o(ab)$. So, a is adjacent to b.

Case 3 If $o(a) = p$ or q and $o(b) = 1$ then b is zero vertex. So, $o(ab) = o(0) = 1$.

$$\gcd(o(a),\ o(b)) = \gcd(o(a),\ 1) = 1$$

0 is adjacent to a.

So, any two distinct vertices $a,\ b$ are adjacent to each other.

Therefore, the graph $O_g(Z_n)$ is complete graph. □

Theorem 8 Degree of any non-zero zero-divisor in $O_g(Z_n)$, where $n = 2^k \cdot p$, is $n - 1$ if the zerodivisor is multiple of 2^k but not multiple of p.

Proof. If a is multiple of 2^k but not multiple of p, then $o(a) = p$.

$$\gcd(o(a),\ o(b)) = \gcd(p,\ o(b))$$

$$\gcd(p,\ o(b)) = p \text{ or } 1$$

Case 1 If $\gcd(p,\ o(b)) = p$.

$o(b)$ is multiple of p.

So, b is not multiple of p.

Therefore, ab is not multiple of p.

$$o(ab) = p$$

So,

$$\gcd(o(a),\ o(b)) = \gcd(p,\ o(b)) = p = o(ab)$$

So, a is adjacent to b.
i.e a is adjacent with every vertex except multiple of p.
Case 2 If $\gcd(p,\ o(b)) = 1$.
$o(b)$ is not a multiple of p. So, b is a multiple of p.
Also, a is multiple of 2^k.
Therefore, ab is multiple of $2^k p$.

$$o(ab) = 1$$

So,

$$\gcd(o(a),\ o(b)) = \gcd(p,\ o(b)) = 1 = o(ab)$$

So, a is adjacent to b.
i.e vertex a is adjacent with every multiple of p.

Also, zero vertex is adjacent with every vertex in the graph. So, from case 1 and case 2 we get, a is adjacent with every element of Z_n except itself.

Hence, degree of any non-zero zero-divisor which is multiple of 2^k is $n-1$. □

Theorem 9 Degree of any non-zero zero-divisor in $O_g(Z_n)$, where $n = 2^k \cdot p$, is $\left(\frac{n}{2} + p\right)$ if the zerodivisor is not a multiple of 2^k or p. Where, p is any odd prime and $k > 1$ is a natural number.

Proof. Let a be any zero-divisor which is not multiple of 2^k or p.
$a,\ b$ be any even vertex except multiple of 2^k.
So, a and b both are multiples of 2. Therefore, ab is always multiple of 4.
So, $\gcd(o(a),\ o(b)) \neq o(ab)$.
$\therefore\ a$ is not adjacent to b.

In this graph unit elements are adjacent with every element. So, a is adjacent with every vertex of the graph except even vertex that are not multiple of 2^k.

In Z_n there are $\left(\frac{n}{2} - 1\right)$ number of non-zero even vertex and there are $(p-1)$ number of multiple of 2^k. So, number of non-adjacent vertex to a is $= \left\{\left(\frac{n}{2} - 1\right) - (p-1)\right\} = \frac{n}{2} - p$.

So, number of adjacent vertices to a is $= \left\{n - \left(\frac{n}{2} - p\right)\right\} = \frac{n}{2} + p$.

Hence, Degree of any non-zero zero-divisor in $O_g(Z_n)$ is $\left(\frac{n}{2} + p\right)$, where $n = 2^k \cdot p$, and the zero-divisor is not a multiple of 2^k or p. □

Theorem 10 The graph $O_g(Z_{2^k})$ is not planar for any odd prime p and $k > 1$.

Proof. In the graph $O_g\left(Z_{2^k p}\right)$ there are p number of even vertices which are not adjacent to each other. So, if we delate $(p-1)$ number of even vertices from these vertices the graph will be complete graph of vertices $2^k p - (p-1)$. For any odd prime p the value of $2^k p - (p-1)$ is always greater than five. Therefore, in the graph $O_g\left(Z_{2^k p}\right)$ there is a subgraph K_5.Hence, the graph $O_g\left(Z_{2^k p}\right)$ is not planar for any odd prime p and $k > 1$. □

Theorem 11 The graph $O_g(Z_{2^k})$ is not Eulerian for any prime p.

Proof. In the graph $O_g(Z_{2^k})$ degree of zero element and any unit element is $n-1$, which is always odd for $n = 2^k p$. Therefore, the graph $O_g\left(Z_{2^k p}\right)$ is not Eulerian. □

Theorem 12 Clique number of the graph $O_g\left(Z_{2^k p}\right)$ is $2^k p - (p-1)$.

Proof. In the graph $O_g(Z_{2^k})$ there are p number of even vertices which are not adjacent to each other. And other zero-divisors are adjacent with every vertex of the graph. Also, unit elements and zero element are adjacent with every element. If we delate $(p-1)$ number of even vertices from the graph, then the graph will be complete graph of vertices $2^k p - (p-1)$, which will be the maximal complete graph. Therefore, Clique number of the graph $O_g\left(Z_{2^k p}\right)$ is $2^k p - (p-1)$. □

Theorem 13 The degree of any zero-divisor of the graph $O_g(Z_{2^k})$ is $2^{k-1}+1$. Where, $k > 1$ is an integer.

Proof. In Z_{2^k} zero-divisors are of the form multiples of 2^r, where, $r(< k)$ is a positive integer. Unit elements of Z_{2^k} are odd elements of Z_{2^k}. Let $a = 2^{k_1} r_1$ and $b = 2^{k_2} r_2$ are any two zero-divisors of Z_{2^k}. Where, r_1 and r_2 are positive odd integers.

Now, $o(a) = 2^{k-k_1}$ and $o(b) = 2^{k-k_2}$

$$\gcd(o(a),\ o(b)) = 2^{k-k_1} \text{ or } 2^{k-k_2}.$$

But,

$$o(ab) = o\left(2^{k_1+k_2} r_1 r_2\right) = 2^{k-(k_1+k_2)}.$$

For any positive integer k_1 and k_2, $\gcd(o(a),\ o(b)) \neq o(ab)$.

So, a is not adjacent to b. Therefore, in the graph $O_g(Z_{2^k})$, zero-divisors are not adjacent to each other.

If $k_1 = 0$ or $k_2 = 0$ then $\gcd(o(a),\ o(b)) = o(ab)$

$\Rightarrow$ if one of $a,\ b$ is odd and other is even then a is adjacent to b.

$\Rightarrow$ if one of $a,\ b$ is unit and other is zero-divisor then a is adjacent to b.

Therefore, in the graph $O_g(Z_{2^k})$, zero-divisors are adjacent to unit element and zero element. Number of unit element in Z_{2^k} is 2^{k-1}.

Hence, degree of any zero-divisor of the graph $O_g(Z_{2^k})$ is $2^{k-1}+1$. Where, $k > 1$ is an integer. □

Theorem 14 The graph $O_g(Z_{2^k})$ is planar if and only if $k = 1$ and 2.

Proof. For $k = 1$ and 2 the graphs $O_g(Z_{2^k})$ are complete graph K_2 and K_4. The graphs K_2 and K_4 are planar. So, the graph $O_g(Z_{2^k})$ is planar if $k = 1$ and 2. Unit vertices of the graph $O_g(Z_{2^k})$ are odd elements of Z_{2^k}. So, for $k > 2$ (k is a positive integer) the graph $O_g(Z_{2^k})$ always has unit vertices $\overline{1},\ \overline{3},\ \overline{5},\ \overline{7}$. Zero vertex and unit vertices are adjacent with every vertex of the graph except itself. So, the graph always has a subgraph K_5 with vertices $\overline{0},\ \overline{1},\ \overline{3},\ \overline{5},\ \overline{7}$. Therefore, the graph $O_g(Z_{2^k})$ is planar if and only if $k = 1$ and 2. □

Theorem 15 Clique number of the graph $O_g(Z_{2^k})$ is $2^{k-1}+1$. Where, k is a positive integer.

Proof. In the graph $O_g(Z_{2^k})$ unit vertices and zero vertex are adjacent with every vertex of the graph except itself. In the graph odd elements of Z_{2^k} are unit elements of Z_{2^k}. In the graph $O_g(Z_{2^k})$ all the unit vertices and zero vertices form a maximal complete subgraph. Number of unit element in Z_{2^k} is 2^{k-1}. Therefore, the maximal complete subgraph is $K_{2^{k-1}+1}$. Hence, Clique number of the graph $O_g(Z_{2^k})$ is $2^{k-1}+1$. Where, k is a positive integer. □

3. Conclusion

For $n = p$, and $n = pq$ the graph $O_g(Z_n)$ is complete graph. Degree of zero vertex and unit elements of the graph $O_g(Z_n)$ is connected with every vertex of the graph except itself. The graph $O_g\left(Z_{p^2}\right)$ is neither Eulerian nor planar for any odd prime p. The degree of any zero-divisor of the graph $O_g(Z_{2^k})$ is equal to the clique number of the graph $O_g(Z_{2^k})$ and which is equal to $2^{k-1}+1$. Where, k is a positive integer. The graph $O_g(Z_{2^k})$ is planar if and only if $k=1$ and 2 . The graph $O_g\left(Z_{2^k p}\right)$ is not planar for any odd prime p and $k>1$ also this graph is not Eulerian. If the zero-divisor of the graph $O_g(Z_{2^k})$ is multiple of 2^k but not multiple of p then a zero-divisor is adjacent with all the vertices of the graph except itself.

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Conflict of interest

The authors declare there is no conflict of interest at any point with reference to research findings.

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