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Research Article

Topological Centers Induced by $L^1(G)^{**}$-Module Actions

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Abstract: Let (π, H) be a unitary representation of a locally compact group G. Recent works on some topological centers induced by π have studied. This paper continues the investigation in this regard. The new topological center induced by the module action related to π, which lies between $L^1(G)$ and the second dual $L^1(G)^{**}$, is studied. As well, some unitary representations are presented whose topological centers in our sense are $L^1(G)$, $L^1(G)^{**}$, or neither.

Keywords: locally compact group, module action, topological center, unitary representation

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1. Introduction

Let G always denote a locally compact group with a fixed left Haar measure dx. The group algebra $L^1(G)$ is defined as in [1] equipped with the convolution product $*$ and $\|\cdot\|_1$-norm. Also, $L^\infty(G)$ denotes the Lebesgue space equipped with the locally essential supremum norm $\|.\|_\infty$. Then $L^\infty(G)$ is the dual of $L^1(G)$ for the pairing

$$\langle f, \phi\rangle = \int_G f(x)\phi(x)\,dx \quad (f \in L^\infty(G),\ \phi \in L^1(G)).$$

Moreover, $L^1(G)^{**}$, the second dual of $L^1(G)$, is a Banach algebra endowed with the first Arens product which is given by making in turn the definitions

$$\langle f\cdot\phi, \psi\rangle = \langle f, \phi*\psi\rangle \quad \langle n\cdot f, \phi\rangle = \langle n, f\cdot\phi\rangle \quad \langle mn, f\rangle = \langle m, n\cdot f\rangle,$$

where $f \in L^\infty(G)$, $\phi, \psi \in L^1(G)$ and $m, n \in L^1(G)^{**}$.

For each complex function f on G and $x \in G$, we use $l_x f$ to denote the left translation of f by x; i.e., $l_x f(y) = f(xy)$ for all $y \in G$. Let $LUC(G)$ mean the space of bounded left uniformly continuous complex-valued functions on G; that is, all $f \in L^\infty(G)$ such that the map $x \mapsto l_x f$ from G into $L^\infty(G)$ is bounded and continuous; see for example [1].

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A continuous unitary representation of G on a Hilbert space H is a group homomorphism π from G into the group of unitary operators on H such that π is continuous with respect to the strong operator topology. In this case, we write (π, H) is a unitary representation of G. Let $B(H)$ be the von-Neumann algebra consisting of all linear and bounded operators on H. Then $B(H)$ is a right G-module by the following action

$$T \cdot_\pi x = \pi(x^{-1})T\pi(x) \quad (T \in B(H),\ x \in G). \tag{1}$$

Note that the above action is not necessarily Banach in the sense of [2]; i.e., the map $x \mapsto T \cdot_\pi x$ from G into $B(H)$ is not continuous in general. An element $T \in B(H)$ is saied to be a G-continuous operator if the mapping $x \mapsto T \cdot_\pi x$ is continuous with respect to the norm topology of $B(H)$. The set of all such operators denote by $UCB(\pi)$ that is a right Banach G-module.

The above action was originally noted by Bekka in [3] when he created the notion of amenable representations. Then it was attended by some authors in terms of amenable group theory; for example [4–6]. Among others, Pak-Keung Chan [7] focused on the bilinear map

$$(M,\ T) \mapsto MT\colon UCB(\pi)^* \times UCB(\pi) \to LUC(G),$$

where $MT(x) = \langle M,\ T \cdot_\pi x\rangle$ for all $x \in G$, and then also defined the bilinear map

$$(m,\ M) \mapsto mM\colon LUC(G)^* \times UCB(\pi)^* \to UCB(\pi)^*,$$

where $mM(T) = \langle m,\ MT\rangle$ for all $T \in UCB(\pi)$. The latter builds $UCB(\pi)^*$ as a left Banach $LUC(G)^*$-module. $Z(\pi)$ denote the induced topological center of the module action; that is, the following set

$$\{m \in LUC(G)^* \mid M \mapsto mM \text{ is weak}^*\text{-weak}^* \text{ continuous on} UCB(\pi)^*\}.$$

In fact, by [7, Proposition 3.1],

$$Z(\pi) = \{m \in LUC(G)^* \mid Tm \in UCB(\pi) \text{ for all} T \in UCB(\pi)\},$$

where Tm is the linear functional on $UCB(\pi)^*$ that given by

$$\langle Tm,\ M\rangle = \langle mM,\ T\rangle \quad (M \in UCB(\pi)^*).$$

According to [7], $Z(\pi)$ is a Banach subalgebra of $LUC(G)^*$ contaning $M(G)$. Also, $Z(\pi)$ is saied to be minimal if $Z(\pi) = M(G)$, and $Z(\pi)$ is called maximal if $Z(\pi) = LUC(G)^*$. The following identification [7, Theorem 4.6] is a crucial result, which is heavily used in [7]. $Z(\pi)$ is minimal if and only if the linear span of the set

$$\{MT \mid M \in UCB(\pi)^*,\ T \in UCB(\pi)\}$$

is dense in $LUC(G)$. By [7, Example 4.10], $Z(\lambda)$ is minimal, where $\lambda: G \longrightarrow B(L^2(G))$ is the left regular representation as defined by $x \mapsto l_x$ for all $x \in G$.

According to [2, pages 24-26] and the fact $l_x MT = (M)(T \cdot x)$ for all $x \in G$, $UCB(\pi)$ is as follows

$$
\begin{aligned}
UCB(\pi) &= \{T \in B(H) \,|\, \text{the map} \, x \mapsto T \cdot_\pi x \, \text{is Haar-measurable}\} \\
&= \{T \in B(H) \,|\, \text{the map} \, x \mapsto T \cdot_\pi x \, \text{is weakly continuous}\} \\
&= \{T \in B(H) \,|\, MT \in LUC(G) \text{ for all } M \in B(H)^*\}.
\end{aligned}
$$

As mentioned in [3], the space $UCB(\pi)$ is corresponded to $LUC(G)$ and $B(H)$ also corresponds to $L^\infty(G)$. We would like to point it out again $MT \in LUC(G)$ for all $M \in B(H)^*$ and $T \in UCB(\pi)$. But from this point of view, often it is not obvious that MT is measurable for all $M \in B(H)^*$ and $T \in B(H)$. It is a strong motivating force for the present research. Noting [7, Remark 2.5], to realize this aim, we must consider a functional approach that coincides with the above argument in the discreteness case. This view extends the notions of Chan's paper [7]; specially, the new topological center can be defined that is a closed subalgebra of $L^1(G)^{**}$ containing $L^1(G)$. We will study this center, and state some examples of unitary representations whose topological centers are maximal, minimal or neither. Note that our results are more than the literature of the topological center that exists in the general texts.

2. The results

Considering any unitary representation (π, H) of G, the space $B(H)$ forms a right Banach $L^1(G)$-module with the action given by

$$\langle (T \cdot_\pi \phi)u, v\rangle = \int_G \phi(x)\langle (T \cdot_\pi x)u, v\rangle \, dx \quad (u, v \in H).$$

As stated in [3, Lemma 3.2],

$$UCB(\pi) = UCB(\pi) \cdot_\pi L^1(G) = B(H) \cdot_\pi L^1(G).$$

Note that an operator $T \in UCB(\pi)$ if and only if $T \cdot_\pi \phi_i \longrightarrow T$ in the weak topology, where (ϕ_i) is a bounded approximate identity of $L^1(G)$.

We comence with the adjoint of the mapping $(T, \phi) \mapsto T \cdot_\pi \phi$ for all $T \in B(H)$ and $\phi \in L^1(G)$ as a preamble material needed in this note; that is, the map $(M, T) \mapsto M \cdot T$: $B(H)^* \times B(H) \to L^\infty(G)$, where

$$\langle M \cdot T, \phi\rangle = \langle M, T \cdot_\pi \phi\rangle \quad (\phi \in L^1(G)). \tag{2}$$

For the latter, we have the map $L^1(G)^{**} \times B(H)^* \longrightarrow B(H)^*$ by $(m, M) \mapsto m \cdot M$, where

$$\langle m \cdot M, T\rangle = \langle m, M \cdot T\rangle \quad (T \in B(H)).$$

So, $B(H)^*$ becomes a left Banach $L^1(G)^{**}$-module, and $\|m\cdot M\| \leq \|m\|\|M\|$. Now, let $Z(\pi,\ L^1(G)^{**})$ denotes the topological center of the $L^1(G)^{**}$-module action induced by π; that is, the collection of all $m \in L^1(G)^{**}$ such that the mapping $M \mapsto m\cdot M$ on $B(H)^*$ is weak*-weak*-continuous. Then the reader can check that $Z(\pi,\ L^1(G)^{**})$ is a closed subalgebra of $L^1(G)^{**}$ containing $L^1(G)$. We say that the center is minimal if $Z(\pi,\ L^1(G)^{**}) = L^1(G)$, and the center is maximal if $Z(\pi,\ L^1(G)^{**}) = L^1(G)^{**}$. Clearly, when G is discrete, $Z(\pi,\ L^1(G)^{**})$ is nothing but $Z(\pi)$. Furthermore, one can easily observe that

$$Z(\pi,\ L^1(G)^{**}) = \{m \in L^1(G)^{**} | T\cdot m \in B(H) \text{ for all } T \in B(H)\},$$

where $T\cdot m$ is the bounded linear functional on $B(H)^*$ defiend by

$$\langle T\cdot m,\ M\rangle = \langle m\cdot M,\ T\rangle \quad (M \in B(H)^*).$$

Also, $Z(\pi,\ L^1(G)^{**})$ is maximal if and only if the map γ_T is weakly compact for all $T \in B(H)$, where $\gamma_T\colon L^1(G) \longrightarrow UCB(\pi)$ is given by

$$\phi \mapsto T\cdot_\pi \phi \quad (\phi \in L^1(G)).$$

Using [8, Proposition 3.4 and Theorem 3.5], the maximality of $Z(\pi)$ follows from the maximality of $Z(\pi,\ L^1(G)^{**})$. One can check that the converse is valid if also $B(H) = UCB(\pi)$.

Now, we demonstrate a characterization of the minimality of the center. Before stating, let us recall the left introverted subspace $L_0^\infty(G)$ of $L^\infty(G)$ as follows.

$$L_0^\infty(G) = \{f \in L^\infty(G)\colon\ \|f\chi_{G\setminus K}\|_\infty \longrightarrow 0, \text{ as compact } K \uparrow G\}.$$

As mentioned in [9], the dual of $L_0^\infty(G)$ can be regarded as a closed subalgebra of $L^1(G)^{**}$. Also, $L^1(G)$ is a closed ideal in $L_0^\infty(G)^*$. Furtheremore, $L^1(G) = L_0^\infty(G)^*$ if and only if G is discrete. We refer the readers to [9] for more details.

Theorem 1 Let G be a non-compact, and let $(\pi,\ H)$ be a unitary representation of G. Then the following statements hold.

(a) $Z(\pi,\ L^1(G)^{**})$ is minimal if and only if the the linear span of the set

$$\{M\cdot T | M \in B(H)^*,\ T \in B(H)\} \tag{3}$$

is dense in $L^\infty(G)$.

(b) $Z(\pi)$ is minimal if $Z(\pi,\ L^1(G)^{**})$ is minimal.

Proof. (a) Let $Z(\pi,\ L^1(G)^{**})$ be minimal, and let there exist a non-zero element $m \in L^1(G)^{**}$ such that vanishing on the linear span of set (3). Then $m\cdot M = 0$ for all $M \in B(H)^*$, and so $m \in Z(\pi,\ L^1(G)^{**}) = L^1(G)$. On the other hand, $L^1(G)^{**}$ can be written as a Banach space direct sum $L_0^\infty(G)^* \oplus L_0^\infty(G)^\perp$, where

$$L_0^\infty(G)^\perp = \{m \in L^1(G)^{**} \,|\, \langle m,\ f\rangle = 0 \ \text{ for all } f \in L_0^\infty(G)\}.$$

Note that $L_0^\infty(G)^\perp$ is a weak*-closed ideal in $L^1(G)^{**}$; see [9] for details. Now, using [10, Theorem 4], pick any non-zero element $n \in L_0^\infty(G)^\perp$ that is right cancellable in $L^1(G)^{**}$. So,

$$(mn)\cdot M = m\cdot(n\cdot M) = 0 \quad (M \in B(H)^*).$$

It follows that

$$mn \in Z(\pi,\ L^1(G)^{**}) \cap L_0^\infty(G)^\perp = L^1(G) \cap L_0^\infty(G)^\perp = \{0\}.$$

Therefore, the right cancellability of n follows that $m = 0$, which is impossible. So, the linear span of set 3 is dense in $L^\infty(G)$ if $Z(\pi,\ L^1(G)^{**})$ is minimal.

Before proving the converse, notice first that if $m \in L^1(G)^{**}$, then for each $f \in L^\infty(G)$, we define a linear functional f_m on $L^1(G)^{**}$ by

$$\langle f_m,\ n\rangle = \langle mn,\ f\rangle \quad (n \in L^1(G)^{**}).$$

Since $L^1(G)$ coincides with the topological center of $L^1(G)^{**}$ by [11, Corollary 5.5], $f_m \in L^\infty(G)$ for all $f \in L^\infty(G)$ if and only if $m \in L^1(G)$. Let now the linear span of set 3 be dense in $L^\infty(G)$, and let $m \in Z(\pi,\ L^1(G)^{**})$ and $f \in L^\infty(G)$. Then we can regard that $f = M\cdot T$, where $M \in B(H)^*$ and $T \in B(H)$. Therefore, for each $n \in L^1(G)^{**}$ and some $S \in B(H)$, we have

$$\langle f_m,\ n\rangle = \langle m,\ (n\cdot M)T\rangle = \langle T\cdot m,\ n\cdot M\rangle$$

$$= \langle S,\ n\cdot M\rangle = \langle M\cdot S,\ n\rangle.$$

It follows that $f_m = M\cdot S \in L^\infty(G)$, and so $m \in L^1(G)$.

(b) Suppose that $f \in LUC(G)$. Then $f = (M\cdot T)\cdot\phi$ for some $M \in B(H)^*$, $T \in B(H)$ and $\phi \in L^1(G)$, by Therorem 1. So, $f = M\cdot(T\cdot_\pi \phi) = (M)(T\cdot_\pi \phi)$. It means that $Z(\pi)$ is minimal by [7, Theorem 4.6]. □

We have the following consequence as might be expected.

Corollary 1 Let $(\lambda,\ L^2(G))$ be the left regular representation of G. Then $Z(\lambda,\ L^1(G)^{**})$ is minimal.

Proof. Let E be a weak*-closure point of the canonical image of the approximate identity of $L^1(G)$, bounded by 1, and let $f \in L^\infty(G)$. Then $f = M_E\cdot T_f$, where M_E is any Hahn-Banach extension of E to $B(L^2(G))$ and T_f is the multiplication operator on $L^2(G)$. It yields that

$$L^\infty(G) = \{M\cdot T \,|\, M \in B(L^2(G))^*,\ T \in B(L^2(G))\}.$$

The claim now follows from Theorem 1. □

As known, G is discrete if and only if $L^\infty(G) = LUC(G)$. Also, for some unitary representations $(\pi,\ H)$ of non-discrete groups, we have $B(H) = UCB(\pi)$; see for instance [6, Example 5.4.1]. As an immediate consequence of Corollary 1 together with [7, Corollary 4.10], we have the following result:

Corollary 2 Let G be a locally compact group. Then the following statements are equivalent.

(a) G is discrete,

(b) $B(H) = UCB(\pi)$ for all unitary representations $(\pi,\ H)$ of G,

(c) $B(L^2(G)) = UCB(\lambda)$ for the left regular representation $(\lambda,\ L^2(G))$.

Proposition 1 Let G be a locally compact non-discrete group, and let $(\pi,\ H)$ be a unitary representation of G. If $Z(\pi,\ L^1(G)^{**})$ is minimal, then $B(H) \neq UCB(\pi)$.

Proof. Suppose contraryly that $B(H) = UCB(\pi)$. We show that $L_0^\infty(G)^*$ is contained in $Z(\pi,\ L^1(G)^{**})$. Let E is a weak*-closure point of a bounded approximate identity of $L^1(G)$ with norm 1. As known the restricted map

$$\theta\colon EL^1(G)^{**} \longrightarrow LUC(G)^*$$

is an isometric isomorphism. Let now $m \in L_0^\infty(G)^*$. Then $\theta(Em) \in M(G)$ by [9, Theorem 2.11]. Since $M(G) \subseteq Z(\pi)$, [7, Proposition 3.1] follows that $T\theta(Em)$ lies in $B(H)$ for all $T \in B(H)$. On the other hand, one can easily check that

$$T\theta(Em) = T \cdot m \quad (T \in B(H)).$$

So, $T \cdot m \in B(H)$ for all $T \in B(H)$. This fact follows

$$L_0^\infty(G)^* \subseteq Z(\pi,\ L^1(G)^{**}) = L^1(G)$$

which is impossible since G is non-discrete. □

Now, we state some examples on the subject. As seen, the topological center of the $L^1(G)^{**}$-module action induced by the left regular representation of G is minimal. Also, one can easily see that the center is always maximal for all finite-dimensional unitary representations of G. In addition, as mentioned earlier, Example 5.4.1 of [6] gives a unitary representation $(\pi,\ L^2(G))$ of the non-discrete locally compact group $G = (\mathbb{R},\ +)$ such that $B(L^2(G)) = UCB(\pi)$. Moreover, $B(L^2(G)) = WAP(\pi)$ by [6, Remark 5.4.2], where $WAP(\pi)$ is the collection of weakly almost G-periodic operatots; see [8] and [6] for more details. Thus, [8, Lemma 3.3] ensuras that γ_T is weakly compact for all $T \in B(L^2(G))$, and so $Z(\pi,\ L^1(G)^{**})$ is maximal. Furthermore, [7, Example 6.1] gives a unitary representation $(\pi,\ l^2(\mathbb{Z}))$ of $G = \mathbb{Z} \times \mathbb{Z}$ such that $Z(\pi)$ is neither of minimal nor maximal. As earlier indicated, when G is a discrete group, our notions and [7] coincide. At the end of the work, we present a unitary representation of a non-discrete group whose topological center is neither minimal nor maximal.

Example 1 Let G be the "$ax + b$" group. We recall that $G = \{(a,\ b)\,|\,a,\ b \in \mathbb{R},\ a > 0\}$ with multiplication $(a_1,\ b_1)(a_2,\ b_2) = (a_1a_2,\ b_1 + a_1b_2)$. Let also, $\pi\colon G \longrightarrow B\big(L^2(\mathbb{R})\big)$ is defined by

$$\big(\pi(a,\ b)g\big)(t) = \exp(ib\exp t)g(t + \log a)$$

for all $(a,\ b) \in G$, $g \in L^2(\mathbb{R})$ and $t \in \mathbb{R}$. Then π is an irreducible infinite dimensional unitary representation of G. For more details, we refer the reader to [4, Example 4.14]. One can directly check that

$$T \cdot_\pi (a,\ 0) = T \quad (T \in B(H),\ a \in \mathbb{R}^+)$$

So, [7, Proposition 4.9] yelids that $Z(\pi)$ is not minimal. It follows that $Z(\pi, L^1(G)^{**})$ is not minimal by part (b) of Theorem 1.

We claim now that $Z(\pi, L^1(G)^{**})$ is not maximal. Noting [4, Example 4.14], $f \in LUC(\mathbb{R})$ if and only if $T_f \in UCB(\pi)$, and also $f \in WAP(\mathbb{R})$ if and only if $T_f \in WAP(\pi)$. Therefore, $UCB(\pi) \neq WAP(\pi)$ since $LUC(\mathbb{R}) \neq WAP(\mathbb{R})$. Now, [8, Theorem 3.5] ensures that $Z(\pi)$ is not maximal, and so $Z(\pi, L^1(G)^{**})$ is not maximal.

Note also that $UCB(\pi) \neq B(H)$ since $LUC(\mathbb{R}) \neq L^\infty(\mathbb{R})$. So, the converse of Proposition 1 is not valid, in general.

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Conflict of interest

The authors declare no competing financial interest.

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