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Research Article

Memory-Efficient Interpolatory Projection Techniques for the Stabilization of Incompressible Navier-Stokes Flows

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Abstract: In this article, we are proposing an updated form of Krylov subspace-based interpolatory projection techniques for the stabilization of incompressible Navier-Stokes flows. In the proposed techniques, we utilize the reduced-order modelling approach implicitly, where reduced-order matrices need not be gathered explicitly. To estimate the aimed optimal feedback matrix, only the factored solution of the desired continuous-time algebraic Riccati equation (CARE) needs to be stored through the classical eigenvalue decomposition. The sparse structure of the target systems will remain invariant within the matrix-vector operations for generating the bases of the projector matrices through Krylov subspace techniques, where a cohesive projection scheme will be incorporated to ensure the potency of the projector matrices. So, the proposed techniques will be feasible for memory allocation and enhance the rapid convergence of the simulation. Analysis of the target systems' transient characteristics, such as eigenvalues and step-responses, will be used to ascertain the competence and reliability of the proposed techniques. Necessary computation will be done numerically through MATLAB. Stabilization of the transient behaviors and minimization of the simulation time are the prime concerns in this work. Eventually, by comparing with the contemporary techniques the advancement of the proposed techniques will be confirmed.

Keywords: incompressible Navier-Stokes flow, Riccati-based optimal feedback stabilization, Krylov subspace techniques, sparsity-preserving projections scheme, eigenvalue decomposition

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1. Introduction

In the present era of engineering advancement and technological development studies of fluid mechanics and its practical implications are widening conspicuously. Analysis of incompressible Navier-Stokes equations is one of the fundamental and indispensable ingredients in the branches of fluid mechanics, which is remarkably contiguous with oceanography and investigation of fluid attributes. Exploration and employment of the incompressible Navier-Stokes equations include the modelling of the physical systems through the matrix-vector formations, known as incompressible Navier-Stokes models. These models can be formed by linearizing incompressible Navier-Stokes equations by the

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mixed finite element method in terms of space and time variables [1]. In this discretization technique, the variable layout within the system will be kept invariant [2]. The linearized incompressible Navier-Stokes models have a sparse input-output structure embedding the block matrix setup as

$$\underbrace{\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix}}_{E}\underbrace{\begin{bmatrix} \dot{v}(t) \\ \dot{p}(t) \end{bmatrix}}_{\dot{x}(t)} = \underbrace{\begin{bmatrix} A_1 & A_2 \\ A_3 & 0 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} v(t) \\ p(t) \end{bmatrix}}_{x(t)} + \underbrace{\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}}_{B} u(t),$$

$$y(t) = \underbrace{\begin{bmatrix} C_1 & C_2 \end{bmatrix}}_{C}\begin{bmatrix} v(t) \\ p(t) \end{bmatrix} + Du(t), \tag{1}$$

where the symmetric positive definite matrix $M \in \mathbb{R}^{n_v \times n_v}$ is for masses of the system, whereas $A_1 \in \mathbb{R}^{n_v \times n_v}$, $A_2 \in \mathbb{R}^{n_v \times n_p}$, and $A_3 \in \mathbb{R}^{n_p \times n_v}$ for the representation of the system components. Input and output frameworks for assessing the velocity pattern at the corresponding nodes are provided by the matrices $B_1 \in \mathbb{R}^{n_v \times n_u}$, $B_2 \in \mathbb{R}^{n_p \times n_u}$ and $C_1 \in \mathbb{R}^{n_y \times n_v}$, $C_2 \in \mathbb{R}^{n_y \times n_p}$, respectively. The amount of flow passing without any diffraction is represented by the direct transform matrix $D \in \mathbb{R}^{n_y \times n_u}$. The discretized velocity, discretized pressure, and boundary control are demonstrated by the modal vectors $v(t) \in \mathbb{R}^{n_v}$, $p(t) \in \mathbb{R}^{n_p}$, and $u(t) \in \mathbb{R}^{n_u}$, respectively, with the initial velocity $v(0) = v_0$, whereas the output vector is $y(t) \in \mathbb{R}^{n_y}$ [3]. Here, $\det(M) \neq 0$, A_2 and A_3 full column ranks. Thus, $A_3 M^{-1} A_2$ has an inverse, and hence the system (1) is a special form of differential-algebraic system, namely, index-2 descriptor system with dimension $n_v + n_p$ [4].

Reynolds number (*Re*) invariably characterizes the flow pattern that the Navier-Stokes equations describe, and as a result, the stability of the Navier-Stokes models. Practical observations show that the Navier-Stokes models become unstable for $Re \geq 300$, and as a result, properties of physical systems exhibit noteworthy deformations [5]. *A* few of the eigenvalues of the matrix pencil (A, E) lie in the positive complex half-plane, where E is a singular matrix. As a consequence, the system (1) becomes unstable [6].

Stabilization of this sort of unstable Navier-Stokes model is one of the major issues in the research of the system and control [7]. To do this, an optimal feedback matrix K^o needs to be exerted, which can be estimated as $K^o = B^T XE$ [8]. Here, X is a symmetric positive definite matrix that can be attained from the Riccati equation linked to the system (1) and defined as

$$A^T XE + E^T XA - E^T XBB^T XE + C^T C = 0. \tag{2}$$

As a result, system (1) can be reformed into an optimally stabilized system by plugging the stable closed-loop matrix $A_s = A - BK^o$. Then, it can be written as

$$E\dot{x}(t) = A_s x(t) + Bu(t),$$

$$y(t) = Cx(t) + Du(t). \tag{3}$$

For the large-scale setting and the demand for the finer meshing in the discretization process, the size of the system (1) is increasing swiftly. As a result, the dimensions of the system matrices grow over time, and hence the simulations incorporating them became infeasible due to massive computational costs involving memory requirement and greedy time-span. So, contemporary computational tools and conventional solvers are not compatible with the simulation of the large dimensional Riccati equation (2). Consequently, the efficiency of the Riccati-based boundary feedback stabilization of the system (1) will be afflicted drastically [9-10]. As the redress of those impairments issues, a competent reduced-order modelling approach needs to be adopted [11]. The desired Reduced-Order Model (ROM) with dimension $r \ll n$ equivalent to the system (1) can be acquired as

$$\hat{E}\dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + \hat{B}\hat{u}(t),$$

$$\hat{y}(t) = \hat{C}\hat{x}(t) + \hat{D}\hat{u}(t), \tag{4}$$

where $\hat{E} = W^T EV$, $\hat{A} = W^T AV$, $\hat{B} = W^T B$, $\hat{C} = CV$, $\hat{D} = D$.

The essential component for understanding a system's input-output node structure is its transfer function. The ROM (4) is designed in such a way that, at some interpolating locations in the system, its transfer function $\hat{G}(s) = \hat{C}(s\hat{E} - \hat{A})^{-1}\hat{B} + \hat{D}$ can accurately mimic the transfer function $G(s) = C(sE - A)^{-1}B + D$ of the system (1).

Here, V and W are the projector matrices that can be constituted by any dexterous matrix-vector algorithm [12]. Accordingly, a reduced-order form of the desired Riccati equation (2) can be defined as

$$\hat{A}^T \hat{X}\hat{E} + \hat{E}^T \hat{X}\hat{A} - \hat{E}^T \hat{X}\hat{B}\hat{B}^T \hat{X}\hat{E} + \hat{C}^T \hat{C} = 0. \tag{5}$$

The reduced-order Riccati equation (5), which is tiny compared to the Riccati equation (2), can be conveniently solved for the symmetric positive-definite matrix $\hat{X}$ using the MATLAB library command care [13-14].

The optimal feedback matrix for the system (1) via $\hat{X}$ can be achieved in plenty of ways. When reduced-order matrices must be stored and used for subsequent manipulations in some of them, optimal feedback matrices can be approximated from the reduced-order feedback matrix using the inverse projection scheme or any other counter approach, such as the Singular-Value Decomposition (SVD)-based Balanced Truncation (BT) [15-18], the Krylov subspace-based Iterative Rational Krylov Algorithm (IRKA) [19-22], and a recently developed hybrid approach Iterative SVD-Krylov Algorithm [23-26]. In those methods, storing the reduced-order matrices claims redundant memory allocation and delays the convergence of the simulation. BT method requires highly time-demanding computation steps with gigantic memory allotment because of the simulation of large dimensional Lyapunov equations, whereas IRKA is comparatively cheaper in computation cost but the stability of the reduced-order matrices is uncertain and there exists no error bound. On the other hand, some more approaches are available where the reduced-order matrices are used implicitly without storing and any suitable matrix factorization techniques are applied to find approximated factored solution of the Riccati equation of the target models, such as, Low-Rank Alternative Direction Implicit (LR-ADI) incorporated Newton-Kleniman (NK) method [27-30] and system-specific priori-based Rational Krylov Subspace Method (RKSM) [31-34]. Newton-Kleinman method consists of the time-consuming multi-layer nested iterative approach but no rigid convergence criteria of this method are well-defined and definiteness of the solution matrix is unrestrained, whereas RKSM is more functional for matrix-vector operations and well-suited for symmetric systems but the essence of adjustable shift parameter and lack of proper convergence measure makes it incompetent for the simulations of the Navier-Stokes models.

We aim to go beyond the incidents mentioned earlier and get the desired optimal feedback matrix for the system (1) with appeasement computational cost. We will derive the reduced-order model by the modified IRKA approach employing a cohesive projection scheme and then apply the eigenvalue-decomposition coupled reversal process to approximate the solution Riccati equation of the target models via $\hat{X}$. The target models are finally stabilized optimally while the requisite optimal feedback matrix is computed.

The stabilization of the eigenvalues and step-responses of the targeted models will be inspected to demonstrate the effectiveness of the proposed approach numerically. The computation time and size of the factored solution to the Riccati equation will be compared in a table with those of the Newton-Kleinman method's output to assess the proposed technique's accuracy.

2. Background

The preliminaries of the present work will be discussed in this section. The LRCF-ADI technique integrated Newton-Kleinman method and classical form of the IRKA approach for solving the Riccati equation arising from a generalized system will be provided at a glance. Generalized systems equivalent to the index-2 descriptor systems of several cases will be provided as well with proper references. Moreover, the extensions for the index-2 descriptor system will be discussed in brief.

The usual input-output structure for the generalized state-space system can be written as

$$E\dot{x}(t) = Ax(t) + Bu(t),$$

$$y(t) = Cx(t) + Du(t), \tag{6}$$

where $E \in \mathbb{R}^{n\times n}$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$, $C \in \mathbb{R}^{m\times n}$ and $D \in \mathbb{R}^{m\times p}$. For the generalized systems, E is non-singular, and in most cases $D = 0$.

Then the Riccati equation corresponding to the system (6) will have the form as (2).

2.1 *Newton-Kleinman method*

The well-known Newton-Kleinman method is a nested iterative approach that is used to find the optimal feedback matrix for a system via the solution of the Riccati equation without forming any subsequent system. Initially, a Lyapunov equation needs to be designed that is equivalent to the Riccati equation connected to the target system needs, and that Lyapunov equation is then to be solved utilizing the LRCF-ADI technique.

The initial guess of the solution of the desired Lyapunov equation needs to be chosen as X_0 and it needs to be updated to a desired level of accuracy by the iterative approach hence corresponding feedback matrix can be estimated as $K_{i+1} = B^T X_{i+1} E$ [35]. The basic structure of the Newton-Kleinman method is given in Algorithm 1.

Algorithm 1: Basic Newton-Kleinman method.

Input: E, A, B, C and X_0 (initial assumption), i_{max} (number of iterations).

Output: Approximate feedback matrix K.

1 while $i \le i_{max}$ do

2 Compute $\tilde{A}_i = A - BB^T X_i E$ and $\mathcal{W}_i = [C^T \quad E^T X_i B]$.

3 For X_{i+1} solve the Lyapunov equation $\tilde{A}_i^T X_{i+1} E + E^T X_{i+1} \tilde{A}_i = -\mathcal{W}_i \mathcal{W}_i^T$.

4 Compute $K_{i+1} = B^T X_{i+1} E$.

5 $i = i + 1$.

6 end while

The iterative LRCF-ADI technique needs to be applied to compute Step-3 of the Algorithm 1. Finding the Cholesky factor Z_i of X_i is the prime task of this scheme [36-38]. Successive steps of the LRCF-ADI technique are provided in Algorithm 2.

To apply Algorithm 2 to the first-order index-2 descriptor system, as per the structure of the system (1) some of the steps in it essentially need to be modified. The sparsity-preserving Newton-Kleinman method for the first-order index-2 descriptor system is derived in [39].

Algorithm 2: LRCF-ADI for generalized systems.

Input: E, A, C, τ (tolerance), i_{max} (number of iterations) and shift parameters $\{\mu_j\}_{j=1}^{i_{max}} \in \mathbb{C}^-$.

Output: Low-rank Cholesky-factor Z such that $X \approx ZZ^T$.

Consider $\mathcal{W}_0 = C^T$, $Z_0 = [\ ]$ and $i = 1$.

while $\|\mathcal{W}_{i-1}\mathcal{W}_{i-1}^T\| \geq \tau$ or $i \leq i_{max}$ do

    Solve $\mathcal{V}_i = (\tilde{A}^T + \mu_i E^T)^{-1}\mathcal{W}_{i-1}$.

    if $\mathrm{Im}(\mu_i) = 0$ then

        Update $Z_i = \left[Z_{i-1} \quad \sqrt{-2\mu_i}\mathcal{V}_i\right]$,

        Compute $\mathcal{W}_i = \mathcal{W}_{i-1} - 2\mu_i E^T \mathcal{V}_i$.

    else

        Assume $\gamma_i = \sqrt{-2\,\mathrm{Re}(\mu_i)}$, $\delta_i = \dfrac{\mathrm{Re}(\mu_i)}{\mathrm{Im}(\mu_i)}$,

        Update $Z_{i+1} = \left[Z_{i-1} \quad \gamma_i(\mathrm{Re}(\mathcal{V}_i) + \delta_i\,\mathrm{Im}(\mathcal{V}_i)) \quad \gamma_i\sqrt{\delta_i^2+1}\ \mathrm{Im}(\mathcal{V}_i)\right]$,

        Compute $\mathcal{W}_{i+1} = \mathcal{W}_{i-1} + 2\gamma_i^2 E^T\left[\mathrm{Re}(\mathcal{V}_i) + \delta_i\,\mathrm{Im}(\mathcal{V}_i)\right]$.

        $i = i + 1$

    end if

    $i = i + 1$

end while

2.2 *Iterative Rational Krylov Algorithm*

A reduced-order model that is analogous to the original model must be obtained to use IRKA to determine the optimal feedback matrix. This can be done by projecting from the left and right sides of the model and then using a typical matrix solver to solve a reduced-order Riccati problem. Using an inverse projection method and the solution to the reduced-order Riccati equation, the intended optimal feedback matrix for the original system can be acquired.

The initial projector matrices are to be put up under an arbitrary set of interpolation points and tangential directions. These interpolation points and the tangential directions must be gradually updated to attain an appropriate level of approximation accuracy [40]. For the reduced-order modelling, the Petrov-Galerkin condition must be satisfied, and the interpolation points and tangential directions must satisfy the Hermite bi-tangential interpolation criteria [41]. Algorithm 3 provides a summary of the IRKA approach's sequential steps.

Due to the requirement of the adjustment with the system structure of the system (1), the construction of the projection matrices in Algorithm 3 needs to be modified. Keeping the sparse form of the system (1) and accordingly modifying the subsequent steps of Algorithm 3 is provided in detail in [42].

Algorithm 3: IRKA for generalized systems.

Input: E, A, B, C, D.

Output: Optimal feedback matrix K^o.

1 Choose the interpolation points $\{\alpha_i\}_{i=1}^r$ and tangential directions $\{b_i\}_{i=1}^r$ and $\{c_i\}_{i=1}^r$ at the outset.

2 Construct the projection matrices $V = \left[(\alpha_1 E - A)^{-1} Bb_1, \cdots, (\alpha_r E - A)^{-1} Bb_r\right]$ and $W = \left[(\alpha_1 E - A)^{-T} C^T c_1, \cdots, (\alpha_r E - A)^{-T} C^T c_r\right]$.

3 while (not converged) do

4 Compute the reduced-order matrices $\hat{E} = W^T EV, \hat{A} = W^T AV, \hat{B} = W^T B, \hat{C} = CV, \hat{D} = D$.

5 Compute $\hat{A}z_i = \lambda_i \hat{E} z_i$ and $y_i^* \hat{A} = \lambda_i y_i^* \hat{E}$ for $\alpha_i \leftarrow -\lambda_i$, $b_i^* \leftarrow -y_i^* \hat{B}$ and $c_i^* \leftarrow \hat{C} z_i^*$, for $i = 1, \cdots, r$.

6 Repeat step 2.

7 Repeat step 4.

8 For $\hat{X}$ solve the reduced-order Riccati equation $\hat{A}^T \hat{X}\hat{E} + \hat{E}^T \hat{X}\hat{A} - \hat{E}\hat{X}\hat{B}\hat{B}^T \hat{X}\hat{E} + \hat{C}^T \hat{C} = 0$.

9 Compute $\hat{K} = \hat{B}^T \hat{X}\hat{E}$ and hence $K^o = \hat{K} V^T E$.

10 end while

2.3 *Derivation of an equivalent generalized system and its associated sparsity-preserving ROM from the index-2 descriptor system*

Because of the distinct structure of the system matrices in the index-2 descriptor system (1) with the singularity of the matrix E, usual matrix computations cannot be possible. To prevail over this adversity, the system (1) needs to be converted to a coequal generalized system or any other commensurate matrix form that should be computable.

Derivation of an equivalent generalized system and its associated sparsity-preserving ROM from the index-2 descriptor system (1) are thoroughly described in [43]. The equivalent generalized system can be configured as

$$\begin{aligned} \mathcal{E}\dot{x}(t) &= \mathcal{A}x(t) + \mathcal{B}u(t), \\ y(t) &= \mathcal{C}x(t) + \mathcal{D}u(t) - C_2 (A_3 M^{-1} A_2)^{-1} B_2 \dot{u}(t), \end{aligned} \tag{7}$$

with the transfigured matrices as

$$x = x_1, \;\; \mathcal{E} = \Pi_l M \Pi_r, \;\; \mathcal{A} = \Pi_l A_1 \Pi_r,$$

$$\mathcal{B} = \Pi_l (B_1 - A_1 M^{-1} A_2 (A_3 M^{-1} A_2)^{-1} B_2),$$

$$\mathcal{C} = (C_1 - C_2 (A_3 M^{-1} A_2)^{-1} A_3 M^{-1} A_1) \Pi_r,$$

$$\mathcal{D} = D - C_2(A_3M^{-1}A_2)^{-1}A_3M^{-1}B_1. \tag{8}$$

Here, Π_l and Π_r are the left and right projectors, respectively, that can be determined as

$$\Pi_l = I - M^{-1}A_2\left(A_3M^{-1}A_2\right)^{-1}A_3,$$

$$\Pi_r = I - A_2\left(A_3M^{-1}A_2\right)^{-1}A_3M^{-1}. \tag{9}$$

The converted generalized system (7) employs projectors Π_l and Π_r in dense form, which hinders the computation convergence. Due to the importance of system structural invariance, the dense form of projectors is significantly detrimental. To resolve this issue, avoid explicitly building the projectors Π_l and Π_r is unavoidable. So, implementing the projectors Π_l and Π_r will be bypassed, and computationally efficient sparsity-preserving techniques the projector matrices V and W are explored and utilized to find the desired ROM [44].

Then the computationally competent ROM that can reasonably approximate the system (1) can be written as

$$\hat{\mathcal{E}}\dot{x}(t) = \hat{\mathcal{A}}x(t) + \hat{\mathcal{B}}u(t),$$

$$y(t) = \hat{\mathcal{C}}x(t) + \hat{\mathcal{D}}u(t), \tag{10}$$

Here, the reduced-order matrices for the ROM defined in (9) can be built in sparsity-preserving form as

$$\hat{\mathcal{E}} = W^TMV,\ \hat{\mathcal{A}} = W^TA_1V,$$

$$\hat{\mathcal{B}} = W^TB_1 - (W^TA_1)M^{-1}A_2(A_3M^{-1}A_2)^{-1}B_2,$$

$$\hat{\mathcal{C}} = C_1V - C_2(A_3M^{-1}A_2)^{-1}A_3M^{-1}(A_1V),$$

$$\hat{\mathcal{D}} = D - C_2(A_3M^{-1}A_2)^{-1}A_3M^{-1}B_1. \tag{11}$$

As a special case, assuming $B_2 = 0$ and $C_2 = 0$, a simplified form of (7) that is analogical to the system (1) can be written as

$$\mathcal{E}\dot{x}(t) = \mathcal{A}x(t) + \mathcal{B}u(t),$$

$$y(t) = \mathcal{C}x(t) + \mathcal{D}u(t), \tag{12}$$

where the associated matrices will have a more convenient form as

$$x = x_1,\ \mathcal{E} = \Pi_lM\Pi_r,\ \mathcal{A} = \Pi_lA_1\Pi_r,$$

$$\mathcal{B} = \Pi_l B_1, \ \mathcal{C} = C_1 \Pi_r, \ \mathcal{D} = D. \tag{13}$$

Then the desired ROM will have the same structure of (9) but the corresponding reduced-order matrices can be formed as

$$\hat{\mathcal{E}} = W^T M V, \ \hat{\mathcal{A}} = W^T A_1 V,$$

$$\hat{\mathcal{B}} = W^T B_1, \ \hat{\mathcal{C}} = C_1 V, \ \hat{\mathcal{D}} = D. \tag{14}$$

Thus, the reduced-order Riccati equation arising from (9) can be reformed as

$$\hat{\mathcal{A}}^T \hat{X} \hat{\mathcal{E}} + \hat{\mathcal{E}}^T \hat{X} \hat{\mathcal{A}} - \hat{\mathcal{E}}^T \hat{X} \hat{\mathcal{B}} \hat{\mathcal{B}}^T \hat{X} \hat{\mathcal{E}} + \hat{\mathcal{C}}^T \hat{\mathcal{C}} = 0. \tag{15}$$

2.4 *Stabilization of index-2 descriptor system*

The unstable incompressible Navier-Stokes flow (1) can be optimally stabilized by implementing the optimal feedback matrix K^o as follows

$$\underbrace{\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix}}_{E} \underbrace{\begin{bmatrix} \dot{v}(t) \\ \dot{p}(t) \end{bmatrix}}_{\dot{x}(t)} = \underbrace{\begin{bmatrix} A_1 - B_1 K^o & A_2 \\ A_3 & 0 \end{bmatrix}}_{A_s} \underbrace{\begin{bmatrix} v(t) \\ p(t) \end{bmatrix}}_{x(t)} + \underbrace{\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}}_{B} u(t),$$

$$y(t) = \underbrace{\begin{bmatrix} C_1 & C_2 \end{bmatrix}}_{C} \begin{bmatrix} v(t) \\ p(t) \end{bmatrix} + Du(t). \tag{16}$$

3. Memory-efficient IRKA for first-order index-2 descriptor system

This section of the article contains its principal objective. Here, techniques for constructing the cohesive projector matrices that preserve the sparsity pattern of the target system will be deduced. The proposed techniques can immensely reduce the requirement of massive memory allocation and computation time.

This topic focuses on techniques to attain the factored solution of the Riccati equation in a way that produces the desired optimal feedback matrix without storing reduced-order matrices, therefore conserving memory. It also permits the factored solution to be stored for future applications.

3.1 *Sparsity-preserving techniques for the construction of cohesive projector matrices*

This section is devoted to constructing the projector matrices V and W deploying the sparsity-preserving techniques. As the eigenvalue decomposition will be utilized for determining the factored solution of the Riccati equation (2), an affinity between the projector matrices V and W needs to be established.

At the outset of this step, at each iteration two pre-projector matrices $\tilde{V}_i$ and $\tilde{W}_i$ need to be estimated implementing the matrix-vector operations as

$$(\alpha_i E - A)^{-1} B b_i = \tilde{V}_i, \text{ and}$$

$$(\alpha_i E - A)^{-T} C^T c_i = \tilde{W}_i, \tag{17}$$

or,

$$\begin{bmatrix} \alpha_i M - A_1 & -A_2 \\ -A_3 & 0 \end{bmatrix} \begin{bmatrix} \tilde{V}_i \\ \Lambda_v \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} b_i, \text{ and}$$

$$\begin{bmatrix} \alpha_i M^T - A_1^T & -A_3^T \\ -A_2^T & 0 \end{bmatrix} \begin{bmatrix} \tilde{W}_i \\ \Lambda_w \end{bmatrix} = \begin{bmatrix} C_1^T \\ C_2^T \end{bmatrix} c_i. \tag{18}$$

Here, Λ_v and Λ_w contained the insignificant components of the system (1), and hence truncated.

After that, with the association of $\tilde{V}_i$ and $\tilde{W}_i$ sparsity-preserving bases of the Krylov subspace for the cohesive projector matrices V and W can be found as

$$V_i = \tilde{V}_i,$$

$$W_i = [M^T (M^{-T} \tilde{W}_i)][\tilde{V}_i^T M^T (M^{-T} \tilde{W}_i)]^{-1}. \tag{19}$$

where V_i and W_i are the generators of V and W, respectively. Here, V and W are non-square projector matrices that are attained from the subspaces having orthogonal columns as their bases. So, V and W have one-sided inverses with the property that $V^T = V^{-1}$ and $W^T = W^{-1}$. Moreover, For an identity matrix I with the proper dimension, V and W confirm the affinity as $W^T V = I$.

3.2 *Estimation of the factored solution of the Riccati equation*

The solution $\hat{X}$ of the reduced-order Riccati equation (5) or equivalent (14) can be applied to approximate the solution X of the Riccati equation (2). Applying the inverse projection and plugging the eigenvalue decomposition, the desired factored solution Z can be attained to approximate X as $X = ZZ^T$. Successive steps of this approximation are given below.

$$\begin{aligned} X &= W^{-T} \hat{X} V^{-1} = W \hat{X} W^T \\ &= W (U \Sigma U^T) W^T = W \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \begin{bmatrix} U_1^T \\ U_2^T \end{bmatrix} W^T \\ &= W U_1 \Sigma_1 U_1^T W^T = (W U_1 \Sigma_1^{\frac{1}{2}})(W U_1 \Sigma_1^{\frac{1}{2}})^T \\ &= ZZ^T. \end{aligned} \tag{20}$$

Here, the eigenvalues of $\hat{X}$ are contained in the diagonal matrix Σ, whereas corresponding linearly independent eigenvectors are contained in U. In the arrangement of the descending order of the magnitude, Σ_1 contains the dominant eigenvalues and Σ_2 contains the negligible eigenvalues. Likewise, U_1 and U_2 maintain the identical preeminence of their corresponding eigenvalues.

Embedding the factored solution $Z = WU_1 \Sigma_1^{\frac{1}{2}}$ intended optimal feedback matrix for the system (1) can be estimated as

$$K^o = B_1^T (ZZ^T) M = (B_1^T Z)(Z^T M). \tag{21}$$

The memory-efficient IRKA for finding the optimal feedback matrix of the system (1) is expounded in Algorithm 4.

Algorithm 4: Memory-efficient IRKA for index-2 descriptor systems.

Input : $M, A_1, A_2, A_3, B_1, B_2, C_1, C_2, D$.

Output: Optimal feedback matrix K^o.

1 Choose the interpolation points $\{\alpha_i\}_{i=1}^r$ and tangential directions $\{b_i\}_{i=1}^r$ and $\{c_i\}_{i=1}^r$ at the outset.

2 Based on the sparse linear systems in (17) and implementing (18), generate the cohesive projection matrices V and W.

3 while (not converged) do

4 Compute the reduced-order matrices as (10).

5 Compute $\hat{\mathcal{A}} z_i = \lambda_i \hat{\mathcal{E}} z_i$ and $y_i^* \hat{\mathcal{A}} = \lambda_i y_i^* \hat{\mathcal{E}}$ for $\alpha_i \leftarrow -\lambda_i$, $b_i^* \leftarrow -y_i^* \hat{\mathcal{B}}$ and $c_i^* \leftarrow \hat{\mathcal{C}} z_i^*$, for $i = 1, \cdots, r$.

6 Repeat step 2.

7 Repeat step 4.

8 For $\hat{X}$ solve the reduced-order Riccati equation (14).

9 Using $\hat{X}$ and executing (19) compute the factored solution Z of X of the Riccati equation (2).

10 Compute the optimal feedback matrix K^o as (20).

11 end while

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4. Numerical results

This section discusses the empirical evidence for the versatility and efficacy of the proposed techniques. MATLAB is used as the simulation tool and investigation involves both figurative and tabular procedures. Computational amelioration through the present techniques will be inspected by comparing it with contem-porary techniques. The cardinal objective is to inquire into the computational time and stabilization of the transient behaviors. We are aiming to stabilize the eigenvalues and step-responses of the target models with minimal computational time. Unstable Navier-Stokes models are to be used as practical applications for validating the proposed techniques.

The MATLAB 8.5.0 (R2015a) program was used to accomplish all the results on a Windows computer with an Intel Xeon Silver 4114 CPU with a clock speed of 2.20 GHz, two cores each, and 64 GB of total RAM.

4.1 *Model description*

In this work, we have taken into account a few unstable Navier-Stokes models with dimensions ranging from 1 to

5, and for all of them, Reynolds numbers are 300, 400, and 500, respectively. Each model has a non-symmetric structure having connections among its 2 inputs and corresponding 7 outputs. For the target models in the first-order index-2 descriptor scheme, B_2 and C_2 are sparse zero matrices. In Table 1, the arrangement of states for the target models is listed.

Table 1. Arrangement of states for the target models

Dimension	1	2	3	4	5
n_v	3,142	8,568	19,770	44,744	98,054
n_p	453	1,123	2,615	5,783	12,566

4.2 *Visual affirmation of stabilization of the target models*

The vivid evidence of the stabilization of the transient of the unstable Navier-Stokes models will be exhibited here. The visual affirmation of stabilizing the eigenvalues and step-responses will be comprised. There are 15 target models having identical input-output structures and system attributes. Thus, analysis of one of the models is ample for the validation of the claim of this work. Here, we are aiming only for the 3-dimensional model with Reynolds number 400 for the forthcoming exploration.

4.2.1 *Analysis for eigenvalues*

Figure 1 depicts the stabilization of the aimed model. Sub-figure 1a displays the unstable eigenvalues of the target model, whereas Sub-figure 1b displays the stabilized eigenvalues of that model. This picturesque ratification essentially accomplished the efficiency of the proposed techniques for stabilizing the eigenvalues of the target models.

4.2.2 *Analysis for step-responses*

Figure 2 and Figure 3 expose the stabilization of the step-responses of the aimed model for 1st to 1st and 2nd to 7th input-output relations, respectively. Sub-figure 2a and Sub-figure 2b exhibit the unstable and stabilized step-responses of 1st-input to 1st-output the target model, whereas Sub-figure 3a and Sub-figure 3b exhibit the unstable and stabilized step-responses of 2nd-input to 7th-output of that model. This exposition reveals the competency of the proposed techniques for stabilizing the step-responses of the target models.

4.3 *Comparative analysis with the existing methods*

There are some existing methods for finding the optimal feedback matrix of unstable systems through reduced-order modelling. However, all of them are not comparable to the proposed techniques, such as the BT method is not under consideration due to the requirement of storing the ROM, and RKSM is not applicable because of the non-symmetric setting of the Navier-Stokes models. LRCF-ADI integrated the Newton-Kleinman method and the proposed techniques have an indistinguishable goal, for instance, attaining the optimal feedback matrix without storing the ROM and achieving the factored solution of the associated Riccati equation. So, the progression of the proposed techniques will be legitimatized by comparing them with the LRCF-ADI integrated Kleinman-Newton method. Since the stabilization of the transient behaviors by both of the approaches is very commensurate, the intended comparative analysis incorporated the computational time and size of the factored solution Z. We aimed to analyze the numerical data of the simulations for the unstable Navier-Stokes models' dimensions 1 to 5 with $Re = 400$. The explanatory analysis of the comparison of the computational time and size of the factored solution Z is manifested in Table 2.

Computational time can be substantially reduced to almost one-tenth of the proposed memory-efficient techniques, which fulfills the prime motive of the work. In the case of the size of Z, it is outstandingly discernible that the proposed technique enhances the approximation of the solution of the Riccati equation with a lower-dimensional factor, and the competency boosts with the dimensions of the target models, which is another essence of this work. As a consequence, it can be claimed that the aim of the work can be attainable by the proposed memory-efficient interpolatory projection techniques.

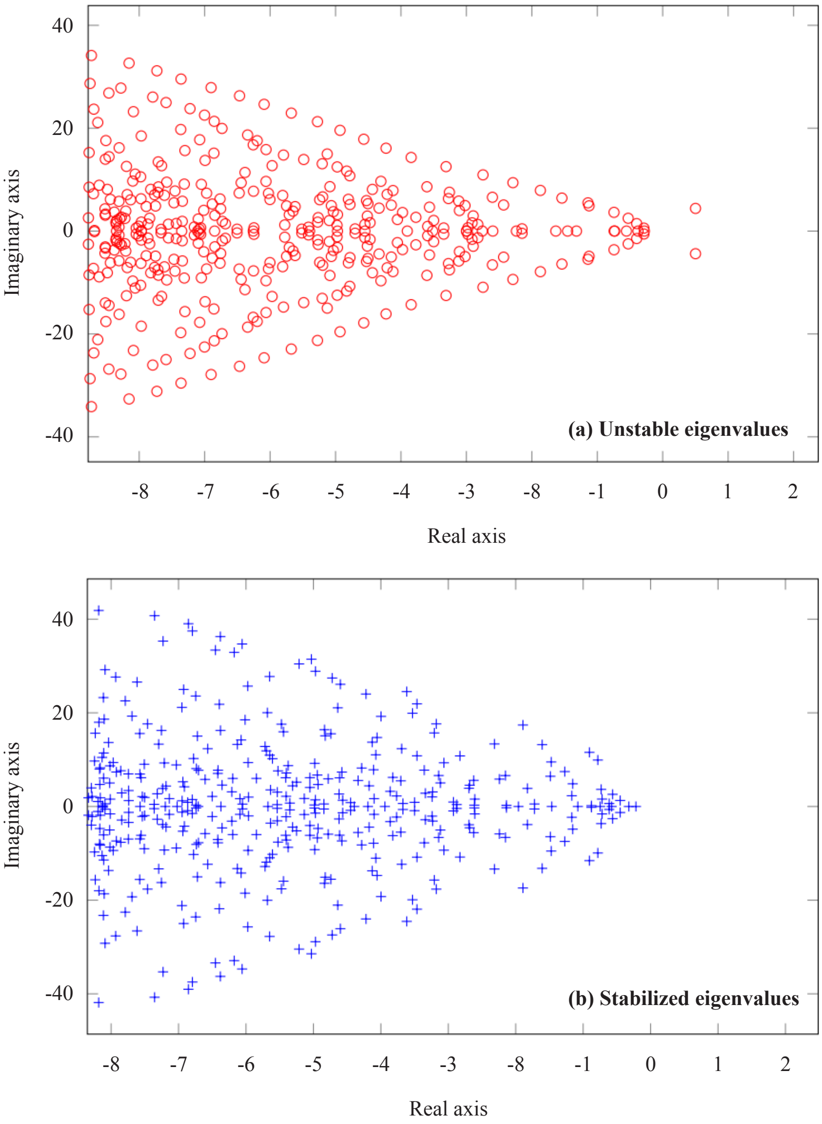

Figure 1. Stabilization of the eigenvalues

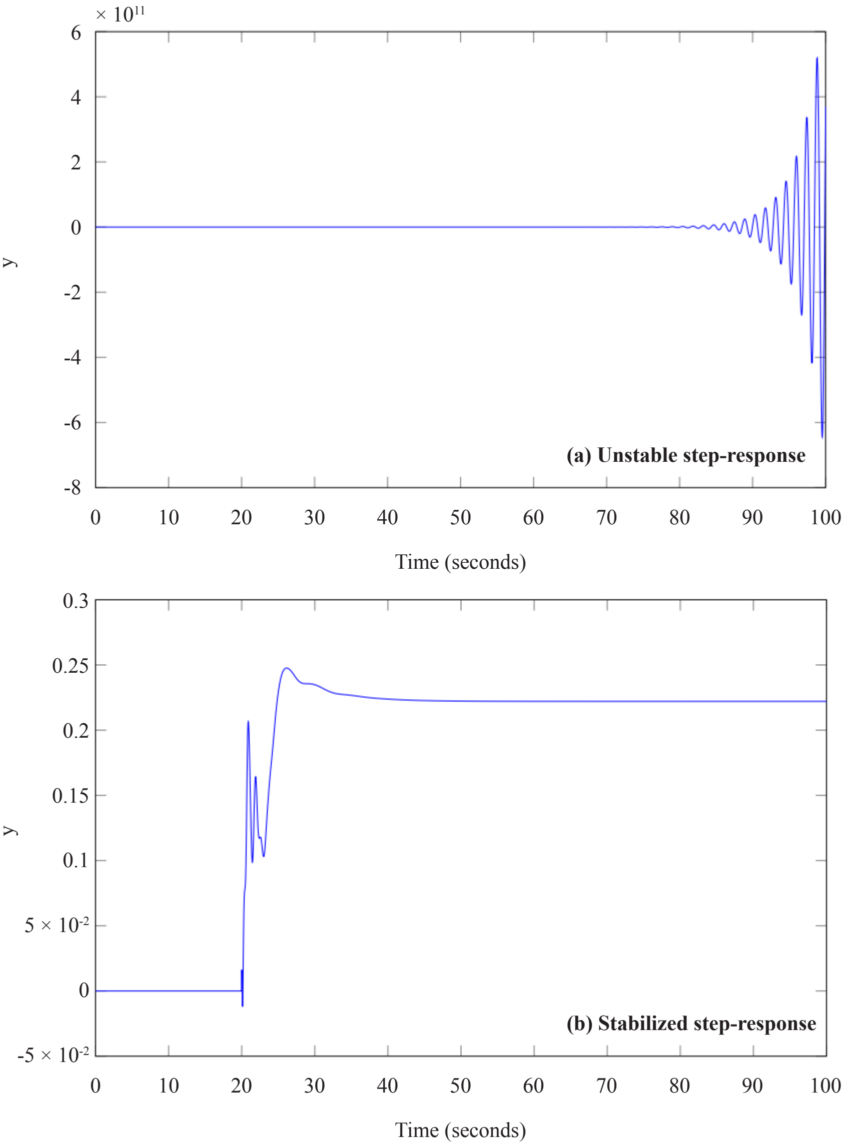

Figure 2. Stabilization of the step-response for 1st input to 1st output

Table 2. Comparison of the memory-efficient IRKA and LRCF-ADI integrated Newton-Kleinman approaches

Dimension	Reynolds number	Computation time (seconds)		Size of Z	
		m-IRKA	NK-LRCF-ADI	m-IRKA	NK-LRCF-ADI
1		3.11×10^2	5.98×10^3	$3{,}142 \times 50$	$3{,}142 \times 784$
2		1.39×10^3	9.41×10^4	$8{,}568 \times 60$	$8{,}568 \times 919$
3	400	4.31×10^3	3.79×10^5	$19{,}770 \times 70$	$19{,}770 \times 1{,}108$
4		1.48×10^4	1.28×10^6	$44{,}744 \times 80$	$44{,}744 \times 1{,}324$
5		3.82×10^4	5.71×10^6	$98{,}054 \times 90$	$98{,}054 \times 1{,}441$

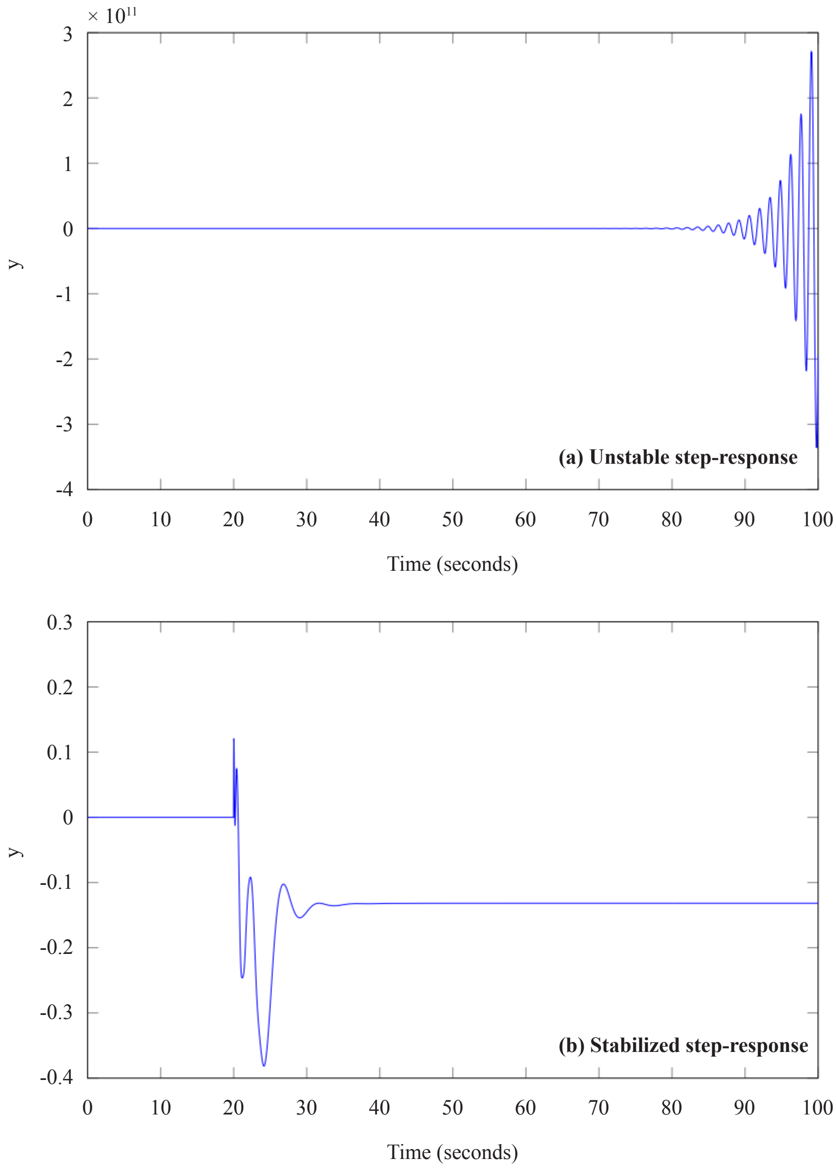

Figure 3. Stabilization of the step-response for 2nd input to 7th output

5. Conclusions

This article proposes an updated Krylov subspace-based interpolatory projection techniques to attain the optimal feedback matrix for the unstable Navier-Stokes models without storing the reduced-order matrices. Enhancing the quick convergence and efficient approximation of the solution of the Riccati equation via factored solution with minimum size were the prime objectives of this work. An effective cohesive projection scheme was incorporated to ensure the potency of the projector matrices and very time-worthy eigenvalue decomposition was implemented within the inverse-projection technique to get the desired factored solution. Reliance on the proposed techniques is investigated numerically through MATLAB simulations and achieved results are demonstrated graphically. Stabilization of the eigenvalues and step-responses are exhibited for a selected model among the target models. Figurative analysis conveys the affirmation of the effectiveness of the proposed techniques. To ascertain the furtherance of the proposed techniques, a comparative analysis with the existing methods, such as the LRCF-ADI integrated Newton-Kleiman method is provided. It is clear from the tabular and descriptive analysis that the proposed strategies outperformed the previous techniques in terms of computational time as well as the size of the factored solution *Z*. Thus, the main takeaway from this article is that the proposed memory-efficient interpolatory projection techniques can be employed to stabilize unstable Navier-Stokes models with better simulation feasibility assimilating minimal computational time and memory allocation.

6. Declarations

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Availability of data and material: Computational data of Navier-Stokes models are available at the respiratory https://github.com/uddinmonir/systemDATA/find/main.

Code availability: Simulation codes of the research work are available to the corresponding author.

Authors' contributions: In this article, all of the authors contributed equally.

Conflict of interest

The authors declare no competing financial interest.

References

[1] Bänsch E, Benner P. Stabilization of incompressible flow problems by riccati-based feedback. In: Leugering G, Engell S, Griewank A, Hinze M, Rannacher R, Schulz V, Ulbrich M, Ulbrich S. (eds.) *Constrained Optimization and Optimal Control for Partial Differential Equations. International Series of Numerical Mathematics, vol 160*. Birkhäuser, Basel; 2012. p.5-20. Available from: https://doi.org/10.1007/978-3-0348-0133-1_1.

[2] Altmann R, Heiland J. Continuous, Semi-discrete, and fully discretised Navier-Stokes equations. In: Campbell S, Ilchmann A, Mehrmann V, Reis T. (eds.) *Applications of Differential-Algebraic Equations: Examples and Benchmarks. Differential-Algebraic Equations Forum*. Springer, Cham; 2018. p.277. Available from: https://doi.org/10.1007/11221_2018_2.

[3] Islam MT. *Computational Techniques for Riccati-Based Feedback Stabilization of Large-Scale Sparse Index-2 Descriptor System*. Bangladesh University of Engineering and Technology; 2021.

[4] Bänsch E, Benner P, Saak J, Weichelt HK. Riccati-based boundary feedback stabilization of incompressible Navier-Stokes flow. *SIAM Journal on Scientific Computing*. 2015; 37(2): A832-A858. Available from: https://doi.org/10.1137/140980016.

[5] Gargano F, Sammartino M, Sciacca V, Cassel KW. Analysis of complex singularities in high-Reynolds-number Navier-Stokes solutions. *Journal of Fluid Mechanics*. 2014; 747: 381-421. Available from: https://doi.org/10. 1017/jfm.2014.153.

[6] Mehrmann V, Stykel T. *Balanced Truncation Model Reduction for Large-Scale Systems in Descriptor Form*. Springer; 2005. Available from: https://doi.org/10.1007/3-540-27909-1_3.

[7] Bänsch E, Benner P, Saak J, Weichelt HK. Optimal control-based feedback stabilization of multi-field flow problems. In: Leugering G, Benner P, Engell S, Griewank A, Harbrecht H, Hinze M, Rannacher R, Ulbrich S. (eds.) *Trends in PDE Constrained Optimization. International Series of Numerical Mathematics, vol 165.* Birkhäuser, Cham; 2014. p.173-188. Available from: https://doi.org/10.1007/978-3-319-05083-6_11.

[8] Uddin MM. *Computational Methods for Model Reduction of Large-Scale Sparse Structured Descriptor Systems*. Otto-von-Guericke Universität Magdeburg; 2015.

[9] Benner P, Bujanović Z, Kürschner P, Saak J. A numerical comparison of different solvers for large-scale, continuous-time algebraic riccati equations and LQR problems. *SIAM Journal on Scientific Computing*. 2020; 42(2): A957-A996. Available from: https://doi.org/10.1137/18M1220960.

[10] Uddin M. *Numerical Study on Continuous-time Algebraic Riccati Equations arising from Large-Scale Sparse Descriptor Systems*. Bangladesh University of Engineering and Technology; 2020.

[11] Antoulas AC. An overview of approximation methods for large-scale dynamical systems. *Annual Reviews in Control*. 2005; 29(2): 181-190. Available from: https://doi.org/10.1016/j.arcontrol.2005.08.002.

[12] Benner P, Stykel T. Model order reduction for differential-algebraic equations: A survey. In: Ilchmann A, Reis T. (eds.) *Surveys in Differential-Algebraic Equations IV. Differential-Algebraic Equations Forum*. Springer, Cham; 2017. p.107. Available from: https://doi.org/10.1007/978-3-319-46618-7_3.
[13] Uddin M, Khan MAH, Uddin MM. Riccati based optimal control for linear quadratic regulator problems. *2019 5th International Conference on Advances in Electrical Engineering (ICAEE)*. IEEE; 2019. p.290-295. Available from: https://doi.org/10.1109/ICAEE48663.2019.8975637.
[14] Uddin M, Khan MAH, Uddin MM. Efficient computation of Riccati-based optimal control for power system models. *2019 22nd International Conference of Computer and Information Technology (ICCIT)*. IEEE; 2019. p.260-265. Available from: https://doi.org/10.1109/ICCIT48885.2019. 9038442.
[15] Uddin MM, Saak J, Kranz B, Benner P. Computation of a compact state space model for an adaptive spindle head configuration with piezo actuators using balanced truncation. *Production Engineering*. 2012; 6(6): 577-586. Available from: https://doi.org/10.1007/s11740-012-0410-x.
[16] Benner P, Kürschner P, Saak J. An improved numerical method for balanced truncation for symmetric second-order systems. *Mathematical and Computer Modelling of Dynamical Systems*. 2013; 19(6): 593-615. Available from: https://doi.org/10.1080/13873954.2013.794363.
[17] Benner P, Saak J, Uddin MM. Balancing based model reduction for structured index-2 unstable descriptor systems with application to flow control. *Numerical Algebra, Control & Optimization*. 2016; 6(1): 1. Available from: https://doi.org/10.3934/naco.2016.6.1.
[18] Pulch R, Narayan A. Balanced truncation for model order reduction of linear dynamical systems with quadratic outputs. *SIAM Journal on Scientific Computing*. 2019; 41(4): A2270-A2295. Available from: https://doi.org/10. 1137/17M1148797.
[19] Sinani K. *Iterative Rational Krylov Algorithm for Unstable Dynamical Systems and Genaralized Coprime Factorizations*. Virginia Tech; 2016.
[20] Rahman MM, Uddin MM, Andallah LS, Uddin M. Interpolatory projection techniques for H_2 optimal structure-preserving model order reduction of second-order systems. *Advances in Science, Technology and Engineering Systems Journal*. 2020; 5(4): 715-723. Available from: https://doi.org/10. 25046/aj050485.
[21] Uddin M, Uddin MM, Khan MAH, Rahman MM. Interpolatory projection technique for Riccati-based feedback stabilization of index-1 descriptor systems. *IOP Conference Series: Materials Science and Engineering. vol. 1070.* IOP Publishing; 2021. p.012022. Available from: https://doi.org/10.1088/1757-899X/1070/1/012022.
[22] Rahman MM, Uddin MM, Andallah L, Uddin M. Tangential interpolatory projections for a class of second-order index-1 descriptor systems and application to mechatronics. *Production Engineering*. 2021; 15(1): 9-19. Available from: https://doi.org/10.1007/s11740-020-00995-4.
[23] Li S, Trevelyan J, Wu Z, Lian H, Wang D, Zhang W. An adaptive SVD-Krylov reduced order model for surrogate based structural shape optimization through isogeometric boundary element method. *Computer Methods in Applied Mechanics and Engineering*. 2019; 349: 312-338. Available from: https://doi.org/10.1016/j.cma.2019.02.023.
[24] Rahman MM, Uddin M, Uddin MM, Andallah L. SVD-Krylov based techniques for structure-preserving reduced order modelling of second-order systems. *Mathematical Modelling and Control*. 2021; 1(2): 79-89. Available from: https://doi.org/10.3934/mmc.2021006.
[25] Uddin M, Uddin MM, Khan MAH, Hossain MT. SVD-Krylov based Sparsity-preserving techniques for Riccati-based feedback stabilization of unstable power system models. *Journal of Engineering Advancements*. 2021; 2(3): 125-131. Available from: https://doi.org/10.38032/jea.2021.03.002.
[26] Uddin M, Uddin MM, Khan MAH. SVD-Krylov-based sparsity-preserving techniques to optimally stabilize the incompressible Navier-Stokes flows. *International Journal of Dynamics and Control*. 2023; 11: 2763-2773. Available from: https://doi.org/10.1007/s40435-023-01159-9.
[27] Mena H, Saak J. On the parameter selection problem in the Newton-ADI iteration for large-scale Riccati equations. *Electronic Transactions on Numerical Analysis*. 2008; 29: 136-149. Available from: https://etna.math.kent.edu/vol.33.2008-2009/pp53-62.dir/pp53-62.html.
[28] Benner P, Li JR, Penzl T. Numerical solution of large-scale Lyapunov equations, Riccati equations, and linear-quadratic optimal control problems. *Numerical Linear Algebra with Applications*. 2008; 15(9): 755-777. Available from: https://doi.org/10.1002/nla.622.
[29] Geromel JC, Deaecto GS. Generalized Kleinman-Newton method. *Optimal Control Applications and Methods*. 2018; 39(2): 1130-1140. Available from: https://doi.org/10.1002/oca.2400.
[30] Palitta D. *The projected Newton-Kleinman method for the algebraic Riccati equation*. arXiv: 190110199. 2019. Available from: https://doi.org/10.48550/arXiv.1901.10199.
[31] Druskin V, Simoncini V. Adaptive rational Krylov subspaces for large-scale dynamical systems. *Systems & Control*

Letters. 2011; 60(8): 546-560. Available from: https://doi.org/10.1016/j.sysconle.2011.04.013.
[32] Stykel T, Simoncini V. Krylov subspace methods for projected Lyapunov equations. *Applied Numerical Mathematics*. 2012; 63(1): 35-50. Available from: https://doi.org/10.1016/j.apnum.2011.09.007.
[33] Lin Y, Simoncini V. A new subspace iteration method for the algebraic Riccati equation. *Numerical Linear Algebra with Applications*. 2015; 22(1): 26-47. Available from: https://doi.org/10.1002/nla.1936.
[34] Uddin M, Uddin MM, Khan MAH. Computationally efficient optimal control for unstable power system models. *Mathematical Problems in Engineering*. 2021; 2021: 8071869. Available from: https://doi.org/10.1155/2021/8071869.
[35] Feitzinger F, Hylla T, Sachs EW. Inexact Kleinman-Newton method for Riccati equations. *SIAM Journal on Matrix Analysis and Applications*. 2009; 31(2): 272-288. Available from: https://doi.org/10.1137/070700978.
[36] Penzl T. A cyclic low-rank Smith method for large sparse Lyapunov equations. *SIAM Journal on Scientific Computing*. 1999; 21(4): 1401-1418. Available from: https://doi.org/10.1137/S1064827598347666.
[37] Li JR. *Model reduction of large linear systems via low rank system gramians*. Massachusetts Institute of Technology; 2000. Available from: https://doi.org/10.1016/B978-008043944-0/50977-X.
[38] Benner P, Kürschner P, Saak J. Efficient handling of complex shift parameters in the low-rank Cholesky factor ADI method. *Numerical Algorithms*. 2013; 62(2): 225-251. Available from: https://doi.org/10.1007/s11075-012-9569-7.
[39] Benner P, Heinkenschloss M, Saak J, Weichelt HK. Efficient solution of large-scale algebraic Riccati equations associated with index-2 DAEs via the inexact low-rank Newton-ADI method. *Applied Numerical Mathematics*. 2020; 152: 338-354. Available from: https://doi.org/10.1016/j.apnum. 2019.11.016.
[40] Beattie C, Gugercin S. Interpolatory projection methods for structure-preserving model reduction. *Systems & Control Letters*. 2009; 58(3): 225-232. Available from: https://doi.org/10.1016/j.sysconle.2008.10.016.
[41] Gugercin S, Antoulas AC, Beattie C. H_2 model reduction for large-scale linear dynamical systems. *SIAM Journal on Matrix Analysis and Applications*. 2008; 30(2): 609-638. Available from: https://doi.org/10.1137/060666123.
[42] Gugercin S, Stykel T, Wyatt S. Model reduction of descriptor systems by interpolatory projection methods. *SIAM Journal on Scientific Computing*. 2013; 35(5): B1010-B1033. Available from: https://doi.org/10.1137/130906635.
[43] Islam MT, Uddin M, Uddin MM, Khan MAH. Sparsity-Preserving Two-Sided Iterative Algorithm for Riccati-Based Boundary Feedback Stabilization of the Incompressible Navier-Stokes Flow. *Mathematical Problems in Engineering*. 2022; 2022: 4435167. Available from: https://doi.org/10.1155/2022/4435167.
[44] Wyatt S. *Issues in interpolatory model reduction: Inexact solves, second-order systems and DAEs*. Virginia Tech; 2012.