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Research Article

Fixed Point Results for Geraghty Type Contractions in Extended Rectangular Fuzzy b-metric Space with an Application

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Abstract: The aim of this paper is to define the notions of extended fuzzy b-metric spaces, fuzzy rectangular b-metric spaces and extended fuzzy rectangular b-metric spaces and to establish certain fixed point results in these settings utilizing Geraghty type contractions. We also furnished an example which illustrates our main result. To show the authenticity of our results we provide an application involving fuzzy integral equation. The obtained results generalize many existing results in the literature.

***Keywords*:** extended b-metric space, fuzzy b-metric space, extended fuzzy b-metric space, fuzzy rectangular b-metric space, extended fuzzy rectangular b-metric space and fixed point theorems

MSC: 46S40, 47H10, 54H25

Abbreviation

bms	b-metric space
ebms	extended b-metric space
fms	fuzzy metric space
fbms	fuzzy b-metric space
efbms	extended fuzzy b-metric space
frms	fuzzy rectangular metric space
frbms	fuzzy rectangular b-metric space
efrbms	extended fuzzy rectangular b-metric space

1. Introduction

The work of Bourbaki [1], Bakhtin [2] and particularly Czerwik [3] motivated many researchers to expand the theory of fixed points for b-Metric spaces (bms). Czerwik [3] introduced a weaker form of the triangle inequality and formally

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presented a generalization of metric space by defining a bms and established a generalization of the famous Banach contraction principle [4] in bms. The examples and fixed point theorems in bms can be seen in [5, 6]. Kamran et al. [7] introduced the notion of extended b-metric space (ebms) and established fixed point theorems in ebms.

In 1965, the definition of fuzzy set was introduced by Zadeh [8]. The concept of fuzzy metric(fms) space was initiated by Kramosil and Michálek [9] in 1975 by using the idea of fuzzy sets Grabiec [10] proved Banach fixed point theorem in fms in 1988. Later on, George and Veermani [11] modified the definition of fms given by Kramosil and Michálek in 1994. Hussain et al. [12] introduced the idea of a fuzzy b-metric space (fbms) in 2015. Mehmood et al. [13] introduced the concept of extended fuzzy b-metric space (efbms) and proved the well known Banach fixed point theorem in 2017. In 2019, Mehmood et al. [14] gave the notion of fuzzy rectangular b-metric spaces (frbms) and established some interesting fixed point results. In 2020, Ashraf et al. [15] established certain fixed point results for Geraghty-type contraction in the setting of G-complete fuzzy b-metric space. In 2022, Batul et al. [16] proved certain fixed point results for Geraghty-type contractions in the setting of G-Complete extended fuzzy b-metric spaces. Sarwar et al. defined the notion of extended fuzzy rectangular b-metric space(efrbms)which generalizes the notions of fms, fbms, efbms, frms and frbms. Recently, [17] proved Banach fixed point theorem and Ciric type contraction theorem in the setting of efrbms.

In this article, we establish some fixed point results using Geraghty type contraction in efrbms. Our results generalize the results of [10, 13–15, 17, 18].

2. Preliminaries

In this section, we recall some basic definitions and results which are related to our main work.

Definition 2.1 [11] A nonempty set M together with a fuzzy set $F: M\times M\times[0, \infty)\to[0, \infty)$ and a continuous t-norm $*$ is said to be a fms if for all $a_1, a_2, a_3\in M$ and $\rho, \rho > 0$ satisfies the following conditions:

F1: $F(a_1, a_2, 0) = 0$

F2: $F(a_1, a_2, \rho) = 1 \iff a_1 = a_2$

F3: $F(a_1, a_2, \rho) = F(a_2, a_1, \rho)$

F4: $F(a_1, a_3, \rho+\rho)\geq F(a_1, a_2, \rho)*F(a_2, a_3, \rho)$

F5: $F(a_1, a_2, .)$: $[0, \infty)\to[0, 1$ is left continuous.

Hussain et al. [12] defined the notion of a fbms by generalizing the definition of a fms given in [9].

Definition 2.2 [12, 18] Consider a nonempty set M and a real number $b\geq 1$ and let $*$ be a continuous t-norm. A fuzzy set F_b: $M\times M\times[0, \infty)\to[0, \infty)$ is called fbms on M if for all $a_1, a_2, a_3\in M$ and $\forall\,\rho > 0$

Fb1: $F_b(a_1, a_2, \rho) > 0$

Fb2: $F_b(a_1, a_2, \rho) = 1 \quad\Leftrightarrow\quad a_1 = a_2$

Fb3: $F_b(a_1, a_2, \rho) = F_b(a_2, a_1, \rho)$

Fb4: $F_b(a_1, a_3, b(\rho+\rho))\geq F_b(a_1, a_2, \rho)*F_b(a_2, a_3, \rho)\ \forall\,\rho, \rho\geq 0$

Fb5: $F_b(a_1, a_2, .)$: $(0, \infty)\to[0, 1$ is continuous, and $\lim\limits_{t\to\infty}F_b(a_1, a_2, \rho) = 1$.

Then the triplet $(M, F_b, *)$ is called a fbms .

In 2017, Mehmood et al. [13] introduced the notion of efbms as follows:

Definition 2.3 [13] Consider a nonempty set M and a self mapping ζ: $M\times M\to[1, \infty)$. Let $*$ be a continuous t-norm. A fuzzy set F_ζ: $M\times M\times[0, \infty)\to[0, \infty)$ is called efbms on M if the following conditions hold for all $a_1, a_2, a_3\in M$:

$[F_\zeta 1]$: $F_\zeta(a_1, a_2, 0) = 0$

$[F_\zeta 2]$: $F_\zeta(a_1, a_2, \rho) = 1, \forall\,\rho > 0 \iff a_1 = a_2$

$[F_\zeta 3]$: $F_\zeta(a_1, a_2, \rho) = F_\zeta(a_2, a_1, \rho)$

$[F_\zeta 4]$: $F_\zeta(a_1, a_3, \zeta(a_1, a_3)(\rho+\rho))\geq F_\zeta(a_1, a_2, \rho)*F_\zeta(a_2, a_3, \rho)\ \forall\,\rho, \rho\geq 0$

$[F_\zeta 5]$: $F_\zeta(a_1, a_2, .)$: $(0, \infty)\to[0, 1]$ is left continuous, and $\lim\limits_{\rho\to\infty}F_\zeta(a_1, a_2, \rho) = 1$.

In 2019, Mehmood et al. [14] introduced the notion of frbms as follows:

Definition 2.4 [14] Let M be a nonempty set and $*$ be a continuous t-norm. A fuzzy set F_{rb}: $M\times M\times[0, \infty)\to[0, \infty)$ is called frbms on M if the following conditions hold for all $x_1, x_2, x_3\in X$ and $\rho, \rho, \varpi > 0$, Frb-1 $F_{rb}(x_1, x_2, 0) = 0$

Frb-2 $F_{rb}(x_1, x_2, \rho) = 1$ if and only if $x_1 = x_2$

Frb-3 $F_{rb}(x_1, x_2, \rho) = F_{rb}(x_2, x_1, \rho)$

Frb-4 $F_{rb}(x_1, x_3, b(\rho+\rho+\varpi)) \geq F_{rb}(x_1, x_2, \rho) * F_{rb}(x_2, u, \rho) * F_{rb}(u, x_3, \varpi)$ for all distinct $x_2, u \in X \setminus \{x_1, x_3\}$.

Frb-5 $F_{rb}(x_1, x_2, .)$: $(0, \infty) \to [0, 1]$ is left continuous, and $\lim_{\rho\to\infty} F_{rb}(x_1, x_2, \rho) = 1$.

Recently, Saleem et al. [17] introduced the notion of efrbms as follows:

Definition 2.5 [17] Consider a nonempty set M, a continuous t-norm $*$ and a self mapping $\zeta: M \times M \to [1, \infty)$. A fuzzy set $F_{rb\zeta}: M \times M \times [0, \infty) \to [0, \infty)$ is called efrbms on M if the following conditions hold for all $a_1, a_2, a_3, a_4 \in M$ and for all $\rho, \rho, \varpi \geq 0$:

$[Frb_\zeta 1]$: $F_{rb\zeta}(a_1, a_2, 0) = 0$

$[Frb_\zeta 2]$: $F_{rb\zeta}(a_1, a_2, \rho) = 1, \forall \rho > 0$ if and only if $a_1 = a_2$

$[Frb_\zeta 3]$: $F_{rb\zeta}(a_1, a_2, \rho) = F_{rb\zeta}(a_2, a_1, \rho)$

$[Frb_\zeta 4]$: $F_{rb\zeta}(a_1, a_4, \zeta(a_1, a_4)(\rho+\rho+\varpi)) \geq F_{rb\zeta}(a_1, a_2, \rho) * F_{rb\zeta}(a_2, a_3, \rho) * F_{rb\zeta}(a_3, a_4, \varpi)$

$[Frb_\zeta 5]$: $F_{rb\zeta}(a_1, a_2, .)$: $(0, \infty) \to [0, 1]$ is left continuous and $\lim_{\rho\to\infty} F_{rb\zeta}(a_1, a_2, \rho) = 1$.

Then $(M, F_{rb\zeta}, *)$ is an efrbms.

Remark 2.1 If we take $\zeta(a_1, a_2) = b$ for $b \geq 1$ in above Definition then the Definition becomes a special case of efrbms.

Example 2.1 Let $M = \{0, 1, 2, 3\}$ and a b-metric d_b: $M \times M \to \mathbb{R}$ defined by $d(a_1, a_2) = (a_1 - a_2)^2$. Define $\zeta: M \times M \to [1, \infty)$ by $\zeta(a_1, a_2) = 1 + a_1^2 + a_2^2$. Let a mapping $F_{rb\zeta}: M \times M \times [0, \infty) \to [0, 1]$ defined by

$$F_{rb\zeta}(a_1, a_2, \rho) = e^{-\frac{(a_1-a_2)^2}{\rho}}$$

where the continuous t-norm $*$ is taken as $t_1 * t_2 = t_1 \wedge t_2 = \min\{t_1, t_2\}$ then $(M, F_{rb\zeta}, \wedge)$ is a efrbms.

Definition 2.6 Let $(M, F_{rb\zeta}, *)$ be an efrbms.

1. A sequence $\{a_\vartheta\}$ in M is called convergent sequence if there exits $a \in M$ and for all $\rho \geq 0$ such that $F_{rb\zeta}(a_\vartheta, a, \rho) = 1$.
2. A sequence $\{a_\vartheta\}$ in M is called G-Cauchy sequence if $\lim_{n\to\infty} M_\zeta(a_n, a_{n+q}, \rho) = 1$ for all $t > 0$ and $q > 0$.
3. An efrbms in which every G-Cauchy sequence is convergent is called a G-complete efrbms.

3. Main results

In this section, we will generalize the results of [14, 16, 19, 20] in the setting of efrbms. Following [20], let $\mathscr{F}_b$ be the class of all functions β: $[0, \infty) \to [0, \frac{1}{b})$ for $b > 1$ satisfying the following condition:

$$\limsup_{n\to\infty} \beta(t_n) = \frac{1}{b} \quad \text{implies} \quad \lim_{n\to\infty} t_n = 1$$

Lemma 3.1 Let $(M, F_{rb\zeta}, *)$ be a G-complete efrbms. If for two elements $a_1, a_2 \in M$ and $\beta(F_{rb\zeta}(a_1, a_2, \rho)) \in \mathscr{F}_b$, $F_{rb\zeta}(a_1, a_2, \beta(F_{rb\zeta}(a_1, a_2, \rho))\rho) \geq F_{rb\zeta}(a_1, a_2, \rho)$ then $a_1 = a_2$.

Proof. It follows from $Frb_\zeta 5$ that

$$\lim_{\rho\to\infty} F_{rb\zeta}(a_1, a_2, \rho) = 1, \quad \Rightarrow \quad F_{rb\zeta}(a_1, a_2, \beta(F_{rb\zeta}(a_1, a_2, \rho))\rho) = 1.$$

It follows from $Frb_\zeta 2$ that $a_1 = a_2$. □

Now we prove the result of [10] in this new setting.

Theorem 3.1 Let $(M, F_{rb\zeta}, *)$ be a G-complete efrbms and $T\colon M \to M$ be a mapping satisfying

$$F_{rb\zeta}(Ta_1, Ta_2, \beta(F_{rb\zeta}(a_1, a_2, \rho))\rho) \geq F_{rb\zeta}(a_1, a_2, \rho) \tag{1}$$

for all $a_1, a_2 \in M$, $\beta(F_{rb\zeta}(a_1, a_2, \rho)) \in \mathscr{F}_b$. Then T has a unique fixed point.

Proof. Let $a_0 \in M$ be a fix arbitrary element and for $\vartheta = 0, 1, 2, \ldots$, taking $a_{\vartheta+1} = Ta_\vartheta$.

For all ϑ, $\rho > 0$, by the application of Inequality (1), we get

$$\begin{aligned}
F_{rb\zeta}(a_\vartheta, a_{\vartheta+1}, \rho) = F_{rb\zeta}(Ta_{\vartheta-1}, Ta_\vartheta, \rho) &\geq F_{rb\zeta}\left(a_{\vartheta-1}, a_\vartheta, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))}\right) \\
&\geq F_{rb\zeta}\left(a_{\vartheta-2}, a_{\vartheta-1}, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho)).\beta(F_{rb\zeta}(a_{\vartheta-2}, a_{\vartheta-1}, \rho))}\right) \\
&\geq F_{rb\zeta}\left(a_{n-3}, a_{\vartheta-2}, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho)).\beta(F_{rb\zeta}(a_{\vartheta-2}, a_{\vartheta-1}, \rho)).\beta(F_{rb\zeta}(a_{n-3}, a_{\vartheta-2}, \rho)}\right) \\
&\;\;\vdots \\
&\geq F_{rb\zeta}\left(a_0, a_1, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho)).\beta(F_{rb\zeta}(a_{\vartheta-2}, a_{\vartheta-1}, \rho))\ldots\beta(F_{rb\zeta}(a_0, a_1, \rho))}\right)
\end{aligned}$$

So, we have

$$F_{rb\zeta}(a_\vartheta, a_{\vartheta+1}, \rho) \geq F_{rb\zeta}\left(a_0, a_1, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho)).\beta(F_{rb\zeta}(a_{\vartheta-2}, a_{\vartheta-1}, \rho))\ldots\beta(F_{rb\zeta}(a_0, a_1, \rho))}\right) \tag{2}$$

Since $(M, F_{rb\zeta}, *)$ is an efrbms, so for the sequence $\{a_n\}$, we write $\rho = \frac{\rho}{3} + \frac{\rho}{3} + \frac{\rho}{3}$ and apply $Frb_\zeta 4$ on $F_{rb\zeta}(a_\vartheta, a_{\vartheta+p}, \rho)$ in the following two cases.

Case-1: If p is odd, say $p = 2n+1$ where $n \in \mathbb{N}$, we have

$$F_{rb\zeta}(a_\vartheta, a_{\vartheta+2n+1}, \rho) \geq F_{rb\zeta}\left(a_\vartheta, a_{\vartheta+1}, \frac{\rho}{3\zeta(a_\vartheta, a_{\vartheta+2n+1})}\right) * F_{rb\zeta}\left(a_{\vartheta+1}, a_{\vartheta+2}, \frac{\rho}{3\zeta(a_\vartheta, a_{\vartheta+2n+1})\zeta(a_{\vartheta+1}, a_{\vartheta+2n+1})}\right)$$

$$* F_{rb\zeta}\left(a_{\vartheta+2}, a_{\vartheta+2n+1}, \frac{\rho}{3\zeta(a_\vartheta, a_{\vartheta+2n+1})\zeta(a_{\vartheta+1}, a_{\vartheta+2n+1})\zeta(a_{\vartheta+2}, a_{\vartheta+2n+1})}\right)$$

$$\geq F_{rb\zeta}\left(a_\vartheta, a_{\vartheta+1}, \frac{\rho}{3\zeta(a_\vartheta, a_{\vartheta+2n+1})}\right) * F_{rb\zeta}\left(a_{\vartheta+1}, a_{\vartheta+2}, \frac{\rho}{3\zeta(a_\vartheta, a_{\vartheta+2n+1})\zeta(a_{\vartheta+1}, a_{\vartheta+2n+1})}\right)$$

$$* F_{rb\zeta}\left(a_{\vartheta+2}, a_{\vartheta+3}, \frac{\rho}{3^2\zeta(a_\vartheta, a_{\vartheta+2n+1})\zeta(a_{\vartheta+1}, a_{\vartheta+2n+1})\zeta(a_{\vartheta+2}, a_{\vartheta+2n+1})}\right)$$

$$* F_{rb\zeta}\left(a_{\vartheta+3}, a_{\vartheta+4}, \frac{\rho}{3^2\zeta(a_\vartheta, a_{\vartheta+2n+1})\zeta(a_{\vartheta+1}, a_{\vartheta+2n+1}), \zeta(a_{\vartheta+2}, a_{\vartheta+2n+1})\zeta(a_{\vartheta+3}, a_{\vartheta+2n+1})}\right)$$

$$* F_{rb\zeta}\left(a_{\vartheta+4}, a_{\vartheta+2n+1}, \frac{\rho}{3^2\zeta(a_\vartheta, a_{\vartheta+2n+1})\zeta(a_{\vartheta+1}, a_{\vartheta+2n+1})\ldots\zeta(a_{\vartheta+4}, a_{\vartheta+2n+1})}\right)$$

$$\geq F_{rb\zeta}\left(a_\vartheta, a_{\vartheta+1}, \frac{\rho}{3\zeta(a_\vartheta, a_{\vartheta+2n+1})}\right) * F_{rb\zeta}\left(a_{\vartheta+1}, a_{\vartheta+2}, \frac{\rho}{3\zeta(a_\vartheta, a_{\vartheta+2n+1}), 3\zeta(a_{\vartheta+1}, a_{\vartheta+2n+1})}\right)$$

$$* F_{rb\zeta}\left(a_{\vartheta+2}, a_{\vartheta+3}, \frac{\rho}{3^2\zeta(a_\vartheta, a_{\vartheta+2n+1}), \zeta(a_{\vartheta+1}, a_{\vartheta+2n+1}), \zeta(a_{\vartheta+2}, a_{\vartheta+2n+1})}\right)$$

$$* F_{rb\zeta}\left(a_{\vartheta+3}, a_{\vartheta+4}, \frac{\rho}{3^2\zeta(a_\vartheta, a_{\vartheta+2n+1}), \zeta(a_{\vartheta+1}, a_{\vartheta+2n+1}), \zeta(a_{\vartheta+2}, a_{\vartheta+2n+1}), \zeta(a_{\vartheta+3}, a_{\vartheta+2n+1})}\right)$$

$$* F_{rb\zeta}\left(a_{\vartheta+4}, a_{\vartheta+5}, \frac{\rho}{3^3\zeta(a_\vartheta, a_{\vartheta+2n+1})\zeta(a_{\vartheta+1}, a_{\vartheta+2n+1})\ldots\zeta(a_{\vartheta+4}, a_{\vartheta+2n+1})}\right)$$

$$* \ldots * F_{rb\zeta}\left(a_{n+2m-2}, a_{n+2m-1}, \frac{\rho}{3^m\zeta(a_\vartheta, a_{\vartheta+2n+1})\zeta(a_{\vartheta+1}, a_{\vartheta+2n+1})\ldots\zeta(a_{n+2m-2}, a_{\vartheta+2n+1})}\right)$$

$$* F_{rb\zeta}\left(a_{n+2m-1}, a_{\vartheta+2n}, \frac{\rho}{3^m\zeta(a_\vartheta, a_{\vartheta+2n+1})\zeta(a_{\vartheta+1}, a_{\vartheta+2n+1})\ldots\zeta(a_{n+2m-1}, a_{\vartheta+2n+1})}\right)$$

$$* \ldots * F_{rb\zeta}\left(a_{\vartheta+2n}, a_{\vartheta+2n+1}, \frac{\rho}{3^m\zeta(a_\vartheta, a_{\vartheta+2n+1})\zeta(a_{\vartheta+1}, a_{\vartheta+2n+1})\ldots\zeta(a_{\vartheta+2n}, a_{\vartheta+2n+1})}\right)$$

Using (1) we get

$$F_{rb\zeta}(a_\vartheta, a_{\vartheta+2n+1}, \rho)$$

$$\geq F_{rb\zeta}\left(a_0, a_1, \frac{\rho}{3\zeta(a_\vartheta, a_{\vartheta+2n+1})\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho)).\beta(F_{rb\zeta}(a_{\vartheta-2}, a_{\vartheta-1}, \rho))\ldots\beta(F_{rb\zeta}(a_0, a_1, \rho))}\right)$$

$$*F_{rb\zeta}\left(a_0, a_1, \frac{\rho}{3\zeta(a_\vartheta, a_{\vartheta+2n+1})\zeta(a_{\vartheta+1}, a_{\vartheta+2n+1})\beta(F_{rb\zeta}(a_\vartheta, a_{\vartheta+1}, \rho)).\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))\ldots\beta(F_{rb\zeta}(a_0, a_1, \rho))}\right)$$

$$*F_{rb\zeta}\left(a_0, a_1, \frac{\rho}{3^2\zeta(a_\vartheta, a_{\vartheta+2n+1})\zeta(a_{\vartheta+1}, a_{\vartheta+2n+1})\zeta(a_{\vartheta+2}, a_{\vartheta+2n+1})\beta(F_{rb\zeta}(a_{\vartheta+1}, a_{\vartheta+2}, \rho)).\beta(F_{rb\zeta}(a_\vartheta, a_{\vartheta+1}, \rho))\ldots\beta(F_{rb\zeta}(a_0, a_1, \rho))}\right)$$

$$*F_{rb\zeta}\left(a_0, a_0, \frac{\rho}{3^2\zeta(a_\vartheta, a_{\vartheta+2n+1})\zeta(a_{\vartheta+1}, a_{\vartheta+2n+1}), \zeta(a_{\vartheta+2}, a_{\vartheta+2n+1})\zeta(a_{\vartheta+3}, a_{\vartheta+2n+1}))\beta(F_{rb\zeta}(a_{\vartheta+2}, a_{\vartheta+3}, \rho))\ldots\beta(F_{rb\zeta}(a_0, a_1, \rho))}\right)$$

$$*F_{rb\zeta}\left(a_0, a_1, \frac{\rho}{3^3\zeta(a_\vartheta, a_{\vartheta+2n+1})\zeta(a_{\vartheta+1}, a_{\vartheta+2n+1})\ldots\zeta(a_{\vartheta+4}, a_{\vartheta+2n+1})\beta(F_{rb\zeta}(a_{\vartheta+3}, a_{\vartheta+4}, \rho))\ldots\beta(F_{rb\zeta}(a_0, a_1, \rho))}\right)$$

$$*\ldots*F_{rb\zeta}\left(a_0, a_1, \frac{\rho}{3^m\zeta(a_\vartheta, a_{\vartheta+2n+1})\zeta(a_{\vartheta+1}, a_{\vartheta+2n+1})\ldots\zeta(a_{n+2m-2}, a_{\vartheta+2n+1})\beta(F_{rb\zeta}(a_{\vartheta+2n-3}, a_{\vartheta+2n-2}, \rho))\ldots\beta(F_{rb\zeta}(a_0, a_1, \rho))}\right)$$

$$*F_{rb\zeta}\left(a_0, a_1, \frac{\rho}{3^m\zeta(a_\vartheta, a_{\vartheta+2n+1})\zeta(a_{\vartheta+1}, a_{\vartheta+2n+1})\ldots\zeta(a_{n+2m-1}, a_{\vartheta+2n+1})\beta(F_{rb\zeta}(a_{\vartheta+2n-2}, a_{\vartheta+2n-1}, \rho))\ldots\beta(F_{rb\zeta}(a_0, a_1, \rho))}\right)$$

$$*\ldots*F_{rb\zeta}\left(a_0, a_0, \frac{\rho}{3^m\zeta(a_\vartheta, a_{\vartheta+2n+1})\zeta(a_{\vartheta+1}, a_{\vartheta+2n+1})\ldots\zeta(a_{\vartheta+2n}, a_{\vartheta+2n+1})\beta(F_{rb\zeta}(a_{\vartheta+2n-1}, a_{\vartheta+2n}, \rho))\rho))\ldots\beta(F_{rb\zeta}(a_0, a_1, \rho))}\right)$$

$$\geq F_{rb\zeta}\left(a_0, a_1, \frac{b^\vartheta\rho}{3\zeta(a_\vartheta, a_{\vartheta+2n+1})}\right)*F_{rb\zeta}\left(a_0, a_1, \frac{b^{\vartheta+1}\rho}{3\zeta(a_\vartheta, a_{\vartheta+2n+1})\zeta(a_{\vartheta+1}, a_{\vartheta+2n+1})}\right)$$

$$*F_{rb\zeta}\left(a_0, a_1, \frac{b^{\vartheta+2}\rho}{3^2\zeta(a_\vartheta, a_{\vartheta+2n+1})\zeta(a_{\vartheta+1}, a_{\vartheta+2n+1})\zeta(a_{\vartheta+2}, a_{\vartheta+2n+1})}\right)$$

$$*F_{rb\zeta}\left(a_0, a_0, \frac{b^{\vartheta+3}\rho}{3^2\zeta(a_\vartheta, a_{\vartheta+2n+1})\zeta(a_{\vartheta+1}, a_{\vartheta+2n+1}), \zeta(a_{\vartheta+2}, a_{\vartheta+2n+1})\zeta(a_{\vartheta+3}, a_{\vartheta+2n+1}))}\right)$$

$$*F_{rb\zeta}\left(a_0, a_1, \frac{b^{\vartheta+4}\rho}{3^3\zeta(a_\vartheta, a_{\vartheta+2n+1})\zeta(a_{\vartheta+1}, a_{\vartheta+2n+1})\ldots\zeta(a_{\vartheta+4}, a_{\vartheta+2n+1})}\right)$$

$$*\ldots*F_{rb\zeta}\left(a_0, a_1, \frac{b^{\vartheta+2n-2}\rho}{3^n\zeta(a_\vartheta, a_{\vartheta+2n+1})\zeta(a_{\vartheta+1}, a_{\vartheta+2n+1})\ldots\zeta(a_{\vartheta+2n-2}, a_{\vartheta+2n+1})}\right)$$

$$*F_{rb\zeta}\left(a_0, a_1, \frac{b^{\vartheta+2n-1}\rho}{3^n\zeta(a_\vartheta, a_{\vartheta+2n+1})\zeta(a_{\vartheta+1}, a_{\vartheta+2n+1})\ldots\zeta(a_{\vartheta+2n-1}, a_{\vartheta+2n+1})}\right)$$

$$* \ldots * F_{rb\zeta}\left(a_0, a_0, \frac{b^{\vartheta+2n}\rho}{3^n\zeta(a_\vartheta, a_{\vartheta+2n+1})\zeta(a_{\vartheta+1}, a_{\vartheta+2n+1})\ldots\zeta(a_{\vartheta+2n}, a_{\vartheta+2n+1})}\right)$$

Case-2: If p is even, say $p = 2n$; $n \in \mathbb{N}$, then we have

$$F_{rb\zeta}(a_\vartheta, a_{\vartheta+2n}, t) \geq F_{rb\zeta}\left(a_\vartheta, a_{\vartheta+1}, \frac{\rho}{3\zeta(a_\vartheta, a_{\vartheta+2n})}\right) * F_{rb\zeta}\left(a_{\vartheta+1}, a_{\vartheta+2}, \frac{\rho}{3\zeta(a_\vartheta, a_{\vartheta+2n})\zeta(a_{\vartheta+1}, a_{\vartheta+2n})}\right)$$

$$* F_{rb\zeta}\left(a_{\vartheta+2}, a_{\vartheta+2n}, \frac{\rho}{3\zeta(a_\vartheta, a_{\vartheta+2n})\zeta(a_{\vartheta+1}, a_{\vartheta+2n})\zeta(a_{\vartheta+2}, a_{\vartheta+2n})}\right)$$

$$\geq F_{rb\zeta}\left(a_\vartheta, a_{\vartheta+1}, \frac{\rho}{3\zeta(a_\vartheta, a_{\vartheta+2n})}\right) * F_{rb\zeta}\left(a_{\vartheta+1}, a_{\vartheta+2}, \frac{\rho}{3\zeta(a_\vartheta, a_{\vartheta+2n})\zeta(a_{\vartheta+1}, a_{\vartheta+2n})}\right)$$

$$* F_{rb\zeta}\left(a_{\vartheta+2}, a_{\vartheta+3}, \frac{\rho}{3^2\zeta(a_\vartheta, a_{\vartheta+2n})\zeta(a_{\vartheta+1}, a_{\vartheta+2n})\zeta(a_{\vartheta+2}, a_{\vartheta+2n})}\right)$$

$$* F_{rb\zeta}\left(a_{\vartheta+3}, a_{\vartheta+4}, \frac{\rho}{3^2\zeta(a_\vartheta, a_{\vartheta+2n})\zeta(a_{\vartheta+1}, a_{\vartheta+2n}), \zeta(a_{\vartheta+2}, a_{\vartheta+2n})\zeta(a_{\vartheta+3}, a_{\vartheta+2n})}\right)$$

$$* F_{rb\zeta}\left(a_{\vartheta+4}, a_{\vartheta+2n}, \frac{\rho}{3^2\zeta(a_\vartheta, a_{\vartheta+2n})\zeta(a_{\vartheta+1}, a_{\vartheta+2n})\ldots\zeta(a_{\vartheta+4}, a_{\vartheta+2n})}\right)$$

$$\geq F_{rb\zeta}\left(a_\vartheta, a_{\vartheta+1}, \frac{\rho}{3\zeta(a_\vartheta, a_{\vartheta+2n})}\right) * F_{rb\zeta}\left(a_{\vartheta+1}, a_{\vartheta+2}, \frac{\rho}{3\zeta(a_\vartheta, a_{\vartheta+2n}), 3\zeta(a_{\vartheta+1}, a_{\vartheta+2n})}\right)$$

$$* F_{rb\zeta}\left(a_{\vartheta+2}, a_{\vartheta+3}, \frac{\rho}{3^2\zeta(a_\vartheta, a_{\vartheta+2n}), \zeta(a_{\vartheta+1}, a_{\vartheta+2n}), \zeta(a_{\vartheta+2}, a_{\vartheta+2n})}\right)$$

$$* F_{rb\zeta}\left(a_{\vartheta+3}, a_{\vartheta+4}, \frac{\rho}{3^2\zeta(a_\vartheta, a_{\vartheta+2n}), \zeta(a_{\vartheta+1}, a_{\vartheta+2n}), \zeta(a_{\vartheta+2}, a_{\vartheta+2n}), \zeta(a_{\vartheta+3}, a_{\vartheta+2n})}\right)$$

$$* F_{rb\zeta}\left(a_{\vartheta+4}, a_{\vartheta+5}, \frac{\rho}{3^3\zeta(a_\vartheta, a_{\vartheta+2n})\zeta(a_{\vartheta+1}, a_{\vartheta+2n})\ldots\zeta(a_{\vartheta+4}, a_{\vartheta+2n})}\right)$$

$$* \ldots * F_{rb\zeta}\left(a_{n+2m-4}, a_{n+2m-3}, \frac{\rho}{3^{m-1}\zeta(a_\vartheta, a_{\vartheta+2n})\zeta(a_{\vartheta+1}, a_{\vartheta+2n})\ldots\zeta(a_{n+2m-4}, a_{\vartheta+2n})}\right)$$

$$* F_{rb\zeta}\left(a_{n+2m-3}, a_{n+2m-2}, \frac{\rho}{3^{m-1}\zeta(a_\vartheta, a_{\vartheta+2n})\zeta(a_{\vartheta+1}, a_{\vartheta+2n})\ldots\zeta(a_{n+2m-3}, a_{\vartheta+2n})}\right)$$

$$* \ldots * F_{rb\zeta}\left(a_{n+2m-2}, a_{\vartheta+2n}, \frac{\rho}{3^{m-1}\zeta(a_\vartheta, a_{\vartheta+2n})\zeta(a_{\vartheta+1}, a_{\vartheta+2n})\ldots\zeta(a_{n+2m-2}, a_{\vartheta+2n})}\right)$$

$$\geq F_{rb\zeta}\left(a_0, a_1, \frac{\rho}{3\zeta(a_\vartheta, a_{\vartheta+2n})\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho)).\beta(F_{rb\zeta}(a_{\vartheta-2}, a_{\vartheta-1}, \rho))\ldots\beta(F_{rb\zeta}(a_0, a_1, \rho))}\right)$$

$$* F_{rb\zeta}\left(a_0, a_1, \frac{\rho}{3\zeta(a_\vartheta, a_{\vartheta+2n})\zeta(a_{\vartheta+1}, a_{\vartheta+2n})\beta(F_{rb\zeta}(a_\vartheta, a_{\vartheta+1}, \rho)).\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))\ldots\beta(F_{rb\zeta}(a_0, a_1, \rho))}\right)$$

$$* F_{rb\zeta}\left(a_0, a_1, \frac{\rho}{3^2\zeta(a_\vartheta, a_{\vartheta+2n})\zeta(a_{\vartheta+1}, a_{\vartheta+2n})\zeta(a_{\vartheta+2}, a_{\vartheta+2n})\beta(F_{rb\zeta}(a_{\vartheta+1}, a_{\vartheta+2}, \rho)).\beta(F_{rb\zeta}(a_\vartheta, a_{\vartheta+1}, \rho))\ldots\beta(F_{rb\zeta}(a_0, a_1, \rho))}\right)$$

$$* F_{rb\zeta}\left(a_0, a_0, \frac{\rho}{3^2\zeta(a_\vartheta, a_{\vartheta+2n})\zeta(a_{\vartheta+1}, a_{\vartheta+2n}), \zeta(a_{\vartheta+2}, a_{\vartheta+2n})\zeta(a_{\vartheta+3}, a_{\vartheta+2n}))\beta(F_{rb\zeta}(a_{\vartheta+2}, a_{\vartheta+3}, \rho))\ldots\beta(F_{rb\zeta}(a_0, a_1, \rho))}\right)$$

$$* F_{rb\zeta}\left(a_0, a_1, \frac{\rho}{3^3\zeta(a_\vartheta, a_{\vartheta+2n})\zeta(a_{\vartheta+1}, a_{\vartheta+2n})\ldots\zeta(a_{\vartheta+4}, a_{\vartheta+2n})\beta(F_{rb\zeta}(a_{\vartheta+3}, a_{\vartheta+4}, \rho))\ldots\beta(F_{rb\zeta}(a_0, a_1, \rho))}\right)$$

$$* \ldots * F_{rb\zeta}\left(a_0, a_1, \frac{\rho}{3^{n-1}\zeta(a_\vartheta, a_{\vartheta+2n})\zeta(a_{\vartheta+1}, a_{\vartheta+2n})\ldots\zeta(a_{\vartheta+2\rho-4}, a_{\vartheta+2n})\beta(F_{rb\zeta}(a_{\vartheta+2n-5}, a_{\vartheta+2n-4}, \rho))\ldots\beta(F_{rb\zeta}(a_0, a_1, \rho))}\right)$$

$$* F_{rb\zeta}\left(a_0, a_1, \frac{\rho}{3^{m-1}\zeta(a_\vartheta, a_{\vartheta+2n})\zeta(a_{\vartheta+1}, a_{\vartheta+2n+1})\ldots\zeta(a_{\vartheta+2n-3}, a_{\vartheta+2n})\beta(F_{rb\zeta}(a_{\vartheta+2n-4}, a_{\vartheta+2n-3}, \rho))\ldots\beta(F_{rb\zeta}(a_0, a_1, \rho))}\right)$$

$$* F_{rb\zeta}\left(a_0, a_0, \frac{\rho}{3^{m-1}\zeta(a_\vartheta, a_{\vartheta+2n})\zeta(a_{\vartheta+1}, a_{\vartheta+2n})\ldots\zeta(a_{\vartheta+2n-2}, a_{\vartheta+2n})\beta(F_{rb\zeta}(a_{\vartheta+2n-3}, a_{\vartheta+2n-2}, \rho))\rho))\ldots\beta(F_{rb\zeta}(a_0, a_1, \rho))}\right)$$

$$\geq F_{rb\zeta}\left(a_0, a_1, \frac{b^\vartheta\rho}{3\zeta(a_\vartheta, a_{\vartheta+2n})}\right) * F_{rb\zeta}\left(a_0, a_1, \frac{b^{\vartheta+1}\rho}{3\zeta(a_\vartheta, a_{\vartheta+2n})\zeta(a_{\vartheta+1}, a_{\vartheta+2n})}\right)$$

$$* F_{rb\zeta}\left(a_0, a_1, \frac{b^{\vartheta+2}\rho}{3^2\zeta(a_\vartheta, a_{\vartheta+2n})\zeta(a_{\vartheta+1}, a_{\vartheta+2n})\zeta(a_{\vartheta+2}, a_{\vartheta+2n})}\right)$$

$$* F_{rb\zeta}\left(a_0, a_0, \frac{b^{\vartheta+3}\rho}{3^2\zeta(a_\vartheta, a_{\vartheta+2n})\zeta(a_{\vartheta+1}, a_{\vartheta+2n}), \zeta(a_{\vartheta+2}, a_{\vartheta+2n})\zeta(a_{\vartheta+3}, a_{\vartheta+2n}))}\right)$$

$$* F_{rb\zeta}\left(a_0, a_1, \frac{b^{\vartheta+4}\rho}{3^3\zeta(a_\vartheta, a_{\vartheta+2n})\zeta(a_{\vartheta+1}, a_{\vartheta+2n})\ldots\zeta(a_{\vartheta+4}, a_{\vartheta+2n})}\right)$$

$$* \ldots * F_{rb\zeta}\left(a_0, a_1, \frac{b^{\vartheta+2n-4}\rho}{3^{n-1}\zeta(a_\vartheta, a_{\vartheta+2n})\zeta(a_{\vartheta+1}, a_{\vartheta+2n})\ldots\zeta(a_{\vartheta+2n-4}, a_{\vartheta+2n})}\right)$$

$$* F_{rb\zeta}\left(a_0, a_1, \frac{b^{\vartheta+2n-3}\rho}{3^{n-1}\zeta(a_\vartheta, a_{\vartheta+2n})\zeta(a_{\vartheta+1}, a_{\vartheta+2n+1})\ldots\zeta(a_{\vartheta+2n-3}, a_{\vartheta+2n})}\right)$$

$$*F_{rb\zeta}\left(a_0,\, a_0,\, \frac{b^{\vartheta+2n-2}\rho}{3^{n-1}\zeta(a_\vartheta,\, a_{\vartheta+2n})\zeta(a_{\vartheta+1},\, a_{\vartheta+2n})\ldots\zeta(a_{\vartheta+2n-2},\, a_{\vartheta+2n})}\right)$$

By applying (1) in both cases, it follows that for all $p \in \mathbb{N}$ we have

$$\lim_{\vartheta\to\infty} F_{rb\zeta}(a_\vartheta,\, a_{\vartheta+p},\, \rho) \geq 1*1*\ldots*1 = 1,$$

hence $\{a_\vartheta\}$ is a G-Cauchy sequence. As $(M,\, F_{rb\zeta},\, *)$ is a G-complete efrbms, so there exists a point $\varpi_1 \in M$ such that

$$\lim_{\vartheta\to\infty} a_\vartheta = \varpi_1.$$

We now show that ϖ_1 is fixed point of T.

$$\begin{aligned}
F_{rb\zeta}(\varpi_1,\, T\varpi_1,\, t) \geq & F_{rb\zeta}\left(\varpi_1,\, a_\vartheta,\, \frac{\rho}{3\zeta(\varpi_1,\, T\varpi_1)}\right) * F_{rb\zeta}\left(a_\vartheta,\, a_{\vartheta+1},\, \frac{\rho}{3\zeta(\varpi_1,\, T\varpi_1)}\right) * F_{rb\zeta}\left(a_{\vartheta+1},\, T\varpi_1,\, \frac{\rho}{3\zeta(\varpi_1,\, T\varpi_1)}\right)\\
\geq & F_{rb\zeta}\left(\varpi_1,\, a_\vartheta,\, \frac{\rho}{3\zeta(\varpi_1,\, T\varpi_1)}\right) * F_{rb\zeta}\left(Ta_{\vartheta-1},\, Ta_\vartheta,\, \frac{\rho}{3\zeta(\varpi_1,\, T\varpi_1)}\right) * F_{rb\zeta}\left(Ta_\vartheta,\, T\varpi_1,\, \frac{\rho}{3\zeta(\varpi_1,\, T\varpi_1)}\right)\\
\geq & F_{rb\zeta}\left(\varpi_1,\, a_\vartheta,\, \frac{\rho}{3\zeta(\varpi_1,\, T\varpi_1)}\right) * F_{rb\zeta}\left(a_{\vartheta-1},\, a_\vartheta,\, \frac{\rho}{3\zeta(\varpi_1,\, T\varpi_1)\beta(a_{\vartheta-1},\, a_\vartheta,\, \rho)}\right)\\
& * F_{rb\zeta}\left(a_\vartheta,\, \varpi_1,\, \frac{\rho}{3\zeta(\varpi_1,\, T\varpi_1)\beta(a_\vartheta,\, \varpi_1,\, \rho)}\right) \quad \longrightarrow 1*1*1 = 1 \text{ as } \vartheta\to\infty.
\end{aligned}$$

$\implies T\varpi_1 = \varpi_1$ is a fixed point.

Uniqueness

Assume $T\varpi_2 = \varpi_2$ for some $\varpi_2 \in M$, then

$$\begin{aligned}
F_{rb\zeta}(\varpi_2,\, \varpi_1,\, \rho) = F_{rb\zeta}(T\varpi_2,\, T\varpi_1,\, t) \geq F_{rb\zeta}\left(\varpi_2,\, \varpi_1,\, \frac{\rho}{\beta(\varpi_2,\, \varpi_1,\, \rho)}\right) = F_{rb\zeta}\left(T\varpi_2,\, T\varpi_1,\, \frac{\rho}{\beta(\varpi_2,\, \varpi_1,\, \rho)}\right)\\
\geq F_{rb\zeta}\left(\varpi_2,\, \varpi_1,\, \frac{\rho}{(\beta(\varpi_2,\, \varpi_1,\, \rho))^2}\right)\ldots \geq F_{rb\zeta}\left(\varpi_2,\, \varpi_1,\, \frac{\rho}{(\beta(\varpi_2,\, \varpi_1,\, \rho))^\vartheta}\right) = F_{rb\zeta}\left(\varpi_2,\, \varpi_1,\, b^\vartheta\rho\right)\\
\longrightarrow 1 \quad \text{as} \quad \vartheta\longrightarrow\infty
\end{aligned}$$

Thus $\varpi_1 = \varpi_2$. Hence the fixed point is unique. □

Remark 3.2

1. If we take $\beta(F_{rb\zeta}(a_1,\, a_2,\, \rho)) = k$, we get the main result of [17].

2. If we take $\zeta(a_1, a_2) = b$ and $\beta(F_{rb\zeta}(a_1, a_2, \rho)) = k$, we get the main result of [14].
3. If we take $\zeta(a_1, a_2) = 1$, we get the same result in frms.
Following example illustrates Theorem 3.1.
Example 3.1 Let $M = [0, 1]$ and define $F_{rb\zeta}\colon M \times M \times [0, \infty) \to [0, 1]$ by

$$F_{rb\zeta}(a_1, a_2, \rho) = \begin{cases} e^{-\dfrac{(a_1 - a_2)^2}{\rho}} & \text{if } \rho > 0 \\ 0 & \text{if } \rho = 0 \end{cases}.$$

It is easy to prove that $(M, F_{rb\zeta}, *)$ is a G-complete efrbms with $b = 3$. Let $k \in (0, 1)$ and define $T\colon M \to M$ by

$$T(a) = \frac{\sqrt{\beta(F_{rb\zeta}(a_1, a_2, \rho))}\, a}{1 + a}.$$

For all $\rho > 0$,

$$F_{rb\zeta}(Ta_1, Ta_2, \beta(F_{rb\zeta}(a_1, a_2, \rho))\rho) = F_{rb\zeta}\left(\frac{\sqrt{\beta(F_{rb\zeta}(a_1, a_2, \rho))} a_1}{1 + a_1}, \frac{\sqrt{k} a_2}{1 + a_2}, \beta(F_{rb\zeta}(a_1, a_2, \rho))\rho \right)$$

$$= e^{-\dfrac{(\sqrt{\beta(F_{rb\zeta}(a_1, a_2, \rho))} a_1 - \sqrt{\beta(F_{rb\zeta}(a_1, a_2, \rho))} a_2)^2}{(1 + a_1)(1 + a_2)\beta(F_{rb\zeta}(a_1, a_2, \rho))\rho}}$$

$$\geq e^{-\dfrac{\beta(F_{rb\zeta}(a_1, a_2, \rho))(a_1 - a_2)^2}{\beta(F_{rb\zeta}(a_1, a_2, \rho))\rho}}$$

$$= e^{-\dfrac{(a_1 - a_2)^2}{\rho}}$$

$$= F_{rb\zeta}(a_1, a_2, \rho)$$

Since all the requirements of Theorem 3.1 are fulfilled and also $a = 0$ is a unique fixed point of T.
Now we establish the main result of [21] in the setting of extended fuzzy rectangular b-metric space.
Theorem 3.2 Let $(M, F_{rb\zeta}, *)$ be a G-complete efrbms and $T\colon M \to M$ be a mapping satisfying

$$F_{rb\zeta}(Ta_1, Ta_2, \beta(F_{rb\zeta}(a_1, a_2, \rho))\rho) \geq \min\left\{ \frac{F_{rb\zeta}(a_2, Ta_2, \rho)\,[1 + F_{rb\zeta}(a_1, Ta_1, \rho)]}{1 + F_{rb\zeta}(a_1, a_2, \rho)}, F_{rb\zeta}(a_1, a_2, \rho) \right\} \tag{3}$$

$\forall\, a_1, a_2 \in M$ and $\beta(F_{rb\zeta}(a_1, a_2, \rho)) \in \mathscr{F}_b$. Then T has a unique fixed point.

Proof. Let $a_0 \in M$ be any fix arbitrary element and for $\vartheta = 0, 1, 2, \ldots$, taking $a_{\vartheta+1} = Ta_\vartheta$. For all ϑ, $\rho > 0$, by the application of Inequality (1), we get

$$
\begin{aligned}
F_{rb\zeta}(a_\vartheta, a_{\vartheta+1}, t) &= F_{rb\zeta}(Ta_{\vartheta-1}, Ta_\vartheta, t) \\
&\geq \min\left\{ \frac{F_{rb\zeta}(a_\vartheta, Ta_\vartheta, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))}) \left[1 + F_{rb\zeta}(a_{\vartheta-1}, Ta_{\vartheta-1}, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))})\right]}{1 + F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))})}, \right. \\
&\qquad \left. F_{rb\zeta}(a_\vartheta, a_{\vartheta-1}, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))}) \right\} \\
&= \min\left\{ \frac{F_{rb\zeta}(a_\vartheta, a_{\vartheta+1}, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))}) \left[1 + F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))})\right]}{1 + F_{rb\zeta}(a_\vartheta, a_{\vartheta-1}, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))})}, \right. \\
&\qquad \left. F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))}) \right\} \\
&= \min\left\{ F_{rb\zeta}(a_{\vartheta+1}, a_\vartheta, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))}), F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))}) \right\}.
\end{aligned}
$$

So, we have

$$
F_{rb\zeta}(a_\vartheta, a_{\vartheta+1}, t) \geq \min\left\{ F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))}), F_{rb\zeta}(a_\vartheta, a_{\vartheta+1}, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))}) \right\} \tag{4}
$$

If

$$
\min\left\{ F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))}), F_{rb\zeta}(a_\vartheta, a_{\vartheta+1}, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))} \right\}
$$

$$
= F_{rb\zeta}(a_\vartheta, a_{\vartheta+1}, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))}),
$$

then, from (4), it slides that

$$
F_{rb\zeta}(a_\vartheta, a_{\vartheta+1}, \rho) \geq F_{rb\zeta}(a_\vartheta, a_{\vartheta+1}, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))}).
$$

Since $\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho)) \in \mathscr{F}_b$, then there is nothing to prove by Lemma 3.

If

$$\min\left\{F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))}), F_{rb\zeta}(a_\vartheta, a_{\vartheta+1}, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))})\right\}$$

$$=F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))}),$$

then from (4),

$$F_{rb\zeta}(a_\vartheta, a_{\vartheta+1}, \rho) \geq F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))})$$

$$\geq F_{rb\zeta}\left(a_{\vartheta-2}, a_{\vartheta-1}, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho)).\beta(F_{rb\zeta}(a_{\vartheta-2}, a_{\vartheta-1}, \rho))}\right)$$

$$\geq F_{rb\zeta}\left(a_{n-3}, a_{\vartheta-2}, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho)).\beta(F_{rb\zeta}(a_{\vartheta-2}, a_{\vartheta-1}, \rho)).\beta(F_{rb\zeta}(a_{n-3}, a_{\vartheta-2}, \rho)}\right)$$

$$\geq \ldots \geq F_{rb\zeta}\left(a_0, a_1, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho)).\beta(F_{rb\zeta}(a_{\vartheta-2}, a_{\vartheta-1}, \rho))\ldots\beta(F_{rb\zeta}(a_0, a_1, \rho))}\right)$$

By adopting the same procedure after inequality (2) as in Theorem 3.1, we can complete the proof. □

Remark 3.3

1.If we take $\zeta(a_1, a_2) = 1$, we get the same result in frms.

2. If we consider efbms instead of efrbms we get Theorem 3 of [16].

3. If we take $\beta(F_{rb\zeta}(a_1, a_2, \rho)) = k$ and $\zeta(a_1, a_2) = 1$ we get the main result of [19].

Theorem 3.3 Let $(M, F_{rb\zeta}, *)$ be a G-complete efrbms and $T\colon M \to M$ be a mapping satisfying

$$F_{rb\zeta}(Ta_1, Ta_2, \beta(F_{rb\zeta}(a_1, a_1, \rho))\rho)$$

$$\geq \min\left\{\frac{F_{rb\zeta}(a_2, Ta_2, \rho)\left[1 + F_{rb\zeta}(a_1, Ta_1, \rho) + F_{rb\zeta}(a_2, Ta_1, \rho)\right]}{2 + F_{rb\zeta}(a_1, a_2, \rho)}, F_{rb\zeta}(a_1, a_2, \rho)\right\} \tag{5}$$

Proof. For $a_0 \in M$, we choose a sequence $\{a_\vartheta\}$ in M such that $Ta_\vartheta = a_{\vartheta+1}$ ($\forall\ \vartheta \in \mathbb{N}$). Now we prove $\{a_\vartheta\}$ is a Cauchy sequence. We take $a_2 = a_{\vartheta-1}$ and $a_1 = a_\vartheta$, then

$$F_{rb\zeta}(a_\vartheta, a_{\vartheta+1}, \rho)$$

$$=F_{rb\zeta}(Ta_{\vartheta-1}, Ta_\vartheta, \rho)$$

$$\geq \min\left\{\frac{F_{rb\zeta}\left(a_\vartheta, Ta_\vartheta, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))}\right)\left[1+F_{rb\zeta}\left(a_{\vartheta-1}, Ta_{\vartheta-1}, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))}\right) + F_{rb\zeta}\left(a_\vartheta, Ta_{\vartheta-1}, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))}\right)\right]}{2+F_{rb\zeta}\left(a_{\vartheta-1}, a_\vartheta, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))}\right)},\right.$$

$$\left. F_{rb\zeta}\left(a_{\vartheta-1}, a_\vartheta, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))}\right)\right\}$$

$$=\min\left\{\frac{F_{rb\zeta}\left(a_\vartheta, a_{\vartheta+1}, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))}\right)\left[1+F_{rb\zeta}\left(a_{\vartheta-1}, a_\vartheta, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))}\right) + F_{rb\zeta}\left(a_\vartheta, a_\vartheta, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))}\right)\right]}{2+F_{rb\zeta}\left(a_{\vartheta-1}, a_\vartheta, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))}\right)},\right.$$

$$\left. F_{rb\zeta}\left(a_{\vartheta-1}, a_\vartheta, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))}\right)\right\}$$

$$=\min\left\{\frac{F_{rb\zeta}\left(a_\vartheta, a_{\vartheta+1}, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))}\right)\left[1+F_{rb\zeta}\left(a_{\vartheta-1}, a_\vartheta, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))}\right)+1\right]}{2+F_{rb\zeta}\left(a_{\vartheta-1}, a_\vartheta, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))}\right)},\right.$$

$$\left. F_{rb\zeta}\left(a_{\vartheta-1}, a_\vartheta, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))}\right)\right\}$$

$$=\min\left\{F_{rb\zeta}\left(a_\vartheta, a_{\vartheta+1}, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))}\right), F_{rb\zeta}\left(a_{\vartheta-1}, a_\vartheta, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))}\right)\right\}$$

So we have,

$$F_{rb\zeta}(a_\vartheta, a_{\vartheta+1}, \rho) \geq \min\left\{F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))}), F_{rb\zeta}(a_\vartheta, a_{\vartheta+1}, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))})\right\} \tag{6}$$

If

$$\min\left\{F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))}), F_{rb\zeta}(a_\vartheta, a_{\vartheta+1}, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))}\right\}$$

$$=F_{rb\zeta}(a_\vartheta, a_{\vartheta+1}, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))})$$

then from (6),

$$F_{rb\zeta}(a_\vartheta, a_{\vartheta+1}, \rho) \geq F_{rb\zeta}(a_\vartheta, a_{\vartheta+1}, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))})$$

Since $\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho)) \in \mathscr{F}_b$, then there is nothing to prove by Lemma 3.

If

$$\min\left\{F_{rb\zeta}\left(a_{\vartheta-1}, a_\vartheta, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))}\right), F_{rb\zeta}\left(a_\vartheta, a_{\vartheta+1}, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))}\right)\right\}$$

$$=F_{rb\zeta}\left(a_{\vartheta-1}, a_\vartheta, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))}\right)$$

then from (6),

$$F_{rb\zeta}(a_\vartheta, a_{\vartheta+1}, \rho) \geq F_{rb\zeta}\left(a_{\vartheta-1}, a_\vartheta, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))}\right).$$

Continuing in this way, we have

$$\begin{aligned}
F_{rb\zeta}(a_\vartheta, a_{\vartheta+1}, \rho) &\geq F_{rb\zeta}\left(a_{\vartheta-1}, a_\vartheta, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))}\right) \\
&\geq F_{rb\zeta}\left(a_{\vartheta-2}, a_{\vartheta-1}, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho)).\beta(F_{rb\zeta}(a_{\vartheta-2}, a_{\vartheta-1}, \rho))}\right) \\
&\geq F_{rb\zeta}\left(a_{n-3}, a_{\vartheta-2}, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho)).\beta(F_{rb\zeta}(a_{\vartheta-2}, a_{\vartheta-1}, \rho)).\beta(F_{rb\zeta}(a_{n-3}, a_{\vartheta-2}, \rho)}\right) \\
&\geq \ldots \geq F_{rb\zeta}\left(a_0, a_1, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho)).\beta(F_{rb\zeta}(a_{\vartheta-2}, a_{\vartheta-1}, \rho))\ldots\beta(F_{rb\zeta}(a_0, a_1, \rho))}\right)
\end{aligned}$$

By adopting the same procedure after inequality (2) as in Theorem 3.1, we can complete the proof. □

Remark 3.4

1.If we take $\zeta(a_1, a_2) = 1$, we get the same result in frms.

2. If we consider efbms instead of efrbms we get Theorem 4 of [16].

3. If we take $\beta(F_{rb\zeta}(a_1, a_2, \rho)) = k$ and $\zeta(a_1, a_2) = 1$ we get the same result in frms

Following result is analogue to main result of [20].

Theorem 3.4 Let $(M, F_{rb\zeta}, *)$ be a G-complete efrbms and $T: M \to M$ be a mapping satisfying

$$F_{rb\zeta}(Ta_1, Ta_2, \beta(F_{rb\zeta}(a_1, a_2, \rho)), \rho)$$

$$\geq \min\left\{\frac{F_{rb\zeta}(a_1, Ta_1, \rho)\left[1+F_{rb\zeta}(a_2, Ta_2, \rho)\right]}{1+F_{rb\zeta}(Ta_1, Ta_2, \rho)}, \frac{F_{rb\zeta}(a_2, Ta_2, \rho)\left[1+F_{rb\zeta}(a_1, Ta_1, \rho)\right]}{1+F_{rb\zeta}(a_1, a_2, \rho)},\right.$$

$$\left.\frac{F_{rb\zeta}(a_1, Ta_1, \rho)\left[2+F_{rb\zeta}(a_1, Ta_2, \rho)\right]}{1+F_{rb\zeta}(a_1, Ta_2, \rho)+F_{rb\zeta}(a_2, Ta_1, \rho)}, F_{rb\zeta}(a_1, a_2, \rho)\right\} \tag{7}$$

$\forall\, a_1, a_2 \in M$ and $\beta(F_{rb\zeta}(a_1, a_2, \rho)) \in \mathscr{F}_b$. Then T has a unique fixed point.

Proof. Let $a_0 \in M$ be an fix arbitrary element and for $\vartheta = 0, 1, 2, \ldots$, taking $a_{\vartheta+1} = Ta_\vartheta$.

For all ϑ, $\rho > 0$, by the application of Inequality (1), we get

$$F_{rb\zeta}(a_\vartheta, a_{\vartheta+1}, \vartheta) = F_{rb\zeta}(Ta_{\vartheta-1}, Ta_\vartheta, \vartheta)$$

$$\geq \min\left\{\frac{F_{rb\zeta}(a_{\vartheta-1}, Ta_{\vartheta-1}, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))})\left[1+F_{rb\zeta}(a_\vartheta, Ta_\vartheta, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))})\right]}{1+F_{rb\zeta}(Ta_{\vartheta-1}, Ta_\vartheta, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))})},\right.$$

$$\frac{F_{rb\zeta}(a_\vartheta, Ta_\vartheta, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))})\left[1+F_{rb\zeta}(a_{\vartheta-1}, Ta_{\vartheta-1}, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))})\right]}{1+F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))})},$$

$$\frac{F_{rb\zeta}(a_{\vartheta-1}, Ta_{\vartheta-1}, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))})\left[2+F_{rb\zeta}(a_{\vartheta-1}, Ta_\vartheta, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))})\right]}{1+F_{rb\zeta}(a_{\vartheta-1}, Ta_\vartheta, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))})+F_{rb\zeta}(a_\vartheta, Ta_{\vartheta-1}, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))})},$$

$$\left.F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))})\right\}$$

$$= \min\left\{ \frac{F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))}) \left[1 + F_{rb\zeta}(a_\vartheta, a_{\vartheta+1}, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))})\right]}{1 + F_{rb\zeta}(a_\vartheta, a_{\vartheta+1}, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))})}, \right.$$

$$\frac{F_{rb\zeta}(a_\vartheta, a_{\vartheta+1}, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))}) \left[1 + F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))})\right]}{1 + F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))})},$$

$$\frac{F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))}) \left[2 + F_{rb\zeta}(a_{\vartheta-1}, a_{\vartheta+1}, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))})\right]}{1 + F_{rb\zeta}(a_{\vartheta-1}, a_{\vartheta+1}, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))}) + F_{rb\zeta}(a_\vartheta, a_\vartheta, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))})},$$

$$\left. F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))}) \right\}$$

$$= \min\left\{ F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))}), F_{rb\zeta}(a_\vartheta, a_{\vartheta+1}, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))}) \right\}$$

So we have,

$$F_{rb\zeta}(a_\vartheta, a_{\vartheta+1}, \rho) \geq \min\left\{ F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))}), F_{rb\zeta}(a_\vartheta, a_{\vartheta+1}, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))}) \right\} \quad (8)$$

If

$$\min\left\{ F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))}), F_{rb\zeta}(a_\vartheta, a_{\vartheta+1}, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))} \right\}$$

$$= F_{rb\zeta}\left(a_\vartheta, a_{\vartheta+1}, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho)} \right)$$

then from (8),

$$F_{rb\zeta}(a_\vartheta, a_{\vartheta+1}, \rho) \geq F_{rb\zeta}\left(a_\vartheta, a_{\vartheta+1}, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho))} \right)$$

Since $\beta(F_{rb\zeta}(a_{\vartheta-1}, a_\vartheta, \rho)) \in \mathscr{F}_b$, then there is nothing to prove by Lemma 3.
If

$$\min\left\{F_{rb\zeta}(a_{\vartheta-1}, a_{\vartheta}, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_{\vartheta}, \rho))}), F_{rb\zeta}(a_{\vartheta}, a_{\vartheta+1}, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_{\vartheta}, \rho))}\right\}$$

$$=F_{rb\zeta}\left(a_{\vartheta-1}, a_{\vartheta}, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_{\vartheta}, \rho))}\right)$$

then from (8),

$$F_{rb\zeta}(a_{\vartheta}, a_{\vartheta+1}, \rho) \geq F_{rb\zeta}\left(a_{\vartheta-1}, a_{\vartheta}, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_{\vartheta}, \rho))}\right)$$

Continuing in this way, we have

$$\begin{aligned} F_{rb\zeta}(a_{\vartheta}, a_{\vartheta+1}, \rho) &\geq F_{rb\zeta}\left(a_{\vartheta-1}, a_{\vartheta}, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_{\vartheta}, \rho))}\right) \\ &\geq F_{rb\zeta}\left(a_{\vartheta-2}, a_{\vartheta-1}, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_{\vartheta}, \rho)).\beta(F_{rb\zeta}(a_{\vartheta-2}, a_{\vartheta-1}, \rho))}\right) \\ &\geq F_{rb\zeta}\left(a_{n-3}, a_{\vartheta-2}, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_{\vartheta}, \rho)).\beta(F_{rb\zeta}(a_{\vartheta-2}, a_{\vartheta-1}, \rho)).\beta(F_{rb\zeta}(a_{n-3}, a_{\vartheta-2}, \rho)}\right) \\ &\geq \ldots \geq F_{rb\zeta}\left(a_0, a_1, \frac{\rho}{\beta(F_{rb\zeta}(a_{\vartheta-1}, a_{\vartheta}, \rho)).\beta(F_{rb\zeta}(a_{\vartheta-2}, a_{\vartheta-1}, \rho))\ldots\beta(F_{rb\zeta}(a_0, a_1, \rho))}\right) \end{aligned}$$

By adopting the same procedure after inequality (2) as in Theorem 3.1, we can complete the proof.
Remark 3.5
1.If we take $\zeta(a_1, a_2) = b$, we get the Theorem 2.3 of [14] .
2. If we consider efbms instead of efrbms we get Theorem 5 of [16].
3. If we take $\beta(F_{rb\zeta}(a_1, a_2, \rho)) = k$ and $\zeta(a_1, a_2) = 1$ we get the same result in frms.

4. Application

Fixed point theory turns out to be a powerful tool for the existence of solution of various kinds of integral and differential equations, for instance see [20, 22]. In this section, a non-linear integral equation has been studied for the existence of the solution as an application of our main fixed point result. Consider,

$$a(\delta) = \Gamma^*(\delta) + \int_0^{\delta} \Xi(\delta, \kappa, a(\kappa))d\kappa, \tag{9}$$

$\forall\, \delta \in [0, a]$ and $\Xi \in C([0, a] \times [0, a] \times \mathbb{R},\ \mathbb{R})$ where $a > 0$. Let $C([0, a], \mathbb{R})$ be the space of all continuous functions and t-norm is defined as $a * b = ab\ \forall\, a, b \in [0, 1]$ and define the G-complete efrbm on $C([0, a], \mathbb{R})$ by

$$F_{rb\zeta}(a_1, a_2, \rho) = e^{-\frac{\sup\limits_{\delta \in [0, a]} |a_1(\delta) - a_2(\delta)|^2}{\rho}}$$

$\forall\, \rho > 0$ and $a_1, a_2 \in C([0, a], \mathbb{R})$.

The solution of the integral equation (9)is given in next result.

Theorem 4.1 Let $T \colon C([0, a], \mathbb{R}) \to C([0, a], \mathbb{R})$ be the integral operator given by

$$T(a_1(\delta)) = \Gamma^*(\delta) + \int_0^\delta \Xi(\delta,\, \kappa,\, a_1(\kappa))d\kappa, \quad \Gamma^* \in C([0, a], \mathbb{R})$$

where $\Xi \in C([0, a] \times [0, a] \times \mathbb{R},\ \mathbb{R})$ satisfies the following condition:
there exists $\Gamma^* \colon [0, a] \times [0, a] \to [0, +\infty]$ such that for all $\kappa, \delta \in [0, a]$, $\Gamma^*(\delta, \kappa) \in L^1([0, a], \mathbb{R})$ and $\forall a_1, a_2 \in C([0, a], \mathbb{R})$, we have

$$|\Xi(\delta,\, \kappa,\, a_1(\kappa)) - \Xi(\delta,\, \kappa,\, a_2(\kappa))|^2 \leq {\Gamma^*}^2(\delta,\, \kappa)|a_1(\kappa) - a_2(\kappa)|^2$$

where

$$\sup_{\delta \in [0, a]} \int_0^\delta {\Gamma^*}^2(\delta,\, \kappa)d\kappa \leq \beta(F_{rb\zeta}(a_1, a_2, \rho)).$$

Then the integral equation has the solution $a_1^* \in C([0, a], \mathbb{R})$.

Proof. For all $a_1, a_2 \in C([0, a], \mathbb{R})$, we have

$$F_{rb\zeta}(T(a_1(\delta)), T(a_2(\delta)), \beta(F_{rb\zeta}(a_1, a_2, \rho))\rho) = e^{-\frac{\sup\limits_{\delta \in [0, a]} |T(a_1(\delta)) - T(a_2(\delta))|^2}{\beta(F_{rb\zeta}(a_1, a_2, \rho))\rho}}$$

$$\geq e^{-\frac{\sup\limits_{s \in [0, a]} \int_0^s |\Xi(\delta,\, \kappa,\, a_1(\kappa)) - \Xi(\delta,\, \kappa,\, a_2(\kappa))|^2 d\kappa}{\beta(F_{rb\zeta}(a_1, a_2, \rho))\rho}}$$

$$\geq e^{-\frac{\sup\limits_{\delta \in [0, a]} \int_0^\delta {\Gamma^*}^2(\delta,\, \kappa)|a_1(\kappa) - a_2(\kappa)|^2 d\kappa}{\beta(F_{rb\zeta}(a_1, a_2, \rho))\rho}}$$

$$\geq e^{-\frac{|a_1(\kappa)-a_2(\kappa)|^2 \sup\limits_{\delta\in[0,\,a]} \int_0^{\delta} \Gamma^{*2}(\delta,\,\kappa)d\kappa}{\beta(F_{rb\zeta}(a_1,\,a_2,\,\rho))\rho}}$$

$$\geq e^{-\frac{\beta(F_{rb\zeta}(a_1,\,a_2,\,\rho))|a_1(\kappa)-a_2(\kappa)|^2}{\beta(F_{rb\zeta}(a_1,\,a_2,\,\rho))\rho}}$$

$$= e^{-\frac{|a_1(\kappa)-a_2(\kappa)|^2}{\rho}}$$

$$\geq e^{-\frac{\sup\limits_{\kappa\in[0,\,a]} |a_1(\kappa)-a_2(\kappa)|^2}{\rho}}$$

$$= F_{rb\zeta}(a_1,\,a_2,\,\rho)$$

Hence $a_1^* \in C([0,\,a],\,\mathbb{R})$ is a fixed point of T, which is the solution of integral equation (9). □

Example 4.1 Consider the differential equation

$$a''(\delta)+a'(\delta)=e^{\delta}, \quad a(0)=0,\ a'(0)=1$$

which gives the integral equation

$$a(\delta)=e^{\delta}-1-\int_0^{\delta}(\delta-\kappa)a(\kappa)d\kappa \tag{10}$$

Now

$$F_{rb\zeta}(T(a_1(\delta)),\,T(a_2(\delta)),\,\beta(F_{rb\zeta}(a_1,\,a_2,\,\rho))\rho) = e^{-\frac{\sup\limits_{\delta\in[0,\,a]} |T(a_1(\delta))-T(a_2(\delta))|^2}{\beta(F_{rb\zeta}(a_1,\,a_2,\,\rho))\rho}}$$

$$F_{rb\zeta}(T(a_1(\delta)),\,T(a_2(\delta)),\,\beta(F_{rb\zeta}(a_1,\,a_2,\,\rho))\rho) = e^{-\frac{\sup\limits_{\delta\in[0,\,a]} |T(a_1(\delta))-T(a_2(\delta))|^2}{\beta(F_{rb\zeta}(a_1,\,a_2,\,\rho))\rho}}$$

$$\geq e^{-\frac{\sup\limits_{s\in[0,\,a]}\int_0^s |(\delta-\kappa)a_1(\kappa))-(\delta-\kappa)a_2(\kappa))|^2 d\kappa}{\beta(F_{rb\zeta}(a_1,\,a_2,\,\rho))\rho}}$$

$$\geq e^{-\frac{\sup\limits_{\delta\in[0,\,a]}\int_0^\delta (\delta-\kappa)^2|a_1(\kappa)-a_2(\kappa)|^2 d\kappa}{\beta(F_{rb\zeta}(a_1,\,a_2,\,\rho))\rho}}$$

$$\geq e^{-\frac{|a_1(\kappa)-a_2(\kappa)|^2\sup\limits_{\delta\in[0,\,a]}\int_0^\delta (\delta-\kappa)^2 d\kappa}{\beta(F_{rb\zeta}(a_1,\,a_2,\,\rho))\rho}}$$

Clearly

$$\sup_{\delta\in[0,\,a]}\int_0^\delta (\delta-\kappa)^2 d\kappa \leq 1.$$

So

$$e^{-\frac{|a_1(\kappa)-a_2(\kappa)|^2\sup\limits_{\delta\in[0,\,a]}\int_0^\delta (\delta-\kappa)^2 d\kappa}{\beta(F_{rb\zeta}(a_1,\,a_2,\,\rho))\rho}} \geq e^{-\frac{|a_1(\kappa)-a_2(\kappa)|^2}{\beta(F_{rb\zeta}(a_1,\,a_2,\,\rho))\rho}}$$

$$\geq e^{-\frac{|a_1(\kappa)-a_2(\kappa)|^2}{\rho}}$$

$$\geq e^{-\frac{\sup\limits_{\kappa\in[0,\,a]}|a_1(\kappa)-a_2(\kappa)|^2}{\rho}}$$

$$= F_{rb\zeta}(a_1,\,a_2,\,\rho)$$

Since all the conditions of Theorem 3.1 are satisfied so integral equation 10 has a unique solution.

Conflict of interest

The authors declare no competing interests in this study.

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