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Meromorphic Functions Sharing Three Values and a Result of G. Brosch

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Abstract: We prove a uniqueness theorem for two non-constant meromorphic functions sharing three values which improves a result of G. Brosch.
Keywords: meromorphic function, uniqueness, weighted sharing

1. Introduction

R. Nevanlinna's five and four value uniqueness theorems[15-16] are the starting points of the modern uniqueness theory of meromorphic functions. These two famous results have been generalized and extended by many authors in different directions. In the paper, we focus on the following improvement of the four value theorem by G. Brosch[2, 16].

Theorem A.[2,16] Let f and g be two distinct non-constant meromorphic functions sharing 0, 1, ∞ counting multiplicities (CM). Let a, b be two complex numbers such that $a, b \notin \{0, 1, \infty\}$. If $f - a$ and $g - b$ share 0 ignoring multiplicities (IM), then f is a bilinear transformation of g.

Let us consider the celebrated functions of G. G. Gundersen[3]: $f(z) = \dfrac{e^z + 1}{(e^z - 1)^2}$ and $g(z) = \dfrac{(e^z + 1)^2}{8(e^z - 1)}$. Then f and g share 0, 1, ∞ IM and $f + \dfrac{1}{2}, g - \dfrac{1}{4}$ share 0 CM, but f is not a bilinear transformation of g. Therefore Theorem A does not hold for IM shared values.

T. C. Alzahary and H. X. Yi[1] used the notion of weighted sharing of values, introduced in [6,7], to relax the hypothesis on value sharing in Theorem A. Before starting the result of Alzahary and Yi, let us recall the definition of weighted sharing of values.

Definition 1.1.[6, 7] Let k be a non-negative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$, we denote by $E_k(a; f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \le k$ and $k + 1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.

The definition implies that if f, g share a value a with weight k, then z_o is a zero of $f - a$ with multiplicity $m(\le k)$ if and only if it is a zero of $g - a$ with multiplicity $m(\le k)$ and z_o is a zero of $f - a$ with multiplicity $m(> k)$ if and only if it is a zero of $g - a$ with multiplicity $n(> k)$ where m is not necessarily equal to n.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for all integers p, $0 \le p < k$. Also, we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

We now state the result of Alzahary and Yi[1].

Theorem B. [1] Let f and g be two non-constant meromorphic functions sharing $(a_1, 1)$, (a_2, ∞) and (a_3, ∞), where $\{a_1, a_2, a_3\} = \{0, 1, \infty\}$. Let a, b be two complex numbers such that $a, b \notin \{0,1,\infty\}$. If $f - a$ and $g - b$ share (0, 0) then f is a bilinear transformation of g. Moreover, f and g satisfy one of the following relations: (i) $f \equiv g$, (ii) $fg \equiv 1$, (iii) $bf \equiv ag$, (iv) $f + g \equiv 1$, (v) $f \equiv ag$, (vi) $f \equiv (1 - a)g + a$, (vii) $(1 - b)f \equiv (1 - a)g + (a - b)$, (viii) $f(a - 1 + g) \equiv ag$, (ix) $f\{(b - a)g + (a - 1)b\} \equiv a(b - 1)g$, (x) $f(g - 1) \equiv g$.

The cases (ii) and (v) may occur if $ab = 1$, cases (iv), (viii) may occur if $a + b = 1$, cases (vi), (x) may occur if $ab = a + b$.

In 2007 the first author and P. Sahoo[13] improved Theorem B in the following manner.

DOI: https://doi.org/10.37256/cm.00037.63-76

Theorem C.[13] Let f and g be two distinct non-constant meromorphic functions sharing $(a_1, 1)$, (a_2, m), (a_3, k), where $(m-1)(mk-1) > (1+m)^2$ and $\{a_1, a_2, a_3\} = \{0, 1, \infty\}$. If for two values $a, b \notin \{0,1,\infty\}$ the functions $f-a$ and $g-b$, share $(0, 0)$, then f g share $(0, \infty)$, $(1, \infty)$, (∞, ∞) and $f-a$, $g-b$ share $(0, \infty)$. Also, there exists a non-constant entire function λ such that f and g are one of the following forms :

(i) $f = ae^{\lambda}$ and $g = be^{-\lambda}$, where $ab = 1$.

(ii) $f = 1 + ae^{\lambda}$ and $g = 1 + (1 - \frac{1}{b})e^{-\lambda}$, where $ab = a + b$.

(iii) $f = \dfrac{a}{a + e^{\lambda}}$ and $g = \dfrac{e^{\lambda}}{1 - b + e^{\lambda}}$, where $a + b = 1$.

(iv) $f = \dfrac{e^{\lambda} - a}{e^{\lambda} - 1}$ and $g = \dfrac{be^{\lambda} - 1}{e^{\lambda} - 1}$, where $ab = 1$.

(v) $f = \dfrac{be^{\lambda} - a}{be^{\lambda} - b}$ and $g = \dfrac{be^{\lambda} - a}{ae^{\lambda} - a}$, where $a \neq b$.

(vi) $f = \dfrac{a}{1 - e^{\lambda}}$ and $g = \dfrac{be^{\lambda}}{e^{\lambda} - 1}$, where $ab = a + b$.

(vii) $f = \dfrac{b - a}{(b-1)(1 - e^{\lambda})}$ and $g = \dfrac{(b-a)e^{\lambda}}{(a-1)(1 - e^{\lambda})}$, where $a \neq b$.

(viii) $f = a + e^{\lambda}$ and $g = b(1 + \frac{1-b}{e^{\lambda}})$, where $a + b = 1$.

(ix) $f = e^{\lambda} - \dfrac{a(b-1)}{a-b}$ and $g = \dfrac{b(a-1)}{a-b}\left\{1 - \dfrac{a(b-1)}{(a-b)e^{\lambda}}\right\}$, where $a \neq b$.

The purpose of the paper is to improve Theorem C, which in turn is an improvement of Brosch's result. To state our result we need the following definition.

Definition 1.2. For a meromorphic function f and $a \in \mathbb{C} \cup \{\infty\}$, we denote by $\overline{E}(a; f)$ the set of distinct a-points of f. If k is a positive integer, then we denote by $\overline{E}_{k)}(a; f)$ the set of those distinct a-points of f whose multiplicities do not exceed k.

We now state the main result of the paper.

Theorem 1.1. Let f and g be two distinct non-constant meromorphic functions sharing $(a_1, 1)$, (a_2, m), (a_3, k), where $(m-1)(mk-1) > (1+m)^2$ and $\{a_1, a_2, a_3\} = \{0,1,\infty\}$. If for two values $a, b \notin \{0,1,\infty\}$, $\overline{E}_{2)}(a; f) \subset \overline{E}(b; g)$ and $\overline{E}_{2)}(b; g) \subset \overline{E}(a; f)$, then the conclusion of Theorem C holds.

Following example shows that the condition $\overline{E}_{2)}(a; f) \subset \overline{E}(b; g)$ and $\overline{E}_{2)}(b; g) \subset \overline{E}(a; f)$ is sharp.

Example 1.1. Let $f(z) = e^{2z} + e^{z} + 1$ and $g(z) = e^{-2z} + e^{-z} + 1$. Then f and g share $(0, \infty)$, $(1, \infty)$ and (∞, ∞). Suppose that $a_1 = \frac{3}{4}$ and $b_1 = 3$. Then $f - a_1 = (e^{z} + \frac{1}{2})^2$ and $g - b_1 = (e^{-z} - 1)(e^{-z} + 2)$. Hence $\overline{E}_{2)}(a_1; f) \subset \overline{E}(b_1; g)$ and $\overline{E}_{2)}(b_1; g) \subset \overline{E}(a_1; f)$.

Next suppose that $a_2 = b_2 = \frac{3}{4}$. Then $f - a_2 = (e^{z} + \frac{1}{2})^2$ and $g - b_2 = (e^{-z} + \frac{1}{2})^2$. Hence $\overline{E}_{1)}(a_2; f) \subset \overline{E}(b_2; g)$ and $\overline{E}_{1)}(b_2; g) \subset \overline{E}(a_2; f)$.

Clearly, we see that f and g do not assume any one of the forms given in Theorem C.

We do not explain the standard definitions and notations of the value distribution theory as those are available in [5]. We, however, explain the following notations used in the paper.

Definition 1.3. Let f be a meromorphic function and $a \in \mathbb{C} \cup \{\infty\}$. For a positive integer p, we denote by $N(r, a; f \mid \leq p)$

($\overline{N}(r, a; f \mid \leq p)$) the counting function (reduced counting function) of those a-points of f whose multiplicities are less than or equal to p.

Definition 1.4. Let f, g be two meromorphic functions and $a \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r, a; f \mid g = b)$ ($\overline{N}(r, a; f \mid \leq g = b)$) the counting function (reduced counting function) of those a-points of f which are b points of g also.

Definition 1.5. Let f, g be two meromorphic functions and $a \in \mathbb{C} \cup \{\infty\}$. For a positive integer p, we denote by $N(r, a; f \mid g = b, \geq p)(\overline{N}(r, a; f \mid g = b, \geq p))$ the counting function (reduced counting function) of those a-points of which are b-points of g with multiplicities not less than p.

Definition 1.6. Let f, g be two meromorphic functions and $a \in \mathbb{C} \cup \{\infty\}$. We denote by $N_{1)}(r, a; f \mid g = b)$ and $N_{1)}(r, a; f \mid g \neq b)$ the counting functions of simple a-points of f which are the b-points of g and are not the b-points of g respectively.

In the paper, we denote by f and g two non-constant meromorphic functions defined in the open complex plane $\mathbb{C}$ unless otherwise stated.

2. Lemmas

In this section, we present some lemmas which are required in the sequel.

Lemma 2.1.[3] If f, g share $(0, 0)$, $(1, 0)$ and $(\infty, 0)$, then $T(r, f) \leq 3T(r, g) + S(r, f)$ and $T(r, g) \leq 3T(r, f) + S(r, g)$.

This shows that $S(r, f) = S(r, g)$ and we denote them by $S(r)$.

Lemma 2.2.[8] Let f, g share $(0, 1)$, $(1, m)$, (∞, k) and $f \not\equiv g$, where $(m-1)(mk-1) > (1+m)^2$. Then for $a = 0, 1, \infty$, $\overline{N}(r, a; f \mid \geq 2) + \overline{N}(r, a; g \mid \geq 2) = S(r)$.

Lemma 2.3.[9, 11] Let f, g share $(0, 1)$, $(1, m)$, (∞, k) and $f \not\equiv g$, where $(m-1)(mk-1) > (1+m)^2$. If f is not a bilinear transformation of g, then each of the following holds:

(i) $T(r, f) + T(r, g) = N(r, 0; f \mid \leq 1) + N(r, 1; f \mid \leq 1) + N(r, \infty; f \mid \leq 1) + N_0(r) + S(r)$,

(ii) $T(r, f) = N(r, 0; g' \mid \leq 1) + N_0(r) + S(r)$,

(iii) $T(r, g) = N(r, 0; f' \mid \leq 1) + N_0(r) + S(r)$,

(iv) $N_1(r) = S(r)$,

(v) $N_0(r, 0; g' \mid \geq 2) = S(r)$,

(vi) $N_0(r, 0; f' \mid \geq 2) = S(r)$,

(vii) $\overline{N}(r, 0; f' \mid \geq 2) = S(r)$,

(viii) $\overline{N}(r, 0; g' \mid \geq 2) = S(r)$,

(ix) $N(r, 0; f-g \mid \geq 2) = S(r)$,

(x) $N(r, 0; f-g \mid g = \infty) = S(r)$,

(xi) $N(r, 0; f-g \mid f = \infty) = S(r)$,

where $N_0(r)(N_1(r))$ denotes the counting function of those simple (multiple) zeros of $f-g$ which are not the zeros of $f(f-1)$ and $\frac{1}{f}$; also, $N_0(r, 0; g' \mid \geq 2)(N_0(r, 0; f' \mid \geq 2))$ is the counting function of those multiple zeros of $g'(f')$ which are not the zeros of $g(g-1)$ and so not of $f(f-1)$.

Lemma 2.4. Let f, g share $(0, 1)$, $(1, m)$, (∞, k) and $f \not\equiv g$, where $(m-1)(mk-1) > (1+m)^2$. If f is not a bilinear transformation of g then each of the following holds:

(i) $N(r, a; f \mid \geq 3) + N(r, a; g \mid \geq 3) = S(r)$,

(ii) $T(r, f) = N(r, a; f \mid \leq 2) + S(r)$,
(iii) $T(r, g) = N(r, a; g \mid \leq 2) + S(r)$.

Proof. By (v) and (vi) of Lemma 2.3, we get

$$N(r, a; f \mid \geq 3) + N(r, a; g \mid \geq 3) \leq 2N_0(r, 0; f' \mid \geq 2) + 2N_0(r, 0; g' \mid \geq 2) = S(r)$$

which is (i).

By the second fundamental theorem, Lemma 2.2, (i), (iii) and (vi) of Lemma 2.3, we get

$$\begin{aligned} 2T(r, f) &\leq 2N(r, a; f) + N(r, 0; f \mid \leq 1) + N(r, 1; f \mid \leq 1) + N(r, \infty; f \mid \leq 1) - N_0(r, 0; f' \mid \leq 1) + S(r) \\ &= N(r, a; f) + T(r, f) + T(r, g) - N_0(r) - N_0(r, 0; f' \mid \leq 1) + S(r) \\ &= N(r, a; f) + T(r, f) + N(r, 0; f' \mid \leq 1) - N_0(r, 0; f' \mid \leq 1) + S(r) \qquad (1) \end{aligned}$$

where $N_0(r, 0; f' \mid \leq 1)$ denotes the counting function of those simple zeros of f' which are not the zeros of

Now by Lemma 2.2, we get

$$\begin{aligned} N(r, 0; f' \mid \leq 1) &\leq N_0(r, 0; f' \mid \leq 1) + \overline{N}(r, 0; f \mid \geq 2) + \overline{N}(r, 1; f \mid \geq 2) \\ &= N_0(r, 0; f' \mid \leq 1) + S(r) \end{aligned}$$

and so $N(r, 0; f' \mid \leq 1) \leq N_0(r, 0; f' \mid \leq 1) + S(r)$. Hence from (1) and (i) of this lemma we get

$$T(r, f) = N(r, a; f \mid \leq 2) + S(r)$$

which is (ii). Similarly we can prove (iii). This proves the lemma.

Lemma 2.5. [10] Let f, g share $(0, 1)$, $(1, m)$, (∞, k) and $f \not\equiv g$, where $(m - 1)(mk - 1) > (1 + m)^2$. If $\alpha = \dfrac{f-1}{g-1}$ and $\beta = \dfrac{g}{f}$ then $\overline{N}(r, a; \alpha) + \overline{N}(r, a; \beta) = S(r)$ for $a = 0, \infty$.

Following lemma is a variant of Lemma 2.7[1] and Theorem 2.5[5].

Lemma 2.6. Let a_1, a_2, a_3 be distinct meromorphic functions such that $T(r, a_j) = S(r, f, g)$ for $j = 1, 2, 3$. Then

$$T(r, f) \leq \overline{N}(r, 0; f - a_1) + \overline{N}(r, 0; f - a_2) + \overline{N}(r, 0; f - a_3) + S(r; f, g)$$

where $S(r; f, g) = o\{T(r, f) + T(r, g)\}$ as $r \to \infty$ possibly outside a set of finite linear measure.

Lemma 2.7. Let f be a non-constant meromororphic function satisfying the Riccati dierential equation

$$f' = a + bf + cf^2 \qquad (2)$$

where $a, b, c (\not\equiv 0)$ are meromorphic functions such that $T(r, a) + T(r, b) + T(r, c) = S(r, f)$.

Further let ρ be a meromorphic function with

(i) If ρ satisfies (2) then $\overline{N}(r, 0; f - \rho) = S(r, f)$.

(ii) If ρ does not satisfy (2) then $\overline{N}(r, 0; f - \rho) = S(r, f)$.

(iii) If ρ does not satisfy (2) then $N(r, 0; f - \rho \mid \geq 2) = S(r, f)$.

Proof. Since (i) and (ii) are proved in Theorem 5.22[16], we prove only (iii).

Putting $f = h + \rho$ in (2) we get

$h' = \mu + (b + 2c\rho)\, h + ch^2$

where $\mu = -\rho' + a + c\rho^2 + b\rho$ and $T(r, \mu) = S(r, f)$.

Since ρ does not satisfy (2), we get $\mu \not\equiv 0$. Let z_0 be a zero of h with multiplicity p (≥ 2) which is not a pole of $b + 2c\rho$ and c. Then from above we see that z_0 is a zero of μ with multiplicity $p - 1$. Therefore

$$N(r, 0; f - \rho \mid \geq 2) = N(r, 0; h \mid \geq 2)$$

$$\leq 2T(r, \mu) + 2T(r, b + 2c\rho) + 2T(r, c)$$

$$= S(r, f)$$

This proves the lemma.

Following lemma is an easy consequence of Theorem 2[14] and the Valiron-Mohonko lemma.

Lemma 2.8. Let f and g be two nonconstant meromorphic functions sharing $(0, \infty)$, $(1, \infty)$, (∞, ∞). If f is not a bilinear transformation of g and $N(r, a; f \mid \leq 1) = S(r, f)$ for some a ($\neq 0, 1, \infty$), then $T(r, g) = T(r, f) + S(r)$.

Lemma 2.9. [12] Let f, g share $(0, 1)$, $(1, m)$, (∞, k) and $f \not\equiv g$, where $(m - 1)(mk - 1) > (1 + m)^2$. If $N_0(r) + N_1(r) \geq \lambda T(r, f) + S(r)$ for some $\lambda > \frac{1}{2}$ then f is a bilinear transformation of g and $N_0(r) + N_1(r) = T(r, f) + S(r) = T(r, g) + S(r)$.

Lemma 2.10. [10] Let f and g be distinct meromorphic functions sharing $(0, 0)$, $(1, 0)$ and $(\infty, 0)$. If f is a bilinear transformation of g, then f and g satisfy one of the following:

(i) $fg \equiv 1$,

(ii) $(f - 1)(g - 1) \equiv 1$,

(iii) $f + g \equiv 1$,

(iv) $f \equiv cg$,

(v) $(f - 1) \equiv c(g - 1)$,

(vi) $\{(c - 1) f + 1\}\{(c - 1) g - c\} + c \equiv 0$, where c ($\neq 0, 1, \infty$) is a constant.

Lemma 2.11. Let f and g be distinct meromorphic functions sharing $(0, 0)$, $(1, 0)$ and $(\infty, 0)$. Further suppose that f is a bilinear transformation of g and $\overline{E}_{1)}(a; f) \subset \overline{E}(b; g)$ and $\overline{E}_{1)}(b; g) \subset \overline{E}(a; f)$, where a, $b \notin \{0, 1, \infty\}$, then there exists a non-constant entire function λ such that f and g are one of the following forms :

(i) $f = ae^{\lambda}$ and $g = be^{-\lambda}$, where $ab = 1$.

(ii) $f = 1 + ae^{\lambda}$ and $g = 1 + (1 - \frac{1}{b})e^{-\lambda}$. where $ab = a + b$.

(iii) $f = \dfrac{a}{a + e^{\lambda}}$ and $g = \dfrac{e^{\lambda}}{1 - b + e^{\lambda}}$, where $a + b = 1$.

(iv) $f = \dfrac{e^{\lambda} - a}{e^{\lambda} - 1}$ and $g = \dfrac{be^{\lambda} - 1}{e^{\lambda} - 1}$, where $ab = 1$.

(v) $f=\dfrac{be^{\lambda}-a}{be^{\lambda}-b}$ and $g=\dfrac{be^{\lambda}-a}{ae^{\lambda}-a}$, where $a\neq b$.

(vi) $f=\dfrac{a}{1-e^{\lambda}}$ and $g=\dfrac{be^{\lambda}}{e^{\lambda}-1}$, where $ab=a+b$.

(vii) $f=\dfrac{b-a}{(b-1)(1-e^{\lambda})}$ and $g=\dfrac{(b-a)e^{\lambda}}{(a-1)(1-e^{\lambda})}$, where $a\neq b$.

(viii) $f=a+e^{\lambda}$ and $g=b(1+\dfrac{1-b}{e^{\lambda}})$, where $a+b=1$.

(ix) $f=e^{\lambda}-\dfrac{a(b-1)}{a-b}$ and $g=\dfrac{b(a-1)}{a-b}\left\{1-\dfrac{a(b-1)}{(a-b)e^{-\lambda}}\right\}$, where $a\neq b$.

Also f, g share $(0,\infty)$, $(1,\infty)$, (∞,∞) and $f-a$, $g-b$ share $(0,\infty)$.

Proof. Clearly, f and g satisfy one of the relations given in Lemma 2.10.

Let $fg\equiv 1$. Then f and g do not assume the values 0 and ∞. Hence there exists a non-constant entire function λ such that $f=ae^{\lambda}$ and $g=\dfrac{1}{a}e^{-\lambda}$. If $f-a$ has no simple zero then $\Theta(a;f)\geq\dfrac{1}{2}$, which is impossible. Hence $f-a$ must have simple zeros. Similarly, $g-b$ must have simple zeros. So by the given condition $ab=1$. Therefore $f=ae^{\lambda}$ and $g=be^{\lambda}$, where $ab=1$. This is the possibility (i).

Suppose that $(f-1)(g-1)\equiv 1$. Then f and g do not assume the values 1 and ∞. Hence there exists a non-constant entire function λ such that $f=1+ae^{\lambda}$ and $g=1+\dfrac{1}{a}e^{-\lambda}$. Since $f-a$ and $g-b$ must have simple zeros, by the given condition we get $ab=a+b$. Therefore $f=1+ae^{\lambda}$ and $g=1+(1-\dfrac{1}{b})e^{-\lambda}$, where $ab=a+b$. This is the possibility (ii).

Suppose that $f+g\equiv 1$. Then f and g do not assume the values 0 and 1. So there exists a non-constant entire function λ such that $f=\dfrac{a}{a+e^{\lambda}}$ and $g=\dfrac{e^{\lambda}}{a+e^{\lambda}}$. Since $f-a$ and $g-b$ must have simple zeros, we get $a+b=1$. Therefore $f=\dfrac{a}{a+e^{\lambda}}$ and $g=\dfrac{e^{\lambda}}{1-b+e^{\lambda}}$, where $a+b=1$. This is the possibility (iii).

Suppose that $f=cg$. Then f does not assume the values 1 and c. Hence there exists a non-constant entire function λ such that $f=\dfrac{e^{\lambda}-c}{e^{\lambda}-1}$ and $g=\dfrac{e^{\lambda}-c}{ae^{\lambda}-a}$.

Suppose that $f-a$ has no simple zero. Then $\Theta(a;f)\geq\dfrac{1}{2}$ and so $c=a$. Hence $f=\dfrac{e^{\lambda}-a}{e^{\lambda}-1}$ and $g=\dfrac{e^{\lambda}-a}{ae^{\lambda}-a}$. If z_0 is a simple zero of $g-b$, by the given condition we get $b=g(z_0)=\dfrac{1}{a}f(z_0)=1$, which is impossible. So $g-b$ has no simple zero and by the second fundamental theorem, we get $bc=1$. Therefore $f=\dfrac{e^{\lambda}-a}{e^{\lambda}-1}$ and $g=\dfrac{be^{\lambda}-1}{e^{\lambda}-1}$, where $ab=1$. This is the possibility (iv).

Suppose that $f-a$ has simple zeros. Then $g-b$ has zeros and we see that $bc=a$. Since $c\neq 1$, where we have $a\neq b$. Therefore from above we get $f=\dfrac{be^{\lambda}-a}{be^{\lambda}-b}$ and $g=\dfrac{be^{\lambda}-a}{ae^{\lambda}-a}$, where $a\neq b$. This is the possibility (v).

Suppose that $f-1\equiv c(g-1)$. Then f does not assume the values 0 and $1-c$. So there exists a non-constant entire function λ such that $f=\dfrac{1-c}{1-e^{\lambda}}$ and $g=\dfrac{(1-c)e^{\lambda}}{c(1-e^{\lambda})}$.

Suppose that $f-a$ has no simple zero. Then $\Theta(a;f)\geq\dfrac{1}{2}$ and so $c=1-a$. Hence $f=\dfrac{a}{1-e^{\lambda}}$ and $g=\dfrac{ae^{\lambda}}{(1-a)(1-e^{\lambda})}$. If

$g - b$ has simple zeros then $a - 1 \equiv c(b - 1)$ and so $b = 0$, which is impossible. Hence $g - b$ has no simple zero and so $bc = c - 1$. Therefore $f = \frac{a}{1-e^{\lambda}}$ and $g = \frac{be^{\lambda}}{e^{\lambda}-1}$, where $ab = a + b$. This is the possibility (vi).

Suppose that $f - a$ has simple zeros. Then $g - b$ has zeros and we get $c(b - 1) = a - 1$ and so $a \neq b$. Therefore $f = \frac{b-a}{(b-1)(1-e^{\lambda})}$ and $g = \frac{(b-a)e^{\lambda}}{(a-1)(1-e^{\lambda})}$, where $a \neq b$. This is the possibility (vii).

Suppose that $\{(c - 1) f + 1\}\{(c - 1)g - c\} + c \equiv 0$. Then f does not assume the values ∞ and $\frac{1}{1-c}$. So there exists a non-constant entire function λ such that $f = \frac{1}{1-c} + e^{\lambda}$ and $g = \frac{c}{c-1}\left\{1 + \frac{1}{(1-c)e^{\lambda}}\right\}$.

Suppose that $f - a$ does not have any simple zero. Then $\Theta(a; f) \geq \frac{1}{2}$ and so $a(1 - c) = 1$. Hence $f = a + e^{\lambda}$ and $g = (1-a)(1+\frac{a}{e^{\lambda}})$. Hence $g - b$ has no simple zero and so $b(c - 1) = c$. Therefore $f = a + e^{\lambda}$ and $g = b(1+\frac{1-b}{e^{\lambda}})$, where $a + b = 1$. This is the possibility (viii).

Suppose that $f - a$ has simple zeros. Then $g - b$ has zeros and so $ca(b - 1) = b(a - 1)$. Since $c \neq 1$, we get $a \neq b$. Therefore

$$f = e^{\lambda} - \frac{a(b-1)}{a-b} \text{ and } g = \frac{b(a-1)}{a-b}\left\{1 - \frac{a(b-1)}{(a-b)e^{\lambda}}\right\}$$

where $a \neq b$. This is the possibility (ix).

Since f and g are one of (i)-(ix), we can easily verify that f and g share $(0, \infty)$, $(1, \infty)$, (∞, ∞) and $f - a$, $g - b$ share $(0, \infty)$. This proves the lemma.

Lemma 2.12. Let f, g share $(0, 1)$, $(1, m)$, (∞, k) and $f \not\equiv g$, where $(m - 1)(mk - 1) > (1 + m)^2$. Then $T(r, \frac{\alpha'}{\alpha}) + T(r, \frac{\beta'}{\beta}) = S(r)$, where α, β are defined as in Lemma 2.5.

Proof. Since $\alpha = \frac{f-1}{g-1}$ and $\beta = \frac{g}{f}$, we get $T(r, \alpha) = O\{T(r, f) + T(r, g)\}$ and $T(r, \beta) = O\{T(r, f) + T(r, g)\}$.

So by Lemma 2.1, we see that $S(r, \alpha)$ and $S(r, \alpha)$ are replaceable by $S(r)$. Now by Lemma 2.5 we get

$$\begin{aligned} T(r, \frac{\alpha'}{\alpha}) &= N(r, \frac{\alpha'}{\alpha}) = m(r, \frac{\alpha'}{\alpha}) \\ &= \overline{N}(r, 0; \alpha) + \overline{N}(r, \infty; \alpha) + S(r, \alpha) \\ &= S(r) \end{aligned}$$

Similarly $T(r, \frac{\beta'}{\beta}) = S(r)$. This proves the lemma.

3. Proof of theorem 1.1

Proof. We show that f is a bilinear transformation of g and so the theorem follows from Lemma 2.11.

First, we suppose that $a_1 = 0$, $a_2 = 1$ and $a_3 = \infty$. We suppose further that f is not a bilinear transformation of g. Then α, β and $\alpha\beta$ are non-constant. We now consider the following cases.

Case I. Let $a = b$. We put $\phi = \frac{f'(f-a)}{f(f-1)} - \frac{g'(g-a)}{g(g-1)}$. Suppose that $\phi \not\equiv 0$. Since $\phi = a\frac{\beta'}{\beta} + (1-a)\frac{\alpha'}{\alpha}$, by Lemma 2.12 we get $T(r, \phi) = S(r)$. By the given condition, we see that $N(r, a; f| \leq 2) \leq 2N(r, 0; \phi) = S(r)$, which contradicts Lemma 2.4 (ii). Therefore $\phi \equiv 0$ and so

$$\frac{f'(f-a)}{f(f-1)} \equiv \frac{g'(g-a)}{g(g-1)} \tag{3}$$

From (3) we see that a double zero of $f - a$ is a common zero of f' and g' and so it is a zero of $\frac{\beta'}{\beta}$. Therefore by **Lemma 2.12** we get

$$N(r,a;f|=2) \le 2N(r,0;\frac{\beta'}{\beta}) = S(r)$$

where by $N(r, a; f| = 2)$ we denote the counting function of double zeros of $f - a$, counted with multiplicities.

Similarly, we get $N(r, a; g| = 2) = S(r)$. From Lemma 2.4 we see in view of the hypotheses that

$$N_0(r) + N_1(r) \ge N(r, a; f \mid \le 1) + S(r) = T(r, f) + S(r)$$

which contradicts Lemma 2.9.

Case II. Let $a \ne b$. We now consider the following subcases.

Subcase (i). Let $N(r, b; g| \ge 2) \ne S(r)$. We define ϕ as in Case I. Since a double zero of $g - b$ is a zero of $f - a$, if $\phi \not\equiv 0$ then by (i) of Lemma 2.4 and Lemma 2.12 we get

$$N(r,a;g|=2) \le 2N(r,0;\frac{\beta'}{\beta}) = S(r)$$

which is a contradiction. Therefore $\phi \equiv 0$ and (3) holds.

From (3) we see that a double zero of $f - a$, is a common zero of f' and g' and so it is a zero of $\frac{\beta'}{\beta}$. Similarly from (3) we see that a double zero of $g - a$ is a zero of $\frac{\beta'}{\beta}$. Therefore from Lemma 2.12 we get

$$N(r,a;f|=2) \le 2N(r,0;\frac{\beta'}{\beta}) = S(r) \tag{4}$$

and

$$N(r,a;g|=2) \le 2N(r,0;\frac{\beta'}{\beta}) = S(r) \tag{5}$$

Since $\overline{E}_{2)}(b;g) \subset \overline{E}(a;f)$, it follows from (3) that $g - b$ has no simple zero. Since $\overline{E}_{2)}(b;g) \subset \overline{E}(a;f)$ and $\overline{E}_{2)}(a;f) \subset \overline{E}(b;g)$, it follows from above and (4) in view of (ii) of Lemma 2.4

$$N(r, b; g| = 2) = 2N(r, a; f| \le 1) + S(r)$$

Therefore by Lemma 2.4 we obtain

$$T(r, g) = 2T(r, f) + S(r) \tag{6}$$

From (3) we can verify that f and g share $(0, \infty)$, $(1, \infty)$, (∞, ∞). Since $g - b$ has no simple zero, by Lemma 2.8 we see that $T(r, g) = T(r, f) + S(r)$, which contradicts (6).

Subcase (ii). Let $N(r, a; f| \ge 2) \ne S(r)$. Supposing

$$\psi = \frac{f'(f-b)}{f(f-1)} - \frac{g'(g-b)}{g(g-1)}$$

and proceeding as Subcase (i), we arrive at a contradiction.

Subcase (iii). Let $N(r, a; f \mid\geq 2) + N(r, b; g \mid\geq 2) = S(r)$. We note that $f = \dfrac{1-\alpha}{1-\alpha\beta}$ and $g = \dfrac{(1-\alpha)\beta}{1-\alpha\beta}$.

We put $F = (f - a)(1 - \alpha\beta) = a\alpha\beta - \alpha + 1 - a$ and $w = \dfrac{F'}{F}$. Since $1-\alpha\beta = \dfrac{g-f}{f(g-1)}$, we get $F = (f-a)\dfrac{g-f}{f(g-1)}$. Since by Lemma 2.5 $\overline{N}(r, \infty; F) = S(r)$ and w has only simple poles, we get

$$T(r, w) = m(r, w) + N(r, w) = \overline{N}(r, 0; F) + S(r) \tag{7}$$

By Lemma 2.2 and (ix) and (xi) of Lemma 2.3 we get

$$\begin{aligned}\overline{N}(r, 0; F \mid \geq 2) &\leq N(r, a; f \mid \geq 2) + N(r, 0; f - g \mid \geq 2) + \overline{N}(r, \infty; f \mid \geq 2) + N(r, 0; f - g \mid f = \infty)\\ &= S(r)\end{aligned} \tag{8}$$

Hence from (7) and (8) we get in view of (ix) of Lemma 2.3

$$\begin{aligned}T(r, w) &= N(r, 0; F \mid \leq 1) + S(r)\\ &= N(r,a;f\mid\leq 1)+N_0(r)+N_2(r)+S(r)\end{aligned} \tag{9}$$

where $N_2(r)$ is the counting function of those simple poles of f which are non-zero regular points of $f - g$.

From the definitions of α and β we get

$$\left\{g-\frac{\alpha'\beta}{(\alpha\beta)'}\right\}\left(\frac{\alpha'}{\alpha}+\frac{\beta'}{\beta}\right)=\frac{f'(g-f)}{f(f-1)} \tag{10}$$

From (10) we see that a simple pole of f which is a non-zero regular point of $f - g$ is a regular point of $\left\{g-\dfrac{\alpha'\beta}{(\alpha\beta)'}\right\}\left(\dfrac{\alpha'}{\alpha}+\dfrac{\beta'}{\beta}\right)$. Hence it is either a pole of $\dfrac{\alpha'\beta}{(\alpha\beta)'}$ or a zero of $\dfrac{\alpha'}{\alpha}+\dfrac{\beta'}{\beta}$. Therefore by Lemma 2.12 and the first fundamental theorem we get

$$\begin{aligned}N_2(r) &\leq T\left(r,\frac{\alpha'\beta}{(\alpha\beta)'}\right)+T\left(r,\frac{\alpha'}{\alpha}+\frac{\beta'}{\beta}\right)\\ &\leq T\left(r,\frac{\alpha'}{\alpha}\right)+T\left(r,\frac{\beta'}{\beta}\right)+T\left(r,\frac{1}{1+\frac{\alpha'}{\alpha}\cdot\frac{\beta'}{\beta}}\right)+o(1)\\ &\leq 2T\left(r,\frac{\alpha'}{\alpha}\right)+2T\left(r,\frac{\beta'}{\beta}\right)+o(1)\\ &= S(r)\end{aligned}$$

So from (9) we get

$$T(r, w) = N(r, a; f \mid\leq 1) + N_0(r) + S(r) \tag{11}$$

By (ii) of Lemma 2.4 we get from (11)

$$T(r, w) = T(r, f) + N_0(r) + S(r) \tag{12}$$

Since $\overline{E}_{2)}(a;f)\subset \overline{E}(b;g)$ and $\overline{E}_{2)}(b;g)\subset \overline{E}(a;f)$, we obtain from (ii) and (iii) of Lemma 2.4

$$T(r,f)=T(r,g)+S(r) \tag{13}$$

Let

$$T_1=\frac{a-1}{b-1}(\gamma-b\delta)$$

$$T_2=\frac{a-1}{2(b-1)}\left\{\gamma'+\gamma^2-b(\delta'+\delta^2)\right\}$$

and $T_3=\dfrac{a-1}{6(b-1)}\left\{\gamma''+3\gamma\gamma'+\gamma^3-b(\delta''+3\delta\delta'+\delta^3)\right\}$,

where $\gamma=\dfrac{\alpha'}{\alpha}$ and $\delta=\dfrac{(\alpha\beta)'}{\alpha\beta}=\dfrac{\alpha'}{\alpha}+\dfrac{\beta'}{\beta}$. Using Lemma 2.12 we can verify that $T(r,\gamma)=S(r)$ and $T(r,\delta)=S(r)$.

If $T_1\equiv 0$, from (10) we see that

$$(g-b)\delta=\frac{f'(g-f)}{f(f-1)} \tag{14}$$

Since $\overline{E}_{2)}(a;f)\subset\overline{E}_{2)}(b;g)$ it follows from (14) that a simple zero of $f-a$, which is neither a zero nor a pole of δ, is a zero of $g-b$ and so a zero of f'. Hence $N(r,a;f\mid\le 1)=S(r)$, which contradicts (ii) of Lemma 2.4. Therefore $T_1\not\equiv 0$.

Let z_0 be a simple zero of $f-a$ and $T_1(z_0)\neq 0$. Then $g(z_0)=b$ and so $\alpha=\dfrac{a-1}{b-1}$ and $\beta(z_0)=\dfrac{b}{a}$. Expanding F around z_0 in Taylor's series we get

$$-F(z)=T_1(z_0)(z-z_0)+T_2(z_0)(z-z_0)^2+T_3(z_0)(z-z_0)^3+O((z-z_0)^4) \tag{15}$$

Hence in some neighbourhood of z_0 we get

$$w(z)=\frac{1}{z-z_0}+\frac{B(z_0)}{2}+C(z_0)(z-z_0)+O((z-z_0)^2) \tag{16}$$

where $B=\dfrac{2T_2}{T_1}$ and $C=\dfrac{2T_3}{T_1}-\left(\dfrac{T_2}{T_1}\right)^2$.

We put

$$H=w'+w^2-Bw-A \tag{17}$$

where $A=3C-\dfrac{B^2}{4}-B'$.

$$T(r,A)+T(r,B)+T(r,C)=S(r)$$

Clearly, $T(r,A)+T(r,B)+T(r,C)=S(r)$ and since $w=\dfrac{F'}{F}$ and $F=(f-a)\dfrac{g-f}{f(g-1)}$, by Lemma 2.1 and (12), we get $S(r,w)=S(r)$.

It is now easy to verify that z_0 is a zero of H. Let $H\not\equiv 0$. Then

$$N(r,a;f\mid\le 1)\le N(r,0;H)\le T(r,H)+O(1)\le N(r,H)+S(r) \tag{18}$$

By (ii) of Lemma 2.4 and (18) we obtain

$$T(r, f) \leq N(r, H) + S(r) \tag{19}$$

Let z_1 be a pole of F. Then z_1 is a simple pole of w. So if z_1 is not a pole of A and B then z_1 is at most a double pole of H. Hence by Lemma 2.5 we get

$$N(r, \infty; H \mid F = \infty) \leq 2\overline{N}(r, \infty; F) + S(r) = S(r) \tag{20}$$

Let z_2 be a multiple zero of F. Then z_2 is a simple pole of w. So if z_2 is not a pole of A and B then z_2 is a pole of H of multiplicity at most two. Hence by (8) we get

$$N(r, \infty; H \mid F = 0, \geq 2) \leq 2\overline{N}(r, 0; F \mid \geq 2) + S(r) = S(r) \tag{21}$$

Let z_3 be a simple zero of F which is not a pole of A and B. Then in some neighbourhood of z_3 we get $f(z) = (z - z_3)\phi_0(z)$, where ϕ_0 is analytic at z_3 and $\phi_0(z_3) \neq 0$. Hence, in some neighborhood of z_3 we obtain

$$H(z) = \left(\frac{2\phi_0'}{\phi_0} - B\right)\frac{1}{z - z_3} + \psi_0,$$

where $\psi_0 = \left(\frac{\phi_0'}{\phi_0}\right)' + \left(\frac{\phi_0'}{\phi_0}\right)^2 - \frac{B\phi_0'}{\phi_0} - A$. This shows that z_3 is at most a simple pole of H.

Since a simple zero of $f - a$ is a zero of H, $N(r, 0; F \mid f = t) \leq N(r, 0; f - g \mid \geq 2)$ for $t = 0, 1$ and $F = \frac{(f-a)(g-f)}{f(g-1)}$, we get from (20) and (21) in view of (ix) of Lemma 2.3

$$\begin{aligned} N(r, H) &= N(r, \infty; H \mid F = \infty) + N(r, \infty, H \mid F = 0) + S(r) \\ &\leq N(r, 0; F \mid \leq 1) - N(r, a, f \mid \leq 0) + S(r) \\ &= N_0(r) + N_2(r) + S(r) \\ &= N_0(r) + S(r) \end{aligned} \tag{22}$$

From (19) and (22) we obtain $T(r, f) \leq N_0(r) + S(r)$, which by (iv) of Lemma 2.3 contradicts Lemma 2.9.

Let $H \equiv 0$, so that w satisfies the Riccati differential equation

$$w' = A + Bw - w^2 \tag{23}$$

From the definitions of F and w we can easily deduce the following

$$F(w - \delta) = (\delta - \gamma)(\alpha - \phi_1) \tag{24}$$

$$F(w - \gamma) = a(\delta - \gamma)(\alpha\beta - \phi_2) \tag{25}$$

$$Fw = a\delta\alpha(\beta - \phi_3) \tag{26}$$

where $\phi_1 = \frac{(1-a)\delta}{\delta - \gamma}, \phi_2 = \frac{(a-1)}{a(\delta - \gamma)}$ and $\phi_3 = \frac{\gamma}{a\delta}$.

Since α, β and $\alpha\beta$ are non-constant, we see that that $\phi_j \not\equiv 0, \infty$ for $j = 1, 2, 3$. Also, since $T(r, \phi_1) = S(r) = S(r, \alpha, \beta)$, we get by Lemma 2.5 and Lemma 2.6

$$T(r,\alpha) \le \overline{N}(r, 0; \alpha) + \overline{N}(r, \infty; \alpha) + \overline{N}(r, 0; \alpha - \phi_1) + S(r; \alpha, \beta)$$

$$= \overline{N}(r, 0; \alpha - \phi_1) + S(r)$$

and so

$$T(r, \alpha) = \overline{N}(r, 0; \alpha - \phi_1) + S(r) = N(r, 0; \alpha - \phi_1) + S(r) \tag{27}$$

From (24) and (8) we get

$$\overline{N}(r, 0; w - \delta) \le \overline{N}(r, 0; \alpha - \phi_1) + \overline{N}(r, 0; \delta - \gamma) + S(r)$$

$$= \overline{N}(r, 0; \alpha - \phi_1) + S(r)$$

$$= \overline{N}(r, 0; F(w - \delta)) + S(r)$$

$$\le \overline{N}(r, 0; w - \delta) + \overline{N}(r, 0; F \mid \ge 2) + S(r)$$

$$\le \overline{N}(r, 0; w - \delta) + S(r)$$

and so from (27) we obtain

$$T(r, \alpha) = \overline{N}(r, 0; w - \delta) + S(r) \tag{28}$$

By Lemma 2.5 and the second fundamental theorem, we get

$$T(r, \alpha) = \overline{N}(r, 1; \alpha) + S(r) \tag{29}$$

Since $\alpha - 1 = \dfrac{f-g}{g-1}$ and by (ix) of Lemma 2.3

$$\overline{N}(r, 0; \frac{f-g}{g-1} \mid g = 1) \le N(r, 0; f - g \mid \ge 2) = S(r)$$

we get by Lemma 2.2 and (iv)and (x) of Lemma 2.3

$$\overline{N}(r, 1; \alpha) = N_0(r) + N(r, 0; f \mid \le 1) + S(r) \tag{30}$$

because $N_2(r) = S(r)$.

Therefore, from (28)-(30) we obtain

$$\overline{N}(r, 0; w - \delta) = N(r, 0; f \mid \le 1) + N_0(r) + S(r) \tag{31}$$

In a similar manner using (25) and (26), we get

$$\overline{N}(r, 0; w - \gamma) = N(r, \infty; f \mid \le 1) + N_0(r) + S(r) \tag{32}$$

$$\overline{N}(r, 0; w) = N(r, 1; f \mid \le 1) + N_0(r) + S(r) \tag{33}$$

$$\overline{N}(r, 1; \alpha\beta) = N(r, \infty; f \mid \le 1) + N_0(r) + S(r) \tag{34}$$

$$\overline{N}(r, 1; \beta) = N(r, 1; f \mid \le 1) + N_0(r) + S(r) \tag{35}$$

$$T(r, \alpha\beta) = \overline{N}(r, 1; \alpha\beta) + S(r) \tag{36}$$

and

$$T(r, \alpha\beta) = \overline{N}(r, 1; \beta) + S(r) \tag{37}$$

If $w = 0$ is a solution of (23) then by (i) of Lemma 2.7, (33), (35) and (37), we get $T(r, \beta) = S(r)$. So

$$N(r, a; f \mid\leq 1) \leq N(r, \frac{b}{a}; \beta) \leq T(r, \beta) + O(1) = S(r)$$

which contradicts (ii) of Lemma 2.4.

If $w = \gamma$ is a solution of (23) then by (i) of Lemma 2.7, (32), (34) and (36), we get $T(r, \alpha\beta) = S(r)$. So

$$N(r, a; f \mid\leq 1) \leq N(r, \frac{b(a-1)}{a(b-1)}; \alpha\beta) \leq T(r, \alpha\beta) + O(1) = S(r)$$

which contradicts (ii) of Lemma 2.4.

If $w = \delta$ is a solution of (23) then by (i) of Lemma 2.7, (29), (30) and (31), we get $T(r, \alpha) = S(r)$. So

$$N(r, a; f \mid\leq 1) \leq N(r, \frac{a-1}{b-1}; \alpha) \leq T(r, \alpha) + O(1) = S(r)$$

which contradicts (ii) of Lemma 2.4.

Therefore $w = 0$, γ and $w = \delta$ are not solutions of (23). Now by (ii) and (iii) of

Lemma 2.7. (12), (31)-(33) we obtain

$$T(r, f) = N(r, 0; f \mid\leq 1) + S(r) \tag{38}$$

$$T(r, f) = N(r, \infty; f \mid\leq 1) + S(r) \tag{39}$$

and

$$T(r, f) = N(r, 1; f \mid\leq 1) + S(r) \tag{40}$$

Now by (i) of Lemma 2.3, (13) and (38)-(40), we get

$$\begin{aligned} 3T(r, f) &= N(r, 0; f \mid\leq 1) + N(r, 1; f \mid\leq 1) + N(r, \infty; f \mid\leq 1) + S(r) \\ &= T(r, f) + T(r, g) - N_0(r) + S(r) \\ &= 2T(r, f) - N_0(r) + S(r) \end{aligned}$$

and so $T(r, f) + N_0(r) = S(r)$, which is a contradiction.

Therefore f is a bilinear transformation of g.

Let $a_1 = 1$, $a_2 = 0$ and $a_3 = \infty$. We put $f_1 = 1 - f$ and $g_1 = 1 - g$. Then f_1, g_1 share $(0, 1)$, $(1, m)$, (∞, k) and $\overline{E}_{2)}(1-a; f_1) \subset \overline{E}(1-b; g_1)$ and $\overline{E}_{2)}(1-b; g_1) \subset \overline{E}(1-a; f_1)$. So f_1 is a bilinear transformation of g_1 and so f is a bilinear transformation of g.

Let $a_1 = \infty$, $a_2 = 1$ and $a_3 = 0$. We put $f_2 = \frac{1}{f}$ and $g_2 = \frac{1}{g}$. Then f_2, g_2 share $(0, 1)$, $(1, m)$, (∞, k) and $\overline{E}_{2)}(\frac{1}{a}; f_2) \subset \overline{E}(\frac{1}{b}; g_2)$ and $\overline{E}_{2)}(\frac{1}{b}; g_2) \subset \overline{E}(\frac{1}{a}; f_2)$. So f_2 is a bilinear transformation of g_2 and so f is a bilinear transformation of g.

Since m and k are interchangeable, we need not consider the other permutations of a_1, a_2 and a_3. This proves the theorem.

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