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Research Article

K^{th} Optimal Solution of the Assignment Problem by Hungarian-Branching Hybrid Method

Elias Munapo[1*], **Santosh Kumar**[2]

[1]Department of Business Statistics and Operations Research, North West University, Mafikeng, South Africa
[2]Department of Mathematical and Geospatial Sciences, RMIT University, Melbourne, VIC, 3000, Australia
E-mail: emunapo@gmail.com

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Abstract: In real-life situations, in addition to an optimal solution, one may be required to find the K^{th} best solution, $K \geq 2$, which usually is a difficult and computationally demanding problem. In this short communication, we develop a method to find the K^{th} best solution for an assignment problem by using the linear integer programming approach. This is done by generating an objective cut and adding it to the current optimal tableau and solve until the desired K^{th} optimal solution is obtained. The proposed method has the advantage that only one problem is solved at every stage or iteration. The assignment problem has real life application in logistics and transportation, workforce scheduling, resource allocation, project assignment, matching algorithms and sports scheduling.

***Keywords*:** optimal solution, K^{th} best solution, assignment model, linear programming

MSC: 90C05

Nomenclature

COT	Continuous Optimal Table
OA	Optimal Allocation
NBVR	Non-Basic Variable Rule
LP	Linear Programming
OC	Objective Cut
CPLEX	Complex Linear Programming Expert

1. Introduction

A need for the K^{th} best, ($K \geq 2$) arises in many different situations, for example, when for some reason, implementation of the optimal solution is considered inappropriate by the decision makers, and they may be interested in getting more information in the form of ordered optimal solutions. A mathematical need for the K^{th} best solution was realized by [1], when they developed a method for analysis of the extreme point mathematical programming problem,

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which was developed by [2-4]. The method by [1] required the K^{th} best solution of a linear program. An ordered K^{th} best optimal solution is defined to be the best after excluding (K-1) best solutions, ($K \geq 2$). In context of an integer programming model, a method to find the K^{th} best integer solution was developed by [5, 6]. This integer programming approach was used by [7] to find non-dominated integer points in a bi-objective integer program. Ordered optimal solutions were used by [8] for determination of the set of non-dominated points in a multi-objective assignment problem. Al-Hasani used the random search method for finding ordered optimal solutions, which in context of mathematical programming was developed by [9, 10] for integer and mixed integer programming models. [11] applied these ideas to solve the extreme point mathematical programming problem, where ordered optimal solutions are required. For more details about the random search method, see Chapter 6 [12].

Similarly, a need for the K^{th} best solution has been realized by many other researchers and in response, they have developed many different approaches. For example, [13] proposed a method for ranking all the assignments in the order of increasing costs. In an assignment problem, [13] obtained the second best by solving n assignment problems by assigning a very high cost to one at a time of the optimal assignments and solved these n different problems. The best of all these n solutions was taken as the 2nd best solution for that assignment problem. [12] resolved [13] example by making an assignment at the least cost element and identified zero cost solution in the remaining $(n-1)\times(n-1)$ assignment problem. They reached the same solution as was obtained by [13].

Determination of the best, in general, is a difficult problem as there is no well-defined theory that exists to find or to identify such a solution. In this paper we consider an assignment problem and develop a linear-integer programming approach to find ordered optimal solutions. These solutions can be obtained sequentially to develop the objective constraint. The method has been illustrated for an assignment problem and 3 ordered optimal solutions have been obtained.

The rest of the paper is structured as follows. The statement of the problem is given in Section 2, the properties used in this paper are explained in Section 3, the proposed algorithm in Section 4, a numerical illustration in Section 5 and the concluding remarks in Section 6.

2. Statement of the problem

Let an assignment problem be as given by Table 1.

Table 1. Assignment problem in general

						Source
	C_{11}	C_{12}	C_{13}	…	C_{1n}	1
	C_{21}	C_{22}	C_{23}	…	C_{2n}	1
	C_{31}	C_{32}	C_{33}	…	C_{3n}	1
	…	…	…	…	…	…
	C_{m1}	C_{m2}	C_{m3}	…	C_{mn}	1
Demand	1	1	1	…	1	D

Where c_{ij} is the cost of the assigning the i^{th} task to the person. j^{th} This is a balanced problem, i.e., number of rows (n) is equal to the number of columns (m) i.e., $D = n = m$. The objective in this problem is to minimize the total assignment cost.

2.1 *Linear Programming (LP) formulation*

The linear programming formulation of the assignment problem is given in (1).
Minimize

$$Z_o = c_{11}z_{11} + c_{12}z_{12} + c_{13}z_{13} + \ldots + c_{1n}z_{1n} + \ldots + c_{m1}z_{m1} + c_{m2}z_{m2} + c_{m3}z_{m3} + \ldots + c_{mn}z_{mn},$$

such that:

$$\begin{aligned} & z_{11} + z_{12} + z_{13} + \ldots + z_{1n} = 1, \\ & z_{21} + z_{22} + z_{23} + \ldots + z_{2n} = 1, \\ & z_{31} + z_{32} + z_{33} + \ldots + z_{3n} = 1, \\ & \ldots \\ & z_{m1} + z_{m2} + z_{m3} + \ldots + z_{mn} = 1, \\ & z_{11} + z_{21} + z_{31} + \ldots + z_{n1} = 1, \\ & z_{21} + z_{22} + z_{23} + \ldots + z_{2n} = 1, \\ & z_{31} + z_{32} + z_{33} + \ldots + z_{3n} = 1, \\ & \ldots \\ & z_{1n} + z_{2n} + z_{3n} + \ldots + z_{mn} = 1. \end{aligned} \tag{1}$$

Where $\text{z}_j = 0$ or 1.

$$\text{z}_{ij} = 0 \text{ or } 1 \text{ for } \forall (ij)$$

2.2 *Optimal simplex table*

Solving the problem in (1) using the simplex method we get Table 2.

Table 2. LP continuous optimal tableau

	z_1	z_2	$\ldots$	z_n	s_1	s_2	$\ldots$	s_k	rhs
z_0	0	0	$\ldots$	0	ω_1	ω_2	$\ldots$	ω_k	R_0
z_1	1	0	$\ldots$	0	ω_{11}	ω_{12}	$\ldots$	ω_{1k}	1
z_2	0	1	$\ldots$	0	ω_{21}	ω_{22}	$\ldots$	ω_{2k}	1
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
z_n	0	0	$\ldots$	1	ω_{n1}	ω_{n2}	$\ldots$	ω_{nk}	1
s_1	1	0	$\ldots$	0	α_{11}	α_{12}	$\ldots$	α_{1k}	0
s_2	0	1	$\ldots$	0	α_{21}	α_{22}	$\ldots$	α_{2k}	0
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
s_m	0	0	$\ldots$	1	α_{m1}	α_{m2}	$\ldots$	α_{mk}	0

Where $z_j = 1$ is the basic variable and $s_i = 0$ is the non-basic variable. The quantitiesare Z_0, ω_{ij}, α_{ij}, R constants. Let the current continuous optimal table be given by

$$COT.$$

Let the Optimal Allocation (OA) be as given by Table 3.

Table 3. Optimal allocation

						Source
	C_{11}	C_{12}	C_{13}	…	C_{1n}	1
	C_{21}	C_{22}	C_{23}	…	C_{2n}	1
	C_{31}	C_{32}	C_{33}	…	C_{3n}	1
	…	…	…	…	…	…
	C_{m1}	C_{m2}	C_{m3}	…	C_{mn}	1
Demand	1	1	1	…	1	D

Suppose the optimal allocation is given in Table 3 then the basic variables are given in (2).

$$z_{12} = \ldots = z_{2n} = z_{33} = \ldots = z_{m1} = 1 \tag{2}$$

3. Properties

3.1 *Property one*

In Table 3 changing any optimal allocation c_{33} will result in a change of another optimal allocation c_{m1} as shown in Table 4.

Table 4. Property one

						Source
	c_{11}	c_{12}	c_{13}	…	c_{1n}	1
	c_{21}	c_{22}	c_{23}	…	c_{2n}	1
	c_{31}	c_{32}	c_{33}	…	c_{3n}	1
	…	$\Uparrow$	$\Downarrow$	…	…	…
	c_{m1}	c_{m2}	c_{m3}	…	c_{mn}	1
Demand	1	1	1	…	1	D

In other words, any change in the optimal allocation will result in at least two non-basic variables z_{31} and z_{m3} entering the new optimal basis. From Table 4, this can be expressed as a mathematical statement as given in (3). This inequality is called the Non-Basic Variable Rule ($NBVR$).

$$z_{11} + z_{13} + \ldots + z_{1n} + z_{21} + z_{22} + \ldots + z_{31} +$$

$$z_{32} + \ldots + z_{3n} + \ldots + z_{m2} + z_{m3} + \ldots + z_{mn} \geq 2 \quad (3)$$

3.2 *Property two*

In property two we assume that we are moving from current optimal solution to the next best optimal solution. Whenever there is a change in the optimal allocation, there is also a change in the objective value. In other words, the objective value increases as given by (4). This inequality is called the Objective Cut (*OC*).

$$c_{11}z_{11} + c_{12}z_{12} + c_{13}z_{13} + \ldots + c_{1n}z_{1n} + \ldots +$$

$$c_{m1}z_{m1} + c_{m2}z_{m2} + c_{m3}z_{m3} + \ldots + c_{mn}z_{mn} \geq R_0 + 1 \quad (4)$$

These two properties are very useful in developing an algorithm that is used to find the K^{th} best solution for an assignment model and is discussed in Section 4.

Alternate optimal allocation.

The approach proposed in this paper requires only one optimal allocation. The alternate optimal allocations are obtained by solving (5).

$$COT$$

$$c_{11}z_{11} + c_{12}z_{12} + c_{13}z_{13} + \ldots + c_{1n}z_{1n} + \ldots +$$

$$c_{m1}z_{m1} + c_{m2}z_{m2} + c_{m3}z_{m3} + \ldots + c_{mn}z_{mn} = z_c. \quad (5)$$

Where z_c is the current optimal value.

4. K^{th} optimal solution for assignment problem

4.1 *Algorithm*

Step 1: Solve the given assignment problem to obtain the Continuous Optimal Tableau (*COT*).

Step 2: Use the optimal solution given by *COT* to come up with an Optimal Allocation (*OA*) of the given assignment problem.

Step 3: Generate a Non-Basic Variable Rule (*NBVR*) from the Optimal Allocation (*OA*) given Step 2.

Step 4: Generate an Objective Cut (*OC*) from the *COT* given in Step 1.

Step 5: Add the *NBVR* and *OC* to the *COT* and use the dual simplex method and branch-cut algorithms to obtain the next optimal solution.

Step 6: Use the solution obtained in Step 5 and return to Step 3 until the desired K^{th} optimal solution is obtained.

CPLEX Software [14] is used for the numerical computations.

5. Numerical illustration

Find the 3^{rd} optimal solution of the assignment problem given in Table 5.

Table 5. Assignment problem for numerical illustration

					Source
	2	8	11	6	1
	18	15	21	18	1
	8	9	22	13	1
	17	16	6	10	1
Demand	1	1	1	1	

The linear programming formulation of assignment problem is given in (6).
Minimize

$$Z_o = 2z_{11} + 8z_{12} + 11z_{13} + 6z_{14} + 18z_{21} + 15z_{22} + 21z_{23} + 18z_{24} + 8z_{31} + 9z_{32} + 22z_{33} + 13z_{34} + 17z_{41} + 16z_{42} + 6z_{43} + 10z_{44},$$

such that:

$$z_{11} + z_{12} + z_{13} + z_{14} = 1,$$

$$z_{21} + z_{22} + z_{23} + z_{24} = 1,$$

$$z_{31} + z_{32} + z_{33} + z_{34} = 1,$$

$$z_{41} + z_{42} + z_{43} + z_{44} = 1,$$

$$z_{11} + z_{21} + z_{31} + z_{41} = 1,$$

$$z_{12} + z_{22} + z_{32} + z_{42} = 1,$$

$$z_{13} + z_{23} + z_{33} + z_{43} = 1,$$

$$z_{14} + z_{24} + z_{34} + z_{44} = 1. \tag{6}$$

Where $z_j = 0$ *or* 1,

$$z_{ij} = 0 \text{ or } 1 \ \forall (ij)$$

Step 1: The continuous optimal solution for (6) is found as *COT* and $R_0 = 42$.
Step 2: The *OA* of the given assignment problem is given in Table 6.

Table 6. *OA* of the given assignment problem

Job 1	Job 2	Job 3	Job 4
2	8	11	6
18	15	21	18
8	9	22	13
17	14	16	10

Step 3: Theis *NBVR* given in (7).

$$z_{12}+z_{13}+z_{14}+z_{21}+z_{22}+z_{24}+$$

$$z_{31}+z_{33}+z_{34}+z_{41}+z_{42}+z_{43}\geq 2 \tag{7}$$

Step 4: The *OC* is given in (8).

$$2z_{11}+8z_{12}+11z_{13}+6z_{14}+18z_{21}+15z_{22}+21z_{23}+18z_{24}+$$

$$8z_{31}+9z_{32}+22z_{33}+13z_{34}+17z_{41}+16z_{42}+6z_{43}+10z_{44}\geq 42+1 \tag{8}$$

Step 5: Using the dual simplex method and branch-cut algorithms on (8) we have Table 7.

$$COT$$

$$z_{12}+z_{13}+z_{14}+z_{21}+z_{22}+z_{24}+$$

$$z_{31}+z_{33}+z_{34}+z_{41}+z_{42}+z_{43}\geq 2$$

$$2z_{11}+8z_{12}+11z_{13}+6z_{14}+18z_{21}+15z_{22}+21z_{23}+18z_{24}+$$

$$8z_{31}+9z_{32}+22z_{33}+13z_{34}+17z_{41}+16z_{42}+6z_{43}+10z_{44}\geq 43. \tag{9}$$

Table 7. Second *OA* of the given assignment problem

Job 1	Job 2	Job 3	Job 4
2	8	11	6
18	15	21	18
8	9	22	13
17	14	16	10

Step 6: Using the solution obtained in Step 5 and return to Step 3 once we have (10) and Table 8.

$$COT$$

$$z_{11}+z_{12}+z_{14}+z_{21}+z_{23}+z_{24}+$$

$$z_{32}+z_{33}+z_{34}+z_{41}+z_{42}+z_{43}\geq 2$$

$$2z_{11}+8z_{12}+11z_{13}+6z_{14}+18z_{21}+15z_{22}+21z_{23}+18z_{24}+$$

$$8z_{31}+9z_{32}+22z_{33}+13z_{34}+17z_{41}+16z_{42}+6z_{43}+10z_{44}\geq 44+1. \tag{10}$$

Table 8. Third *OA* of the given assignment problem

Job 1	Job 2	Job 3	Job 4
2	8	11	6
18	15	21	18
8	9	22	13
17	14	16	10

This gives $z = 45$. The alternative solution(s) are obtained by solving (11).

$$COT$$

$$2z_{11}+8z_{12}+11z_{13}+6z_{14}+18z_{21}+15z_{22}+21z_{23}+18z_{24}+$$

$$8z_{31}+9z_{32}+22z_{33}+13z_{34}+17z_{41}+16z_{42}+6z_{43}+10z_{44}=45. \tag{11}$$

In this case we obtain the alternate solution as given in Table 9.

Table 9. Alternate third *OA* of the given assignment problem

Job 1	Job 2	Job 3	Job 4
2	8	11	6
18	15	21	18
8	9	22	13
17	14	16	10

6. Concluding remarks

Munapo [15] introduced the idea of acceleration in the context of an assignment problem. The given problem can be modified in any form using acceleration ideas, if desired. The procedure developed in this paper is a general approach, which is applicable to any problem that has a structure to develop COT and BNVR constraints. At every iteration of this procedure a single sub-problem is solved. Unlike the approach proposed by [13], computational load does not increase unduly for determination of an ordered optimal solution. The assignment problem has real life application in logistics and transportation, workforce scheduling, resource allocation, project assignment, matching algorithms and sports scheduling.

For delivery companies to make profit, there is a need to assign a set of delivery routes to a set of vehicles at their disposal. This is done by formulated the problem as an assignment problem and the delivery companies can then optimize the allocation of routes to vehicles. The optimization in this case is based on factors such as distance travelled, time taken, vehicle capacity etc. In every transportation business, customer satisfaction and profit must be maximized. Also, the total transportation costs must be pushed to the lowest level possible.

Workforce Scheduling is another area where the assignment problem has practical application. For example, a very big school may need to assign teachers to different subject time-periods, and this is based on the teacher's availability and subject expertise. The workforce scheduling problem can be formulated as an assignment problem and has the advantage that teacher individual preferences and constraints are catered for.

In resource allocation, the assignment model also plays a pivotal role. Resource allocation requires minimization of downtime and costs and maximization of profit and productivity. To achieve this, the resource allocation problem is formulated as an assignment problem.

Project management is another field where the assignment model has played a crucial role. To ensure a project success, there is a need to assign project team members to specific tasks or activities. This is formulated as an assignment problem. Factors such as expertise, availability or salary grade are incorporated to maximize the overall project efficiency.

The assignment model is also applied in matching algorithms. In matching algorithms, the objective is to match individuals or entities based on their preferences and characteristics. For example, in job matching platforms the matching algorithm uses an assignment model to match job seekers with the available job vacancies or job opportunities.

The other common application of the assignment problem is in sports scheduling. For example, the recent soccer World Cup of 2022 in Qatar, there were 32 teams. There was a limited number of soccer facilities at Qatar 2022 and there will always be limited soccer facilities given the fact that the number teams at the next soccer World Cup (2026) are being increased to 48. At the soccer World Cup the assignment problem is used to determine the optimal allocation of teams to fixtures and venues.

Conflict of interest

The authors declare no competing financial interest.

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