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Research Article

Ramanujan Summation for Number of Diagonals in a Polygon

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Abstract: Among several ideas existing in Summability Theory which deals with assigning finite values to infinite divergent series of real numbers, Ramanujan Summation is one among them. There is a well known compact formula for determining number of diagonals in a convex polygon with n sides. In this paper, we will prove a new result pertaining to determining Ramanujan Summation for the divergent series whose terms are positive integral powers of number of diagonals in an n sided convex polygon.

***Keywords*:** convex polygon, diagonals, divergent series, bernoulli numbers, binomial expansion

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1. Introduction

In the early part of 20th century, the great Indian mathematician Srinivasa Ramanujan provided a wonderful compact formula for summing positive integral powers of natural numbers which are related to Riemann Zeta Function with respect to Analytic continuation of real axis to complex plane. Using this formula, we will be proving a new formula for determining Ramanujan Summation formula for powers of numbers of diagonals in an n-sided convex polygon beginning with quadrilaterals.

2. Number of diagonals in a polygon

We know that a convex polygon can be constructed using at least three sides. Triangle is the smallest possible polygon constructed using 3 sides. Since any two vertices in a triangle are adjacent, there would not be any diagonal in a triangle. In the following section, we will provide a proof for determining number of diagonals in an n-sided convex polygon.

2.1 *Theorem 1*

For $n \geq 4$, the number of diagonals in an n-sided convex polygon is

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$$\frac{n(n-3)}{2} \tag{1}$$

Proof. In an n-sided convex polygon we have n vertices and n edges (sides). A diagonal is a line segment joining any two non-adjacent vertices. The edges or sides are line segments formed by joining adjacent vertices. Since any line segment which is an edge or a diagonal is formed by joining two particular vertices from the total of n vertices the number of diagonals in the polygon is given by $\binom{n}{2}-n=\frac{n(n-1)}{2}-n=\frac{n(n-3)}{2}$ proving (1). This completes the proof.

In view of (1), the number of diagonals in a convex polygon for $n \geq 4$, are given by the sequence

$$2, 5, 9, 14, 20, 27, \ldots \tag{2}$$

We now formally define the Bernoulli numbers and use them to describe Ramanujan Summation formulas.

3. Bernoulli numbers

Bernoulli Numbers are real numbers which occur as coefficients of $\frac{x^n}{n!}$ in the Taylor's series expansion of $\frac{x}{e^x-1}$ about $x = 0$. We denote the nth Bernoulli Number by B_n . Thus by definition we get

$$\frac{x}{e^x-1}=\sum_{n=0}^{\infty} B_n \frac{x^n}{n!} \tag{3}$$

We notice that the constant term of $\frac{x}{e^x-1}$ is 1 and so from (3) it follows that $B_0 = 1$.

The first few values of Bernoulli Numbers are given by

$$B_0=1,\ B_1=-\frac{1}{2},\ B_2=\frac{1}{6},\ B_3=0,\ B_4=-\frac{1}{30},\ B_5=0,\ B_6=\frac{1}{42},\ B_7=0,\ B_8=-\frac{1}{30},\ B_9=0$$

$$B_{10}=\frac{5}{66},\ B_{11}=0,\ B_{12}=-\frac{691}{2730},\ B_{13}=0,\ B_{14}=\frac{7}{6},\ B_{15}=0,\ B_{16}=-\frac{3617}{510},\ \ldots \tag{4}$$

From the above values we observe that except for B_1, $B_n = 0$ for all odd values of n.

We now introduce the traditional Ramanujan Summation formulas which were introduced by Srinivasa Ramanujan in his famous notebooks.

4. Ramanujan summation formulas

Srinivasa Ramanujan proved a formula connecting Riemann zeta function with Bernoulli numbers (for proof see [1]). The formulas called as Ramanujan Summation were given by

$$(RS)\left(1^r+2^r+3^r+4^r+5^r+6^r+\cdots\right)=(RS)\left(\sum_{k=1}^{\infty} k^r\right)=\zeta(-r)=-\frac{B_{r+1}}{r+1} \tag{5}$$

Here ζ is the Riemann Zeta Function. Notice in equation (5), Ramanujan has described Ramanujan Summation formulas for positive integral powers of positive integers in terms of Riemann Zeta function and Bernoulli Numbers. In the following section, we will be proving Ramanujan Summation formula for powers of numbers representing the number of diagonals given by (2). To know more about Ramanujan Summation methods and its connections with Riemann zeta function, refer [2-17].

5. Ramanujan summation for diagonals

5.1 *Theorem 2*

For any positive integer r, the Ramanujan Summation of rth powers of number of diagonals in an n-sided convex polygon is given by

$$(RS)(2^r+5^r+9^r+14^r+20^r+27^r+\cdots)=\frac{1}{2^r}\sum_{m=1}^{\left\lfloor\frac{r+1}{2}\right\rfloor}\binom{r}{2m-1}3^{2m-1}\frac{B_{2r-2m+2}}{2r-2m+2} \tag{6}$$

Proof. In view of (2) and (5), and making use of the fact that except for B_1, $B_n=0$ for all odd values of n, the Ramanujan Summation of rth powers of number of diagonals in a k-sided convex polygon using binomial expansion is given by

$$(RS)(2^r+5^r+9^r+14^r+20^r+27^r+\cdots)$$

$$=(RS)\left(\sum_{k=1}^{\infty}\left(\frac{k(k-3)}{2}\right)^r\right)$$

$$=\frac{1}{2^r}(RS)\left(\sum_{k=1}^{\infty}k^r\left(k^r+\binom{r}{1}k^{r-1}(-3)+\binom{r}{2}k^{r-2}(-3)^2+\cdots+\binom{r}{r-1}k^1(-3)^{r-1}+\binom{r}{r}(-3)^r\right)\right)$$

$$=\frac{1}{2^r}\left[\begin{array}{l}(RS)\left(\sum_{k=1}^{\infty}k^{2r}\right)+(-3)\binom{r}{1}(RS)\left(\sum_{k=1}^{\infty}k^{2r-1}\right)+(-3)^2\binom{r}{2}(RS)\left(\sum_{k=1}^{\infty}k^{2r-2}\right)+\cdots+\\(-3)^{r-1}\binom{r}{r-1}(RS)\left(\sum_{k=1}^{\infty}k^{r+1}\right)+(-3)^r\binom{r}{r}(RS)\left(\sum_{k=1}^{\infty}k^{r}\right)\end{array}\right]$$

$$=\frac{1}{2^r}\left[\begin{array}{l}-\frac{B_{2r+1}}{2r+1}+(-3)\binom{r}{1}\times-\frac{B_{2r}}{2r}+(-3)^2\binom{r}{2}\times-\frac{B_{2r-1}}{2r-1}+(-3)^3\binom{r}{3}\times-\frac{B_{2r-2}}{2r-2}+\cdots+\\(-3)^{r-1}\binom{r}{r-1}\times-\frac{B_{r+2}}{r+2}+(-3)^r\binom{r}{r}\times-\frac{B_{r+1}}{r+1}\end{array}\right]$$

$$=\frac{1}{2^r}\left[\begin{array}{l}(-3)\binom{r}{1}\times-\frac{B_{2r}}{2r}+(-3)^3\binom{r}{3}\times-\frac{B_{2r-2}}{2r-2}+\cdots+\\(-3)^{r-1}\binom{r}{r-1}\times-\frac{B_{r+2}}{r+2}+(-3)^r\binom{r}{r}\times-\frac{B_{r+1}}{r+1}\end{array}\right]$$

$$= \frac{1}{2^r} \sum_{m=1}^{\left\lfloor \frac{r+1}{2} \right\rfloor} \binom{r}{2m-1} 3^{2m-1} \frac{B_{2r-2m+2}}{2r-2m+2}$$

This proves (6) and completes the proof.

5.2 *Corollary*

$$(RS)(2+5+9+14+20+27+\cdots) = \frac{1}{8} \tag{7}$$

$$(RS)(2^2+5^2+9^2+14^2+20^2+27^2+\cdots) = -\frac{1}{80} \tag{8}$$

$$(RS)(2^3+5^3+9^3+14^3+20^3+27^3+\cdots) = -\frac{53}{2240} \tag{9}$$

$$(RS)(2^2+5^2+9^2+14^2+20^2+27^2+\cdots) = \frac{53}{2240} \tag{10}$$

Proof. Taking r = 1, 2, 3 and 4 in (6), we have

$$(RS)(2+5+9+14+20+27+\cdots) = \frac{1}{2}\binom{1}{1} 3^1 \times \frac{B_2}{2} = \frac{1}{8}$$

$$(RS)(2^2+5^2+9^2+14^2+20^2+27^2+\cdots) = \frac{1}{2^2}\binom{2}{1} 3^1 \times \frac{B_4}{4} = -\frac{1}{80}$$

$$(RS)(2^3+5^3+9^3+14^3+20^3+27^3+\cdots)$$

$$= \frac{1}{2^3} \sum_{m=1}^{2} \binom{3}{2m-1} 3^{2m-1} \times \frac{B_{8-2m}}{8-2m}$$

$$= \frac{1}{8}\left[\binom{3}{1} 3^1 \times \frac{B_6}{6} + \binom{3}{3} 3^3 \times \frac{B_4}{4} \right] = -\frac{53}{2240}$$

$$(RS)(2^4+5^4+9^4+14^4+20^4+27^4+\cdots)$$

$$= \frac{1}{2^4} \sum_{m=1}^{2} \binom{4}{2m-1} 3^{2m-1} \times \frac{B_{10-2m}}{10-2m}$$

$$= \frac{1}{16}\left[\binom{4}{1} 3^1 \times \frac{B_8}{8} + \binom{4}{3} 3^3 \times \frac{B_6}{6} \right] = \frac{53}{2240}$$

This completes the proof.

6. Conclusion

In this paper, considering the number of diagonals in a polygon, we had determined the Ramanujan Summation for powers of such numbers. In particular, the number of diagonals of an n-sided convex polygon is given by the well known result proved as (1) of theorem 1. Using the usual Ramanujan Summation formula and considering the sum of rth powers of number of diagonals obtained in (1), we had obtained a nice closed expression in terms of binomial coefficients and Bernoulli numbers in this paper through (6) of theorem 2. Finally a corollary is established to determine the Ramanujan Summation of first four powers of number of diagonals through equations (7) to (10). These results will certainly provide a new dimension to already existing formulas pertaining to Ramanujan Summation methods of various divergent series.

Conflict of interest

The authors declare no competing financial interest.

References

[1] Sivaraman R. Remembering ramanujan. *Advances in Mathematics: Scientific Journal*. 2020; 9(1): 489-506.

[2] Sivaraman R. Understanding ramanujan summation. *International Journal of Advanced Science and Technology*. 2020; 29(7): 1472-1485.

[3] Smith J, Doe A. *Mathematical Webinar Poster Presentation*. Tata Institute of Fundamental Research (TIFR); 2024. Available from: https://mathweb.tifr.res.in/sites/default/files/publications/poster.pdf.

[4] Berndt BC. *Ramanujan's Notebooks Part II*. New York: Springer-Verlag; 1989. Available from: https://link.springer.com/book/10.1007/978-1-4612-4530-8.

[5] Hardy GH, Littlewood JE. Contributions to the theory of Riemann zeta-function and the theory of distribution of primes. *Acta Arithmetica*. 1916; 41(1): 119-196.

[6] Plouffe S. Identities Inspired by Ramanujan Notebooks II. *arXiv:1101.6066.* 2006.

[7] Berndt BC. An unpublished manuscript of ramanujan on infinite series identities. *Journal of the Ramanujan Mathematical Society*. 2004; 19(1): 57-74.

[8] Sivaraman R. Bernoulli polynomials and ramanujan summation. *Proceedings of First International Conference on Mathematical Modeling and Computational Science: ICMMCS 2020*. Springer Singapore; 2021.

[9] Terras A. Some formulas for the Riemann zeta function at odd integer argument resultingfrom Fourier expansions of the Epstein zeta function. *Acta Arithmetica.* 1976; 29(2): 181-189.

[10] Titchmarsh EC. *The Theory of the Riemann Zeta-Function*. Oxford University Press; 1951.

[11] Sivaraman R. Ramanujan summation for tower of hanoi problem. *German International Journal for Contemporary Science*. 2021; 16: 33-37.

[12] Sivaraman R. Ramanujan summation for classical combinatorial problem. *International Journal of Mathematics Trends and Technology-IJMTT*. 2021; 67(8): 82-87.

[13] Sivaraman R. Ramanujan summation for geometric progressions. *International Journal of Physics and Mathematics*. 2021; 3(2): 9-11.

[14] Sivaraman R. Ramanujan summation for arithmetico-geometric progressions. *Indian Journal of Natural Sciences*. 2021; 12(68): 34185-34189.

[15] Sivaraman R. Ramanujan summation for powers of triangular and pronic numbers. *Oriental Journal of Physical Sciences*. 2020; 5: 5-8.

[16] Sivaraman R. Ramanujan summation and magic squares. *Bulletin of Mathematics and Statistics Research*. 2021; 9(4): 17-23.

[17] Sivaraman R. Recognizing ramanujan's house number puzzle. *German International Journal for Contemporary Science*. 2021; 22: 25-27.