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Research Article

Modelling Climate Change in a Stochastic Extreme Values Framework

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Abstract: This article presents a contribution to modelling of rainfall hazards in a stochastic framework. The independence of the realizations of a process conditionally on the latent random effect, which also allowing us to calculate the likelihood of the observed annual maxima. We also showed the predominance of the moment estimator over the Hill estimator. Furthernore, we used a stochastic model to provide a temporal framework for modeling extreme events, as well as the evolution of extreme rainfall in west Africa from 2019 to 2169, which shows a significant decrease over the entire range of the different stations.

Keywords: modeling, stochastic process, extreme values, climate change

MSC: 93E03, 60-XX, 92D30, 62G32

1. Introduction

Our planet is increasingly polluted by human activities activities (industry, transport, agriculture...) see [1]. The Intergovernmental Panel on Climate Change (IPCC), unequivocally states that human activities are the cause of current global warming, which is having an indisputable impact on extreme extreme events [2], several models exist in the pluviometric literature Regional Climate Model [2] which uses simulations of a global simulations of a global model to define boundary conditions. This model simulates, over a limited area, all atmospheric atmospheric circulations. The reduced domain enables us to a seasonal Markov Model [3, 4] this model was formalized by Baum and Petrie, who also studied its asymptotic properties, this model and their generations have since given rise to an abundant literature. literature, both theoretical and applied. Chain models [5], the schadex model for ungauged sites [6]. Sawadogo and Barro [7] proposes a new method for estimating extreme precipitation at points where we have no observations, using information from marginal distributions and dependence structure.

Extreme Values Theory (EVT) focuses on the analysis of and inference about extreme events, i.e. events with a very low probability of occurrence. And the study of rainfall and hydrological droughts in tropical Africa [8] Panthou Geremy, for example, highlights recent rainfall trends and explains the hydrological and agronomic dynamics of the West African region, using simple statistical methods in two sub-regions, Senegambia and the Niger Basin. Furthernore, the Reich Shaby model is defined

$$Z(s) = U_\alpha(s) \times \vartheta_\alpha(s). \tag{1}$$

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where Z is the spatial process, U_α an independent process of marginal laws $GEV(1,\ \alpha,\ \alpha)$ and ϑ_α a process composed of independent variables of positive law and positive deterministic functions.

Since observations began in 1854, three main dry periods have been recognized. periods have been recognized up to the present day (see [2]). This has particularly harmful effects on human activity and the environment. human activities and on the environment [9, 10]. Recently, following rainy years, some authors are wondering whether the events of the 1990s are not a sign of a return a gradual return to more favorable rainfall conditions (see [11]).

While modeling in extreme values theory is one of the best the Pickands estimator strong convergence and asymptotic normality have been demonstrated by Dekkers and de Haan. Pickands' estimator is defined for $\xi \in \mathbb{R}$,

$$\hat{\xi}^P = \hat{\xi}^P(k) = \frac{1}{\log 2} \log\left(\frac{X_{n-k+1:n} - X_{n-2k+1:n}}{X_{n-2k+1:n} - X_{n-4k+1:n}}\right) \tag{2}$$

where $X_1,\ X_2,\ \ldots,\ X_n$ are n independent and indetically distributed random variables, k is an integer sequence $1 < k < n$.

The Hill estimator was introduced by Hill in 1975, to estimate the tail parameter of distributions belonging to the Frechet attraction domain $D\left(\Phi_{\frac{1}{\xi}}\right)$, i.e., when the tail of the distribution has a Pareto shape. A great deal of theoretical work has been devoted to studying the properties of Hill's estimator. Weak consistency was established [12], strong consistency [13–15] and more recently [16].

Hill's estimator is defined by

$$\hat{\xi}^H = \hat{\xi}^H(k) = \frac{1}{k} \sum \log X_{n+i-1:n} - \log X_{n-k:n} \tag{3}$$

where $X_1,\ X_2,\ \ldots,\ X_n$ are n independent and indetically distributed random variables, k is an integer sequence $1 < k < n$.

One drawback of Hill's estimator is that it is designed only for heavy-tailed distributions. In 1989, Dekkers et al. proposed an extension for all types of distribution, called the Moments estimator.

$$\hat{\xi}^M = \hat{\xi}^M(k) = M_n^{(1)} + T_n = M_n^{(1)} + 1 - \frac{1}{2}\left(\frac{\left(M_n^{(1)}\right)^2}{M_n^{(2)}}\right) \tag{4}$$

with

$$M_n^r = M_n^r(k) = \frac{1}{k} \sum_{i=1}^{k} \left(\log X_{n-i+1:n} - \log X_{n-k:n}\right),\quad r = 1,\ 2$$

where M_n^1 is the Hill estimator $\hat{\xi}^H$.

The contribution of this article is to show, first of all, the independence of the realizations of the process Y conditional on the latent random effect A. Thus, is it possible to reduce to a process Y whose marginal law at a site s is a GEV of parameters $\mu(s)$, $\sigma(s)$ and $\xi(s)$ knowing that the margins of Z have a unit Frechet law? Next, the error of the estimators is illustrated by showing which of Hill's estimator and the Moment estimator wins the most, while a number of interesting theorems, definitions and propositions are established. The rest of the article is organized as follows: in section 2, we recall the concepts essential to the study, in section 3, we present the main results obtained, and in section 4, we provide a conclusion and a discussion.

2. Preliminaries

In this section, we collect some important definitions and themes on extreme value modelling and extreme quantiles, which prove necessary for our approach. We refer the reader to the references to detailed introductions to the subject (see [17–21]).

2.1 *Extreme values results*

The Generalized Extreme Value (GEV) and Generalized Pareto Distributions (GPD) are widely used in extreme value analysis in hydrology. The calibration of these distributions beyond a threshold, however, poses a number of in particular that of bias in the tails of the distributions. As a reminder, a variable X follows the generalized extreme law if its distribution function F_X is given by assumption the c.d.f. H_t of the process is max-stable. That means in particular that, for all site x, there exist constants $a_{n,\, x} > 0$ and $b_{n,\, x} \in \mathbb{R}$ such that, for any realization $y(x)$ of $\{Y_x,\, x \in \chi\}$ the one-dimensional margin of G is a spatial GEV-distribution defined at all spatial site $x \in \chi$ by

$$H^k(a_{n,\,x}y(x) + b_{n,\,x}) = GEV_{(\mu,\,\sigma,\,\xi)}(y) = \begin{cases} \exp\left\{-\left[1+\xi_x\left(\dfrac{y(x)-\mu}{\sigma}\right)\right]_+^{\frac{-1}{\xi}}\right\} & \text{if } \xi \neq 0 \\ \exp\left\{-\exp\left(-\dfrac{y(x)-\mu}{\sigma}\right)\right\} & \text{if } \xi = 0 \end{cases}; \tag{5}$$

with $u_+ = \max(u,\, 0)$, where μ is a real location parameter, σ a positive scale parameter and ξ a real shape parameter for the site x. While modelling spatial snow depth [2] considered these parameters as respectively the longitude, latitude and elevation of the site x_i. This means in particulary that, there is no loss of generality by making such a assumption since every one-dimensional extreme values distribution can be obtained by a functional transformation of another. Note that, if for a given site x_i,

$$Z(x_i) \sim Fréchet(x_i) \implies Y(x_i) = \mu_{x_i} + \frac{\sigma_{x_i}}{\xi_{x_i}}\left[Z(x_i)^{\xi_{x_i}} - 1\right] \sim GEV_{(\sigma_{x_i},\, \mu_{x_i},\, \xi_{x_i})}(x_i). \tag{6}$$

2.2 *GEV's catchment area*

Theorem 2.1 The F distribution function of the random variable X belongs to the max-domain of attraction of the generalized extreme value distribution GEV G_ξ if and only if, there exists a measurable function δ such that:

$$\lim_{t \to x_F} \frac{\tilde{F}(t + x\delta(t))}{\tilde{F}(t)} = \begin{cases} (1+\xi x)^{-1/\xi} & \text{if } \xi \neq 0 \\ e^{-x} & \text{if } \xi = 0 \end{cases};$$

with $1 + \xi x > 0$ and $x \in \mathbb{R}$.

We are looking for a parametric model to describe the shape of the distribution function F of a random variable X above a high level u. According to Fisher and Tippet, it is natural to assume that for n large

$$P(\max(X_1,\, \ldots,\, X_n \leqslant x)) = F^n(x) \approx \exp\left(-\left[1 + k\frac{x-\mu}{\sigma}\right]^{-1/k}\right)$$

and therefore, we obtain, the survival function

$$P(X \geqslant x) = 1 - F(x) \approx \frac{1}{n}\left[1 + k\frac{x-\mu}{\sigma}\right]^{-1/k}$$

Moreover, for a conditional distribution of $X - u$ (the level u) knowing that $X \geqslant u$ (the level u is exceeded)

$$P(X - u \leq y \setminus X \geq u) \approx 1 - \left[1 + \frac{ky}{\hat{\sigma}}\right]^{-1/k}$$

furthermore

$$P[X - u \setminus X \geqslant u] \sim GPD(\sigma + ku,\ k)$$

The Pickands-Balkema-de Haan theorem is the second theorem of extreme value theory.

Theorem 2.2 (Pickands-Balkema-de Haan). A distribution function F belongs to the domain of maximum attraction of H_ξ, if and only if there exists a positive function $\beta(u)$ such that:

$$\lim_{u \to x_F} \sup_{o \leq y \leq x_F - u} |F_u(y) - G_{\xi,\ \beta(u)}(y)| = 0$$

where $F_u(y)$ is the conditional excess distribution function for high u, x_F is the terminal point of F, is the conditional excess distribution function for high u. $x_F = \{x \in \mathbb{R} : F(x) < 1\}$ and $G_{\xi,\ \beta(u)}(y)$ is the GPD given by :

$$G_{\xi,\ \beta(u)}(x) = \begin{cases} 1 - \left(1 + \xi\dfrac{y}{\beta(u)}\right)^{\frac{-1}{\xi}} & \text{if } \xi \neq 0 \\ 1 - \exp\left(-\dfrac{y}{\beta(u)}\right) & \text{if } \xi = 0 \end{cases} \tag{7}$$

where $y \geq 0$ for $\xi \geq 0$ and $0 \leq y \leq -\dfrac{\beta(u)}{\xi}$ for $\xi < 0$.

2.3 *Extreme quantile estimation methods*

While estimating extreme quantiles for climatic application three essentials methods are used: The Pickands estimator, Hill and Moments one.

Let $X_1,\ X_2,\ \ldots,\ X_n$ n be independent and identically distributed random variables with distribution function $F \in D\left(H_\xi\right)$, where $\xi \in \mathbb{R}$. Let $K = K_n$ be a sequence of integers with $1 < K < n$, Pickands' estimator is defined by

$$\hat{\xi}^P = \hat{\xi}^P(K) = \frac{1}{\log 2}\log\left(\frac{X_{n-k+1:n} - X_{n-2K+1:n}}{X_{n-2k+1:n} - X_{n-4K+1:n}}\right).$$

For Hil estimator, we estimate in a non-parametric way the tail parameter. So, let $F \in D\left(\Phi_{\frac{1}{\xi}}\right)$, or $\xi < 0$. Let $k = k_n$ be a sequence of integers with $1 < k < n$, Hill's estimator is defined by

$$\hat{\xi}^H = \hat{\xi}^H(K) = \frac{1}{k}\sum \log X_{n+i-1:n} - \log X_{n-k:n}.$$

If we consider the distribution functions belonging to Frechet's domain of maximum attraction ($\xi > 0$), then the survival function can be written as $F = x^{-1/\xi}L(x)$ where L is a slowly varying function. So we can write

$$\frac{\hat{(x)}}{\hat{F}(X_{n-k})} = \frac{L(x)}{L(X_{n-k})}\left(\frac{x}{X_{n-k}}\right)^{-1/\xi}$$

and if we consider that the ratio of slowly varying functions is close to 1, we find

$$\hat{F}(x) \simeq \hat{F}(X_{n-k})\left(\frac{x}{X_{n-k}}\right)^{-1/\xi}$$

For $\xi \in \mathbb{R}$, the estimator of moments is:

$$\hat{\xi}^M = \hat{\xi}^M(k) = M_n^{(1)} + T_n$$

with

$$M_n^r = M_n^r(k) = \frac{1}{k}\sum_{i=1}^{k}\left(\log X_{n-i+1:n} - \log X_{n-k:n}\right), \quad r = 1,\ 2$$

$$T_n = 1 - \frac{1}{2}\left(\frac{\left(M_n^{(1)}\right)^2}{M_n^{(2)}}\right).$$

where M_n^1 is the Hill estimator $\hat{\xi}^H$.

3. Main results

In this section, we propose a Reich and Shaby model appropriate to the data we're simulating. appropriate to the data, i.e. the simulated data at our disposal. our objective. Let Z be a spatial process defined for all $s \in \mathbb{S}$.

Proposition 3.1 Let U_α be an independent process with marginal laws $GEV(1,\ \alpha,\ \alpha)$. Suppose Z is the spatial process defined for all $s \in S$ by the product $Z(s) = U_\alpha(s) \times \vartheta_\alpha(s)$ then there is a process built by $\vartheta_\alpha(s) =$

$\left(\sum_{l=1}^{L} A_{lw_l}(s)^{1/\alpha}\right)^{\alpha}$ such that $A_1, \ldots, A_L$ are independent variables with a positive stable law PS(α), with characteristic exponent α, and $\omega_1, \ldots, \omega_L$ are positive deterministic functions vérifying the condition $\sum_{l=1}^{L} \omega_l(s) = 1$ for all $s \in S$.

Proof. Let Y be a marginal law process at a site s.

We have

$$Y(s)/\mu,\ \sigma,\ \xi,\ \alpha,\ \vartheta_\alpha \sim^{indep} GEV(\mu^\star(s),\ \sigma^\star(s),\ \xi^\star(s)).$$

Therefore

$$\mu^\star(s) = \mu(s) + \frac{\sigma(s)}{\xi(s)}(\vartheta_\alpha(s)^{\xi(s)} - 1),$$

and moreover

$$\sigma^\star(s) = \alpha\sigma(s)\vartheta_\alpha(s)^{\xi(s)},$$

and

$$\xi^\star(s) = \alpha\xi(s),$$

or

$$\vartheta_\alpha(s) = \left(\sum_{l=1}^{L} A_{l\omega_l^{1/\alpha}(s)}\right)^{\alpha},$$

from which

$$A_1, \ldots, A_L \sim^{iid} PS(\alpha).$$

□

The independence of the realizations of the Y process conditional on the the latent random effect A, we can calculate the likelihood of the observed observed annual maxima, by taking the product of the density functions of the GEV laws for each site.

Shaby and Reich differentiate between two types of prediction that different in the choice of positive stable variables $A_t = \{A_{1t}, \ldots, A_{Lt}\}$ used to calculate $\vartheta_{\alpha,\, t}(s^\star)$. Among these predictions, the one of interest is the climatological one, which predicts what might have happened with the model's parameters, by sampling A_t independently according to the positive stable law PS(α).

Proposition 3.2 Let Z be the distribution function evaluated at positions $\{s_1, \ldots, s_d\}$.

The function of répartition évaluée on the set of sites $\{s_1,\ ...,\ s_d\}$ is written as:

$$Pr(Z(s_1) \leqslant z_1,\ ...,\ Z(s_d) \leqslant z_d) = \exp\left(-\sum_{l=1}^{L}\left[\sum_{j=1}^{d}\left\{\frac{\omega_l(s_j)}{z_j}\right\}^{1/\alpha}\right]^{\alpha}\right).$$

As a result,

$$Z(s) = \left[1 + \xi(s)\frac{Y(s) - \mu(s)}{\sigma(s)}\right]^{1/\xi(s)}.$$

Proof. First, the Laplace transform of a random variable A of positive stable law with characteristic exponent α is written as:

$$\mathbb{E}\left[e^{-tA}\right] = \int_0^{\infty} e^{-tA} f_{PS(\alpha)}(A)\,dA = e^{-t^{\alpha}},\ \ \forall t \in \mathbb{R}$$

where $f_{PS(\alpha)}$ is the density function of the positive stable law $PS(\alpha)$. This densit'e is not explicit for $\alpha \in (0,\ 1)$. Thereafter, we denote by $A = (A_1,\ ...,\ A_L)$. The random vector whose components are i.i.d. of stable laws positive $PS(\alpha)$ and by $a = (a_1,\ ...,\ a_L)$ a particular realization of this vector. The multidimensional multidimensional $f_{PS(\alpha)}(a)$ is the product of product, by independence of a_l, of the marginal functions $_{\alpha}(S) = \left(\sum_{l=1}^{L} a_l w_l(S)^{1/\alpha}\right)^{\alpha}$ to designate $v_{\alpha}(S)\backslash A = a$.

The joint distribution function G of the vector $(Z(s_1),\ ...,\ Z(s_d))$ is written:

$$\begin{aligned} G(z_1,\ ...,\ z_d) &= Pr(Z(s_1) \leq z_1,\ ...,\ Z(s_d) \leq z_d) \\ &= \int_{\mathbb{R}_+^{\mathbb{L}}} Pr\left(Z(s_1) \leq z_1,\ ...,\ Z(s_d) \leq z_d \backslash A = a\right) f_{PS(\alpha)(a)}(a)\,da \end{aligned} \tag{8}$$

According to equation (8), we obtain

$$G(z_1,\ ...,\ z_d) = \int_{\mathbb{R}_+^{\mathbb{L}}} Pr\left(U_{\alpha}(s_1) \leq \frac{z_1}{v_{\alpha}(s_1)},\ ...,\ U_{\alpha}(s_d) \leq \frac{z_d}{v_{\alpha}(s_d)} \backslash A = a\right) f_{PS(\alpha)(a)}(a)\,da \tag{9}$$

however

$$Pr(Z(s_1) \leq z_1,\ ...,\ Z(s_d) \leq z_d) = \Pi_{j=1}^{d} Pr\left(u_{\alpha}(s_j) \leq \frac{z_j}{\vartheta_{\alpha}(s_1)} \backslash A = a\right) \tag{10}$$

Replacing equation (10) in equation (9), we obtain

$$G(z_1,\ \ldots,\ z_d) = \int_{\mathbb{R}_+^{\mathbb{L}}} \Pi_{j=1}^{d} Pr\left(U_\alpha\left(S_j\right) \leq \frac{z_j}{v_\alpha\left(s_1\right)} \backslash A = a\right) f_{PS(\alpha)(a)}\left(a\right) da \tag{11}$$

We have

$$Pr\left(U_\alpha\left(S_j\right) \leq \frac{z_j}{v_\alpha\left(s_1\right)} \backslash A = a\right) = \exp\left[-\left(\frac{\theta_\alpha(s_j)}{z_j}\right)^{1/\alpha}\right]$$

$$= -\frac{\sum_{l=1}^{L} a_l w_l(s_j)^{1/\alpha}}{z_j}$$

Furthermore, we have

$$Pr\left(U_\alpha\left(S_j\right) \leq \frac{z_j}{v_\alpha\left(s_1\right)} \backslash A = a\right) = -\sum_{l=1}^{L} a_l \left(\frac{w_l\left(s_j\right)}{z_j}\right)^{1/\alpha}$$

Which gives

$$G(z_1,\ \ldots,\ z_d) = \int_{\mathbb{R}_+^{\mathbb{L}}} \exp\left(-\sum_{l=1}^{L} a_l \left[\sum_{j=1}^{d}\left(\frac{w_l\left(s_j\right)}{z_j}\right)^{1/\alpha}\right]\right) f_{PS(\alpha)(a)}\left(a\right) da \tag{12}$$

Assuming that

$$\sum_{j=1}^{d}\left(\frac{w_l\left(s_j\right)}{z_j}\right)^{1/\alpha} = t_l \tag{13}$$

Then we have

$$Pr(Z(s_1) \leq z_1,\ \ldots,\ Z(s_d) \leq z_d) = \int_{\mathbb{R}_+^{\mathbb{L}}} \exp\left(-\sum_{l=1}^{L} a_l t_l\right) f_{PS(\alpha)(a)}\left(a\right) da$$

Using the Laplace transformation, we obtain

$$
\begin{aligned}
Pr(Z(s_1) \leq z_1, \; \ldots, \; Z(s_d) \leq z_d) &= \mathbb{E}\left[e^{-\sum_{l=1}^{L} t_l A_l} \right] \\
&= \Pi_{l=1}^{L} \mathbb{E}\left[e^{-t_l A_l} \right] \\
&= \Pi_{l=1}^{L} e^{-t_l^{\alpha}} \\
&= e^{-\sum_{l=1}^{L} t_l^{\alpha}}
\end{aligned} \tag{14}
$$

By introducing (14) and (13) into (12), we finally obtain

$$
Pr(Z(s_1) \leq z_1, \; \ldots, \; Z(s_d) \leq z_d) = \exp\left(-\sum_{l=1}^{L} \left[\sum_{j=1}^{d} \left(\frac{w_l(s_j)}{z_j} \right)^{1/\alpha} \right]^{\alpha} \right)
$$

From this result, it is easy to show that the Z process is of margin $GEV(1, 1, 1)$ using the assumption of normalization hypothesis.

$$
\begin{aligned}
Pr(Z(s) \leqslant z) &= \exp\left(-\sum_{l=1}^{L} \frac{w_l(s)}{z} \right) \\
&= \exp(-1/z)
\end{aligned}
$$

The exponent function V of the Reich and Shaby model evaluated at positions $s_1, \; \ldots, \; s_d$ of the form:

$$
V(z_1, \; \ldots, \; z_d) = \sum_{l=1}^{L} \left[\sum_{j=1}^{d} \left\{ \frac{\omega_l(s_j)}{z_j} \right\}^{1/\alpha} \right]^{\alpha}
$$

□

Proposition 3.3 The hierarchical formulation of Reich and Shaby's max-stable model uses, on the one hand, the form of the product $Z(s) = U_{\alpha}(s)\vartheta_{\alpha}(s)$ and the relationship between Z and Y:

$$
Z(s) = \left[1 + \xi(s) \frac{Y(s) - \mu(s)}{\sigma(s)} \right]^{1/\xi(s)}
$$

Proof. Since the process U_α is spatially independent and of marginal laws $GEV(1,\ \alpha,\ \alpha)$, we have the product $U_\alpha \vartheta_\alpha$

$$Z(s)/\vartheta_\alpha \sim^{indep} GEV(\vartheta_\alpha,\ \alpha\vartheta_\alpha,\ \alpha)$$

and since Z has unit Frechet margins, we have:

$$Pr\left(Z\left(s\right) \leq z/\vartheta_\alpha\left(s\right)\right) = \exp\left(-\left[1+\alpha\frac{z-\vartheta_\alpha\left(s\right)}{\alpha\vartheta_\alpha\left(s\right)}\right]^{-1/\alpha}\right)$$

$$= \exp\left(-\vartheta_\alpha\left(s\right)^{1/\alpha_z - 1/\alpha}\right)$$

The marginal laws of the process Y conditioned by ϑ_α are given for all s by:

$$Pr\left(Y\left(s\right) \leq y/\vartheta_\alpha\left(s\right)\right) = Pr\left(\mu\left(s\right) + \frac{\sigma\left(s\right)}{\xi\left(s\right)}\left[Z\left(s\right)^{\xi\left(s\right)} - 1\right] \leq y/\vartheta_\alpha\left(s\right)\right)$$

$$= Pr\left(Z\left(s\right) \leq \left[1+\xi\left(s\right)\frac{y-\mu\left(s\right)}{\sigma\left(s\right)}\right]^{1/\xi\left(s\right)} /\vartheta_\alpha\left(s\right)\right)$$

Furthermore

$$\begin{aligned} &Pr\left(Z\left(s\right) \leq \left[1+\xi\left(s\right)\frac{y-\mu\left(s\right)}{\sigma\left(s\right)}\right]^{1/\xi\left(s\right)} /\vartheta_\alpha\left(s\right)\right) \\ &= \exp\left(-\vartheta_\alpha\left(s\right)^{1/\alpha_z}\left[1+\xi\left(s\right)\frac{y-\mu\left(s\right)}{\sigma\left(s\right)}\right]^{-1/\alpha\xi\left(s\right)}\right) \end{aligned} \tag{15}$$

According to (15)

$$Pr\left(Y\left(s\right) \leq y/\vartheta_\alpha\left(s\right)\right) = \exp\left(-\vartheta_\alpha\left(s\right)^{1/\alpha_z}\left[1+\xi\left(s\right)\frac{y-\mu\left(s\right)}{\sigma\left(s\right)}\right]^{-1/\alpha\xi\left(s\right)}\right)$$

posing for $s \in S$

$$\begin{cases} \mu^{\star}(s) = \mu(s) + \dfrac{\sigma(s)}{\xi(s)}(\vartheta_{\alpha}(s)^{\xi(s)} - 1) \\ \\ \sigma^{\star}(s) = \alpha\sigma(s)\vartheta_{\alpha}(s)^{\xi}(s) \\ \\ \xi^{\star}(s) = \alpha\xi(s) \end{cases} ;$$

by successive identifications of $\xi^{\star}(s)$, $\sigma^{\star}(s)$ and $\mu^{\star}(s)$, it comes that:

$$Pr(Y(s) \leq y/\vartheta_{\alpha}(s)) = \exp\left(-\left[\vartheta_{\alpha}(s)^{-\xi(s)} + \xi^{\star}(s)\,\frac{y-\mu(s)}{\alpha\vartheta_{\alpha}(s)^{\xi(s)}\sigma(s)}\right]^{-1/\xi^{\star}(s)}\right);$$

which gives

$$Pr(Y(s) \leq y/\vartheta_{\alpha}(s)) = \exp\left(-\left[\vartheta_{\alpha}(s)^{-\xi(s)} + \xi^{\star}(s)\,\frac{y-\mu(s)}{\sigma^{\star}(s)}\right]^{-1/\xi^{\star}}\right) \tag{16}$$

However, taking equation (16), we have

$$\begin{aligned} &\exp\left(-\left[\vartheta_{\alpha}(s)^{-\xi(s)} + \xi^{\star}(s)\,\frac{y-\mu(s)}{\sigma^{\star}(s)}\right]^{-1/\xi^{\star}}\right) \\ = &\exp\left(-\left[1+\vartheta_{\alpha}(s)^{-\xi(s)} - 1 + \xi^{\star}(s)\,\frac{y-\mu(s)}{\sigma^{\star}(s)}\right]^{-1/\xi^{\star}(s)}\right) \\ = &\exp\left(-\left[1+\frac{\xi^{\star}(s)\,\sigma^{\star}(s)}{\sigma^{\star}(s)\,\xi^{\star}(s)}\left(\vartheta_{\alpha}(s)^{-\xi(s)} - 1\right) + \xi^{\star}(s)\,\frac{y-\mu(s)}{\sigma^{\star}(s)}\right]^{-1/\xi^{\star}(s)}\right) \end{aligned}$$

Let's put

$$\mathbb{K}_{\xi^{\star},\,\sigma^{\star},\,\mu^{\star}}(y) = \exp\left(-\left[1+\frac{\xi^{\star}(s)\,\sigma^{\star}(s)}{\sigma^{\star}(s)\,\xi^{\star}(s)}\left(\vartheta_{\alpha}(s)^{-\xi(s)} - 1\right) + \xi^{\star}(s)\,\frac{y-\mu(s)}{\sigma^{\star}(s)}\right]^{-1/\xi^{\star}(s)}\right) \tag{17}$$

Which gives

$$\mathbb{K}_{\xi^\star,\ \sigma^\star,\ \mu^\star}(y) = \exp\left(-\left[1+\frac{\xi^\star(s)}{\sigma^\star(s)}\left(y-\left[\mu(s)-\frac{\xi^\star(s)}{\sigma^\star(s)}\left(\vartheta_\alpha(s)^{-\xi(s)}-1\right)\right]\right)\right]^{-1/\xi^\star(s)}\right)$$

$$= \exp\left(-\left[1+\xi^\star(s)\frac{y-\mu^\star(s)}{\sigma^\star(s)}\right]^{-1/\xi^\star(s)}\right)$$

As a result, we obtain

$$Pr(Y(s)\leq y/\vartheta_\alpha(s)) = \exp\left(-\left[1+\xi^\star(s)\frac{y-\mu^\star(s)}{\sigma^\star(s)}\right]^{-1/\xi^\star(s)}\right)$$

□

At the current rate of warming, the earth is experiencing dangerous climate shifts. To this end, we define the return level as the value x_t such that we expect to detect on average a single overshoot of this quantity at the end of t period, i.e.

$$\mathbb{E}\left(\sum_{i=1}^{t} I_{X_i>X_Y}\right) = 1 \Longleftrightarrow Pr(X_i > x_t) = \frac{1}{t} \Longleftrightarrow 1-F(x_t) = \frac{1}{t}$$

The estimator of a return level of order t amounts to the estimation of an extreme quantile of order $p_t = 1-\dfrac{1}{t}$.

For the Pickands estimator, let $(k_n)_{n>0}$ be an integer sequence such that $1<k_n<n$, $k_n\to+\infty$ and $\lim_{n\to\infty}\dfrac{k_n}{n}=0$ then: $\hat{\xi}^H_{k_n}$ converges in probability to ξ.

Furthermore, if $\lim_{n\to\infty}\dfrac{k_n}{\log\log n}=+\infty$ then: $\hat{\xi}^H_{k_n}$ almost surely converges to ξ.

Under additional conditions on the sequence k_n and the repartition function F, we will have:

$$\sqrt{k_n}\left(\hat{\xi}^P_{k_n}-\xi\right) \text{ converges in law } \mathscr{N}\left(0,\ \frac{\xi^2\left(2^{2\xi+1}+1\right)}{4(\log 2)^2\left(2^{\xi}-1\right)^2}\right).$$

Turning to Hill's estimator, let $(k_n)_{n>0}$ be an integer sequence such that $1<k_n<n,\ k_n\to+\infty$ and $\lim_{n\to\infty}\dfrac{k_n}{n}=0$ then: $\hat{\xi}^H_{k_n}$ converges in probability to ξ .

Furthermore, if $\lim_{n\to\infty}\dfrac{k_n}{\log\log n}=+\infty$ then: $\hat{\xi}^H_{k_n}$ converges almost surely to ξ.

Also, if other variation conditions are verified with $\sqrt{k_n}\varepsilon\left(\dfrac{n}{k}\right)\to 0$ then: $\sqrt{k_n}\left(\hat{\xi}^H_{k_n}-\xi\right)$ converges in law to $\mathscr{N}\left(0,\ \xi^2\right)$ (asymptotic normality).

As far as moments are concerned, suppose $F\in D\left(H_\gamma\right)$, $\gamma\in\mathbb{R}$, $k\to\infty$ and $\dfrac{k}{n}\to 0$ when $n\to\infty$ we have: $\hat{\gamma}^M_n$ converges in probability to γ and almost certainly converges to γ moreover, if $\gamma\geq 0$ then $\sqrt{k}\left(\hat{\gamma}^M_n-\gamma\right)\longrightarrow^d\mathscr{N}\left(0,\ 1+\gamma^2\right)$ in Figure 1 and 2.

The precision behavior of the different estimators presented and illustrated in the graphs below highlights the error of the three estimators as a function of sample size for two levels 1% and 3% in Figure 3 and 4.

Our findings focus primarily on the Hill and moments estimators.

We note that:

- Hill's estimator is relatively more volatile and less efficient than moments', especially for $k < 0.02 \times n$.
- The moments estimator starts with a significant deviation, but it gets closer as the sample size increases.
- The Hill estimator varies between -0.0425 and 0.0154 for a level of 1% and 3% and between -0.038 and 0.029 for a level not exceeding 1%. For the Moment estimator, it varies between -0.026 and 0.0003 for a level of 1% and 3% and between -0.0446 and 0.0019 for a level of 1%. So, as we go down, the curves get closer together, and on the other hand, the curves become more rapidly linear and aligned with 0, meaning that the estimator converges more rapidly to the true value. In addition, the dominance of the moments estimator is clearly visible.

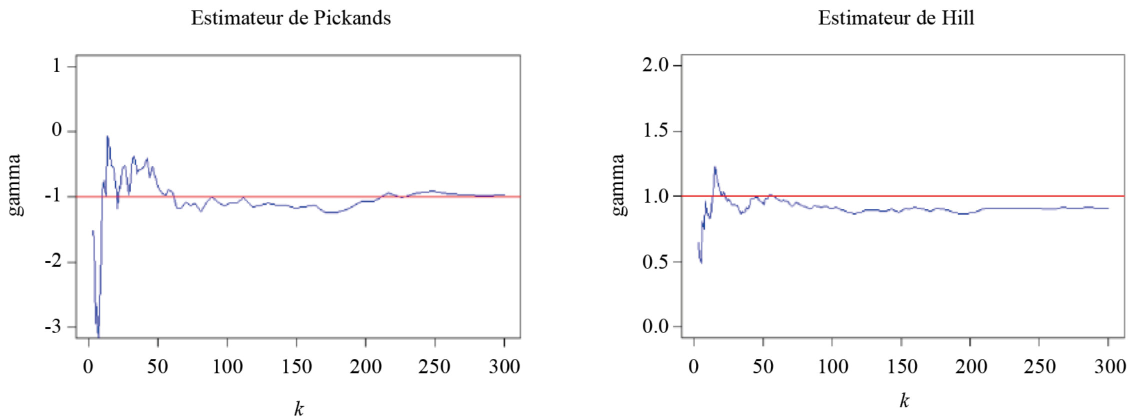

Figure 1. To the left: graphical representation of the Pickands estimator, to the righ: graphical representation of the Hill estimator

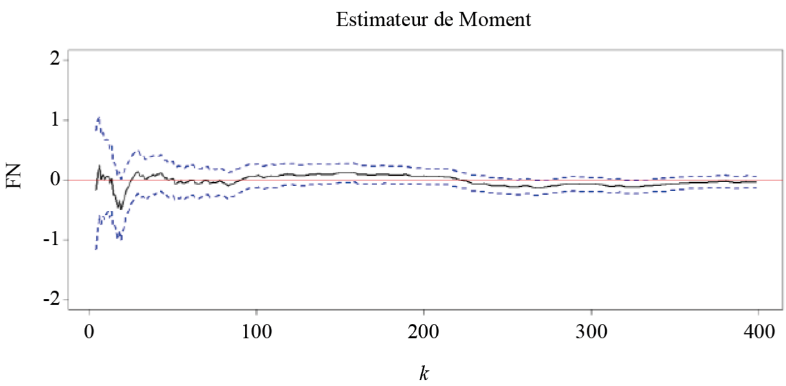

Figure 2. Graphical representation of the moment estimator

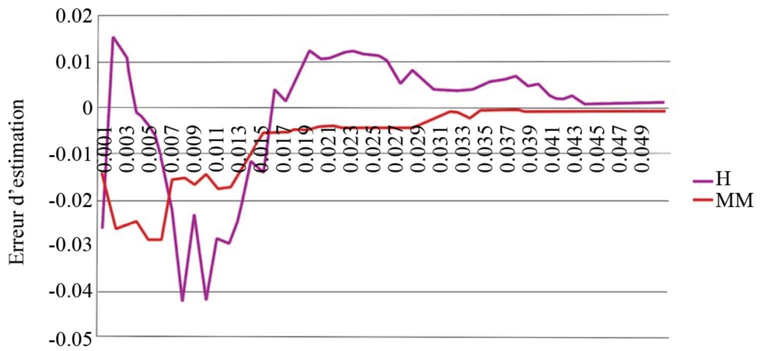

Figure 3. Comparison of estimation error a 1%

Figure 4. Comparison of estimation error a 3%

The Hill estimator makes it possible to estimate the tail parameter of the distributions belonging to the Frchet domain of attraction, that is to say, when the tail of the distribution has a Pareto shape. Its disadvantage is that it is designed only for heavy tailed distributions. Thus, the Moments estimator is an extension of all types of distribution. The Hill estimator is more volatile and less efficient, however the School Moments estimator has a significant deviation.

Using simulated data generated by our model, we can explore different precipitation estimation methods and compare their performance. We can test various estimators, adjust the parameters and evaluate their effectiveness in terms of estimating the amount of water fallen. This allows us to identify the most appropriate method to use in real-world studies, based on statistical criteria such as precision and bias.

The simulated data allows us to test the robustness of the estimators in the face of hypothetical situations. For example, we can simulate different extreme weather conditions, such as intense storms or prolonged periods of drought. By evaluating the performance of the estimators in these fictitious scenarios, we can determine their ability to provide estimators with reliable and consistent estimates, taking into account the statistical notions of reliability and stability.

The use of simulated nopus data allows the design of more effective empirical studies on precipitation.

We use the function $\hat{A}$ (hkevp.rand $\hat{A}$) of the hkevp package of the R software which allows to simulate Hierarchical Kernel Extreme Value Process (HKEVP) realizations on a set of sites $\{s_1, \ldots, s_d\}$ given. To achieve our objectives, we first estimate the parameters of shape, location and scale of the GEV law for each site, using the $\hat{A}$ function (extrapol.gev $\hat{A}$) $\hat{A}$ (extrapol.return.level $\hat{A}$) package $\hat{A}$ hkevp $\hat{A}$ software R.

α and τ describe the dependency structure.

The generalized extreme value distribution models the behavior of a sample maximum.

To determine the distribution and estimate these parameters, 1,000 observations are programmed in *R* software.

The results in *R* language are as follows (Table 1).

As part of our research, we generated nine random samples, each consisting of a thousand observations, which gave us nine means $\{\mu_1, \ldots, \mu_9\}$ and nine standard deviation $\{\sigma_1, \ldots, \sigma_9\}$. In order to estimate the overall sample mean, we obtained mean estimators from each sample (Table 2). In this context, we plan to take the mean of these mean estimators as an aggregate estimator, subject to compliance with the rules for reducing variability, the size of the samples (which are all identical in our case) and the necessary precautions for avoid bias.

We then set the parameters of the HKEVP model.

The mean is statistically significant at the 5%, because they belong to the confidence interval. According to the results obtained, the 3 parameters are statistically significant at the $\alpha = 5\%$.

The location parameter estimator is included in the confidence interval for a threshold of $\alpha = 5\%$.

The interval width is 0.0531, so the smaller the interval, the higher the quality.

The dispersion (scale) parameter is included in a confidence interval [1.3946; 1.4488].

The most important parameter in extreme distributions is the tail index, and the estimator of this parameter $\hat{\xi}$ is included in a confidence interval and tends towards zero.

Table 1. The results in *R* language

μ	σ	ξ
1.790064	0.4116456	0.08547153
1.113826	0.3196874	0.08547153
1.575862	0.4909657	0.08547153
1.689209	0.4035377	0.08547153
1.005482	0.3358132	0.08547153
1.214773	0.3776105	0.08547153
1.952399	0.5632503	0.08547153
1.222140	0.5355208	0.08547153
1.231882	0.4192192	0.08547153

Table 2. The result of aggregate estimator

$\mu^\star$	$\sigma^\star$	$\xi^\star$
1.421737	0.4285833	0.08547153

The first phase, from 2019 to 2029, during which the projections show a slight increase in precipitation over West Africa reaching a maximum of around 3,500 mm in 2029. Then a trend reversal in the second phase from 2029 to 2079 during which we see a significant drop in precipitation, reaching a minimum reaching a minimum water level of around 1,550 mm in 2079 (Figure 5).

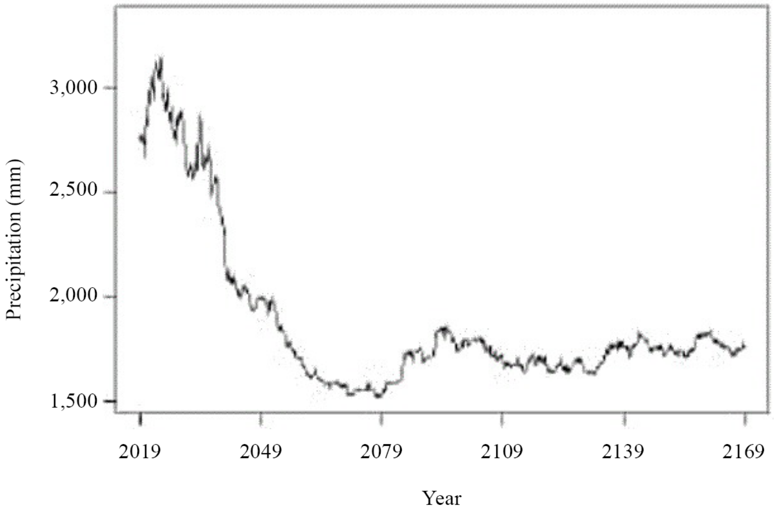

Figure 5. Evolution of a precipitation time series in West Africa from 2019 to 2169

This, of course, can be explained by the production of CO_2 through the activities of major industries that pollute the atmosphere, the release of which has an impact on global rainfall, and many other factors. The third phase (2079-2169) saw a slight increase in rainfall before maintaining a relatively stable trend fluctuating between between 1,500 and 1,700 mm for the rest of the period.

4. Conclusion and discussion

In the present work, we have made a contribution to this theme. In particular, the results allow us to model and describe the independence of the realizations of the Y process conditional on the latent random effect A allows us to calculate the likelihood of the observed annual maxima, by taking the product of the density functions of the GEV laws for each site.

The HKEVP model is a model co-designed by Reich Shaby in 2012. This model is both hierarchical and max-stable. It is a spatial model which makes it possible to consider information on an entire region around the ungauged position concerned and to produce a prediction of the risk measure. We can also predict the value of the observed phenomenon having a spatial character.

In addition, the Reich Shaby model gives us a temporal framework for modeling extreme events (extreme precipitation), and we have also used it to study the evolution of extreme precipitation in West Africa from 2019 to 2169 using simulated data in Figure 6. We also estimated the extreme quantiles, which led to the estimation of the return period and the return level. The most interesting fact is that they verify the max-stability property, which makes them related to max-stable processes. Fortunately, the latter are used in statistical modeling.

	[, 1]	[, 2]	[, 3]	[, 4]	[, 5]	[, 6]	[, 7]	[, 8]	[, 9]
[1,]	8.108420	18.49269	29.44826	11.097666	18.83944	32.65889	10.496451	19.74966	30.8796
[2,]	14.313110	26.29352	39.66581	12.773951	24.16402	37.80536	12.066507	24.33997	33.5244
[3,]	13.027656	24.77852	34.58589	12.345308	25.70013	35.67988	12.801413	26.28803	34.9723
[4,]	12.227611	26.06495	32.95405	11.904525	21.48754	32.60804	14.872884	21.54240	33.4289
[5,]	27.568955	39.30990	44.88471	27.428609	37.06056	46.69871	27.138240	35.63714	45.5562
[6,]	16.039673	30.21164	41.16860	20.398975	29.82575	42.42711	23.642201	33.43062	49.1399
[7,]	21.557188	30.79765	42.15663	21.532783	33.86522	41.60522	20.935267	32.11514	39.9384
[8,]	12.841213	19.01094	30.04119	9.935146	21.18551	27.97317	16.063373	18.76786	32.5525
[9,]	9.544167	22.67959	32.63755	8.748154	24.43465	33.39169	10.761766	23.15470	32.6201
[10,]	12.935858	21.38235	30.77612	10.361468	21.51879	28.05179	8.646094	16.39985	28.0338
[11,]	11.618198	18.00726	29.89859	10.853827	20.60877	28.74446	11.116566	22.42694	30.5724
[12,]	14.063472	20.98810	25.88163	13.147826	25.79983	31.09857	14.437855	22.67469	29.6508
[13,]	15.948667	25.74883	39.41256	14.809824	28.08518	36.58472	12.373487	20.84123	31.7984
[14,]	11.014865	23.77349	37.27770	14.726301	24.58729	37.47665	10.302986	23.83354	37.1537
[15,]	12.899863	20.41718	31.99663	9.488307	19.87235	33.23836	7.590983	20.22626	34.7146

Figure 6. Predictive volume construction

Looking ahead, it would be interesting to construct coherent estimators for predictive measures in the context of extreme values for possible applications to real data.

Conflict of interest

The authors claim that there is no conflict of interest.

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