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Research Article

Fekete-Szegö Inequality for a Subclass of Bi-Univalent Functions Linked to q-Ultraspherical Polynomials

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Abstract: In this study, we introduce a new class of bi-univalent functions using q-Ultraspherical polynomials. We derive the Fekete and Szegö functional problems for functions in this new subclass, as well as estimates for the Taylor-Maclaurin coefficients $|\alpha_2|$ and $|\alpha_3|$. Furthermore, a collection of fresh outcomes is presented by customizing the parameters employed in our initial discoveries.

***Keywords*:** analytic functions, subordination, orthogonal polynomial, q-Ultraspherical polynomials, Fekete-Szegö problem, quantum calculus, univalent, bi-univalent functions

MSC: 30C45, 30C50

1. Introduction

The field of quantum calculus, also known as q-calculus, extends traditional calculus to incorporate the principles of quantum mechanics. The branch of mathematics known as q-calculus is characterized by the introduction of a novel parameter, q, which serves to generalize classical calculus concepts and techniques. This field has been observed to possess a wide range of applications across diverse domains of mathematics, physics, and engineering. The theory of q-orthogonal polynomials (q-op) is a highly significant and extensively researched domain within the field of q-calculus.

The start of the q-op theory can be traced back to the research conducted by Leonard Carlitz and his contemporaries during the 1940s and 1950s. In his work, Carlitz [1] introduced a novel class of polynomials referred to as q-polynomials. These polynomials are characterized by a specific recurrence relation that involves the q-analog of the factorial function. Askey and Wilson extended the theory of q-op by generalizing the aforementioned polynomials [2].

q-op are a family of polynomials that are orthogonal with respect to a certain weight function that depends on the parameter q. These polynomials have been found to have many applications in various areas of mathematics and physics, including number theory, combinatorics, statistical mechanics, and quantum mechanics.

There are several types of q-op, including q-Hermite, q-Jacobi, q-Laguerre, and q-Gegenbauer polynomials, among others. Each type of q-op has its own recurrence relation, weight function, and orthogonality properties, for comprehensive study see ([3–8]).

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The study of q-op has led to the development of many important results and techniques in q-calculus, including the q-analog of the binomial theorem, q-difference equations, and q-special functions. The theory of q-op has also been used to study q-integrals and q-series, which are important tools in the study of q-calculus. In recent times, Quesne [8] has reformulated Jackson's q-exponential as a self-contained series of regular exponentials with well-defined coefficients. This development carries significant implications for the theory of q-op in the given context and should be duly noted.

The theory of orthogonal polynomials has been extensively studied due to its numerous applications in many fields of mathematics and physics. In recent years, the use of orthogonal polynomials and their analogs has become an important tool for studying analytic functions in the complex plane, particularly bi-univalent functions.

2. Preliminaries

Let $\mathscr{A}$ be the class of functions $\digamma$ of the form

$$\digamma(\zeta) = \zeta + \sum_{n=2}^{\infty} \alpha_n \zeta^n, \ \ (\zeta \in \mathbb{U}). \tag{1}$$

which are analytic in the disk $\mathbb{U} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ that satisfy the normalization condition of $\digamma'(0) - 1 = 0 = \digamma(0)$. The subclass of $\mathscr{A}$ that consists of functions of Eq. (1) and are univalent in $\mathbb{U}$ is denoted by $\mathscr{S}$. For every function $\digamma$ belonging to the set $\mathscr{S}$, there exists an inverse function denoted by $\digamma^{-1}$ and defined by

$$\zeta = \digamma^{-1}(\digamma(\zeta)), \quad \varpi = \digamma(\digamma^{-1}(\varpi)) \quad \left(r_0(\digamma) \geq \frac{1}{4};\ |\varpi| < r_0(\digamma);\ \zeta \in \mathbb{U}\right)$$

where

$$\hbar(\varpi) = \digamma^{-1}(\varpi) = \varpi\left(1 - \varpi^3(\alpha_4 + 5\alpha_2^3 - 5\alpha_3\alpha_2) + \varpi^2(-\alpha_3 + 2\alpha_2^2) + \alpha_2\varpi + \cdots\right). \tag{2}$$

A function is regarded as bi-univalent in a given domain $\mathbb{U}$ if both $\digamma(\zeta)$ and its inverse function $\digamma^{-1}(\zeta)$ are univalent, or one-to-one, within $\mathbb{U}$.

The subclass denoted by Σ in $\mathscr{S}$ is defined by indicating the class of bi-univalent functions in $\mathbb{U}$ as given by (1). Examples of the class Σ functions include

$$\digamma_1(\zeta) = \frac{\zeta}{1-\zeta}, \ \digamma_2(\zeta) = -\log(1-\zeta) \ \text{ and } \ \digamma_3(\zeta) = \frac{1}{2}\log\left(\frac{1+\zeta}{1-\zeta}\right).$$

The inverse functions that correspond to the aforementioned functions:

$$\hbar_1(\varpi) = \frac{\varpi}{1+\varpi}, \ \hbar_2(\varpi) = \frac{e^{2\varpi}-1}{e^{2\varpi}+1} \ \text{ and } \ \hbar_3(\varpi) = \frac{e^{\varpi}-1}{e^{\varpi}}.$$

The utilization of differential subordination of analytical functions can yield significant advantages for the field of geometric function theory. The initial differential subordination problem was formulated by Miller and Mocanu [9], which has been further discussed in [10]. The book authored by Miller and Mocanu [11] presents a comprehensive overview of the advancements made in the field, accompanied by their respective dates of publication.

This article presents an overview of q-calculus, initially introduced by Jackson and subsequently explored by numerous mathematicians [12–16]. It focuses on introducing key concepts and definitions within the realm of q-calculus. Additionally, it highlights the significance of the q-difference operator, widely employed in scientific disciplines such as geometric function theory. Emphasizing the assumption that q lies within the interval $(0,\ 1)$, the study extensively relies on fundamental definitions and properties of q-calculus, as extensively documented by Gasper and Rahman in their work [7].

Definition 1 Let $0 < q < 1$. The symbol $[\kappa]_q$ represents the fundamental quantity known as the q-number. It is formally defined as such

$$[\kappa]_q = \begin{cases} \dfrac{1-q^{\kappa}}{1-q}, & \text{if} \quad \kappa \in \mathbb{C}\backslash\{0\} \\ q^{n-1}+q^{n-2}+\cdots+q+1 = \sum\limits_{i=0}^{n-1} q^i, & \text{if} \quad \kappa = n \in \mathbb{N} \\ 1, & \text{if} \quad q \to 0^+,\ \kappa \in \mathbb{C}\backslash\{0\} \\ \kappa, & \text{if} \quad q \to 1^-,\ \kappa \in \mathbb{C}\backslash\{0\} \end{cases}. \tag{3}$$

Definition 2 [17] The q-derivative, also known as the q-difference operator, of a function $\digamma$ is defined by

$$\partial_{\mathrm{q}} \digamma(\zeta) = \begin{cases} \dfrac{\digamma(\zeta)-\digamma(\mathrm{q}\zeta)}{\zeta-\mathrm{q}\zeta}, & \text{if} \quad 0<\mathrm{q}<1,\ \zeta \neq 0, \\ \digamma'(0), & \text{if} \quad \zeta = 0, \\ \digamma'(\zeta), & \text{if} \quad \mathrm{q} \to 1^-,\ \zeta \neq 0. \end{cases}$$

and $\partial_{\mathrm{q}}^n \digamma(\zeta) = \partial_{\mathrm{q}}(\partial_{\mathrm{q}}^{n-1} \digamma(\zeta))$ for $n > 1$ where $\partial_{\mathrm{q}}^2 \digamma(\zeta) = \partial_{\mathrm{q}}(\partial_{\mathrm{q}} \digamma(\zeta))$.

In view of Definition 2, for $\digamma$ of the form (1), we have

$$\partial_{\mathrm{q}} \digamma(\zeta) = 1 + \sum_{n=2}^{\infty} [n]_{\mathrm{q}} \alpha_n \zeta^{n-1}$$

$$\partial_{\mathrm{q}}^2 \digamma(\zeta) = \partial_{\mathrm{q}} \left(1 + \sum_{n=2}^{\infty} [n]_{\mathrm{q}} \alpha_n \zeta^{n-1}\right) = \sum_{n=2}^{\infty} [n]_{\mathrm{q}} [n-1]_{\mathrm{q}} \alpha_n \zeta^{n-2}$$

In 1996, Koekoek and Swarttouw [4] introduced the q-Gegenbauer polynomials (q-gp), also known as the continuous q-ultraspherical polynomials, and they are related to the q-analogue of the Laplace operator and q-special functions. The q-gp have been studied in several areas of mathematics and physics, including harmonic analysis, number theory, quantum groups, and statistical mechanics (see [5, 6]).

The $\mathfrak{G}_q^{(\eth)}(\aleph,\ \zeta)$, referred to as the q-ultraspherical polynomials, are a set of orthogonal polynomials that are defined on the interval $[-1,\ 1]$. These polynomials are defined with respect to the weight function $(1-\aleph^2)^{\eth-\frac{1}{2}}$ on the same interval, and feature a q-analog. Askey and Ismail [5] identified a category of q-generalized polynomials, commonly referred to as q-gp, which are essentially the following polynomials

$$\mathfrak{G}_q^{(\eth)}(\aleph,\ \zeta)=\sum_{n=0}^{\infty}\mathscr{C}_n^{(\eth)}(\aleph;\ q)\zeta^n,\quad (\aleph\in[-1,\ 1],\ \zeta\in\mathbb{U}). \tag{4}$$

Chakrabarti et al. [6] in 2006, discovered the initial terms of GP's q-analog in 2006, which listed below:

$$\begin{aligned}
&\mathscr{C}_0^{(\eth)}(\aleph;\ q)=1\\
&\mathscr{C}_1^{(\eth)}(\aleph;\ q)=2[\eth]_q\aleph\\
&\mathscr{C}_2^{(\eth)}(\aleph;\ q)=2\left([\eth]_{q^2}+[\eth]_q^2\right)\aleph^2-[\eth]_{q^2}
\end{aligned} \tag{5}$$

Orthogonal polynomials have played a crucial role in investigating bi-univalent functions, leading to significant advancements and perspectives in the field of geometric function theory. Recently, there has been a notable surge of interest in exploring subsets of bi-univalent functions associated with orthogonal polynomials, particularly Gegenbauer and Chebyshev polynomials. Through this research, estimations for the initial coefficients of these functions have been derived. However, the exact bounds for the coefficients $|\alpha_n|$ for $n=3,\ 4,\ 5,\ \ldots$ still remain unresolved, as indicated in various studies [18–22] and more recently in [23–26].

However, in the year 2023, Amourah et al. [27] and Alsoboh et al. [28] developed different classifications of analytic bi-univalent functions by employing q-ultraspherical polynomials, also commonly known as q-Gegenbauer polynomials. This current investigation establishes the Fekete-Szegö inequalities and determines the coefficient bounds $|\alpha_2|$ and $|\alpha_3|$ for functions belonging to the mentioned subclasses.

The primary objective of this study is to initiate an investigation into the characteristics of bi-univalent functions that are associated with q-ultraspherical polynomials. In order to attain this objective, the definitions enumerated below are taken into consideration.

3. Definition and examples

Within this section, novel subclasses of bi-univalent functions are presented. These subclasses are subordinate to the q-gp.

Definition 3 When $\beta\geq 0$. A function $\digamma$ in the set Σ, defined by equation (1), is considered to be a member of the class $\mathscr{B}_\Sigma(\beta,\ \mathfrak{G}_q^{(\eth)}(\aleph,\ \zeta))$ if certain subordinations are met:

$$(1-\beta)\frac{\zeta\partial_q\digamma(\zeta)}{\digamma(\zeta)}+\beta\left(1+\frac{\zeta\partial_q^2\digamma(\zeta)}{\partial_q\digamma(\zeta)}\right)\prec\mathfrak{G}_q^{(\eth)}(\aleph,\ \zeta), \tag{6}$$

and

$$(1-\beta)\frac{\varpi\partial_q\hbar(\varpi)}{\hbar(\varpi)}+\beta\left(1+\frac{\varpi\partial_q^2\hbar(\varpi)}{\partial_q\hbar(\varpi)}\right)\prec\mathfrak{G}_q^{(\eth)}(\aleph,\ \varpi), \tag{7}$$

where $\aleph \in \left(\frac{1}{2}, 1\right]$ and $q \in (0, 1)$, the function $\hbar(\varpi)$ is defined as the inverse of $\digamma^{-1}(\varpi)$ of the form (2) and $\mathfrak{G}_q^{(\eth)}$ is the generating function of the ordinary (gp) given by (4).

Example 1 Let $\beta = 0$. We say that a function $\digamma \in \Sigma$ given by (1) belongs to the class $\mathscr{B}_\Sigma(0, \mathfrak{G}_q^{(\eth)}(\aleph, \zeta))$ if it satisfies the following subordinations:

$$\frac{\zeta \partial_q \digamma(\zeta)}{\digamma(\zeta)} \prec \mathfrak{G}_q^{(\eth)}(\aleph, \zeta), \text{ and } \frac{\varpi \partial_q \hbar(\varpi)}{\hbar(\varpi)} \prec \mathfrak{G}_q^{(\eth)}(\aleph, \varpi),$$

where $\aleph \in \left(\frac{1}{2}, 1\right]$ and $q \in (0, 1)$, the function $\hbar(\varpi)$ is defined as the inverse of $\digamma^{-1}(\varpi)$ of the form (2) and $\mathfrak{G}_q^{(\eth)}$ is the generating function of the ordinary (gp) given by (4).

Example 2 Let $\beta = 1$. We say that a function $\digamma \in \Sigma$ given by (1) belongs to the class $\mathscr{B}_\Sigma(1, \mathfrak{G}_q^{(\eth)}(\aleph, \zeta))$ if it satisfies the following subordinations:

$$1 + \frac{\zeta \partial_q^2 \digamma(\zeta)}{\partial_q \digamma(\zeta)} \prec \mathfrak{G}_q^{(\eth)}(\aleph, \zeta) \text{ and } 1 + \frac{\varpi \partial_q^2 \hbar(\varpi)}{\partial_q \hbar(\varpi)} \prec \mathfrak{G}_q^{(\eth)}(\aleph, \varpi),$$

where $\aleph \in \left(\frac{1}{2}, 1\right]$ and $q \in (0, 1)$, the function $\hbar(\varpi)$ is defined as the inverse of $\digamma^{-1}(\varpi)$ of the form (2) and $\mathfrak{G}_q^{(\eth)}$ is the generating function of the ordinary (gp) given by (4).

Example 3 For $q \to 1^-$, we say that a function $\digamma \in \Sigma$ given by (1) belongs to the class $\mathscr{B}_\Sigma(\beta, \mathfrak{G}_1^{(\eth)}(\aleph, \zeta))$ if it satisfies the following subordinations:

$$(1-\beta)\frac{\zeta \digamma'(\zeta)}{\digamma(\zeta)} + \beta\left(1 + \frac{\zeta \digamma''(\zeta)}{\digamma'(\zeta)}\right) \prec \mathfrak{G}_1^{(\eth)}(\aleph, \zeta),$$

and

$$(1-\beta)\frac{\varpi \hbar'(\varpi)}{\hbar(\varpi)} + \beta\left(1 + \frac{\varpi \hbar''(\varpi)}{\hbar'(\varpi)}\right) \prec \mathfrak{G}_1^{(\eth)}(\aleph, \varpi),$$

where $\aleph \in \left(\frac{1}{2}, 1\right]$ and the function $\hbar(\varpi)$ is defined as the inverse of $\digamma^{-1}(\varpi)$ of the form (2) and $\mathfrak{G}_q^{(\eth)}$ is the generating function of the ordinary (gp) given by (4).

4. The bounds of the coefficients within the $\mathscr{B}_\Sigma(\beta, \mathfrak{G}_q^{(\eth)}(\aleph, \zeta))$ class

Initially, the estimates for the coefficients of the class $\mathscr{B}_\Sigma(\beta, \mathfrak{G}_q^{(\eth)}(\aleph, \zeta))$, as defined in Definition 4, are provided.

Theorem 1 If $\digamma$ is an element of Σ defined by (1), it can be said that $\digamma$ is a member of the class $\mathscr{B}_\Sigma(\beta, \mathfrak{G}_q^{(\eth)}(\aleph, \zeta))$, as per the following statement:

$$|\alpha_2| \leq \frac{2|[\eth]_q|\aleph . \sqrt{2[\eth]_q \aleph}}{\sqrt{4[\eth]_q^2\left[q^2[2]_q\beta + q^2(1-\beta)\right]\aleph^2 - \left(q+\beta\right)^2\left(2\left([\eth]_{q^2} + [\eth]_q^2\right)\aleph^2 - [\eth]_{q^2}\right)}},$$

and

$$|\alpha_3| \leq \frac{4[\eth]_q^2\aleph^2}{\left(q+\beta\right)^2} + \frac{2\left|[\eth]_q\right|\aleph}{\left|[2]_q[3]_q\beta + \left([3]_q - 1\right)(1-\beta)\right|}.$$

If $\digamma$ belongs to the class $\mathscr{B}_{\Sigma}(\beta,\ \mathfrak{G}_q^{(\eth)}(\aleph,\ \zeta))$, according to Definition 3, there exist some analytic functions ω and v such that $\omega(0) = v(0) = 0$, and $|\omega(\zeta)| < 1$, $|v(\varpi)| < 1$ for all $\zeta,\ \varpi \in \mathbb{U}$. Under these conditions, we can express $\digamma$ as follows

$$(1-\beta)\frac{\zeta\partial_q\digamma(\zeta)}{\digamma(\zeta)} + \beta\left(1 + \frac{\zeta\partial_q^2\digamma(\zeta)}{\partial_q\digamma(\zeta)}\right) = \mathfrak{G}_q^{(\eth)}(\aleph,\ \omega(\zeta)) \tag{8}$$

and

$$(1-\beta)\frac{\varpi\partial_q\hbar(\varpi)}{\hbar(\varpi)} + \beta\left(1 + \frac{\varpi\partial_q^2\hbar(\varpi)}{\partial_q\hbar(\varpi)}\right) = \mathfrak{G}_q^{(\eth)}(\aleph,\ v(\varpi)), \tag{9}$$

From Definition 2, it is clear that

$$\begin{aligned}(1-\beta)\frac{\zeta\partial_q\digamma(\zeta)}{\digamma(\zeta)} + \beta\left(1 + \frac{\zeta\partial_q^2\digamma(\zeta)}{\partial_q\digamma(\zeta)}\right) &= 1 + \left(q+\beta\right)\alpha_2\zeta + \\ \left(\left\{\beta[3]_q[2]_q + \left([3]_q - 1\right)(1-\beta)\right\}\alpha_3 - \left\{[2]_q^2\beta + (1-\beta)q\right\}\alpha_2^2\right)\zeta^2 + \cdots,\end{aligned} \tag{10}$$

and

$$\begin{aligned}&(1-\beta)\frac{\varpi\partial_q\hbar(\varpi)}{\hbar(\varpi)} + \beta\left(1 + \frac{\varpi\partial_q^2\hbar(\varpi)}{\partial_q\hbar(\varpi)}\right) = \\ &\left\{\begin{array}{l} 1 - \left(q+\beta\right)\alpha_2\omega - \left(\left\{[2]_q[3]_q\beta + \left([3]_q - 1\right)(-\beta+1)\right\}\alpha_3 + \right. \\ \left.\left\{2\left[[3]_q[2]_q\beta + ([3]_q - 1)(-\beta+1)\right] - \left[[2]_q^2\beta + (1-\beta)q\right]\right\}\alpha_2^2\right)\omega^2 + \cdots \end{array}\right\}.\end{aligned} \tag{11}$$

By utilizing equations (8) and (9), we can derive the following expression.

$$
\begin{aligned}
&(1-\beta)\frac{\zeta\partial_q F(\zeta)}{F(\zeta)}+\beta\left(1+\frac{\zeta\partial_q^2 F(\zeta)}{\partial_q F(\zeta)}\right)\\
&=1+C_1^{(\Game)}(\aleph;\, q)c_1\zeta+\left[C_1^{(\Game)}(\aleph;\, q)c_2+C_2^{(\Game)}(\aleph;\, q)c_1^2\right]\zeta^2+\cdots,
\end{aligned}
\tag{12}
$$

and

$$
\begin{aligned}
&(1-\beta)\frac{\varpi\partial_q \hbar(\varpi)}{\hbar(\varpi)}+\beta\left(1+\frac{\varpi\partial_q^2 \hbar(\varpi)}{\partial_q \hbar(\varpi)}\right)\\
&=1+C_1^{(\Game)}(\aleph;\, q)d_1\varpi+\left[C_1^{(\Game)}(\aleph;\, q)d_2+C_2^{(\Game)}(\aleph;\, q)d_1^2\right])\varpi^2+\cdots.
\end{aligned}
\tag{13}
$$

It is generally understood that if

$$
|\omega(\zeta)|=\left|c_1\zeta+c_2\zeta^2+c_3\zeta^3+\cdots\right|<1, \quad (\zeta\in\mathbb{U})
$$

and

$$
|v(\omega)|=\left|d_1\varpi+d_2\varpi^2+d_3\varpi^3+\cdots\right|<1, \quad (\varpi\in\mathbb{U}),
$$

then

$$
|c_j|\leq 1 \text{ and } |d_j|\leq 1 \text{ for all } j\in\mathbb{N}. \tag{14}
$$

In view of (1), (2), from (12) and (13), we obtain

$$
\begin{aligned}
&1+\big(q+\beta\big)\alpha_2\zeta+\big(\big\{\beta[3]_q[2]_q+\big([3]_q-1\big)(1-\beta)\big\}\alpha_3-\big\{[2]_q^2\beta+(1-\beta)q\big\}\alpha_2^2\big)\zeta^2+\cdots\\
&=1+C_1^{(\Game)}(\aleph;\, q)c_1\zeta+\left[C_1^{(\Game)}(\aleph;\, q)c_2+C_2^{(\Game)}(\aleph;\, q)c_1^2\right]\zeta^2+\cdots
\end{aligned}
$$

and

$$
\begin{aligned}
&1-\left(q+\beta\right)\alpha_2\omega-\Bigg(\Big\{[2]_q[3]_q\beta+\left([3]_q-1\right)\left(-\beta+1\right)\Big\}\alpha_3\\
&+\Big\{2\Big[[3]_q[2]_q\beta+([3]_q-1)\,(-\beta+1)\Big]-\Big[[2]_q^2\beta+(1-\beta)q\Big]\Big\}\,\alpha_2^2\Bigg)\omega^2+\cdots\\
&=1+C_1^{(\backepsilon)}(\aleph;\,q)d_1\omega+\Big[C_1^{(\backepsilon)}(\aleph;\,q)d_2+C_2^{(\backepsilon)}(\aleph;\,q)d_1^2\Big]\,\omega^2+\cdots.
\end{aligned}
$$

By comparing the pertinent coefficients in (12) and (13), we arrive at the following.

$$
(q+\beta)\alpha_2=C_1^{(\backepsilon)}(\aleph;\,q)c_1, \tag{15}
$$

$$
-(q+\beta)\alpha_2=C_1^{(\backepsilon)}(\aleph;\,q)d_1, \tag{16}
$$

$$
\begin{aligned}
&\left(\left\{[2]_q[3]_q\beta+\left([3]_q-1\right)(1-\beta)\right\}\alpha_3-\left\{[2]_q^2\beta+q(1-\beta)\right\}\alpha_2^2\right)\\
&=C_1^{(\backepsilon)}(\aleph;\,q)c_2+C_2^{(\backepsilon)}(\aleph;\,q)c_1^2,
\end{aligned} \tag{17}
$$

and

$$
\begin{aligned}
&-\Big\{[2]_q[3]_q\beta+\left([3]_q-1\right)(1-\beta)\Big\}\alpha_3+\Big\{2\Big[[2]_q[3]_q\beta+([3]_q-1)\,(1-\beta)\Big]\\
&-\Big[[2]_q^2\beta+q(1-\beta)\Big]\Big\}\,\alpha_2^2=C_1^{(\backepsilon)}(\aleph;\,q)d_2+C_2^{(\backepsilon)}(\aleph;\,q)d_1^2.
\end{aligned} \tag{18}
$$

It follows from (15) and (16) that

$$
c_1=-d_1 \tag{19}
$$

and

$$
\begin{aligned}
&2\left(q+\beta\right)^2\alpha_2^2=\Big[C_1^{(\backepsilon)}(\aleph;\,q)\Big]^2\left(c_1^2+d_1^2\right)\\
&\alpha_2^2=\frac{\Big[C_1^{(\backepsilon)}(\aleph;\,q)\Big]^2}{2\left(q+\beta\right)^2}\left(c_1^2+d_1^2\right)\Longleftrightarrow c_1^2+d_1^2=\frac{2\left(q+\beta\right)^2}{\Big[C_1^{(\backepsilon)}(\aleph;\,q)\Big]^2}\alpha_2^2.
\end{aligned} \tag{20}
$$

Adding (17) and (18), we get

$$2\Big[[2]_q\big(-[2]_q+[3]_q\big)\beta+([3]_q-q-1)(-\beta+1)\Big]\ \alpha_2^2=C_1^{(\eth)}(\aleph;\,q)(c_2+d_2)+C_2^{(\eth)}(\aleph;\,q)(c_1^2+d_1^2). \tag{21}$$

Substituting the value of $(c_1^2+d_1^2)$ from (20), we obtain

$$\alpha_2^2=\frac{\Big[C_1^{(\eth)}(\aleph;\,q)\Big]^3(c_2+d_2)}{2\left(\Big[[2]_q([3]_q-[2]_q)\beta+([3]_q-1-q)(1-\beta)\Big]\Big[C_1^{(\eth)}(\aleph;\,q)\Big]^2-(q+\beta)^2\,C_2^{(\eth)}(\aleph;\,q)\right)}. \tag{22}$$

Applying for the coefficients c_2 and d_2 and using (5), we obtain

$$|\alpha_2|\leq\frac{2\big|[\eth]_q\big|\aleph.\sqrt{2[\eth]_q\aleph}}{\sqrt{\Big|4[\eth]_q^2\Big[q^2[2]_q\beta+q^2(1-\beta)\Big]\aleph^2-(q+\beta)^2\left(2\left([\eth]_{q^2}+[\eth]_q^2\right)\aleph^2-[\eth]_{q^2}\right)\Big|}}.$$

By subtracting (18) from (17), we get

$$2\Big\{[2]_q[3]_q\beta+\big([3]_q-1\big)(1-\beta)\Big\}(\alpha_3-\alpha_2^2)=C_1^{(\eth)}(\aleph;\,q)\,(c_2-d_2)+C_2^{(\eth)}(\aleph;\,q)\,\big(c_1^2-d_1^2\big). \tag{23}$$

Then, in view of (19) and (20), Eq. (23) becomes

$$\alpha_3=\frac{\Big[C_1^{(\eth)}(\aleph;\,q)\Big]^2}{2(\beta+q)^2}\big(c_1^2+d_1^2\big)+\frac{C_1^{(\eth)}(\aleph;\,q)}{2\Big\{[2]_q[3]_q\beta+\big([3]_q-1\big)(1-\beta)\Big\}}(c_2-d_2)$$

Thus applying (5), we conclude that

$$|\alpha_3|\leq\frac{4[\eth]_q^2\aleph^2}{(q+\beta)^2}+\frac{2\big|[\eth]_q\big|\aleph}{\big|[2]_q[3]_q\beta+\big([3]_q-1\big)(1-\beta)\big|}.$$

This completes the proof of Theorem.

5. Fekete and Szegö functional

Fekete and Szegö established a precise limit for the functional $\eta\alpha_2^2-\alpha_3$ in their 1933 publication [29]. The limit was derived using real values of η $(0\leq\eta\leq1)$ and has been commonly known as the classical Fekete and Szegö outcome. Establishing precise boundaries for a given function within a compact family of functions $\mathcal{F}\in\mathscr{A}$, and for any complex η, poses a formidable challenge. The Fekete-Szegö inequality for functions belonging to the class $\mathscr{B}_\Sigma(\beta,\,\mathfrak{G}_q^{(\eth)}(\aleph,\,\zeta))$ is examined in view of Zaprawa's [30] finding.

Theorem 2 Given that $\digamma$ is an element of Σ defined by (1) and belongs to the class $\mathscr{B}_{\Sigma}(\beta,\ \mathfrak{G}_q^{(\eth)}(\aleph,\ \zeta))$, and η is a real number, we can state the following

$$\left|\alpha_3-\eta\alpha_2^2\right| \leq \begin{cases} \dfrac{2\left|[\eth]_q\right|\aleph}{[2]_q[3]_q\beta+([3]_q-1)(1-\beta)}, & |1-\eta| \leq \vartheta(\beta,\ \eth,\ \aleph;\ q), \\ \\ 4\left|[\eth]_q\right|\aleph\left|\mathscr{H}(\eta)\right|, & |1-\eta| \geq \vartheta(\beta,\ \eth,\ \aleph;\ q), \end{cases}$$

where

$$\mathscr{H}(\eta)=\frac{(1-\eta)\left[C_1^{(\eth)}(\aleph;\ q)\right]^2}{2\left(\left[q^2[2]_q\beta+q^2(1-\beta)\right]\left[C_1^{(\eth)}(\aleph;\ q)\right]^2-(q+\beta)^2\ C_2^{(\eth)}(\aleph;\ q)\right)}$$

and

$$\vartheta(\beta,\ \eth,\ \aleph;\ q)=\left|\frac{\left[q^2[2]_q\beta+q^2(1-\beta)\right]\left[C_1^{(\eth)}(\aleph;\ q)\right]^2-(q+\beta)^2\ C_2^{(\eth)}(\aleph;\ q)}{[[2]_q[3]_q\beta+([3]_q-1)(1-\beta)]\left[C_1^{(\eth)}(\aleph;\ q)\right]^2}\right|.$$

If $\digamma\in\mathscr{B}_{\Sigma}(\beta,\ \mathfrak{G}_q^{(\eth)}(\aleph,\ \zeta))$ is given by (1), from (22) and (23), we have

$$\alpha_3-\eta\alpha_2^2=\frac{C_1^{(\eth)}(\aleph;\ q)}{2\left\{\beta[3]_q[2]_q+([3]_q-1)(1-\beta)\right\}}(c_2-d_2)$$

$$+\frac{(1-\eta)\left[C_1^{(\eth)}(\aleph;\ q)\right]^3(c_2+d_2)}{2\left(\left[[2]_q([3]_q-[2]_q)\beta+([3]_q-1-q)(-\beta+1)\right]\left[C_1^{(\eth)}(\aleph;\ q)\right]^2-(\beta+q)^2\ C_2^{(\eth)}(\aleph;\ q)\right)}$$

$$=C_1^{(\eth)}(\aleph;\ q)\left(\left[\mathscr{H}(\eta)+\frac{1}{2\left\{[2]_q[3]_q\beta+([3]_q-1)(1-\beta)\right\}}\right]c_2\right.$$

$$\left.+\left[\mathscr{H}(\eta)-\frac{1}{2\left\{[2]_q[3]_q\beta+([3]_q-1)(-\beta+1)\right\}}\right]d_2\right)$$

where

$$\mathscr{H}(\eta)=\frac{(1-\eta)\left[C_1^{(\partial)}(\aleph;\,q)\right]^2}{2\left(\left[q^2[2]_q\beta+q^2(-\beta+q)\right]\left[C_1^{(\partial)}(\aleph;\,q)\right]^2-(\beta+q)^2C_2^{(\partial)}(\aleph;\,q)\right)}$$

Then, we conclude that

$$\left|\alpha_3-\eta\alpha_2^2\right|\le\begin{cases}\dfrac{\left|C_1^{(\partial)}(\aleph;\,q)\right|}{[2]_q[3]_q\beta+([3]_q-1)(1-\beta)}, & |\mathscr{H}(\eta)|\le\dfrac{1}{2\{[2]_q[3]_q\beta+([3]_q-1)(1-\beta)\}},\\[2ex] 2\left|C_1^{(\partial)}(\aleph;\,q)\right|\left|\mathscr{H}(\eta)\right|, & |\mathscr{H}(\eta)|\ge\dfrac{1}{2\{[2]_q[3]_q\beta+([3]_q-1)(1-\beta)\}}.\end{cases}$$

Which completes the proof of Theorem 2.

6. Corollaries

The following corollaries, which roughly match Examples 1, 2 and 3, are produced by Theorems 1 and Theorems 2.

Corollary 1 If $\digamma$ is an element of Σ defined by (1) and belongs to the class $\mathscr{B}_\Sigma(0,\,\mathfrak{G}_q^{(\partial)}(\aleph,\,\zeta))$, then we can state the following

$$|\alpha_2|\le\frac{2|[\partial]_q|\aleph.\sqrt{2[\partial]_q\aleph}}{q\sqrt{2\left([\partial]_{q^2}-[\partial]_q^2\aleph^2\right)+[\partial]_{q^2}}},\quad |\alpha_3|\le\frac{4[\partial]_q^2\aleph^2}{q^2}+\frac{2|[\partial]_q|\aleph}{q(1+q)},$$

and

$$\left|\alpha_3-\eta_1\alpha_2^2\right|\le\begin{cases}\dfrac{|[\partial]_q|\aleph}{q(1+q)}, & |1-\eta_1|\le q\left|\dfrac{2\left([\partial]_q^2-[\partial]_{q^2}\right)\aleph^2+[\partial]_{q^2}}{4(1+q)[\partial]_q\aleph^3}\right|,\\[2ex] \dfrac{8|[\partial]_q\aleph|^3|1-\eta_1|}{q^2\left[2\left([\partial]_q^2-[\partial]_{q^2}\right)\aleph^2+[\partial]_{q^2}\right]}, & |1-\eta_1|\ge q\left|\dfrac{2\left([\partial]_q^2-[\partial]_{q^2}\right)\aleph^2+[\partial]_{q^2}}{4(1+q)[\partial]_q\aleph^3}\right|.\end{cases}$$

Corollary 2 If $\digamma$ is an element of Σ defined by (1) and belongs to the class $\mathscr{B}_\Sigma(1,\,\mathfrak{G}_q^{(\partial)}(\aleph,\,\zeta))$, then we can state the following

$$\left|\alpha_2\right| \leq \frac{2[\eth]_q \aleph \sqrt{2|[\eth]_q| \aleph}}{\sqrt{[2]_q\left(\left((2[3]_q-3[2]_q)[\eth]_q^2-2[2]_q[\eth]_{q^2}\right)\aleph^2+[2]_q[\eth]_{q^2}\right)}},$$

$$|\alpha_3| \leq \frac{4[\eth]_q^2 \aleph^2}{(q+1)^2}+\frac{2\left|[\eth]_q\right| \aleph}{[2]_q[3]_q},$$

and

$$\left|\alpha_3-\eta_2 \alpha_2^2\right| \leq \begin{cases} \dfrac{2[\eth]_q \aleph}{[2]_q[3]_q}, & |1-\eta_2| \leq \vartheta(1,\ \eth,\ \aleph;\ q), \\ 4[\eth]_q \aleph\left|\mathscr{H}(\eta_2)\right|, & |1-\eta_2| \geq \vartheta(1,\ \eth,\ \aleph;\ q), \end{cases}$$

where

$$\mathscr{H}(\eta_2)=\frac{(1-\eta_2)\left[C_1^{(\eth)}(\aleph;\ q)\right]^2}{2\left(q^2[2]_q\left[C_1^{(\eth)}(\aleph;\ q)\right]^2-(q+1)^2 C_2^{(\eth)}(\aleph;\ q)\right)}$$

and

$$\vartheta(1,\ \eth,\ \aleph;\ q)=\left|\frac{q^2[2]_q\left[C_1^{(\eth)}(\aleph;\ q)\right]^2-(q+1)^2 C_2^{(\eth)}(\aleph;\ q)}{[2]_q[3]_q\left[C_1^{(\eth)}(\aleph;\ q)\right]^2}\right|.$$

Corollary 3 If $\digamma$ is an element of Σ defined by (1) and belongs to the class $\mathscr{B}_\Sigma(\beta,\ \mathfrak{G}_1^{(\eth)}(\aleph,\ \zeta))$, then we can state the following

$$|\alpha_2| \leq \frac{2|\eth| \aleph \sqrt{2\eth \aleph}}{\sqrt{4\eth^2[\beta+1]\aleph^2-(1+\beta)^2\left(2\eth(1+\eth)\aleph^2-\eth\right)}},$$

$$|\alpha_3| \leq \frac{4\eth^2 \aleph^2}{(1+\beta)^2}+\frac{|\eth| \aleph}{|2\beta+1|}.$$

and

$$\left|\alpha_3-\eta_3\alpha_2^2\right| \leq \begin{cases} \dfrac{|\eth|\aleph}{2\beta+1}, & |1-\eta_3| \leq \vartheta(\beta,\, \eth,\, \aleph), \\ \\ 4|\eth|\aleph\left|\mathscr{H}(\eta_3)\right|, & |1-\eta_3| \geq \vartheta(\beta,\, \eth,\, \aleph), \end{cases}$$

where

$$\mathscr{H}(\eta_3)=\frac{4\eth^2\aleph^2(1-\eta_3)}{2[\beta+1]\left(2\left[2\eth^2-\eth\ (1+\eth)(1+\beta)\right]\aleph^2+(1+\beta)\eth\right)}$$

and

$$\vartheta(\beta,\, \eth,\, \aleph)=\left|\frac{4(\beta+1)\eth^2\aleph^2-(1+\beta)^2(2\eth(1+\eth)\aleph^2+\eth)}{8(2\beta+1)\eth^2\aleph^2}\right|.$$

7. Conclusion

In this study, we have investigated the coefficient problems associated with each of the novel subclasses of bi-univalent functions defined in Definitions 3 within the disk $\mathbb{U}$. These subclasses include $\mathscr{B}_\Sigma(\beta,\ \mathfrak{G}_q^{(\eth)}(\aleph,\ \zeta))$, $\mathscr{B}_\Sigma(\beta,\ \mathfrak{G}_1^{(\eth)}(\aleph,\ \zeta))$, $\mathscr{B}_\Sigma(0, \mathfrak{G}_q^{(\eth)}(\aleph,\ \zeta))$ and $\mathscr{B}_\Sigma(1,\ \mathfrak{G}_q^{(\eth)}(\aleph,\ \zeta))$. For functions belonging to each of these bi-univalent function classes, we have estimated the Taylor-Maclaurin coefficients $|\alpha_2|$ and $|\alpha_3|$, as well as the Fekete-Szegö functional problem estimates.

Upon the parameter specialization of our main findings, we have identified several additional new results. It is expected that the q-differential operator will have wide-ranging applications in various scientific fields, including mathematics and technology.

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Confilict of interest

The authors declare no competing financial interest.

References

[1] Carlitz L. Some polynomials related to theta functions. *Annali di Matematica pura ed Applicata*. 1956; 41(1): 359-373.

[2] Askey R, Wilson JA. *Some Basic Hypergeometric Orthogonal Polynomials that Generalize Jacobi Polynomials*. vol. 319. American Mathematical Soc.; 1985.

[3] Kiepiela K, Naraniecka I, Szynal J. The Gegenbauer polynomials and typically real functions. *Journal of Computational and Applied Mathematics*. 2003; 153(1-2): 273-282.

[4] Koekoek R, Swarttouw RF. The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue. *arXiv*. 1996. Available from: https://doi.org/10.48550/arXiv.math/9602214.

[5] Askey R, Ismail MEH. *A Generalization of Ultraspherical Polynomials*. Springer; 1983. Available from: https://doi.org/10.1007/978-3-0348-5438-2_6.

[6] Chakrabarti R, Jagannathan R, Mohammed SSN. New connection formulae for the q-orthogonal polynomials via a series expansion of the q-exponential. *Journal of Physics A: Mathematical and General*. 2006; 39(40): 12371.

[7] Gasper G, Rahman M. *Basic Hypergeometric Series*. vol. 96. Cambridge: Cambridge university press; 2011.

[8] Quesne C. Disentangling q-exponentials: a general approach. *International Journal of Theoretical Physics*. 2004; 43: 545-559.

[9] Miller SS, Mocanu PT. Second order differential inequalities in the complex plane. *Journal of Mathematical Analysis and Applications*. 1978; 65(2): 289-305.

[10] Miller SS, Mocanu PT. Differential subordinations and univalent functions. *Michigan Mathematical Journal*. 1981; 28(2): 157-172.

[11] Miller SS, Mocanu PT. *Differential Subordinations: Theory and Applications*. Florida: CRC Press; 2000.

[12] Aldweby H, Darus M. On a subclass of bi-univalent functions associated with the q-derivative operator. *Journal of Mathematics and Computer Science*. 2019; 19(1): 58-64.

[13] Al-Salam WA. Some fractional q-integrals and q-derivatives. *Proceedings of the Edinburgh Mathematical Society*. 1966; 15(2): 135-140.

[14] Alsoboh A, Darus M. New subclass of analytic functions defined by q-differential operator with respect to k-symmetric points. *International Journal of Mathematics and Computer Science*. 2019; 14: 761-773.

[15] Al-Hawary T, Amourah A, Alsoboh A, Ogilat O, Harny I, Darus M. Applications of q-Ultraspherical polynomials to bi-univalent functions defined by q-Saigo's fractional integral operators. *AIMS Mathematics*. 2024; 9(7): 17063-17075.

[16] Alsoboh A, Amourah A, Darus M, Rudder CA. Studying the harmonic functions associated with quantum calculus. *Mathematics*. 2023; 11(10): 2220.

[17] Ahuja O, Bohra N, Cetinkaya A, Kumar S. Univalent functions associated with the symmetric points and cardioid-shaped domain involving (p, q)-calculus. *Kyungpook Mathematical Journal*. 2021; 61(1): 75-98.

[18] Buyankara M, Çaglar M. On Fekete-Szegö problem for a new subclass of bi-univalent functions defined by Bernoulli polynomials. *Acta Universitatis Apulensis*. 2022; 71: 137-145.

[19] Çağlar M. Chebyshev polynomial coefficient bounds for a subclass of bi-univalent functions. *Comptes rendus de l'Academie bulgare des Sciences*. 2019; 72(12): 1608-1615.

[20] Çaglar M, Orhan H, Kamali M. Fekete-Szego problem for a subclass of analytic functions associated with chebyshev polynomials. *Bolem Sociedade Paranaemse de matematica*. 2022; 40: 1-6.

[21] Çağlar M, Cotîrlă LI, Buyankara M. Fekete-Szegö inequalities for a new subclass of bi-univalent functions associated with Gegenbauer polynomials. *Symmetry*. 2022; 14(8): 1572.

[22] Hu Q, Shaba TG, Younis J, Khan B, Mashwani WK, Çağlar M. Applications of q-derivative operator to subclasses of bi-univalent functions involving Gegenbauer polynomials. *Applied Mathematics in Science and Engineering*. 2022; 30(1): 501-520.

[23] Kamali M, Çağlar M, Deniz E, Turabaev M. Fekete-Szegö problem for a new subclass of analytic functions satisfying subordinate condition associated with Chebyshev polynomials. *Turkish Journal of Mathematics*. 2021; 45(3): 1195-1208.

[24] Atshan WG, Rahman IAR, Lupaş AA. Some results of new subclasses for bi-univalent functions using quasi-subordination. *Symmetry*. 2021; 13(9): 1653.

[25] Alnajar O, Ogilat O, Amourah A, Darus M, Alatawi MS. The Miller-Ross Poisson distribution and its applications to certain classes of bi-univalent functions related to Horadam polynomials. *Heliyon*. 2024; 10(7): e28302.

[26] Tayyah AS, Atshan WG. New Results on (r, k, μ)-riemann-liouville fractional operators in complex domain with applications. *Fractal and Fractional*. 2024; 8(3): 165.

[27] Amourah A, Alsoboh A, Ogilat O, Gharib GM, Saadeh R, Al Soudi M. A generalization of Gegenbauer polynomials and bi-univalent functions. *Axioms*. 2023; 12(2): 128.

[28] Alsoboh A, Amourah A, Darus M, Rudder CA. Investigating new subclasses of bi-univalent functions associated with q-pascal distribution series using the subordination principle. *Symmetry*. 2023; 15(5): 1109.
[29] Fekete M, Szegö G. Eine Bemerkung über ungerade schlichte Funktionen. *Journal of the London Mathematical Society*. 1933; 1(2): 85-89.
[30] Zaprawa P. On the Fekete-Szegö problem for classes of bi-univalent functions. *Bulletin of the Belgian Mathematical Society-Simon Stevin*. 2014; 21(1): 169-178.