Contemporary Mathematics
https://ojs.wiserpub.com/index.php/CM/

Research Article

The Approximate Numerical Solutions to First Order Non-Linear Differential Equations and Their Connections to the Orthogonal Double Cover in Graph Theory

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Received: 6 October 2023; **Revised:** 31 October 2023; **Accepted:** 2 November 2025

Abstract: The main objective of this study is to make a linkage between Bernoulli's differential equations and graph theory using simple technique. Firstly, we transform an Orthogonal Double Cover (briefly, ODC) to metric graph. Then, we use the Generalized Fibonacci Polynomials (GFP) to transform the non-linear differential equation into a system of equations with undetermined constants. As a conclusion, some numerical examples were solved and the error was evaluated which prove the accuracy of the studied method.

***Keywords*:** orthogonal double cover, symmetric starter, generalized Fibonacci polynomials, collocation method

MSC: 05C70, 05B30

1. Introduction

Throughout this work, we make use of usual notations: As the following $K_{m,\, n}$ denotes the complete bipartite graph with m-vertices that are labeled by $a_{1_0},\, a_{2_0},\, \ldots,\, a_{m_0}$ in one partite set and n-vertices that are labeled by $a_{1_1},\, a_{2_1},\, \ldots,\, a_{n_1}$ in the other partite set [1].

For the complete bipartite graph $K_{n,\, n}$ with $2n$ vertices, the family $\mathscr{G} = \{G_0,\, G_1,\, \ldots,\, G_{(2n-1)}\}$ of $2n$ subgraphs (briefly, pages) of $K_{n,\, n}$. This family is an Orthogonal Double Cover (ODC) of $K_{n,\, n}$ if it fit the following properties:

1. Every edge of $K_{n,\, n}$ is contained in exactly two pages in $\mathscr{G}$ (double cover property).
2. For any two distinct pages G_c and G_d in $\mathscr{G}$, $|E(G_c) \cap E(G_d)| = 1$ if and only if c and d are adjacent in $K_{n,\, n}$ (orthogonality property).

If $G_c \cong G$ for all $c \in \{0,\, 1,\, \ldots,\, 2n-1\}$, then $\mathscr{G}$ is an ODC of $K_{n,\, n}$ by G [2].

Recently, the spectral methods evaluate the approximated solution of the differential equations. These methods have a lot of advantages such as small error and small number of unknowns. The algorithm of these methods depends on the solution which can be expressed as the expansion of the polynomials. In addition, theses methods investigate different kinds of differential and integral equations in the form of fractional differential equations such as: solving

DOI: https://doi.org/10.37256/cm.6420253740

fractional pantograph differential equation, multi-term initial value problems and Volterra-Fredholm integral equation using generalized Lucas polynomials [3–7]. Different attempts of solving the variable order Space-Fractional diffusion equations, system of fractional differential equations, Volterra-Fredholm integral equations and Volterra-Fredholm integral differential equations where performed via Generalized Fibonacci Polynomials (GFP) [8–13]. The framework of our study is to reformulate an ODC graphs to metric graphs, then transform it to matrices which are used to solve the non-linear differential equations. Also, we present and analyze a spectral algorithm for solving these equations. Finally, the numerical solutions are evaluated in a series of Fibonacci polynomials, then we apply the spectral collocation method.

Consider the following differential equation

$$P(y)\upsilon^{'}(y)+Q(y)\upsilon(y)=g(y)\upsilon^{m}(y) \tag{1}$$

Where $\upsilon(y)$ is an unknown function. $P(y)$, $Q(y)$ and $g(y)$ are continuous known functions. The organizations of the next sections are summarized as follows: section 2 aims to solve the differential equations using the graph theory as a unique technique away from the exact and the numerical ones. In section 3, the recurrence relation, properties of generalized Fibonacci polynomials and the algorithm of the methods are explored in details. Section 4, we satisfied our theorem for different function of the right-hand-side. In addition, we give some examples which prove the accuracy and efficiency of our method in section 5. At the end, some conclusions are introduced in section 6.

2. The method of the problem

Let us suppose that $G=(V,\,E)$ where $V(G)=\{a_1,\,a_2,\,...,\,a_n\}$ and $E(G)=\{e_1,\,e_2,\,...,\,e_n\}$. If G represents an ODC graph, then one can reformulate this ODC in the following way. Define the outgoing incidence matrix (mapping) Φ^- by

$$\phi_{ij}^{-}=\begin{cases}1 & \text{if } x_j(0)=a_i\\ 0 & \text{otherwise.}\end{cases} \tag{2}$$

and the incoming incidence matrix (mapping) Φ^+ by

$$\phi_{ij}^{+}=\begin{cases}1 & \text{if } x_j(l_j)=a_i\\ 0 & \text{otherwise.}\end{cases} \tag{3}$$

Obviously, $\Phi^+=(\phi_{ij}^{+})$ and $\Phi^-=(\phi_{ij}^{-})$ are $m\times n$ matrices, and they have exactly one nonzero entry in each column if G has no isolated vertex.

If G_i is one of the pages. The new suggested formula for this page takes the vertices a_{n_i} of G_i as $a_{1_i},\,a_{2_i},\,...,\,a_{n_i}$ [14].

Now, look at each one of this vertices and edges between them in G_i. If one of them is connected to itself, this will make a loop, and if it connected to another vertex, we can draw a directed edge starting from the first to the second.

When a vertex is not connected to any other vertex so we shall call it isolated vertex.

The following example illustrates the above discussion in Figure 1.

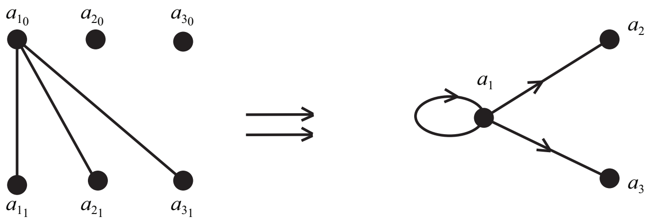

Figure 1. Convert of an ODC of $K_{3,\,3}$ by $K_{1,\,3}$ to new suggested graph G_1

Now, let us turn our attention to impose the non-linear differential equation (1). Where

$$\upsilon(1) = (\phi^+)^T d,\ \upsilon(0) = (\phi^-)^T d \tag{4}$$

and

$$\upsilon(y) = S(y,\ 0)\upsilon(0) + \int_0^y S(y,\ r)P^{-1}(r)g(r)dr$$

where

$$S(y,\ r) = -\int_r^y \frac{Q(t)}{P(t)}dt \tag{5}$$

From conditions in (2). Then, we have

$$[(\phi^+)^T - S(1,\ 0)(\phi^-)^T]d = \breve{g}(1) \tag{6}$$

The function $g(y)$ and $S(y,\ r)$ are given by the information ϕ^+ and ϕ^-. The solution becomes

$$\begin{pmatrix} 1 & 0 & 0 & . & 0 & K_1 \\ 0 & 1 & 0 & . & 0 & K_2 \\ 0 & 0 & 1 & . & 0 & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & 1 & K_n \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ . \\ . \\ d_m \end{pmatrix} = \begin{pmatrix} \breve{g}_1 \\ \breve{g}_2 \\ . \\ . \\ . \\ \breve{g}_n \end{pmatrix} \tag{7}$$

We can choose the right hand side is one of the following basic functions:
1. Polynomial function.
2. sine or cosine functions.
3. Exponential function.

3. Properties and used formulas

In this section, some definitions and properties for the generalized Fibonacci polynomials are stated [9]. These relations have been used in the following section [15–18].

If α and β are non zero real numbers. The recurrence relation for generalized Fibonacci:

$$\zeta_k^{\alpha,\,\beta}(y) = \alpha\, y\, \zeta_{k-1}^{\alpha,\,\beta}(y) + \beta\ \zeta_{k-2}^{\alpha,\,\beta}(y)\,,\ k \geq 2 \tag{8}$$

with the initial condition

$$\zeta_0^{\alpha,\,\beta}(y) = 1,\ \ \zeta_1^{\alpha,\,\beta}(y) = \alpha\, y.$$

it has the Binet's form:

$$\zeta_k^{\alpha,\,\beta}(y) = \frac{\left(\alpha\, y + \sqrt{\alpha^2 y^2 + 4\beta}\right)^k - \left(\alpha\, y - \sqrt{\alpha^2 y^2 + 4\beta}\right)^k}{2^k \sqrt{\alpha^2 y^2 + 4\beta}},\ \ k \geq 0. \tag{9}$$

We evaluate $\upsilon(y)$ as terms in generalized Fibonacci polynomials which has the following form

$$\nu(y) = \sum_{k=0}^{\infty} c_k \zeta_k^{\alpha,\,\beta}(y)$$

Which has the approximate solution

$$\upsilon(y) \approx \upsilon_M(y) \ = \sum_{k=0}^{M} c_k\ \ \zeta_k^{\alpha,\,\beta}(y) = C^T\ \Im(y), \tag{10}$$

Where

$$\Im(y) = \left[\zeta_0^{\alpha.\beta}(y),\ \zeta_1^{\alpha.\beta}(y),\ \ldots,\ \zeta_M^{\alpha.\beta}(y)\right]^T,$$

and the constants which must be determined have the following form

$$C^T = [c_0,\ c_1,\ \ldots,\ c_M].$$

we will discuss the algorithm of the method:

$$\upsilon'(y) = A\Im(y),$$

$$a_{k,\, l} = \begin{cases} (-1)^{\frac{l-k+1}{2}} l\alpha\beta^{\frac{l-k+1}{2}}\gamma_k & \text{if } l > k,\ (l+k) \text{ odd,} \\ 0 & \text{otherwise.} \end{cases}$$

and

$$\gamma_k = \begin{cases} \dfrac{1}{2}, & k = 0 \\ 1, & \text{otherwise.} \end{cases}$$

Substituting these relations in equation (1), −

$$\frac{d}{dy}\left(\sum_{k=0}^{M} c_k\ \zeta_k^{\alpha,\,\beta}(y)\right) + \sum_{k=0}^{M} c_k\ \zeta_k^{\alpha,\,\beta}(y) = g(y).$$

So, we have the following

$$\sum_{k=0}^{M} c_k\ u_k(y) = g(y).$$

where

$$u_k(y) = \frac{d(\zeta_k^{\alpha,\,\beta}(y))}{dy} + \ \zeta_k^{\alpha,\,\beta}(y),$$

By collecting, we have a system of equations

$$\sum_{k=0}^{M} c_k\ u_k(y) = g(y_k). \tag{11}$$

We have the matrix form of these equations:

$$U^T C = G.$$

So, the constant has the form

$$C = \left(U^T\right)^{-1} G,$$

$G = (g(y_k)) = [g(y_0),\ g(y_1),\ \ldots,\ g(y_M)]^T$ with the initial condition

$$C^T\ \Im(0) = 0.$$

Algorithm 1 Coding algorithm for the proposed schema.
Input 1 $P(y),\ Q(y)$ and $g(y)$ continuous functions.
Step 1. Define Fibonacci polynomials by (8).
Step 2. Evaluate the basis function of Fibonacci polynomials by (9).
Step 3. Define the function vector $\Im(y)$ by (10).
Step 4. Collocating Eq. (11) in $(M+1)$ roots.
Step 5. Use Nsolve command in Mathematica software to solve these equations.

In the coming section, we will use the forgoing relations which are defined by the ODC to evaluate the exact of numerical solutions of the given differential equation.

4. Main result

We shall use the next important theorem in ODC for the coming work [19].

Theorem 1 Let $n > 2$ be an even integer, $n \not\equiv 0\ (mod\ 6)$ and $n \not\equiv 0\ (mod\ 10)$. Then there is a symmetric starter of an ODC of $K_{n,\ n}$ by disjoint union of paths.

Theorem 2 By using Theorem 11 in [19], we can explain the solution of (1) using graph theory. We have the form of D in each case.

Case 1 Suppose that $g(y) = y^2$, then the 1_{st} order differential equation has this form

$$P(y)\upsilon^{'}(y) + Q(y)\upsilon(y) = y^2\upsilon^{m}(y)$$

Proof. Using the definition of $\phi^-,\ \phi^+$ in equations (2), (3) to substitute in equation (7), we conclude that

$$\begin{pmatrix} \dfrac{3n+1}{3n}d_1 \\ . \\ . \\ . \\ . \\ . \end{pmatrix} = \begin{pmatrix} \dfrac{((n^4-2n^3+3n^2-2n+2)e^{-n(n-1)})-2}{(n^5-2n^4+n^3)} \\ . \\ . \\ . \\ . \\ . \end{pmatrix}$$

by solving the above matrices, we have conclude that

$$D = \frac{3(((n^4-2n^3+3n^2-2n+2)e^{-n(n-1)})-2)}{(3n^5-5n^4+n^3+n^2)}$$

Example 1 At $n = 4$, $V(G) = (0,\ 0,\ 1,\ 2)$. In the following figure, we convert of an ODC of $K_{4,\ 4}$ by $P_4 \cup P_2$ to new suggested graph G_0 in Figure 2.

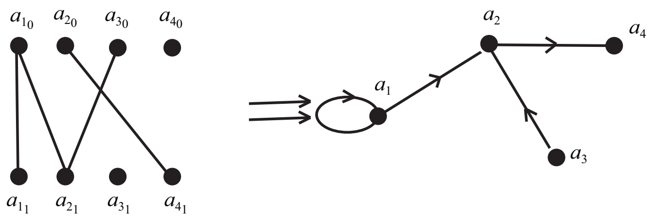

Figure 2. The deformed of $P_4 \cup P_2$ to G_0

$$\phi = \begin{pmatrix} 1,\ -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and then we conclude

$$\phi^+ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},\ \phi^- = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

from (5),

$$S(1,\ 0)I = \begin{pmatrix} \frac{-1}{12} & 0 & 0 & 0 \\ 0 & \frac{-1}{12} & 0 & 0 \\ 0 & 0 & \frac{-1}{12} & 0 \\ 0 & 0 & 0 & \frac{-1}{12} \end{pmatrix}$$

$$\breve{g}(1) = \int_0^1 -3r^2 e^{-12r} dr = \frac{\left(85e^{-12} - 1\right)}{288}$$

then by using equation (6).

$$\begin{pmatrix} \frac{13}{12} & 0 & 0 & 0 \\ \frac{1}{12} & 1 & 0 & 0 \\ 0 & \frac{1}{12} & 0 & 1 \\ 0 & 1 & \frac{1}{12} & 0 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{pmatrix} = \begin{pmatrix} \frac{(85e^{-12}-1)}{288} \\ \frac{(85e^{-12}-1)}{288} \\ \frac{(85e^{-12}-1)}{288} \\ \frac{(85e^{-12}-1)}{288} \end{pmatrix}$$

So, we find that

$$d_1 = d_2 = d_3 = d_4 = \frac{(85e^{-12}-1)}{312}.$$

Example 2 At $n = 8$, $V(G) = (0,\ 0,\ 5,\ 2,\ 7,\ 4,\ 1,\ 6)$. In the following figure, we convert of an ODC of $K_{8,\ 8}$ by $P_4 \cup 2P_3 \cup P_2$ to new suggested graph G_1 in Figure 3.

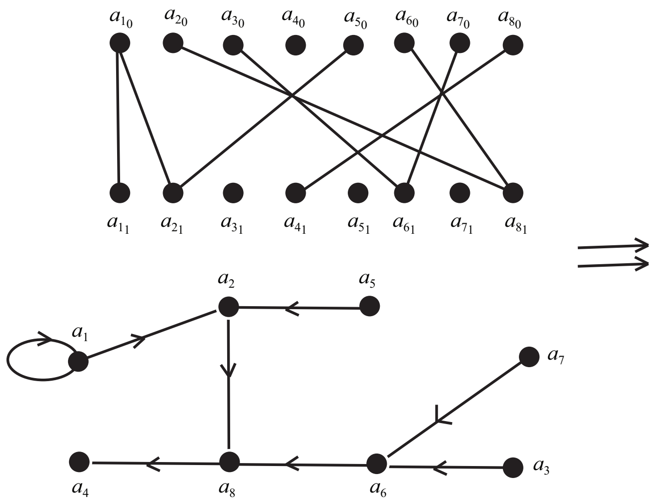

Figure 3. The deformed of $P_4 \cup 2P_3 \cup P_2$ to G_1

$$
\phi = \begin{pmatrix}
1,\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 & -1 & 0 & 1 & 0
\end{pmatrix}
$$

and then we conclude

$$
\phi^{+} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
$$

$$
\phi^{-} = \begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}
$$

from (5),

$$S(1,\ 0)I=\begin{pmatrix} \frac{-1}{24} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{-1}{24} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-1}{24} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-1}{24} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-1}{24} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{-1}{24} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{24} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{24} \end{pmatrix}$$

$$\breve{g}(1)=\int_0^1 -7r^2e^{-56r}dr=\frac{1,625e^{-56}-1}{12,544}$$

then by using equation (6).

$$\begin{pmatrix} \frac{25}{24} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{24} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{24} & 0 & 1 \\ 0 & 0 & \frac{1}{24} & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{24} \\ 0 & 1 & 0 & 0 & \frac{1}{24} & 0 & 0 & 0 \\ 0 & \frac{1}{24} & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{24} & 0 \end{pmatrix}\begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \\ d_7 \\ d_8 \end{pmatrix}=\begin{pmatrix} \frac{1,625e^{-56}-1}{12,544} \\ \frac{1,625e^{-56}-1}{12,544} \\ \frac{1,625e^{-56}-1}{12,544} \\ \frac{1,625e^{-56}-1}{12,544} \\ \frac{1,625e^{-56}-1}{12,544} \\ \frac{1,625e^{-56}-1}{12,544} \\ \frac{1,625e^{-56}-1}{12,544} \\ \frac{1,625e^{-56}-1}{12,544} \end{pmatrix}$$

So, we find that

$$d_1 = d_2 = d_3 = d_4 = d_5 = d_6 = d_7 = d_8 = \frac{3(1,625e^{-56} - 1)}{39,200}.$$

□

Case 2 Let $g(y) = e^y$, then D has the following form

$$D = \frac{(n^2 - n)(e^{-n^2+n+1} - 1)}{(n^2 - n - 1)(e + n - 1)}.$$

Proof. Using the definition of ϕ^-, ϕ^+ in equations (2), (3) similarly as case 2, we conclude that

$$\begin{pmatrix} \frac{1}{n}(e+n-1)d_1 \\ . \\ . \\ . \\ . \\ . \end{pmatrix} = \begin{pmatrix} \frac{(n-1)(e^{-n^2+n+1} - 1)}{(n^2 - n - 1)} \\ . \\ . \\ . \\ . \\ . \end{pmatrix}$$

So, if we equal two matrices. Then, we prove this case. □

Case 3 Let $g(y) = \cos y$, as the same.

Proof. We conclude that

$$\begin{pmatrix} \left(1 + \frac{1}{n}\sin(1)\right) d_1 \\ . \\ . \\ . \\ . \\ . \end{pmatrix} = \begin{pmatrix} \frac{(1-n)\left[e^{-n^2+n}(\sin(1) - (n^2 - n)\cos(1)) + n^2 - n\right]}{n^4 - 2n^3 + n^2 + 1} \\ . \\ . \\ . \\ . \\ . \end{pmatrix}$$

So,

$$D = \frac{(1-n)\left[e^{-n^2+n}(\sin(1) - (n^2 - n)\cos(1)) + n^2 - n\right]}{(n^4 - 2n^3 + n^2 + 1)\left(1 + \frac{1}{n}\sin(1)\right)}.$$

□

Case 4 Suppose that $g(y) = \sin y$, similarly.

Proof. We conclude that

$$\begin{pmatrix} \left(\frac{1}{n}(n+1-\cos(1))\right) d_1 \\ . \\ . \\ . \\ . \\ . \end{pmatrix} = \begin{pmatrix} \frac{(n-1)\left[e^{-n^2+n}(\cos(1)+(n^2-n)\sin(1))-1\right]}{n^4-2n^3+n^2+1} \\ . \\ . \\ . \\ . \\ . \end{pmatrix}$$

So

$$D = \frac{(n^2-n)\left[e^{-n^2+n}(\cos(1)+(n^2-n)\sin(1))-1\right]}{(n^4-2n^3+n^2+1)(n+1-\cos(1))}.$$

□

5. Numerical examples

In this section, we solve some numerical examples.

Example 3 We consider the nonlinear differential equation on case (4) when the function $g(y) = y^2$, as follows:

$$\upsilon'(y) + \upsilon(y) = y^2 \upsilon^m(y) \qquad \upsilon(0) = 0$$

The exact solution of this equation is

$$\upsilon(y) = y^2 - 2y + 2 - 2e^{-y}.$$

In Table 1, 2, lists the numerical results obtained by the proposed method at $n = 2,\ 3,\ 4,\ 5,\ m = 4$ and for different values of a and b. The absolute errors of this method are plotted in Figure 4. We observe from the figure that the convergence is exponential and the errors are better when the values of n and m are large in Table 3, 4.

Table 1. Maximum absolute errors with various values of n and $m = 4$

a	b	$n=2$	$n=3$	$n=4$	$n=5$
1	1	7.3×10^{-3}	6.5×10^{-4}	1.1×10^{-4}	6.6×10^{-5}
2	1	7.3×10^{-3}	6.5×10^{-4}	1.2×10^{-4}	6.6×10^{-5}
2	−1	7.3×10^{-3}	6.5×10^{-4}	1.2×10^{-4}	6.6×10^{-5}
3	−2	7.3×10^{-3}	6.5×10^{-4}	1.2×10^{-4}	6.6×10^{-5}

Table 2. Maximum absolute errors with various values of n and $m = 8$

a	b	$n = 2$	$n = 3$
1	1	5.3×10^{-3}	6.1×10^{-4}
2	1	5.3×10^{-3}	6.1×10^{-4}
2	-1	5.3×10^{-3}	6.1×10^{-4}
3	-2	5.3×10^{-3}	6.1×10^{-4}

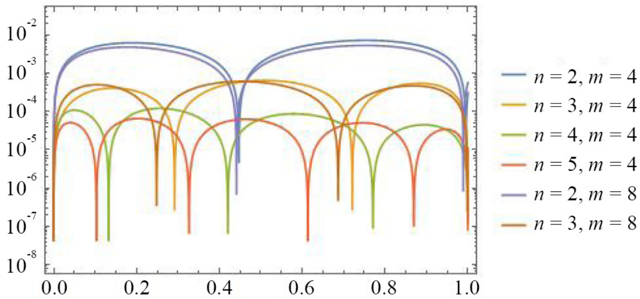

Figure 4. Graph of the error at $N = 3$, 6, 8 and 9

Table 3. CPU time at $n = 2,\ 3,\ 4,\ 5$ and $m = 4$

n	CPU time
2	5.812
3	30.875
4	216.249
5	377.344

Table 4. CPU time at $n = 2,\ 3$ and $m = 8$

n	CPU time
2	27
3	237.219

Example 4 We consider the nonlinear differential equation on case (9) when the function $g(y) = \sin y$, as follows:

$$\upsilon'(y) + \upsilon(y) = (\sin y)\upsilon^m(y) \qquad \upsilon(0) = 0$$

The exact solution of this equation is

$$\upsilon(y) = \frac{1}{2}\sin y - \frac{1}{2}\cos y + \frac{1}{2}e^{-y}.$$

Table 5, 6, there is a comparison between the absolute errors of the present method at different values of m, n, a and b. Figure 5 displays the absolute errors at the same values. The figure shows that the convergence is exponential. The values at $m = 4$ are better than at $m = 8$ in Table 7, 8.

Table 5. Comparison between absolute errors with different values of n and $m = 4$

a	b	$n = 2$	$n = 3$	$n = 4$	$n = 5$
1	1	8.5×10^{-3}	2.5×10^{-3}	2×10^{-3}	3.8×10^{-3}
2	1	8.5×10^{-3}	2.5×10^{-3}	4.7×10^{-3}	3.9×10^{-2}
2	-1	8.5×10^{-3}	2.5×10^{-3}	1.5×10^{-3}	2.4×10^{-3}
3	-2	8.5×10^{-3}	2.5×10^{-3}	1.5×10^{-3}	2.5×10^{-3}

Table 6. Maximum absolute errors with various values of n and $m = 8$

a	b	$n = 2$	$n = 3$
1	1	4.2×10^{-3}	3.4×10^{-2}
2	1	4.2×10^{-3}	3.3×10^{-2}
2	-1	4.2×10^{-3}	3.3×10^{-2}
3	-2	4.2×10^{-3}	4×10^{-3}

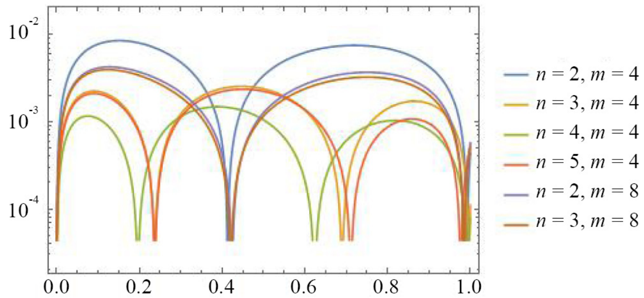

Figure 5. Graph of the absolute error at $N = 3, 6, 8$ and different values of v_1 and v_2

Table 7. CPU time at $n = 2, 3, 4, 5$ and $m = 4$

n	CPU time
2	41.438
3	152.624
4	1,319.11
5	2,467.98

Table 8. CPU time at $n = 2, 3$ and $m = 8$

n	CPU time
2	143.844
3	1,990.11

Example 5 We consider the nonlinear differential equation on case (8) when the function $g(y) = \cos y$, as follows:

$$\upsilon'(y) + \upsilon(y) = (\cos y)\upsilon^m(y) \qquad \upsilon(0) = 0$$

The exact solution of this equation is

$$\upsilon(y) = \frac{1}{2}\cos y + \frac{1}{2}\sin y - \frac{1}{2}e^{-y}.$$

In Table 9, 10, we observe that the absolute errors obtained by our method at $m = 4$ are better than obtained at $m = 8$. In Figure 6, we illustrate the results of the present method at $n = 2, 3, 4, 5$ and $m = 4, 8$ in Table 11, 12.

Table 9. Comparison between absolute errors with different values of n and $m = 4$

a	b	$n = 2$	$n = 3$	$n = 4$	$n = 5$
1	1	3.5×10^{-2}	6.2×10^{-3}	5×10^{-3}	7.1×10^{-3}
2	1	3.5×10^{-2}	6.2×10^{-3}	6.3×10^{-2}	1.6×10^{-1}
2	-1	3.5×10^{-2}	6.2×10^{-3}	5.6×10^{-3}	7.1×10^{-3}
3	-2	3.5×10^{-2}	6.2×10^{-3}	6.9×10^{-3}	3.3×10^{-1}

Table 10. Maximum absolute errors with various values of n and $m = 8$

a	b	$n = 2$	$n = 3$
1	1	1.5×10^{-1}	1.5×10^{-1}
2	1	1.5×10^{-1}	1.5×10^{-1}
2	-1	1.5×10^{-1}	1.5×10^{-1}
3	-2	1.5×10^{-1}	2×10^{-1}

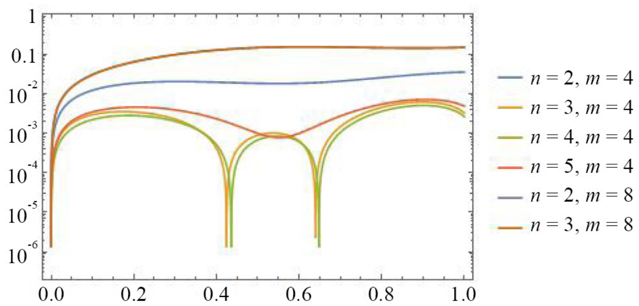

Figure 6. Graph of the error at $N = 3$, 6 and 8

Table 11. CPU time at $n = 2, 3, 4, 5$ and $m = 4$

n	CPU time
2	78.14
3	183.64
4	1,419.19
5	2,928.42

Table 12. CPU time at $n = 2, 3$ and $m = 8$

n	CPU time
2	211.359
3	2,352.86

When increasing the values of $n-$steps, the error increases. So, the method is accurate for small values of n. In this problem

$$\upsilon'(y) + \upsilon(y) = e^y \upsilon^m(y)$$

the solution can not be evaluated. So this function is not convergent.

6. Conclusion

In this article, we studied the conversion of ODC graph to a metric graph then represent it in the form of matrices. From the obtained matrices, we can solve Bernoulli's differential equations. As a result, some exact and numerical examples (which modified to a system of linear equation) can be computed using Mathematica software. Due to the fact that the numerical approximation depends on the generalized Fibonacci polynomials to evaluate errors in each case.The theoretical analysis and effectiveness of the recently created algorithm are supported by numerical evidence. As a conclusion, Results proved that the studied method is strong adequacy and viability. We plan to generalize the offered algorithm to more general differential problems and study the stability in the near future.

Data availability

The data used to support the findings of this study are included within the article.

Conflict of interest

The authors declare that there are not conflict of interest.

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