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Research Article

On the Endomorphism Monoid of Certain Ultrametric Speces

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Abstract: In this work we investigate the endomorphism monoid of certain ultrametric spaces. According to our main result, if $\mathcal{X} = \langle X, \varrho \rangle$ is an ultrametric space such that the range of ϱ is finite, then the set of locally finite endomorphisms is dense in the endomorphism monoid of $\mathcal{X}$ and the endomorphism monoid of $\mathcal{X}$ has a dense, locally finite submonoid. This can be regarded as a homomorphism oriented counterpart of some recently obtained results about the existence of dense, locally finite subgroups of the automorphism group of certain homogeneous structures. Further, as a byproduct, we obtain Hrushovski style extension theorems for the ages of certain ultrametric spaces, but here, instead of partial isomorphisms we extend partial homomorphisms.

***Keywords*:** ultrametric spaces, extension property for homomorphisms, locally finite monoids

MSC: 03C13, 03C15, 20M30

1. Introduction

Continuing investigations initiated in [1–3] we study some model theoretic properties of ultrametric spaces. More concretely, we study extension properties of finite partial endomorphisms of certain ultrametric spaces. This area is still active, and valuable related results can be found e.g., in [4–6]. For further details on recent results about ultrametric spaces we refer to [7]. This work has two motivations.

Our first motivation originates from theoretical computer science. More concretely, assume

- X is a set of instances of an abstract data type and
- ϱ is a distance function on X measuring similarity of elements of X (that is, if $\varrho\ (a,\ b)$ is smaller, then a and b are "more similar").

Suppose we are given a fixed set $A \subseteq X$ and the question is to find all elements $a \in A$ which are "similar enough" to the input x. Versions of this question are called "*similarity detecting problems*". Usually, X may be infinite and often, the distance function ϱ is an ultrametric. For further details we refer to [2, 8]. The crucial point for designing an efficient similarity detecting algorithm is to find a clever representation for A.

The second motivation of this work comes from the model theory of homogeneous structures in general, and from the model theory of metric spaces in particular (see e.g., [5, 6, 9]). Related investigations provide a better understanding how a "nice" countably infinite structure can be built up from its finite substructures. In case of metric or ultrametric spaces,

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such information may help to construct representations of certain (ultra) metric spaces as we mentioned at the end of the previous paragraph. To be more concrete, we recall the following model theoretical results. In his celebrated paper [10], Hrushovski proved that each finite graph $\mathcal{G}$ can be embedded into another finite graph $\mathcal{H}$ such that isomorphisms between subgraphs of $\mathcal{G}$ can be extended to automorphisms of $\mathcal{H}$. With contemporary terminology that result often cited as the class of finite graphs have the *extension property* (EP for short). Since then, many different proofs has been found for the fact that the class of finite graphs have the EP and instead of graphs, and the EP has been proved to many other classes of finite structures. For further details we refer to [11–17] and to the excellent survey [18]. In [19], a particularly interesting proof has been presented for the EP of finite graphs: it was shown that the automorphism group of the Rado graph (i.e. the countable random graph) has a dense, locally finite subgroup. Related investigations received renewed impetus, see [20, 21] where the existence of locally finite, dense subgroups of the automorphism group of certain structures has been thoroughly studied.

The main results of the present work are as follows. As in [4–6, 22, 23], instead of the automorphism group, we will investigate the endomorphism monoid of certain ultrametric spaces. In Theorem 3 we show that the endomorphism monoid of an ultrametric space of finite spectrum always has a dense, locally finite submonoid. Further, as a byproduct, in Theorems 1 and 2 we present Hrushovski style extension theorems for finite substructures of certain ultrametric spaces, but at this time, instead of partial isomorphisms we are extending partial homomorphisms. At the technical level these proofs are short and elementary, but the results can be regarded as a homomorphism oriented counterpart of some results in [16, 17, 21].

The structure of the rest of this paper is rather simple. At the end of this section we sum up our system of notation. Section 2 contains the proofs of the main results of the paper.

1.1 *Notation*

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Our notation is mostly standard, but the following list may be helpful.

Throughout ω denotes the set of natural numbers and for every $n \in \omega$ we have $n = \{0, 1, ..., n-1\}$. Let A and B be sets. Then ${}^{A}B$ denotes the set of functions whose domain is A and whose range is a subset of B. In addition, $|A|$ denotes the cardinality of A.

Throughout we use function composition in such a way that the rightmost factor acts first. That is, for functions f, g we define $f \circ g(x) = f(g(x))$. Further, Id_A is the identity function on A and if $C \subseteq A$ then $f|_C$ denotes the restriction of f to C.

If $\mathcal{G}$ is a group or semigroup (with underlying set G) and $f_0, ..., f_{n-1} \in G$ then $\langle f_0, ..., f_{n-1}\rangle$ denotes the subgroup (or subsemigroup) of $\mathcal{G}$ generated by $\{f_0, ..., f_{n-1}\}$. We warn the reader that sometimes $\langle f_0, ..., f_{n-1}\rangle$ simply denotes the sequence with terms $f_0, ..., f_{n-1}$. It will always be clear from the context if we mean the substructure generated by the f_i.

If $\mathcal{A}$ and $\mathcal{B}$ are structures, then $A \leq \mathcal{B}$ denotes the fact that $\mathcal{A}$ is a substructure of $\mathcal{B}$. Structures will be denoted by calligraphic letters and their underlying sets will be denoted by the corresponding latin letter (in the case of groups, monoids and semigroups, sometimes, we don't make such a strict distinction between the structure itself and its underlying set and simply denote both by latin letters).

2. Proofs

In this section we present proofs of our main results. We start by some technical preparations.

Let $\mathcal{A}$ be an arbitrary structure. The endomorphism monoid *Endo* $(\mathcal{A})$ and the automorphism group *Aut*$(\mathcal{A})$ can be endowed with a topology as follows. Endow A with the discrete topology and ^{A}A with the product topology (or equivalently, with the pointwise convergence topology) τ. Throughout this paper *Endo*$(\mathcal{A})$ (respectively *Aut*$(\mathcal{A})$) will be endowed with the subspace topology inherited from τ. Further, if $n \in \omega$ and $f_i : A \to A$ are finite partial functions for all $i < n$ then, for $\overline{f} = \langle f_0, ..., f_{n-1}\rangle$, the elementary open set $N_{\overline{f}}$ in the product space $^{n}Endo(\mathcal{A})$ is defined to be

$$N_{\overline{f}} = \{\overline{g} = \langle g_0, ..., g_{n-1}\rangle \in {}^{n}Endo(\mathcal{A}) : (\forall i < n)(f_i \subseteq g_i)\}.$$

In order to apply model theoretic methods to metric spaces we recall a standard method associating a relational structure $\mathcal{A}(\mathcal{X})$ to a metric space $\mathcal{X} = \langle X, \varrho\rangle$. The universe of $\mathcal{A}(\mathcal{X})$ is X and for each $d \in$ *ran* (ϱ) (range of ϱ) there is a binary relation symbol R_d in the language of $\mathcal{A}(\mathcal{X})$ denoting the relation

$$R_d = \{\langle a, b\rangle \in X^2 : \varrho\ (a, b) \leq d\}.$$

The range of ϱ is called the spectrum of $\mathcal{X}$. In this work, by a slight abuse of notation, we do not make a strict distinction between $\mathcal{A}(\mathcal{X})$ and $\mathcal{X}$; when we mention a model theoretic property of $\mathcal{X}$ we tacitly mean $\mathcal{A}(\mathcal{X})$ in place of $\mathcal{X}$.

Definition 1 Let $\mathcal{X} = \langle X, \varrho\rangle$ be a metric space, if the distance function ϱ also satisfies the following, rather strong form of the triangle inequality: for all $a, b, c \in X$ we have

$$\varrho(a, b) \leq max\{\varrho\ (a, c), \varrho\ (c, b)\}$$

then $\mathcal{X} = \langle X, \varrho\rangle$ is defined to be an ultrametric space.

Definition 2 Let $\mathcal{X} = \langle X, \varrho\rangle$ be a metric space and let $Y \subseteq X$. A function $f : X \to X$ is defined to be a homomorphism iff

$$(\forall a, b \in X)(\varrho\ (f(a), f(b)) \leq \varrho\ (a, b)).$$

In addition, f is defined to be a retraction over Y iff f is a homomorphism, $f|_Y$ is the identity function of Y and *ran*$(f) \subseteq Y$.

In the context of metric spaces homomorphisms are the same as "nonexpansive mappings" or "1-Lipschitz functions". We also recall that a structure $\mathcal{A}$ is defined to be *homomorphism homogeneous* iff homomorphisms between finite substructures of $\mathcal{A}$ can be extended to endomorphisms of $\mathcal{A}$.

In Lemma 1 below we establish the existence of several retractions. Although we are using different terminology, the essence of our proof is similar to that of Theorem 3.2 in [5] (however, we obtained it independently). The main difference is that Theorem 3.2 of [5] applies to finite or countably infinite ultrametric spaces (of arbitrary spectrum), while our Lemma 1 applies to all (even uncountable) ultrametric spaces with finite spectrum.

Lemma 1 Let $\mathcal{X} = \langle X, \varrho\rangle$ be an ultrametric space with a finite spectrum. Let $\emptyset \neq Y \subseteq X$ be arbitrary. Then there exists a function $f : X \to Y$ which is a retraction over Y.

Proof. Let $\kappa = |Y|$ and let $\{y_i : i < \kappa\}$ be a (possibly transfinite) enumeration of Y. Let $x \in X$ be arbitrary. Since $\mathcal{X}$ has finite spectrum, the set $\{\varrho\ (x,\ y_i) : i < \kappa\}$ has a minimum m_x. Define $f(x) = y_i$ where i is the smallest element of κ for which $\varrho\ (x,\ y_i) = m_x$. Clearly, $f|_Y = Id_Y$ and $ran(f) \subseteq Y$. It remains to show that f is a homomorphism. Let $a,\ b \in X$, we shall show

$$\varrho\ (f(a),\ f(b)) \leq \varrho\ (a,\ b). \tag{1}$$

We apply a case distinction.

Case 1 $\varrho\ (a,\ f(b)) > \varrho\ (b,\ f(b))$. On one hand, $\varrho\ (a,\ b) \geq \varrho\ (a,\ f(b))$ (because otherwise $\varrho\ (a,\ f(b)) \leq max\{\varrho\ (a,\ b),\ \varrho\ (b,\ f(b))\}$ would not hold). On the other hand,

$$\varrho\ (f(a),\ f(b)) \leq \max\{\varrho\ (f(a),\ a),\ \varrho(a,\ f(b))\} \overset{\text{def. of } f(a)}{=} \varrho\ (a,\ f(b)).$$

combining these observations, one obtains (1).

Case 2 $\varrho\ (b,\ f(a)) > \varrho\ (a,\ f(a))$. It is similar to Case 1, by symmetry.

Case 3 $\varrho\ (a,\ f(b)) \leq \varrho\ (b,\ f(b))$ and $\varrho\ (b,\ f(a)) \leq \varrho\ (a,\ f(a))$. By construction, there are $i,\ j < \kappa$ such that $f(a) = y_i$ and $f(b) = y_j$. Observe that

$$\varrho\ (a,\ f(a)) \geq \varrho\ (b, f(a)) \overset{\text{def. of } f(b)}{\geq} \varrho\ (b,\ f(b)) \geq \varrho\ (a,\ f(b)) \overset{\text{def. of } f(a)}{\geq} \varrho\ (a,\ f(a)).$$

Hence, in the previous line equality holds everywhere. In particular, $\varrho\ (a,\ f(a)) = \varrho\ (a,\ f(b))$. Completely similarly, $\varrho\ (b,\ f(b)) = \varrho\ (b,\ f(a))$. Further,

$$m_a = \varrho\ (a,\ f(a)) = \varrho\ (a,\ f(b)).$$

Hence $i \leq j$. Similarly, $m_b = \varrho\ (b,\ f(b)) = \varrho\ (b,\ f(a))$, hence $j \leq i$. It follows, that $i = j$, that is, $f(a) = f(b)$. Therefore $\varrho\ (f(a),\ f(b)) = 0$, so $(*)$ holds, as desired. □

Now we present our first result. It states that the endomorphism monoid of an ultrametric space of finite spectrum is smoothly approximated in a strong sense. (We use the expression "smoothly approximated" in its group theoretic and semigroup theoretic meaning).

Theorem 1 Suppose $\mathcal{X} = \langle X,\ \varrho \rangle$ is an ultrametric space with a finite spectrum. Then *Endo*$(\mathcal{X})$ is smoothly approximated in the following sense. For all nonempty, finite $A \subseteq X$, if $\overline{a},\ \overline{b} \in A$ are finite tuples such that there exists $f \in$ *Endo*$(\mathcal{X})$ with $f(\overline{a}) = f(\overline{b})$, then there exists $f' \in$ *Endo*$(\mathcal{X})$ which maps A into itself (in fact, *ran*$(f') \subseteq A$) and still $f'(\overline{a}) = f'(\overline{b})$.

Proof. Applying Lemma 1 we obtain a function $g : X \to A$ which is a retraction over A. Assume $\overline{a},\ \overline{b} \in A$ and $f \in$ *Endo*$(\mathcal{X})$ with $f(\overline{a}) = f(f'(\overline{b}))$. Then $f' := g \circ f$ maps $\overline{a}$ onto $\overline{b}$ and maps A into itself, as desired. □

Let X be an arbitrary set and let G be a monoid acting on X. The *G-orbit* $O_G(a)$ of $a \in X$ is defined to be

$$O_G(a) = \{g(a) : g \in G\}.$$

For $Id \in G$ we have $Id(a) = a$, hence $a \in O_G(a)$ holds always (in particular, the orbit of an element is never empty). However, if G is not a group, then the G-orbits of elements of X may not form a partition of X because different orbits may not be disjoint.

Let $\mathcal{A}$ be any first order structure and let $n \in \omega$. Then the set $Endo_n^{Fin}(\mathcal{A})$ of finitary (or locally finite) n-tuples of endomorphisms of $\mathcal{A}$ is defined to be

$$Endo_n^{Fin}(\mathcal{A}) = \{\overline{g} \in {}^nG : (\forall a \in A)(|O_{\langle \overline{g} \rangle}(a)| < \aleph_0)\}.$$

Our next result has been motivated by Theorem of [24], see also Lemma and Theorem of [16].

Theorem 2 Suppose $\mathcal{X} = \langle X, \varrho \rangle$ is an ultrametric space with a finite spectrum. Then $Endo_n^{Fin}(\mathcal{X})$ is dense in ${}^nEndo(\mathcal{X})$ for each $n \in \omega$.

Proof. Fix $n \in \omega$ and let $N_{\overline{f}}$ be a nonempty elementary open subset of $Endo(\mathcal{X})$. It is enough to show that

$$\text{there exists } \overline{f}' \in N_{\overline{f}} \cap Endo_n^{Fin}(\mathcal{X}). \tag{2}$$

.

Let

$$A = \bigcup_{i<n} \Big(dom(f_i) \cup ran(f_i)\Big).$$

Applying the proof of Theorem 1, we obtain that for each $i < n$ there exists $f_i' \in Endo(\mathcal{X})$ mapping $dom(f_i)$ onto $ran(f_i)$ and $ran(f_i') \subseteq A$. Hence, $\overline{f}' := \langle f_i' : i < n \rangle$ is an element of $N_{\overline{f}}$. In addition, since $ran(f_i') \subseteq A$ holds for all $i < n$, it follows that $\overline{f}' \in Endo_n^{Fin}(\mathcal{X})$. Thus, (2) holds, as desired. □

As usual, for a structure $\mathcal{C}$, the set of finitely generated substructures of $\mathcal{C}$ will be denoted by $Age(\mathcal{C})$. If $Aut_n^{Fin}(\mathcal{C})$ is dense in ${}^nAut(\mathcal{C})$ for all n, then there is a standard method for deriving Hrushovski style extension theorems for $Age(\mathcal{C})$ (for more information we refer to Corollary 2.4 of [15] and Remark 4.5 of [16]). This method can be adapted to $Endo_n^{Fin}(\mathcal{C})$ in a straightforward way. Hence Theorem 1 implies Hrushovski style extension theorems for partial homomorphisms in the context of ultrametric spaces which can be regarded as a homomorphism oriented counterpart of Theorem 2.1 in [17] (see also Corollary 1.13 in [21]). However, for an ultrametric space $\mathcal{X}$, there is a drastically simpler way to obtain Hrushovski style theorems for $Age(\mathcal{X})$ and for their partial homomorphisms: by Theorem 3.2 of [5], each finite ultrametric space $\mathcal{A}$ is homomorphism homogeneous (that is, partial endomorphisms of $\mathcal{A}$ can be extended to endomorphisms of $\mathcal{A}$; this also can be quickly derived from Lemma 1. Therefore for any $\mathcal{A} \in Age(\mathcal{X})$, a Hrushovski extension of $\mathcal{A}$ is $\mathcal{A}$ itself. This establishes the next Corollary.

Proposition 1 Suppose $\mathcal{X} = \langle X, \varrho \rangle$ is an ultrametric space with a finite spectrum. Then $Age(\mathcal{X})$ satisfies $EPPE$ (the extension property of partial endomorphisms), that is, for all $\mathcal{A} \in Age(\mathcal{X})$ there exists $\mathcal{B} \in Age(\mathcal{X})$ such that the following holds. If f is a homomorphism between substructures of $\mathcal{A}$ that extends to an endomorphism of $\mathcal{X}$ then there exists an endomorphism of $\mathcal{B}$ which extends f.

In fact, $\mathcal{B}$ can be chosen to be $\mathcal{B} = \mathcal{A}$ so the same conclusion holds, if f is a homomorphism between substructures of $\mathcal{B}$.

Motivated by [19] and the more recent [20, 21], in the last theorem of this work we deal with the existence of locally finite, dense submonoids of the endomorphism monoid of certain ultrametric spaces. It turns out, that modulo some mild technical conditions, such submonoids always exist.

Theorem 3 Suppose $\mathcal{X} = \langle X, \varrho \rangle$ is an ultrametric space with finite spectrum. Then there exists a dense, locally finite submonoid $G \leq Endo(\mathcal{X})$.

Proof. Let f be an arbitrary finite partial endomorphism of $\mathcal{X}$. Applying the proof of Theorem 1 to $A := dom(f) \cup ran(f)$ we obtain $f' \in Endo(\mathcal{X})$ such that $ran(f') \subseteq A$ and f' maps $dom(f)$ onto $ran(f)$. It follows that each finite endomorphism of $\mathcal{X}$ can be extended to an endomorphism $f' \in Endo(\mathcal{X})$ such that $ran(f')$ is finite. Let $\mathcal{G}$ be the submonoid of $Endo(\mathcal{X})$ generated by $\{f' : f \text{ is a finite endomorphism of } \mathcal{X}\}$. Clearly, $\mathcal{G}$ is dense in $Endo(\mathcal{X})$ and each element of $\mathcal{G}$ has finite range.

In order to complete the proof, we prove a little bit more: we show that if $\mathcal{H}$ is any submonoid of $Endo(\mathcal{X})$ such that all elements of $\mathcal{H}$ has finite range, then $\mathcal{H}$ is locally finite. To do so, assume $\{h_0, \ldots, h_{n-1}\}$ is a finite subset of $\mathcal{H}$. Let

$$A = \bigcup_{i=0}^{n-1} ran(h_i) \quad \text{and} \quad M = \{Id_X\} \cup \{f \circ h_i : f \in {}^A A, \, i < n\}.$$

Clearly, $h_i \in M$ holds for all $i < n$. Further, M is closed under composition because of the following. If $i, j < n$ and $f, g \in {}^A A$ then $f \circ h_i \circ g \in {}^A A$ (that is, $dom(f \circ h_i \circ g) = A$ and $ran(f \circ h_i \circ g) \subseteq A$) because $ran(h_i) \subseteq A$. Hence

$$(f \circ h_i) \circ (g \circ h_j) = (f \circ h_i \circ g) \circ h_j$$

shows that M is indeed closed under composition. It follows that the submonoid $\mathcal{H}_0$ of $\mathcal{H}$ generated by $\{h_0, \ldots, h_{n-1}\}$ is contained in M. But obviously, M is finite, hence $\mathcal{H}_0$ is finite, as desired. □

3. Conclusion

As we discussed in the Introduction, investigating the model theoretic properties of ultrametric space is a rather active research topic recently. In the present work we focused our attention to the endomorphism monoids of ultrametric spaces and obtained the following results. According to Theorem 3, the endomorphism monoid of an ultrameric space of finite spectrum always contains a dense, locally finite submonoid. As a byproduct of this result, in Theorems 1 and 2 we established Hrushovski style extension theorems for finite substructures of certain ultrametric spaces.

Conflict of interest

Author declares there is no conflict of interest at any point with reference to research findings.

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