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Research Article

Fixed Point Results in Generalized Bi-2-Metric Space Using θ-Type Contractions

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Abstract: The main purpose of this manuscript is to provide a generalization of the concept of a 2-metric and prove some fixed point results for θ-type contractions. In this paper, we proved that a mapping T that is orbitally continuous, satisfying a θ-type contraction as a result of a relation between a pair of generalized 2-metrics and if one of the spaces is T-orbitally complete, then the mapping T has a fixed point.

Keywords: 2-metric, θ-type contraction, orbitally continuous

MSC: 47H10, 54H25

1. Introduction

Banach's contraction principle is one of the fundamental results in nonlinear analysis. The principle is of great value in applications [1] and many authors have obtained interesting extensions and generalizations of Banach's contraction principle [2–4]. Some authors have introduced generalizations of contractions while others have introduced generalizations of the underlying space [5, 6]. Recently, Jleli et al. [7, 8], introduced a new type of contractions which they called the θ-contraction and established some new fixed point theorems for such a contraction in the context of a generalized metric space.

The concept of a 2-metric space was introduced by Gähler [9, 10]. The space has a unique nonlinear structure and very different from that of a metric space and has been investigated by various authors. In [11, 12] obtained the basic results on fixed point of mappings on 2-metric spaces. Following Iseki, many authors have extended and generalized fixed point theorems in 2-metric spaces for different types of mappings [13, 14].

Definition 1 [15] Let X be a non-empty set and $d\colon X \times X \times X \to [0, \infty)$ be a map satisfying the following properties:
(i) If $x, y, z \in X$ such that $d(x, y, z) = 0$ only if at least two of the three points are the same.
(ii) For $x, y \in X$ such that $x \neq y$, there exists a point $z \in X$ such that $d(x, y, z) \neq 0$.
(iii) Symmetry property: for $x, y, z \in X$,

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$$d(x, y, z) = d(x, z, y) = d(y, x, z) = d(y, z, x) = d(z, x, y) = d(z, y, x).$$

(iv) Rectangle inequality:

$$d(x, y, z) \le d(x, y, t) + d(y, z, t) + d(z, x, t),$$

for $x, y, z, t \in X$.

Then d is a 2-metric and (X, d) is a 2-metric space.

In a paper by Mustafa et al., the authors established the structure of a b_2-metric space, as a generalization of the 2-metric space. Some fixed point results for various contraction type mappings in the context of an ordered b_2-metric spaces are presented [16]. To establish the structure the authors have weakened the rectangle inequality for a 2-metric by a constant $s \ge 1$ to form a s-rectangle inequality. In this paper, we have adopted a similar approach, by weakening the rectangle inequality for the 2-metric by introducing different weights $\alpha, \beta, \gamma \ge 1$, to form a modified rectangle inequality. In the special case that $\alpha = \beta = \gamma = 1$, we have a 2-metric. If $\alpha = \beta = \gamma = s$, then we have a b_2-metric. If we take $s = \max\{\alpha, \beta, \gamma\}$ or $s = \text{average}\{\alpha, \beta, \gamma\}$, then we have a b_2-metric.

Further to Mustafa et al. [17], the authors concentrated on the existence and uniqueness of common fixed points of various mappings in b_2-metric under generalized $(\phi, f)_\lambda$-expansive conditions and implicit contractive condition. The concept of a bi-2-metric space is a space endowed with a pair of generalized 2-metrics.

2. Preliminaries

We present the definition of a θ-type contraction introduced by Samet et al. [7, 8], on a 2-metric space.

Definition 2 Let (X, d) be a 2-metric space and a mapping $T\colon X \to X$ is a θ-type contraction if there exists $r \in (0, 1)$ such that

$$x, y, z \in X, d(Tx, Ty, z) \ne 0 \implies \theta(d(Tx, Ty, z)) \le [\theta(c(d(x, y, z)))]^r, \tag{1}$$

where $c \ge 1$ and θ: $(0, \infty) \to (1, \infty)$ is a function satisfying the following conditions:

(i) The function θ is continuous and non-decreasing.

(ii) For each sequence $\{t_n\} \subset (0, \infty)$, $\lim_{n\to\infty} \theta(t_n) = 1 \iff \lim_{n\to\infty} t_n = 0$.

(iii) There exists $r_1 \in (0, 1)$ and $l \in [0, \infty)$ such that $\lim_{t\to 0^+} \frac{\theta(t)-1}{t^{r_1}} = l$.

Definition 3 [15] Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a 2-metric space (X, d).

a) The sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be convergent to $x \in X$ iff for all $\xi \in X$,

$$\lim_{n\to\infty} d(x_n, x, \xi) = 0.$$

b) The sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be a Cauchy sequence in X iff for all $\xi \in X$,

$$\lim_{n, m\to\infty} d(x_n, x_m, \xi) = 0.$$

Definition 4 [18] Let (X, d) be a 2-metric space and $T: X \to X$ be a self map, then a set $\mathscr{O}(x_0, T) = \{T^n x_0, n = 0, 1, 2, \cdots\}$ is called the orbit of T at x_0 and T is orbitally continuous if $u = \lim_{n\to\infty} T^n x_0$ implies $Tu = \lim_{n\to\infty} T(T^n x_0)$.

Every continuous self-mapping is orbitally continuous, but not conversely [19]. A 2-metric space is T orbitally complete iff every Cauchy sequence which is contained in $\mathscr{O}(x)$ for some $x \in X$ converges in X [3]. For additional information in literature on fixed point results based on the orbit and orbital continuity, one can consult papers in [20, 21].

3. Main results

We begin by introducing the concept of a generalized 2-metric by weakening the rectangle inequality found in Definition 1.

Definition 5 Let X be a non-empty set and $d: X \times X \times X \to [0, \infty)$ be a map satisfying the following properties:

(i) If $x, y, z \in X$ such that $d(x, y, z) = 0$ only if at least two of the three points are the same.

(ii) For $x, y \in X$ such that $x \neq y$ there exists a point $z \in X$ such that $d(x, y, z) \neq 0$.

(iii) Symmetry property: for $x, y, z \in X$,

$$d(x, y, z) = d(x, z, y) = d(y, x, z) = d(y, z, x) = d(z, x, y) = d(z, y, x).$$

(iv) Modified rectangle inequality: there exists $\alpha, \beta, \gamma \geq 1$ such that

$$d(x, y, z) \leq \alpha d(x, y, t) + \beta d(y, z, t) + \gamma d(z, x, t),$$

for $x, y, z, t \in X$.

Then d is a generalized 2-metric and (X, d) is a generalized 2- metric space.

We extend the definition of a θ-type contraction introduced by Samet et al. [7], to a generalized 2-metric space.

Definition 6 Let (X, d) be a generalized 2-metric space and a mapping $T: X \to X$ is a θ-type contraction if there exists $r \in (0, 1)$ such that

$$x, y, z \in X, d(Tx, Ty, z) \neq 0 \implies \theta(d(Tx, Ty, z)) \leq [\theta(c(d(x, y, z))]^r, \tag{2}$$

where $c \geq 1$ and $\theta: (0, \infty) \to (1, \infty)$ is a function satisfying the following conditions:

(i) The function θ is continuous and non-decreasing.

(ii) For each sequence $\{t_n\} \subset (0, \infty)$, $\lim_{n\to\infty} \theta(t_n) = 1 \iff \lim_{n\to\infty} t_n = 0$.

(iii) There exists $r_1 \in (0, 1)$ and $l \in [0, \infty)$ such that $\lim_{t\to 0^+} \frac{\theta(t)-1}{t^{r_1}} = l$.

Definition 7 Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a generalized 2-metric space (X, d).

a) The sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be convergent to $x \in X$ iff for all $\xi \in X$,

$$\lim_{n\to\infty} d(x_n, x, \xi) = 0.$$

b) The sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be a Cauchy sequence in X iff for all $\xi \in X$,

$$\lim_{n,\, m\to\infty} d(x_n,\, x_m,\, \xi) = 0.$$

Definition 8 Let $(X,\, d)$ be a generalized 2-metric space and $T\colon X \to X$ be a self map, then a set $\mathscr{O}(x_0,\, T) = \{T^n x_0,\, n = 0,\, 1,\, 2,\, \cdots\}$ is called the orbit of T at x_0 and T is orbitally continuous if $u = \lim_{n\to\infty} T^n x_0$ implies $Tu = \lim_{n\to\infty} T\,(T^n x_0)$.

Definition 9 Let $(X,\, d)$ be a generalized 2-metric space. Let $x,\, y \in X$ and $\varepsilon > 0$. Then the subset

$$B_\varepsilon(x,\, y) = \{z \in X;\, d(x,\, y,\, z) < \varepsilon\}$$

of X is called a generalized 2-ball centered at x, y with radius ε. A topology can be generated in X by taking the collection of all generalized 2-balls as a subbasis, which we call the generalized 2-metric topology and is denoted by τ. Thus $(X,\, \tau)$ is a generalized 2-metric topological space. Members of τ are called 2-open sets. From the property of the metric it can easily be seen that $B_\varepsilon(x,\, y) = B_\varepsilon(y,\, x)$ for $\varepsilon > 0$.

Lemma 1 Every generalized 2-metric topological space $(X,\, \tau)$ is a T_1 space, (a topological space in which, every pair of distinct points, each has a neighborhood not containing the other point).

Proof. Let $x,\, y \in X$ with $x \neq y$. Then there is exists an element $z \in X$ such that $d(x,\, y,\, z) > 0$. If $\varepsilon = \frac{d(x,\, y,\, z)}{2} > 0$ then $B_\varepsilon(x,\, z)$ and $B_\varepsilon(y,\, z)$ are 2-open sets with $x \in B_\varepsilon(x,\, z)$ and $y \in B_\varepsilon(y,\, z)$ but $x \notin B_\varepsilon(y,\, z)$ and $y \notin B_\varepsilon(x,\, z)$. □

We provide an example to substantiate the generalization in Definition 5.

Example 1 Let $X = (0,\, 1)$ and define $d(x,\, y,\, z) = 0$ only if at least two of the three points are the same and $d(x,\, y,\, z) = e^{|x-y|+|y-z|+|z-x|}$, otherwise. Since properties (i)-(iii) of Definition 5, can be easily verified. It suffices to verify the modified rectangle inequality: For $x,\, y,\, z \in X$ and using Jensen's inequality, we get

$$\begin{aligned} d(x,\, y,\, z) &= e^{|x-y|+|y-z|+|z-x|} \\ &= e^{\frac{1}{2}|x-y|+\frac{1}{3}|y-z|+\frac{1}{6}|z-x|} e^{\frac{1}{2}|x-y|+\frac{2}{3}|y-z|+\frac{5}{6}|z-x|} \\ &\leq e^2 e^{\frac{1}{2}|x-y|+\frac{1}{3}|y-z|+\frac{1}{6}|z-x|} \\ &\leq e^2 \left\{ \frac{1}{2} e^{|x-y|} + \frac{1}{3} e^{|y-z|} + \frac{1}{6} e^{|z-x|} \right\} \\ &\leq e^2 \left\{ \frac{1}{2} e^{|x-y|+|y-t|+|t-x|} + \frac{1}{3} e^{|z-y|+|y-t|+|t-z|} + \frac{1}{6} e^{|z-x|+|x-t|+|t-z|} \right\} \\ &= \alpha d(x,\, y,\, t) + \beta d(z,\, y,\, t) + \gamma d(z,\, x,\, t), \end{aligned}$$

where $\alpha = \frac{1}{2} e^2 \geq 1$, $\beta = \frac{1}{3} e^2 \geq 1$ and $\gamma = \frac{1}{6} e^2 \geq 1$. It follows that d is a generalized 2-metric.

The following example gives a formula to generate a generalized 2-metric.

Example 2 Let X be a nonempty set with a generalized 2-metric $d\colon X \times X \times X\colon \to [0,\, \infty)$ then define

$$\rho(x, y, z) = \frac{d(x, y, z)}{1+d(x, y, z)}.$$

To show that ρ is a 2-metric, properties (i)-(iii) of Definition 5, can easily be shown since d is a 2-metric. We shall verify property (iv).

Let $x, y, z, t \in X$ then $d(x, y, z) \leq \alpha d(x, y, t)+\beta d(y, z, t)+\gamma d(z, x, t)$ for some $\alpha, \beta, \gamma \geq 1$. It follows that

$$\begin{aligned}
&\rho(x, y, z) \\
=&\frac{d(x, y, z)}{1+d(x, y, z)} \\
\leq&\frac{\alpha d(x, y, t)+\beta d(y, z, t)+\gamma d(z, x, t)}{1+\alpha d(x, y, t)+\beta d(y, z, t)+\gamma d(z, x, t)} \\
\leq&\alpha \frac{d(x, y, t)}{1+\alpha d(x, y, t)+\beta d(y, z, t)+\gamma d(z, x, t)}+\beta \frac{d(y, z, t)}{1+\alpha d(x, y, t)+\beta d(y, z, t)+\gamma d(z, x, t)} \\
&+\gamma \frac{d(z, x, t)}{1+\alpha d(x, y, t)+\beta d(y, z, t)+\gamma d(z, x, t)} \\
\leq&\alpha \frac{d(x, y, t)}{1+d(x, y, t)}+\beta \frac{d(y, z, t)}{1+d(y, z, t)}+\gamma \frac{d(z, x, t)}{1+d(z, x, t)} \\
=&\alpha \rho(x, y, t)+\beta \rho(y, z, t)+\gamma \rho(z, x, t).
\end{aligned}$$

Thus (X, ρ) is a generalized 2-metric space.

In the theorem which follows, we show that if the distance between the image points a mapping by a metric is related to the distance of the points by another metric under some distance function and if these metrics satisfy some kind of contractive type relation under the distance function then the mapping is a θ-type contraction. If the generalized 2-metric space is T-orbitally complete and satisfying some additional contraction conditions, then we can be prove the existence of a fixed point for mapping T.

Theorem 1 Let X be a nonempty set with generalized 2-metrics, d_1, d_2: $X \times X \times X \to [0, \infty)$ and T: $X \to X$ be orbitally continuous mapping which satisfies:

(i)

$$\theta(d_1(Tx, Ty, z)) \leq [\theta(c(d_2(x, y, z)))]^{r_1},$$

for $x, y, z \in X$, $c \geq 1$ and $r_1 \in (0, 1)$.

(ii) (X, d_1) is T-orbitally complete.

(iii)

$$\frac{\theta(min\{d_1(Tx, Ty, z), d_1(x, Tx, z), d_1(y, Ty, z)\})}{\theta(min\{d_1(x, Ty, z), d_1(y, Tx, z)\})} \leq [\theta(c(d_2(x, y, z)))]^{r_2},$$

for $c \geq 1$ $r_2 \in (0, 1)$.

(iv)

$$\theta(c(d_2(x, y, z))) \leq [\theta(c(d_1(x, y, z)))]^{r_3},$$

for $x, y, z \in X$, $c \geq 1$ and $r_3 \in (0, 1)$.

Then T has a fixed point in X.

Proof. Let $x, y, z \in X$ such that $d_1(Tx, Ty, z) \neq 0$ then

$$\theta(d_1(Tx, Ty, z)) \leq [\theta(c(d_2(x, y, z)))]^{r_1} \leq [\theta(c(d_1(x, y, z)))]^{r_1 r_3}$$

Using $0 < r_1 r_3 < 1$, it follows that T is a θ-type contraction.

Let $x_0 \in X$ be arbitrary. Then, we claim that the sequence $\{x_n\}_{n\in\mathbb{N}}$, where $x_n = Tx_{n-1}$ is a Cauchy sequence in X. If $x_{n-1} = x_n$ for some $n \in \mathbb{N}$ then $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. To prove that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence we suppose that $x_{n-1} \neq x_n$ for $n \in \mathbb{N}$ and let $x = x_{n-1}$ and $y = x_n$ in (iii) of the assumptions, then we get

$$\begin{aligned}
&\frac{\theta(min\{d_1(Tx_{n-1}, Tx_n, z), d_1(x_{n-1}, Tx_{n-1}, z), d_1(x_n, Tx_n, z)\})}{\theta(min\{d_1(x_{n-1}, Tx_n, z), d_1(x_n, Tx_{n-1}, z)\})} \\
=&\frac{\theta(min\{d_1(x_n, x_{n+1}, z), d_1(x_{n-1}, x_n, z), d_1(x_n, x_{n+1}, z)\})}{\theta(min\{d_1(x_{n-1}, x_{n+1}, z), d_1(x_n, x_n, z)\})} \\
=&\frac{\theta(min\{d_1(x_n, x_{n+1}, z), d_1(x_{n-1}, x_n, z)\})}{\theta(min\{d_1(x_{n-1}, x_{n+1}, z), 0\})} \\
\leq&[\theta(c(d_2(x_{n-1}, x_n, z)))]^{r_2}.
\end{aligned} \tag{3}$$

It follows that inequality (3), reduces to

$$\begin{aligned}
&\theta(min\{d_1(x_n, x_{n+1}, z), d_1(x_{n-1}, x_n, z)\}) \\
\leq&[\theta(c(d_2(x_{n-1}, x_n, z)))]^{r_2}.
\end{aligned} \tag{4}$$

From (4) using assumption (iv), we obtain

$$\theta(min\{d_1(x_n, x_{n+1}, z), d_1(x_{n-1}, x_n, z)\})$$

$$\leq[\theta(c(d_1(x_{n-1}, x_n, z)))]^{r_2r_3}, \tag{5}$$

and $0 < r_2r_3 < 1$. Now, suppose that

$$min\{d_1(x_n, x_{n+1}, z), d_1(x_{n-1}, x_n, z)\} = d_1(x_{n-1}, x_n, z)$$

then

$$\theta(d_1(x_{n-1}, x_n, z)) \leq [\theta(c(d_1(x_{n-1}, x_n, z)))]^{r_2r_3}$$

which is a contradiction. Thus we conclude that $min\{d_1(x_n, x_{n+1}, z), d_1(x_{n-1}, x_n, z)\} = d_1(x_n, x_{n+1}, z)$ which implies that

$$\theta(d_1(x_n, x_{n+1}, z) \leq [\theta(c(d_1(x_{n-1}, x_n, z)))]^{r_2r_3}. \tag{6}$$

Recursively, using (6), we get

$$\begin{aligned}
&\theta(d_1(x_n, x_{n+1}, z)\\
&\leq[\theta(c(d_1(x_{n-1}, x_n, z)))]^{r_2r_3}\\
&\leq[\theta(c(d_1(x_{n-2}, x_{n-1}, z)))]^{(r_2r_3)^2}\\
&\vdots\\
&\leq[\theta(c(d_1(x_0, x_1, z)))]^{(r_2r_3)^n}.
\end{aligned} \tag{7}$$

It follows that

$$1 \leq \theta(d_1(x_n, x_{n+1}, z) \leq [\theta(c(d_1(x_0, x_1, z)))]^{(r_2r_3)^n} \tag{8}$$

Since $0 < r_2r_3 < 1$ and letting $n \to \infty$, we get $\lim_{n\to\infty} \theta(d_1(x_n, x_{n+1}, z)) = 1$. By the property of the function θ, we get $\lim_{n\to\infty} d_1(x_n, x_{n+1}, z) = 0$.

We now show that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. It follows from the property of θ that from $r_1 \in (0, 1)$ and $l \in (0, \infty)$ that

$$\lim_{n\to\infty} \frac{\theta(d_1(x_n, x_{n+1}, z)) - 1}{[d_1(x_n, x_{n+1}, z)]^{r_1}} = l.$$

For $0 < \lambda < l$, by the definition of a limit there exists $n_1 \in \mathbb{N}$ such that

$$\lambda < \frac{\theta(d_1(x_n, x_{n+1}, z)) - 1}{[d_1(x_n, x_{n+1}, z)]^{r_1}},$$

and this implies that

$$\lambda[d_1(x_n, x_{n+1}, z)]^{r_1} < \theta(d_1(x_n, x_{n+1}, z)) - 1.$$

From inequality (7), we get

$$\begin{aligned} n[d_1(x_n, x_{n+1}, z)]^{r_1} &< n\lambda^{-1}(\theta(d_1(x_n, x_{n+1}, z)) - 1) \\ &\leq n\lambda^{-1}\left([\theta(c(d_1(x_0, x_1, z)))]^{(r_2 r_3)^n} - 1\right), \end{aligned}$$

for all $n > n_1$ which yields that

$$\lim_{n\to\infty} n[d_1(x_n, x_{n+1}, z)]^{r_1} = 0.$$

Hence there exists $n_2 \in \mathbb{N}$ such that for $n \geq n_2$, we get

$$n[d_1(x_n, x_{n+1}, z)]^{r_1} \leq 1,$$

which implies that

$$d_1(x_n, x_{n+1}, z) \leq \frac{1}{n^{\frac{1}{r_1}}}, \tag{9}$$

for all $n > n_2$. Using inequality (9) for $m \in \mathbb{N}$, we obtain

$$d_1(x_n, x_{n+m}, z)$$

$$\leq \alpha d_1(x_n, x_{n+m}, x_{n+1}) + \beta d_1(x_{n+m}, z, x_{n+1}) + \gamma d_1(z, x_n, x_{n+1})$$

$$\leq \max\{\alpha, \beta, \gamma\} (d_1(x_n, x_{n+m}, x_{n+1}) + d_1(x_{n+m}, z, x_{n+1}) + d_1(z, x_n, x_{n+1}))$$

$$\leq \max\{\alpha, \beta, \gamma\} \left(\frac{2}{n^{\frac{1}{r_1}}} + d_1(x_{n+m}, z, x_{n+1}) \right)$$

$$\leq \max\{\alpha, \beta, \gamma\} \left(\frac{2}{n^{\frac{1}{r_1}}} + \alpha d_1(x_{n+m}, z, x_{n+2}) + \beta d_1(z, x_{n+1}, x_{n+2}) + \gamma d_1(x_{n+1}, x_{n+m}, x_{n+2}) \right)$$

$$\leq \max\{\alpha, \beta, \gamma\} \left(\max\{\alpha, \beta, \gamma\} \frac{2}{n^{\frac{1}{r_1}}} + \max\{\alpha, \beta, \gamma\} \left(\frac{2}{(n+1)^{\frac{1}{r_1}}} + d_1(x_{n+m}, z, x_{n+2}) \right) \right)$$

$$= (\max\{\alpha, \beta, \gamma\})^2 \left(\frac{2}{n^{\frac{1}{r_1}}} + \frac{2}{(n+1)^{\frac{1}{r_1}}} + d_1(x_{n+m}, z, x_{n+2}) \right)$$

$$\leq (\max\{\alpha, \beta, \gamma\})^{m+1} \left(\frac{2}{n^{\frac{1}{r_1}}} + \frac{2}{(n+1)^{\frac{1}{r_1}}} + \cdots + \frac{2}{(n+m)^{\frac{1}{r_1}}} \right)$$

$$= (\max\{\alpha, \beta, \gamma\})^{m+1} \sum_{j=n}^{n+m} \frac{2}{j^{\frac{1}{r_1}}}$$

$$\leq (\max\{\alpha, \beta, \gamma\})^{m+1} 2 \sum_{j=n}^{\infty} \frac{1}{j^{\frac{1}{r_1}}}.$$

Based on the convergence of the series $\sum_{j=n}^{\infty} \frac{1}{j^{\frac{1}{r_1}}}$, since $0 < \frac{1}{j} < 1$, we conclude that $\{T^n x_0\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X. Since (X, d_1) is T a orbitally complete generalized 2-metric space there exist $u \in X$ such that $u = \lim_{n\to\infty} T^n x_0$. From the orbital continuity of T, we get $Tu = \lim_{n\to\infty} T^{n+1} x_0 = u$. □

Example 3 Let $X = [1, 2]$ and define the generalized 2-metrics as follows:

$$d_1(x, y, z) = \begin{cases} 0, & \text{if at least two of the three points are the same} \\ \frac{e^{|x-y|+|y-z|+|z-x|}}{(\gamma_1+k)}, & \text{otherwise} \end{cases}$$

and

$$d_2(x, y, z) = \begin{cases} 0, & \text{if at least two of the three points are the same} \\ \frac{\left(e^{|x-y|+|y-z|+|z-x|}\right)^2}{\gamma_2}, & \text{otherwise} \end{cases}$$

where $\gamma_1 = \max_{x, y, z \in X} e^{|x-y|+|y-z|+|z-x|}$, $\gamma_2 = \max_{x, y, z \in X} \left(e^{|x-y|+|y-z|+|z-x|}\right)^2$ and constant $k > 0$ is chosen such that $\gamma_1 + k \geq \gamma_2$. Define a mapping $T: X \to X$ by

$$Tx = \sqrt{x}.$$

We shall show that T is a $\theta-$type contraction where $\theta(t) = e^t$:
For $x \neq y \neq z \in X$ such that $Tx \neq Ty \neq z \in X$, we get

$$d_1(Tx, Ty, z)$$

$$= \frac{e^{|Tx-Ty|+|Ty-z|+|z-Tx|}}{(\gamma_1 + k)}$$

$$= \frac{e^{|\sqrt{x}-\sqrt{y}|+|\sqrt{y}-z|+|z-\sqrt{x}|}}{(\gamma_1 + k)}$$

$$\leq \frac{e^{|x-y|+|y-z|+|z-x|}}{(\gamma_1 + k)}$$

$$= d_1(x, y, z)$$

If $\theta(t) = e^t$ then $\theta(d_1(Tx, Ty, z)) \leq [\theta(r(d_1(x, y, z))]^{\frac{1}{r}}$ for some $r > 1$. It follows that T is a θ-type contraction.
Condition (i): For $x \neq y \neq z \in X$, such that $Tx \neq Ty \neq z \in X$,

$$d_1(Tx, Ty, z)$$

$$= \frac{e^{|Tx-Ty|+|Ty-z|+|z-Tx|}}{(\gamma_1 + k)}$$

$$= \frac{e^{|\sqrt{x}-\sqrt{y}|+|\sqrt{y}-z|+|z-\sqrt{x}|}}{(\gamma_1 + k)}$$

$$\leq \frac{e^{|x-y|+|y-z|+|z-x|}}{(\gamma_1 + k)}$$

$$\leq \frac{\left(e^{|x-y|+|y-z|+|z-x|}\right)^2}{\gamma_2}$$

$$\leq \frac{\left(e^{|x-y|+|y-z|+|z-x|}\right)^2}{\gamma_2}$$

$$= d_2(x, y, z)$$

Since θ is an increasing function it follows that

$$\theta(d_1(Tx, Ty, z)) \leq [\theta(r_1 d_2(x, y, z))]^{\frac{1}{r_1}}$$

for some $r_1 > 1$.

Condition (iii):

$$\frac{\theta(min\{d_1(Tx, Ty, z), d_1(x, Tx, z), d_1(y, Ty, z)\})}{\theta(min\{d_1(x, Ty, z), d_1(y, Tx, z)\})} = 1$$

For $x, y, z \in X$, we get $d_2(x, y, z) \geq 0$. It follows that $\theta((d_2(x, y, z))) \geq 1$. Thus we get $[\theta(r_2 d_2(x, y, z))]^{\frac{1}{r_2}} > 1$.

Condition (iv): For $x \neq y \neq z \in X$, we obtain

$$d_2(x, y, z) = \frac{\left(e^{|x-y|+|y-z|+|z-x|}\right)^2}{\gamma_2}$$

$$= \frac{(\gamma_1 + k)^2}{\gamma_2} d_1^2(x, y, z)$$

$$\leq cd_1(x, y, z)$$

where $c = \frac{(\gamma_1+k)^2}{\gamma_2}$. Then it follows that

$$\theta(d_2(x, y, z)) \leq \theta(c(d_1(x, y, z)) = [\theta(r_3 cd_1(x, y, z))]^{\frac{1}{r_3}}.$$

for some $r_3 > 1$. By applying Theorem 1 it follows that T has a fixed point in X.

Corollary 1 Let X be a nonempty set with a generalized 2- metric $d: X \times X \times X \to [0, \infty)$ and $T: X \to X$ be orbitally continuous mapping which satisfies:

(i)

$$\theta(d(Tx, Ty, z)) \leq [\theta(d(x, y, z))]^{r_1},$$

for $x, y, z \in X$ and $r_1 \in (0, 1)$

(ii) (X, d) is T orbitally complete.

(iii)

$$\frac{\theta(min\{d(Tx, Ty, z), d(x, Tx, z), d(y, Ty, z)\})}{\theta(min\{d(x, Ty, z), d(y, Tx, z)\})} \leq [\theta(d(x, y, z))]^{r_2},$$

for $r_2 \in (0, 1)$

(iv)

$$\theta(d(x, y, z)) \leq [\theta(d(x, y, z))]^{r_3},$$

for $x, y, z \in X$ and $r_3 \in (0, 1)$.

Then T has a fixed point in X.

Proof. The proof follows that of Theorem 1, with $d_1 = d_2$ □

Corollary 2 Let X be a nonempty set with a generalized 2- metric $d_2: X \times X \times X \to [0, \infty)$, a 2-metric $d_1: X \times X \times X \to [0, \infty)$ and $T: X \to X$ be orbitally continuous mapping which satisfies:

(i)

$$\theta(d_1(Tx, Ty, z)) \leq [\theta(d_2(x, y, z))]^{r_1},$$

for $x, y, z \in X$ and $r_1 \in (0, 1)$

(ii) (X, d_1) is T orbitally complete.

(iii)

$$\frac{\theta(min\{d_1(Tx, Ty, z), d_1(x, Tx, z), d_1(y, Ty, z)\})}{\theta(min\{d_1(x, Ty, z), d_1(y, Tx, z)\})} \leq [\theta(d_2(x, y, z))]^{r_2},$$

for $r_2 \in (0, 1)$

(iv)

$$\theta(d_2(x, y, z)) \leq [\theta(d_1(x, y, z))]^{r_3},$$

for $x, y, z \in X$ and $r_3 \in (0, 1)$

Then T has a unique fixed point in X.

Proof. The proof follows in line with Theorem 1 by taking $\alpha = \beta = \gamma = 1$. □

Theorem 2 Let X be a nonempty set with generalized 2-metrics d_1, d_2: $X \times X \times X \to [0, \infty)$ and T: $X \to X$ be orbitally continuous mapping which satisfies:

(i)

$$\theta(d_1(Tx, Ty, z)) \leq [\theta\,(max\{d_2(x, y, z), d_2(x, Tx, z), d_2(y, Ty, z), d_2(x, Ty, z) + d_2(y, Tx, z)\})]^{r_1},$$

for $x, y, z \in X$ and $r_1 \in (0, 1)$.

(ii) (X, d_1) is T orbitally complete.

(iii)

$$\frac{\theta(min\{d_1(Tx, Ty, z), d_1(x, Tx, z), d_1(y, Ty, z)\})}{\theta(min\{d_1(x, Ty, z), d_1(y, Tx, z)\})} \leq [\theta(d_2(x, y, z))]^{r_2}$$

for $r_2 \in (0, 1)$.

(iv)

$$\theta\,(max\{d_2(x, y, z), d_2(x, Tx, z), d_2(y, Ty, z), d_2(x, Ty, z) + d_2(y, Tx, z)\})$$

$$\leq [\theta(d_1(x, y, z)]^{r_3},$$

for $x, y, z \in X$ and $r_3 \in (0, 1)$.

Then T has a fixed point in X.

Proof. Let $x_0 \in X$ be arbitrary. Then, we claim that the sequence $\{x_n\}_{n\in\mathbb{N}}$, where $x_n = Tx_{n-1}$ is a Cauchy sequence in X. If $x_{n-1} = x_n$ for some $n \in \mathbb{N}$ then $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. To prove that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence we suppose that $x_{n-1} \neq x_n$ for $n \in \mathbb{N}$ and let $x = x_{n-1}$ and $y = x_n$ in (iii) of the assumptions, then we get

$$\frac{\theta(min\{d_1(Tx_{n-1}, Tx_n, z), d_1(x_{n-1}, Tx_{n-1}, z), d_1(x_n, Tx_n, z)\})}{\theta(min\{d_1(x_{n-1}, Tx_n, z), d_1(x_n, Tx_{n-1}, z)\})}$$

$$= \frac{\theta(min\{d_1(x_n, x_{n+1}, z), d_1(x_{n-1}, x_n, z), d_1(x_n, x_{n+1}, z)\})}{\theta(min\{d_1(x_{n-1}, x_{n+1}, z), d_1(x_n, x_n, z)\})}$$

$$= \frac{\theta(min\{d_1(x_n, x_{n+1}, z), d_1(x_{n-1}, x_n, z)\})}{\theta(min\{d_1(x_{n-1}, x_{n+1}, z), 0\})}$$

$$\leq [\theta(d_2(x_{n-1}, x_n, z))]^{r_2}.$$

It follows that

$$\theta(min\{d_1(x_n, x_{n+1}, z), d_1(x_{n-1}, x_n, z)\})$$

$$\leq [\theta(d_2(x_{n-1}, x_n, z))]^{r_2}$$

$$\leq [\theta\,(max\{d_2(x_{n-1}, x_n, z), d_2(x_n, x_{n+1}, z), d_2(x_{n-1}, x_{n+1}, z) + d_2(x_n, x_n, z)\})]^{r_2}.$$

From assumption (iv), we obtain

$$\theta(min\{d_1(x_n, x_{n+1}, z), d_1(x_{n-1}, x_n, z)\})$$

$$\leq [\theta(d_1(x_{n-1}, x_n, z))]^{r_2 r_3},$$

and $0 < r_2 r_3 < 1$. Now, suppose that

$$min\{d_1(x_n, x_{n+1}, z), d_1(x_{n-1}, x_n, z)\} = d_1(x_{n-1}, x_n, z)$$

then

$$\theta(d_1(x_{n-1}, x_n, z)) \leq [\theta(d_1(x_{n-1}, x_n, z))]^{r_2 r_3}$$

which is a contradiction. Thus, we conclude that $min\{d_1(x_n, x_{n+1}, z), d_1(x_{n-1}, x_n, z)\} = d_1(x_n, x_{n+1}, z)$ which implies that

$$\theta(d_1(x_n, x_{n+1}, z) \leq [\theta(d_1(x_{n-1}, x_n, z))]^{r_2 r_3}. \tag{10}$$

Recursively using (10), we get

$$
\begin{aligned}
&\theta(d_1(x_n, x_{n+1}, z)\\
\leq&[\theta(d_1(x_{n-1}, x_n, z))]^{r_2r_3}\\
\leq&[\theta(d_1(x_{n-2}, x_{n-1}, z))]^{(r_2r_3)^2}\\
&\vdots\\
\leq&[\theta(d_1(x_0, x_1, z))]^{(r_2r_3)^n}.
\end{aligned}
\tag{11}
$$

It follows that

$$
1 < \theta(d_1(x_n, x_{n+1}, z) \leq [\theta(d_1(x_0, x_1, z))]^{(r_2r_3)^n}. \tag{12}
$$

Since $0 < r_1r_2 < 1$ and letting $n \to \infty$, we get $\lim_{n\to\infty} \theta(d_1(x_n, x_{n+1}, z) = 1$. By the property of the function θ we get $\lim_{n\to\infty} d_1(x_n, x_{n+1}, z) = 0$. We now show that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. It follows from the property of θ that from $r_1 \in (0, 1)$ and $l \in (0, \infty)$ that

$$
\lim_{n\to\infty} \frac{\theta(d_1(x_n, x_{n+1}, z)) - 1}{[d_1(x_n, x_{n+1}, z)]^{r_1}} = l.
$$

For $0 < \lambda < l$, by the definition of a limit there exists $n_1 \in \mathbb{N}$ such that

$$
\lambda < \frac{\theta(d_1(x_n, x_{n+1}, z)) - 1}{[d_1(x_n, x_{n+1}, z)]^{r_1}}
$$

$$
\lambda[d_1(x_n, x_{n+1}, z)]^{r_1} < \theta(d_1(x_n, x_{n+1}, z)) - 1.
$$

From inequality (12), we get

$$
\begin{aligned}
n[d_1(x_n, x_{n+1}, z)]^{r_1} &< n\lambda^{-1}(\theta(d_1(x_n, x_{n+1}, z)) - 1)\\
&\leq n\lambda^{-1}\left([\theta(d_1(x_0, x_1, z))]^{(r_2r_3)^n} - 1\right),
\end{aligned}
$$

for all $n > n_1$ which yields that

$$\lim_{n\to\infty} n[d_1(x_n, x_{n+1}, z)]^{r_1} = 0.$$

Hence there exists $n_2 \in \mathbb{N}$ such that

$$n[d_1(x_n, x_{n+1}, z)]^{r_1} \leq 1,$$

which implies that

$$d_1(x_n, x_{n+1}, z) \leq \frac{1}{n^{\frac{1}{r_1}}}, \tag{13}$$

for all $n > n_2$. Using inequality (13), for $m \in \mathbb{N}$, we obtain

$$\begin{aligned}
& d_1(x_n, x_{n+m}, z) \\
\leq & \alpha d_1(x_n, x_{n+m}, x_{n+1}) + \beta d_1(x_{n+m}, z, x_{n+1}) + \gamma d_1(z, x_n, x_{n+1}) \\
\leq & \max\{\alpha, \beta, \gamma\} \left(d_1(x_n, x_{n+m}, x_{n+1}) + d_1(x_{n+m}, z, x_{n+1}) + d_1(z, x_n, x_{n+1})\right) \\
\leq & \max\{\alpha, \beta, \gamma\} \left(\frac{2}{n^{\frac{1}{r_1}}} + d_1(x_{n+m}, z, x_{n+1})\right) \\
\leq & \max\{\alpha, \beta, \gamma\} \left(\frac{2}{n^{\frac{1}{r_1}}} + \alpha d_1(x_{n+m}, z, x_{n+2}) + \beta d_1(z, x_{n+1}, x_{n+2}) + \gamma d_1(x_{n+1}, x_{n+m}, x_{n+2})\right) \\
\leq & \max\{\alpha, \beta, \gamma\} \left(\max\{\alpha, \beta, \gamma\} \frac{2}{n^{\frac{1}{r_1}}} + \max\{\alpha, \beta, \gamma\} \left(\frac{2}{(n+1)^{\frac{1}{r_1}}} + d_1(x_{n+m}, z, x_{n+2})\right)\right) \\
= & (\max\{\alpha, \beta, \gamma\})^2 \left(\frac{2}{n^{\frac{1}{r_1}}} + \frac{2}{(n+1)^{\frac{1}{r_1}}} + d_1(x_{n+m}, z, x_{n+2})\right) \\
\leq & (\max\{\alpha, \beta, \gamma\})^{m+1} \left(\frac{2}{n^{\frac{1}{r_1}}} + \frac{2}{(n+1)^{\frac{1}{r_1}}} + \cdots + \frac{2}{(n+m)^{\frac{1}{r_1}}}\right)
\end{aligned}$$

$$= (\max\{\alpha, \beta, \gamma\})^{m+1} \sum_{j=n}^{n+m} \frac{2}{j^{\frac{1}{r_1}}}$$

$$\leq (\max\{\alpha, \beta, \gamma\})^{m+1} 2 \sum_{j=n}^{\infty} \frac{1}{j^{\frac{1}{r_1}}}.$$

Based on the convergence of the series $\sum_{j=n}^{\infty} \frac{1}{j^{\frac{1}{r_1}}}$ we conclude that $\{T^n x_0\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X. Since (X, d_1) is T-orbitally complete there exist $u \in X$ such that $u = \lim_{n\to\infty} T^n x_0$. From the orbitally continuity of T, we get $Tu = \lim_{n\to\infty} T^{n+1} x_0 = u$. □

4. Conclusion

In this paper, we have provided a generalization of the concept of a 2-metric with an example to justify the generalization. We have proved that an orbitally continuous mapping T on a generalized 2-metric space that is a θ-type contraction and that if the space is T-orbitally complete then the mapping T has a fixed point. Our future direction in this study is to investigate if the continuity property of the mapping can be discarded, if one imposes further properties on the underlying space and the goal is to provide applications in the field of science and engineering.

Conflict of interest

The authors declare no competing financial interest.

References

[1] Banach S. On operations in abstract sets and their application to integral equations. *Fundamenta Mathematicae*. 1922; 3(1): 133-181.

[2] Vetro F. A generalization of Nadler fixed point theorem. *Carpathian Journal of Mathematics*. 2015; 31: 403-410.

[3] Ćirić LB. A generalization of Banach's contraction principle. *Proceedings of the American Mathematical Society*. 1974; 45(2): 267-273.

[4] Zhilong L, Jiang S. Fixed point theorems of JS-quasi-contractions. *Journal of Fixed Point Theory and Applications*. 2016; 40: 1-11.

[5] Van An T, Van Dung N, Le Hang VT. General fixed point theorems on metric spaces and 2-metric spaces. *Filomat*. 2014; 28(10): 2037-2045.

[6] Singh P, Singh V. Fixed point results in a complex valued generalized G_b-metric space. *Italian Journal of Pure and Applied Mathematics*. 2022; 48: 1181-1189.

[7] Jleli M, Samet B. A new generalization of the Banach contraction principle. *Journal of Inequalities and Application*. 2014; 48: 1-8.

[8] Ahmad J, Al-Mazrooei AE, Cho YJ, Yang Y. Fixed point results for generalized θ contractions. *Journal of Nonlinear Sciences and Applications*. 2017; 10: 2350-2358.

[9] Gähler S. Lineare 2-normierte Räume. *Mathematische Nachrichten*. 1965; 28: 1-43.

[10] Gähler S. Über die uniformisierbarkeit 2-metrischer Räume. *Mathematische Nachrichten*. 1965; 28: 235-244.

[11] Iseki K, Sharma PL, Sharma BK. Contraction type mappings on 2-metric spaces. *Mathematica Japonica*. 1976; 21: 67-70.

[12] Iseki K. Fixed point theorems in 2-metric spaces. *Mathematics Seminar Notes Kobe University*. 1975; 3: 133-136.

[13] Imdad M, Khan MS, Khan MD. A common fixed point theorem in 2-metric spaces. *Mathematica Japonica*. 1991; 36(5): 907-914.
[14] Liu Z, Feng C, Chun S. Fixed and periodic point theorems in 2-metric spaces. *Nonlinear Functional Analysis and Applications*. 2003; 8(4): 497-505.
[15] Naidu SVR. Some fixed point theorems in metric and 2-metric spaces. *International Journal of Mathematics and Mathematical Sciences*. 2001; 28: 625-636.
[16] Mustafa Z, Parvaneh V, Roshan JR, Kadelburg Z. b_2-Metric spaces and some fixed point theorems. *Fixed Point Theory and Applications*. 2014; 144: 1-23. Available from: https://doi.org/10.1186/1687-1812-2014-144.
[17] Zhang M, Zhu C. Theorems of common fixed points for some mappings in b_2-metric spaces. *Multidisciplinary Digital Publishing Institute*. 2022; 10(18): 1-19.
[18] Pant A, Pant RP. Orbital continuity and fixed points. *Multidisciplinary Digital Publishing Institute*. 2017; 31: 3495-3499.
[19] Ćirić LB. On contraction type mappings. *Mathematica Balkanica*. 1971; 1: 52-57.
[20] Zeng LC. Asymptotic behavior for commutative semigroups of almost asymptotically nonexpansive type mappings. *Acta Mathematica Scientia Series B English Edition*. 2006; 26: 246-254.
[21] Zeng LC. Ergodic retraction theorem and weak convergence theorem for reversible semigroups of non-Lipschitzian mappings. *Acta Mathematica Scientia Series B English Edition*. 2006; 22: 407-416.