Contemporary Mathematics
http://ojs.wiserpub.com/index.php/CM/

Research Article

An Extension of Nobel's Multiplying Factor Method for Solutions of Dual and Triple q-Integral Equations

Maryam AL-Towailb

Department of Computer Science and Engineering, College of Applied Studies and Community Service, King Saud University, P.O. Box 22459, Riyadh 11495, KSA
E-mail: mtowaileb@ksu.edu.sa

Received: 12 October 2023; **Revised:** 31 October 2023; **Accepted:** 2 November 2023

Abstract: In this paper, we convert certain dual q-integral equations involving third Jackson q-Bessel functions to the first and second kind Fredholm q-integral equations by extending the multiplying-factor method introduced by Nobel. We also apply these results to convert a certain triple q-integral equations to two simultaneous Fredholm q-integral equations.

***Keywords*:** q-Bessel functions, dual q-integral equations, triple q-integral equations

MSC: 45F10, 31B10, 26A33, 33D45

Abbreviation

MDPI	Multidisciplinary Digital Publishing Institute
DOAJ	Directory of open access journals
TLA	Three letter acronym
LD	Linear dichroism
MDPI	Multidisciplinary Digital Publishing Institute
DOAJ	Directory of open access journals
TLA	In this paper, we determine a single server retrial

1. Introduction

Fredholm integral equations play a significant role in various scientific applications. These equations can be found in fields such as engineering and mathematical physics, and provide a powerful mathematical tool for studying phenomena in various scientific disciplines. They can be used to model a wide range of physical systems and to analyze their behavior. Many researchers and scientists heavily rely on these equations to solve complex problems, one of them to find efficient and accurate methods to solve the problems of multiple integral equations.

Wang and Zhou [1] have recently studied the numerical solution of Fredholm integral equations using the Nyström method. Moreover, they discussed a two-grid iterative method for solving this class of equations based on the radial basis function interpolation (see [2]).

DOI: https://doi.org/10.37256/cm.5220243771

In [3], Noble presented a multiplying-factor method to transfer some dual and triple integral equations to Fredholm integral equations of the second kind. Cheshmehkani and Ghadi [4] extended Noble's method to solve the following system:

$$\int_0^\infty t^\alpha D(t)B(t)J_\nu(\rho t)\,dt = f(\rho), \qquad 0 \le \rho \le a, \tag{1}$$

$$\int_0^\infty B(t)J_\nu(\rho t)\,dt = g(\rho), \qquad \rho > a, \tag{2}$$

with some conditions imposed on the functions f, g and D. They converted (1)-(2) to the first and second kind Fredholm integral equations.

In [5], the authors considered and solved three different systems of dual q-integral equations where the kernel is the third Jackson q-Bessel functions by different methods. They used the q-Mellin transform, the fractional q-calculus and applied the multiplying factor method. Mansour and AL-Towailb [6] employed the fractional q-calculus in solving a triple system of q-integral equations (see also [7–10]).

Our aim to study a q-analogue of the system (1)-(2), where the kernel is the third Jackson q-Bessel functions, and we apply the results to convert a certain triple q-integral equations to two simultaneous Fredholm q-integral equations.

The paper is organized as follows: In the next section, we provide a recap of the fundamental concepts and principles of q-analysis that are essential for our investigations. Section 3 focuses on the study of a special system of dual q-integral equations. We analyze different cases that encompass a wider scope of dual q-integral equations involving third Jackson q-Bessel functions. Section 4 outlines the process of converting the dual q-integral equations into the first and second kind Fredholm q-integral equations. Additionally, we present an illustrative example to exemplify this conversion. Section 5 centers around the consideration of a certain system of triple q-integral equations. We utilize our findings from earlier sections to successfully reduce this system to Fredholm q-integral equations. The conclusions are presented in the last section.

2. Preliminaries

Throughout this paper, we follow Gasper and Rahman [11] for the definitions of the q-shifted factorial, q-gamma function and Jackson q-integrals, where $0 < |q| < 1$. For $t > 0$, let $\mathbb{R}_{q,t,+}$, $A_{q,t}$ and $B_{q,t}$ be the following sets:

$$\mathbb{R}_{q,t,+} := \{tq^k : k \in \mathbb{Z}\}, \quad A_{q,t} := \{tq^n : n \in \mathbb{N}_0\}, \quad B_{q,t} := \{tq^{-n} : n \in \mathbb{N}\}.$$

Note that, for $t = 1$ we write $\mathbb{R}_{q,+}$, A_q and B_q, and we will use following spaces:

$$L_{q,\eta}(\mathbb{R}_{q,+}) := \left\{ f : \|f\|_{q,\eta} := \int_0^\infty |t^\eta f(t)|\, d_qt < \infty \right\},$$

$$L_{q,\eta}(A_q) := \left\{ f : \|f\|_{A_q,\eta} := \int_0^1 |t^\eta f(t)|\, d_qt < \infty \right\},$$

$$L_{q,\eta}(B_q) := \left\{ f : \|f\|_{B_q,\eta} := \int_1^\infty |t^\eta f(t)|\, d_qt < \infty \right\},$$

where $\eta \in \mathbb{C}$ and $L_{q,\,\eta}(\mathbb{R}_{q,\,+}) = L_{q,\,\eta}(A_q) \cap L_{q,\,\eta}(B_q)$.

In [12], the authors introduced inverse pair of q-Hankel integral transforms:

$$f(x) = \int_0^\infty \lambda g(\lambda) J_\nu(\lambda x;\, q^2)\, d_q\lambda;$$

$$g(\lambda) = \int_0^\infty x f(x) J_\nu(\lambda x;\, q^2)\, d_q x, \tag{3}$$

where $f,\ g \in L_q^2(\mathbb{R}_{q,\,+})$, and $\lambda,\ x \in \mathbb{R}_{q,\,+}$.

A set $A \subseteq \mathbb{R}$ is called a q-geometric set if $qx \in A$ for any $x \in A$. If f is a function defined on a q-geometric set A, the q-derivative $D_q f$ is defined by

$$D_q f(z) = \begin{cases} \dfrac{f(z) - f(qz)}{(1-q)z}, & z \in A - \{0\}; \\ \\ f'(0), & z = 0, \end{cases} \tag{4}$$

provided that f is differentiable at zero (see [13]).

The third Jackson q-Bessel function is defined by

$$J_\nu(z;\, q) := \frac{(q^{\nu+1};\, q)_\infty}{(q;\, q)_\infty} \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n+1)/2} z^{2n+\nu}}{(q;\, q)_n (q^{\nu+1};\, q)_n} \quad (z \in \mathbb{C}), \tag{5}$$

and satisfies the following relations:

$$D_q \left[(.)^{-\nu} J_\nu(.;\, q^2) \right](z) = -\frac{q^{1-\nu} z^{-\nu}}{1-q} J_{\nu+1}(qz;\, q^2), \tag{6}$$

$$D_q \left[(.)^{\nu} J_\nu(.;\, q^2) \right](z) = \frac{z^{\nu}}{1-q} J_{\nu-1}(z;\, q^2) \tag{7}$$

(see [14, 15]). Also, for $\Re(\nu) > -1$, the q-Bessel function $J_\nu(.;\, q^2)$ satisfies:

$$\left| J_\nu(q^n;\, q^2) \right| \le \frac{(-q^2;\, q^2)_\infty (-q^{2\nu+2};\, q^2)_\infty}{(q^2;\, q^2)_\infty} \begin{cases} q^{n\nu}, & \text{if } n \ge 0; \\ \\ q^{n^2-(\nu+1)n}, & \text{if } n < 0, \end{cases} \tag{8}$$

(see [12, 16]). We need the following results (see [13]):

Lemma 1 Let x, ν and γ be complex numbers and $u \in \mathbb{R}_{q,\,+}$. Then, for $\Re(\gamma) > -1$ and $\Re(\nu) > -1$ the following identity holds:

$$\int_0^x \rho^{\nu+1}(q^2\rho^2/x^2;\,q^2)_\gamma J_\nu(u\rho;\,q^2)\,d_q\rho$$

$$=x^{\nu-\gamma+1}u^{-\gamma-1}(1-q)(1-q^2)^\gamma\Gamma_{q^2}(\gamma+1)J_{\gamma+\nu+1}(ux;\,q^2). \tag{9}$$

Moreover, if $\Re(\gamma) > 0$ and $\Re(\nu) > -1$, then

$$\int_x^\infty \rho^{2\gamma-\nu-1}(x^2/\rho^2;\,q^2)_{\gamma-1}J_\nu(u\rho;\,q^2)\,d_q\rho$$

$$=x^{\gamma-\nu}u^{-\gamma}(1-q)q^\gamma\frac{(q^2;\,q^2)_\infty}{(q^{2\gamma};\,q^2)_\infty}J_{\nu-\gamma}(\frac{ux}{q};\,q^2). \tag{10}$$

Now, we recall some of q-fractional operators. The Riemann-Liouville fractional q-integral operator is defined in [17] by

$$I_q^\alpha f(x):=\frac{x^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^x (qt/x;\,q)_{\alpha-1}f(t)\,d_qt,$$

where $\alpha \notin \{-1,\,-2,\,...\}$. Agarwal [18] defined the q-fractional derivative to be

$$D_q^\alpha f(x):=I_q^{-\alpha}f(x)=\frac{x^{-\alpha-1}}{\Gamma_q(-\alpha)}\int_0^x (qt/x;\,q)_{-\alpha-1}f(t)\,d_qt. \tag{11}$$

In [17], Al-Salam defined a two parameter q-fractional operator by

$$K_q^{\eta,\,\alpha}\phi(x):=\frac{q^{-\eta}x^\eta}{\Gamma_q(\alpha)}\int_x^\infty \left(x/t;\,q\right)_{\alpha-1}t^{-\eta-1}\phi(tq^{1-\alpha})\,d_qt,$$

$\alpha \neq -1,\,-2,\,\ldots$. In [5], the authors introduced a slight modification of the operator $K_q^{\eta,\,\alpha}$. This operator is denoted by $\mathscr{K}_q^{\eta,\,\alpha}$ and defined by

$$\mathscr{K}_q^{\eta,\,\alpha}\phi(x):=\frac{q^{-\eta}x^\eta}{\Gamma_q(\alpha)}\int_x^\infty (x/t;\,q)_{\alpha-1}t^{-\eta-1}\phi(qt)d_qt,$$

where $\alpha \neq -1,\,-2,\,\ldots$. In case of $\eta = -\alpha$, we set

$$\mathcal{K}_q^{\alpha} f(x) := q^{-\alpha} x^{\alpha} q^{\alpha(\alpha-1)/2} \mathcal{K}_q^{-\alpha,\,\alpha} f(x)$$

$$= \frac{q^{-\alpha(\alpha-1)/2}}{\Gamma_q(\alpha)} \int_x^{\infty} t^{\alpha-1} (x/t;\, q)_{\alpha-1} f(qt)\, d_q t. \tag{12}$$

3. Extension of Noble's method for wider range of dual q-integral equations

We consider the following dual q-integral equations:

$$\int_0^{\infty} u^{2\alpha} D(u) \psi(u) J_\nu(u\rho;\, q^2)\, d_q u = f(\rho), \quad \rho \in A_{q,\,a}, \tag{13}$$

$$\int_0^{\infty} \psi(u) J_\nu(u\rho;\, q^2)\, d_q u = g(\rho), \quad \rho \in B_{q,\,a}, \tag{14}$$

where $0 < a < \infty$, $D(u)$ is a bounded function with the condition

$$\lim_{n\to\infty} D(aq^n) = 0 \text{ or } \lim_{n\to\infty} D(aq^{-n}) = 1,$$

α and ν are complex numbers such that $\Re(\nu) > -1$, $\psi \in L_{q,\,\nu}(\mathbb{R}_{q,\,+}) \cap L_{q,\,\nu+2\alpha}(\mathbb{R}_{q,\,+})$ is an unknown function to be determined, and the functions f and g are known functions defined on $A_{q,\,a}$ and $B_{q,\,a}$ respectively.

We study the double q-integral Equations (13) and (14) in several cases for variations of α and ν.

We start by multiplying both sides of (13) by

$$\rho^{\nu+1} (q^2\rho^2/x^2;\, q^2)_\gamma,$$

and then integrating with respect to ρ from 0 to x ($x \in A_{q,\,a}$), we get

$$\int_0^x \rho^{\nu+1} (q^2\rho^2/x^2;\, q^2)_\gamma \Big[\int_0^{\infty} u^{2\alpha} D(u) \psi(u) J_\nu(u\rho;\, q^2)\, d_q u \Big] d_q \rho$$

$$= \int_0^x \rho^{\nu+1} (q^2\rho^2/x^2;\, q^2)_\gamma f(\rho)\, d_q \rho. \tag{15}$$

Notice, the double q-integral on the left hand side of (15) is absolutely convergent for $\Re(\nu) > -1$ provided that $\psi \in L_{q,\,\nu}(\mathbb{R}_{q,\,+})$ (see Equation (8)). So, by interchanging the order of integration, and using (9), we get

$$\int_0^\infty u^{2\alpha-\gamma-1} D(u)\psi(u)J_{\nu+\gamma+1}(xu;q^2)\,d_qu$$

$$=\frac{x^{\gamma-\nu-1}}{(1-q)(1-q^2)^{\gamma}\Gamma_{q^2}(\gamma+1)}\int_0^x \rho^{\nu+1}(\frac{q^2\rho^2}{x^2};q^2)_\gamma f(\rho)\,d_q\rho \quad (x\in A_{q,a}). \tag{16}$$

Similarly, multiplying both sides of (14) by

$$\rho^{2\beta-\nu-1}(x^2/\rho^2;q^2)_{\beta-1}$$

and integrating with respect to ρ from x to ∞ $(x\in B_{q,a})$, and then using (10) we obtain the following:

$$\int_0^\infty u^{-\beta-1}\,\psi(u)J_{\nu-\beta-1}(\frac{xu}{\rho};q^2)\,d_qu$$

$$=\frac{(q^{2(\beta+1)};q^2)_\infty}{(1-q)q^{\beta+1}(q^2;q^2)_\infty}x^{\nu-\beta-1}\int_x^\infty \rho^{2\beta-\nu+1}(\frac{x^2}{\rho^2};q^2)_\beta\, g(\rho)\,d_q\rho \quad (x\in B_{q,a}). \tag{17}$$

From the Equations (16) and (17), we will get

$$\int_0^\infty u^\alpha D(u)\psi(u)J_{\nu+\alpha}(xu;q^2)\,d_qu=F_i(x), \quad x\in A_{q,a}, \tag{18}$$

$$\int_0^\infty u^\alpha \psi(u)J_{\nu+\alpha}(xu;q^2)\,d_qu=G_i(qx), \quad x\in B_{q,a}, \tag{19}$$

where, i in the functions F_i and G_i denotes to the case number considered for the range of α and ν.

Case [i]: $0<\Re(\alpha)<1$, $\Re(\nu+\alpha)>0$.

In this case we assume that $\gamma=\alpha-1$ and $\beta=-\alpha$. Replacing these parameters in Equations (16) and (17), we obtain

$$\int_0^\infty u^\alpha D(u)\psi(u)J_{\nu+\alpha}(xu;q^2)\,d_qu$$

$$=\frac{x^{\alpha-\nu-2}}{(1-q)(1-q^2)^{\alpha-1}\Gamma_{q^2}(\alpha)}\int_0^x \rho^{\nu+1}(\frac{q^2\rho^2}{x^2};q^2)_{\alpha-1} f(\rho)\,d_q\rho, \tag{20}$$

$$\int_0^\infty u^{\alpha-1}\,\psi(u)J_{\nu+\alpha-1}(\frac{xu}{q};q^2)\,d_qu$$

$$=\frac{(q^{2(1-\alpha)};q^2)_\infty}{(1-q)q^{1-\alpha}(q^2;q^2)_\infty}x^{\nu+\alpha-1}\int_x^\infty \rho^{-2\alpha-\nu+1}(\frac{x^2}{\rho^2};q^2)_{-\alpha}\, g(\rho)\,d_q\rho. \tag{21}$$

Multiplying both sides of (21) by $x^{1-\nu-\alpha}$, and then calculating the q-derivative of the two sides with respect to x and using Equation (6), we obtain

$$\int_0^\infty u^\alpha\,\psi(u) J_{\nu+\alpha}(xu;\,q^2)\,d_q u \tag{22}$$

$$= -\,q^{-(1+\nu)}\,\frac{(q^{2(1-\alpha)};\,q^2)_\infty}{(q^2;\,q^2)_\infty}\,x^{\nu+\alpha-1}\,D_{q,\,x}\int_x^\infty \rho^{-2\alpha-\nu+1}(\frac{x^2}{\rho^2};\,q^2)_{-\alpha}\,g(\rho)\,d_q\rho. \tag{23}$$

Therefore, the Equations (20) and (22) can be written in the form of Equations (18) and (19), where

$$F_1(x) = \frac{x^{\alpha-\nu-2}}{(1-q)(1-q^2)^{\alpha-1}\Gamma_{q^2}(\alpha)}\int_0^x \rho^{\nu+1}(\frac{q^2\rho^2}{x^2};\,q^2)_{\alpha-1}\,f(\rho)\,d_q\rho,$$

$$G_1(qx) = -q^{-(1+\nu)}\,\frac{(q^{2(1-\alpha)};\,q^2)_\infty}{(q^2;\,q^2)_\infty}\,x^{\nu+\alpha-1}\,D_{q,\,x}\int_x^\infty \rho^{-2\alpha-\nu+1}(\frac{x^2}{\rho^2};\,q^2)_{-\alpha}\,g(\rho)\,d_q\rho.$$

By using (12), we get

$$G_1(qx) = -\frac{q^{\frac{\alpha^2-\alpha}{2}}}{q^{\nu+1}}(1-q^2)^\alpha\,x^{\nu+\alpha-1}D_{q,\,x}\Big[\mathscr{K}_{q^2}^{1-\alpha}[(.)^{-\frac{\nu}{2}}g(\sqrt{.})](\frac{x}{q^2})\Big] \quad (x\in B_{q,\,a}).$$

Case [ii]: $-1<\Re(\alpha)<0,\,\Re(\nu+\alpha)>-1$.

In this case we assume that $\gamma=\alpha$ and $\beta=-1-\alpha$. By the same argument in Case [i], the Equations (16) and (17) are transformed to (18) and (19), where

$$F_2(x) = \frac{x^{-(\alpha+\nu-1)}}{(1-q^2)^\alpha\Gamma_{q^2}(\alpha+1)}D_{q,\,x}\Big[x^{2\alpha}\int_0^x \rho^{\nu+1}(\frac{q^2\rho^2}{x^2};\,q^2)_\alpha\,f(\rho)\,d_q\rho\Big],$$

$$G_2(x) = \frac{q^\alpha(1-q^2)^{1+\alpha}}{(1-q)\Gamma_{q^2}(-\alpha)}\,x^{\nu+\alpha}\int_x^\infty \rho^{-2\alpha-\nu-1}(\frac{x^2}{\rho^2};\,q^2)_{-\alpha-1}\,g(\rho)\,d_q\rho.$$

By using (11), $F_2(x)$ can be rewritten in the following form:

$$F_2(x) = \frac{x^{-(\alpha+\nu-1)}}{(1-q^2)^\alpha}D_{q,\,x}\Big[D_{q^2,\,x}^{(-\alpha)}[(.)^{\frac{\nu}{2}}f(\sqrt{.})](x)\Big] \quad (x\in A_{q,\,a}).$$

Remark 1 Case [i] and [ii] are q-analogous of the results introduced by Noble in [3].

Case [iii]: $-1<\alpha<1,\,\Re(\nu+\alpha)>-1$. Replacing γ by α and β by $-\alpha$ in Equations (16) and (17), we obtain

$$\int_0^\infty u^{\alpha-1} D(u)\psi(u) J_{\nu+\alpha+1}(xu;\, q^2)\, d_qu$$

$$=\frac{x^{\alpha-\nu-1}}{(1-q)(1-q^2)^\alpha \Gamma_{q^2}(\alpha+1)} \int_0^x \rho^{\nu+1}(\frac{q^2\rho^2}{x^2};\, q^2)_\alpha f(\rho)\, d_q\rho \quad (x\in A_{q,\,a}); \tag{24}$$

$$\int_0^\infty u^{\alpha-1}\, \psi(u) J_{\nu+\alpha-1}(\frac{xu}{q};\, q^2)\, d_qu$$

$$=\frac{(q^{2-\alpha};\, q^2)_\infty}{(1-q)q^{1-\alpha}\,(q^2;\, q^2)_\infty} x^{\nu+\alpha-1} \int_x^\infty \rho^{-2\alpha-\nu+1}(\frac{x^2}{\rho^2};\, q^2)_{-\alpha}\, g(\rho)\, d_q\rho \quad (x\in B_{q,\,a}). \tag{25}$$

Multiplying both sides of (24) and (25) by $x^{1+\nu+\alpha}$ and $x^{1-\nu-\alpha}$ respectively, and then calculating the q-derivative of the two sides of each equation with respect to x and using Equation (6) and Equation (7), we get

$$\int_0^\infty u^\alpha D(u)\psi(u) J_{\nu+\alpha}(xu;\, q^2)\, d_qu \tag{26}$$

$$=\frac{x^{-(1+\alpha+\nu)}}{(1-q^2)^\alpha \Gamma_{q^2}(\alpha+1)} D_{q,\,x}\Big[x^{2\alpha}\int_0^x \rho^{\nu+1}(\frac{q^2\rho^2}{x^2};\, q^2)_\alpha\, f(\rho)\, d_q\rho\Big] \quad (x\in A_{q,\,a}); \tag{27}$$

$$\int_0^\infty u^\alpha\, \psi(u) J_{\nu+\alpha}(xu;\, q^2)\, d_qu \tag{28}$$

$$=-\, q^{-(1+\nu)}\frac{(q^{2-\alpha};\, q^2)_\infty}{(q^2;\, q^2)_\infty} x^{\nu+\alpha-1} D_{q,\,x}\int_x^\infty \rho^{-2\alpha-\nu+1}(\frac{x^2}{\rho^2};\, q^2)_{-\alpha}\, g(\rho)\, d_q\rho \quad (x\in B_{q,\,a}). \tag{29}$$

The order of the third Jackson q-Bessel functions and the power of u in Equations (27) and (29) are equal. Therefore, these equations are transformed to Equations (18) and (19) respectively, where

$$F_3(x) = \frac{x^{-(\alpha+\nu+1)}}{(1-q^2)^\alpha \Gamma_{q^2}(\alpha+1)} D_{q,\,x}\Big[x^{2\alpha}\int_0^x \rho^{\nu+1}(\frac{q^2\rho^2}{x^2};\, q^2)_\alpha\, f(\rho)\, d_q\rho\Big]$$

$$=\frac{x^{-(\alpha+\nu-1)}}{(1-q^2)^\alpha} D_{q,\,x}\Big[D^{(-\alpha)}_{q^2,\,x}[(.)^{\frac{\nu}{2}} f(\sqrt{.})](x)\Big] \quad (x\in A_{q,\,a});$$

$$G_3(qx) = -\, q^{-(1+\nu)}\frac{(q^{2-\alpha};\, q^2)_\infty}{(q^2;\, q^2)_\infty} x^{\nu+\alpha-1} D_{q,\,x}\int_x^\infty \rho^{-2\alpha-\nu+1}(\frac{x^2}{\rho^2};\, q^2)_{-\alpha}\, g(\rho)\ d_q\rho$$

$$=-\, \frac{q^{\frac{\alpha^2-\alpha}{2}}}{q^{\nu+1}}(1-q^2)^\alpha\, x^{\nu+\alpha-1} D_{q,\,x}\Big[\mathscr{K}^{1-\alpha}_{q^2}[(.)^{-\frac{\nu}{2}} g(\sqrt{.})](\frac{x}{q^2})\Big] \quad (x\in B_{q,\,a}).$$

Remark 2 Case [iii] is an extension to the cases [i] and [ii] which presents one single formulation for the whole range of $-1 < \alpha < 1$ including $\alpha = 0$.

4. The reduction of dual q-integral equations to Fredholm q-integral equations

Our aim to convert the dual q-integral Equations (18) and (19) to the Fredholm q-integral equations, where $D(u)$ is a given function with the condition of either $\lim\limits_{n\to\infty} D(aq^{-n}) = 1$ or $\lim\limits_{n\to\infty} D(aq^{n}) = 0$.

Case [a]: $\lim\limits_{n\to\infty} D(aq^{-n}) = 1$.

For getting the solution, we assume that

$$D(u) = 1 + w(u) \text{ where } \lim_{n\to\infty} w(aq^{-n}) = 0.$$

Then, the dual q-integral Equations (18) and (19) can be represented as

$$\int_0^\infty u^\alpha \left[1 + w(u)\right] \psi(u) J_{\nu+\alpha}(xu; q^2)\, d_q u = F_i(x), \quad x \in A_{q,a}, \tag{30}$$

$$\int_0^\infty u^\alpha \psi(u) J_{\nu+\alpha}(xu; q^2)\, d_q u = G_i(qx), \quad x \in B_{q,a}, \tag{31}$$

where $F_i(x)$ and $G_i(x)$ are defined in Section 3 for different cases.

By the argument in the previous section, we can assume that

$$\int_0^\infty u^\alpha \psi(u) J_{\nu+\alpha}(ux; q^2)\, d_q u = \begin{cases} \Phi(x), & x \in A_{q,a}, \\ G_i(qx), & x \in B_{q,a}, \end{cases} \tag{32}$$

where, $\Phi(x)$ is an unknown function to be determined. Applying the inverse pair of q-Hankel integral transforms (3) on (32), we obtain

$$\psi(u) := u^{1-\alpha} \left[\int_0^a y \Phi(y) J_{\nu+\alpha}(uy; q^2)\, d_q y + \int_a^\infty y G_i(qx) J_{\nu+\alpha}(uy; q^2)\, d_q y \right]. \tag{33}$$

Substituting from (32) and (33) into (30), we get

$$\Phi(x) + \int_0^a y \Phi(y) \left[\int_0^\infty u\, w(u) J_{\nu+\alpha}(uy; q^2) J_{\nu+\alpha}(ux; q^2)\, d_q u \right] d_q y +$$

$$\int_a^\infty y G_i(qy) \left[\int_0^\infty u\, w(u) J_{\nu+\alpha}(uy; q^2) J_{\nu+\alpha}(ux; q^2)\, d_q u \right] d_q y = F_i(x) \quad (x \in A_{q,a}). \tag{34}$$

This equation can be written in the form

$$\Phi(x)+\int_0^a K(x,\,y)\Phi(y)d_qy=\Theta(x)\quad (x\in A_{q,\,a}), \tag{35}$$

where,

$$K(x,\,y)=y\int_0^\infty u\,w(u)J_{\nu+\alpha}(uy;\,q^2)J_{\nu+\alpha}(ux;\,q^2)\,d_qu;$$

$$\Theta(x):=F_i(x)-\int_a^\infty K(x,\,y)G_i(qx)\,d_qy.$$

Notice:

1. $\Phi(x)$ satisfies the Fredholm q-integral equation of the second kind, and we can solve Equation (35) numerically in the general case.

2. $x^{-\nu-\alpha}\Phi(x)$ is bounded function in $A_{q,\,a}$.

Remark 3 Case [a] with $D(u)=1$ and $g(\rho)=0$ was considered and solved with some examples in [5].

Case [b]: $\lim\limits_{n\to\infty} D(aq^n)=0$

Assume that $D(u)=w(u)$ with $\lim\limits_{n\to\infty} w(aq^n)=0$. Then the dual q-integral Equations (18) and (19) can be represented as

$$\int_0^\infty u^\alpha\,w(u)\psi(u)J_{\nu+\alpha}(xu;\,q^2)\,d_qu=F_i(x),\quad x\in A_{q,\,a}, \tag{36}$$

$$\int_0^\infty u^\alpha\,\psi(u)J_{\nu+\alpha}(xu;\,q^2)\,d_qu=G_i(qx),\quad x\in B_{q,\,a}. \tag{37}$$

Hence, similar to Case [a], we assumed that $\widehat{\Phi}(x)$ is the function defined by

$$\int_0^\infty u^\alpha\,\psi(u)J_{\nu+\alpha}(ux;\,q^2)\,d_qu=\widehat{\Phi}(x),\quad x\in A_{q,\,a}, \tag{38}$$

provided that $\Re(\nu+\alpha)>0$. Then $\widehat{\Phi}(x)$ satisfies the first kind Fredholm q-integral equation of the form

$$\int_0^a \widehat{K}(x,\,y)\widehat{\Phi}(y)d_qy=\widehat{\Theta}(x),\quad x\in A_{q,\,a};$$

where,

$$\widehat{K}(x,\,y)=y\int_0^\infty u\,w(u)J_{\nu+\alpha}(uy;\,q^2)J_{\nu+\alpha}(ux;\,q^2)\,d_qu;$$

$$\widehat{\Theta}(x):=F_i(x)-\int_a^\infty \widehat{K}(x,\,y)G_i(qx)\,d_qy.$$

5. Converting certain triple q-integral equations to Fredholm q-integral equations

The problem of the change distribution on a circular annulus can be formulated as a triple integral equations of the form

$$\int_0^\infty \phi(\xi) J_\nu(\xi x)\, d\xi = 0, \qquad 0 < x < a, \tag{39}$$

$$\int_0^\infty \xi^{-1} \phi(\xi) J_\nu(\xi x)\, d\xi = f(x), \qquad a < x < b, \tag{40}$$

$$\int_0^\infty \phi(\xi) J_\nu(\xi x)\, d\xi = 0, \qquad b < x < \infty. \tag{41}$$

The solution of the integral Equations (39)-(41) given by Noble in [3].

In this section, we consider a general case of this system in q-calculus, and we will reduce the equations to two simultaneous Fredholm q-integral equations by using the results in Sections 3 and 4.

Consider the following system:

$$\int_0^\infty \psi(u) J_\nu(u\rho; q^2)\, d_q u = f(\rho), \quad \rho \in A_{q,a}; \tag{42}$$

$$\int_0^\infty u^{2\alpha} D(u) \psi(u) J_\nu(u\rho; q^2)\, d_q u = g(\rho), \quad \rho \in A_{q,b} \cap B_{q,a}; \tag{43}$$

$$\int_0^\infty \psi(u) J_\nu(u\rho; q^2)\, d_q u = h(\rho), \quad \rho \in B_{q,b}, \tag{44}$$

where $0 < a < b < \infty$, α and ν are complex numbers such that $\Re(\nu) > -1$, $\psi \in L_{q,\nu}(\mathbb{R}_{q,+}) \cap L_{q,\nu+2\alpha}(\mathbb{R}_{q,+})$ is an unknown function to be determined, the functions f, g and h are known functions, and $D(u)$ is a bounded function with the following conditions:

$$\lim_{n\to\infty} D(aq^n) = 0 = \lim_{n\to\infty} D(bq^n) \quad \text{or} \quad \lim_{n\to\infty} D(aq^{-n}) = 1 = \lim_{n\to\infty} D(bq^{-n}). \tag{45}$$

Since the function $g(\rho)$ is defined in $A_{q,b} \cap B_{q,a}$, we can write

$$g(\rho) = g_1(\rho) + g_2(\rho),$$

g_1 and g_2 defined in $A_{q,b}$ and $B_{q,a}$ respectively. So, we may assume that $\psi = \psi_1 + \psi_2$. Then, we rewrite the Equations (42)-(44) in the following form

$$\int_0^\infty u^{2\alpha} D(u)\psi_1(u)J_\nu(u\rho;\, q^2)\, d_qu = g_1(\rho), \quad \rho \in A_{q,\,b},$$

$$\int_0^\infty u^{2\alpha} D(u)\psi_2(u)J_\nu(u\rho;\, q^2)\, d_qu = g_2(\rho), \quad \rho \in B_{q,\,a},$$

$$\int_0^\infty (\psi_1+\psi_2)(u)J_\nu(u\rho;\, q^2)\, d_qu = \begin{cases} f(\rho),\ \rho \in A_{q,\,a}, \\ h(\rho),\ \rho \in B_{q,\,b}. \end{cases}$$

Now, we can consider the following pairs of dual q-integral equations:

$$\begin{cases} \int_0^\infty u^{2\alpha} D(u)\psi_1(u)J_\nu(u\rho;\, q^2)\, d_qu = g_1(\rho), \quad \rho \in A_{q,\,b}, \\ \int_0^\infty \psi_1(u)J_\nu(u\rho;\, q^2)\, d_qu = h(\rho) - h_1(\rho), \quad \rho \in B_{q,\,b}, \end{cases} \tag{46}$$

$$\begin{cases} \int_0^\infty u^{2\alpha} D(u)\psi_2(u)J_\nu(u\rho;\, q^2)\, d_qu = g_2(\rho), \quad \rho \in B_{q,\,a}, \\ \int_0^\infty \psi_2(u)J_\nu(u\rho;\, q^2)\, d_qu = f(\rho) - f_1(\rho), \quad \rho \in A_{q,\,a}, \end{cases} \tag{47}$$

where the functions f_1 and h_1 defined by

$$f_1(\rho) = \int_0^\infty \psi_1(u)J_\nu(\rho u;\, q^2)\, d_qu \quad \rho \in A_{q,\,a};$$

$$h_1(\rho) = \int_0^\infty \psi_2(u)J_\nu(\rho u;\, q^2)\, d_qu \quad \rho \in B_{q,\,b}.$$

Case [a]: $\lim\limits_{n\to\infty} D(bq^{-n}) = 1 = \lim\limits_{n\to\infty} D(aq^{-n})$.

We assume that $D(u) = 1 + w(u)$, where $\lim\limits_{n\to\infty} w(aq^{-n}) = 0 = \lim\limits_{n\to\infty} w(bq^{-n})$. For the first pair (46), by the same argument in Section 3, we obtain

$$\int_0^\infty u^{\alpha} D(u)\psi_1(u)J_{\nu+\alpha}(xu;\, q^2)\, d_qu = G_i(x), \quad x \in A_{q,\,b}, \tag{48}$$

$$\int_0^\infty u^{\alpha} \psi_1(u)J_{\nu+\alpha}(xu;\, q^2)\, d_qu = H_i(qx), \quad x \in B_{q,\,b}, \tag{49}$$

with the condition $\Re(\nu+\alpha) > -1$, and the functions G_i and H_i depend on g_1 and $(h-h_1)$ respectively.

From Section 4, we can convert the system (48)-(49) to the Fredholm q-integral equations of the second kinds, that is

$$\psi_1(u) := u^{1-\alpha}\Big[\int_0^b y\Phi_1(y)J_{\nu+\alpha}(uy;\,q^2)\,d_qy + \int_b^\infty yH_i(qy)\,J_{\nu+\alpha}(uy;\,q^2)\,d_qy\Big],$$

$$\Phi_1(x) = \int_0^\infty u^\alpha\psi_1(u)J_{\nu+\alpha}(xu;\,q^2)\,d_qu, \quad x\in A_{q,\,b}, \quad \text{and}$$

$$\Phi_1(x) + \int_0^b K_1(x,\,y)\Phi_1(y)d_qy = \Theta_1(x), \quad x\in A_{q,\,b}, \tag{50}$$

where,

$$K_1(x,\,y) = y\int_0^\infty u\,w(u)J_{\nu+\alpha}(uy;\,q^2)J_{\nu+\alpha}(ux;\,q^2)\,d_qu;$$

$$\Theta_1(x) := G_i(x) - \int_b^\infty K_1(x,\,y)H_i(qy)\,d_qy.$$

We next apply the results for second pair of dual q-integral Equations (47). Proceeding exactly as before, we obtain

$$\psi_2(u) := u^{1-\alpha}\Big[\int_0^a yF_i(y)J_{\nu+\alpha}(uy;\,q^2)\,d_qy + \int_a^\infty y\Phi_2(qy)\,J_{\nu+\alpha}(uy;\,q^2)\,d_qy\Big];$$

$$\Phi_2(qx) = \int_0^\infty u^\alpha\psi_2(u)J_{\nu+\alpha}(xu;\,q^2)\,d_qu, \quad x\in B_{q,\,a};$$

$$\Phi_2(qx) + \int_0^a K_2(x,\,y)F_i(y)d_qy = \Theta_2(x), \quad x\in A_{q,\,a}, \tag{51}$$

where,

$$K_2(x,\,y) = y\int_0^\infty u\,w(u)J_{\nu+\alpha}(uy;\,q^2)J_{\nu+\alpha}(ux;\,q^2)\,d_qu;$$

$$\Theta_2(x) := \widetilde{G}_i(x) - \int_a^\infty K_2(x,\,y)\Phi_2(qy)\,d_qy. \tag{52}$$

Notice, Equations (50) and (51) are two simultaneous Fredholm q-integral equations of the second kinds which may be solved numerically.

Case [b]: $\lim\limits_{n\to\infty} D(bq^{-n}) = 0 = \lim\limits_{n\to\infty} D(aq^{-n})$.

In this case, we assume that $D(u) = w(u)$ with $\lim\limits_{n\to\infty} w(aq^n) = 0 = \lim\limits_{n\to\infty} w(bq^n)$.

Similarly, from Section 4, we can convert the dual q-integral Equations (48) and (49) to the Fredholm q-integral equations of the first kinds. Assume that $\widehat{\Phi}_1(x)$ is the function defined by

$$\int_0^\infty u^\alpha \psi_1(u) J_{\nu+\alpha}(ux; q^2)\, d_q u = \widehat{\Phi}_1(x), \quad x \in A_{q,b},$$

provided that $\Re(\nu+\alpha) > 0$. Then $\widehat{\Phi}_1(x)$ satisfies the first kind Fredholm q-integral equation of the form

$$\int_0^a \widehat{K}_1(x, y)\widehat{\Phi}_1(y) d_q y = \widehat{\Theta}_1(x), \quad x \in A_{q,b}; \tag{53}$$

where,

$$\widehat{K}_1(x, y) = y \int_0^\infty u\, w(u) J_{\nu+\alpha}(uy; q^2) J_{\nu+\alpha}(ux; q^2)\, d_q u;$$

$$\widehat{\Theta}_1(x) := G_i(x) - \int_a^\infty \widehat{K}_1(x, y) H_i(qy)\, d_q y.$$

For system (47), we get

$$\int_0^a \widehat{K}_2(x, y) F_i(y) d_q y = \widehat{\Theta}_2(x), \quad x \in A_{q,b}; \tag{54}$$

where,

$$\widehat{K}_2(x, y) = y \int_0^\infty u\, w(u) J_{\nu+\alpha}(uy; q^2) J_{\nu+\alpha}(ux; q^2)\, d_q u;$$

$$\widehat{\Theta}_2(x) := \widetilde{G}_i(x) - \int_a^\infty \widehat{K}_2(x, y)\widehat{\Phi}_2(qy)\, d_q y;$$

$$\widehat{\Phi_2}(qx) = \int_0^\infty u^\alpha \psi_2(u) J_{\nu+\alpha}(xu; q^2)\, d_q u, \quad x \in B_{q,a}.$$

Remark 4 • If $f = h = 0$, $D(u) = 1$ and $\alpha = -\frac{1}{2}$ we get a q-analogue of the integral Equations (39)-(41).
• Similar system of the triple q-integral Equations (42)-(44) is considered and solved analytically in [6].

6. Conclusion

In this paper, we extended the Noble's multiplying-factor method to transfer the system:

$$\int_0^\infty u^{2\alpha} D(u)\psi(u)J_\nu(u\rho;\, q^2)\, d_q u = f(\rho), \quad \rho \in A_{q,\,a},$$

$$\int_0^\infty \psi(u)J_\nu(u\rho;\, q^2)\, d_q u = g(\rho), \quad \rho \in B_{q,\,a},$$

to Fredholm integral equations of the second kind. We used also this method to convert a certain triple q-integral equations to two simultaneous Fredholm q-integral equations. Another study to provide some applications and numerically solutions of these systems is in progress.

Acknowledgments

The author is grateful to the referees for their valuable comments and remarks, which have improved the manuscript in its present form. Also, the author thanks Prof. Zeinab S. Mansour for many discussions and for her interest in this work.

Conflict of interest

The author declares no competing financial interest.

References

[1] Wang Q, Zhou H. Two-grid iterative method for a class of fredholm functional integral equations based on the radial basis function interpolation. *Fundamental Journal of Mathematics and Applications.* 2019; 2: 117-122. Available from: https://doi.org/10.33401/fujma.633905.

[2] Zhou H, Wang Q. The nyström method and convergence analysis for system of fredholm integral equations. *Fundamental Journal of Mathematics and Applications.* 2019; 2: 28-32. Available from: https://doi.org/10.33401/fujma.486878.

[3] Noble B. The solution of Bessel function dual integral equations by a multiplying-factor method. *Proceedings of the Cambridge Philosophical Society.* 1963; 59: 351-362. Available from: https://doi.org/10.1017/S0305004100036987.

[4] Cheshmehkani S, Eskandari-Ghadi M. The solution of Bessel function dual integral equations by converting to the first kind Fredholm integral equation: An extension of Noble's solution. *Mathematics and Mechanics of Solids.* 2019; 24(8): 2536-2557. Available from: https://doi.org/10.1177/1081286518776452.

[5] Ashour O, Ismail M, Mansour ZS. On certain dual q-integral equations. *Pacific Journal of Mathematics.* 2015; 275(1): 63-102. Available from: https://doi.org/10.2140/pjm.2015.275.63.

[6] Mansour ZS, AL-Towailb M. On certain triple q-integral equations involving the third Jackson q-Bessel functions as kernel. *Advances in Differential Equations.* 2016; 81: 1-18. Available from: https://doi.org/10.1186/s13662-016-0809-3.

[7] Ashour O, Ismail M, Mansour ZS. Dual and triple equations and q-orthogonal polynomials. *Journal of Differential Equations and their Applications.* 2016; 22 (7): 1-16. Available from: https://doi.org/10.1080/10236198.2016.1167889.

[8] Jleli M, Mursaleen M, Samet B. q-integral equations of fractional orders. *Electronic Journal of Differential Equations.* 2016; 17: 1-14. Available from: http://ejde.math.txstate.edu.

[9] AL-Towailb M. The solution of certain triple q-integral equations in fractional q-calculus approach. *Arab Journal of Mathematical Sciences.* 2019; 25(1): 17-28. Available from: https://doi.org/10.1016/j.ajmsc.2018.10.001.

[10] Smetanin B. On an integral equation and its application to problems of thin detached inclusions in elastic bodies. *Journal of Applied Mathematics and Mechanics.* 1985; 49(5): 602-607. Available from: https://doi.org/10.1016/0021-8928(85)90078-4.
[11] Gasper G, Rahman M. *Basic hypergeometric series*. Cambridge university Press, Cambrdge; 2004.
[12] Koornwinder H, Swarttouw F. On a q-analoge of fourier and hankel transforms. *Transactions of the American Mathematical Society.* 1992; 333: 445-461. Available from: https://doi.org/10.2307/2154118.
[13] Annaby MH, Mansour ZS. *q-Fractional Calculus and Equations*. Lecture Notes in Mathematics 2056, Springer-Verlag, Berlin; 2012.
[14] Ismail M. *Classical and quantum orthogonal polynomials in one variable*. Cambridge University Press; 2005.
[15] Jackson F. The basic gamma function and elliptic functions. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences.* 1905; 76: 127-144. Available from: https://doi.org/10.1098/rspa.1905.0011.
[16] Swarttouw R. *The Hahn-Exton q-Bessel Function*. Ph.D. Thesis, The Technical University of Delft; 1992.
[17] Al-Salam W. Some fractional q-integrals and q-derivatives. *Proceedings of the Edinburgh Mathematical Society.* 1967; 2 (15): 135-140. Available from: https://doi.org/10.1017/S0013091500011469.
[18] Agarwal RP. Certain fractional q-integrals and q-derivatives. *Proceedings of the Cambridge Philosophical Society.* 1969; 66: 365-370. Available from: https://doi.org/10.1017/S0305004100045060.