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Research Article

Collocation Technique Based on Chebyshev Polynomial to Solve Lane-Emden-Fowler Boundary Value Problem

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Abstract: We present an innovative technique to find numerical solutions of the Lane-Emden-Fowler singular-type BVPs which plays a crucial role in comprehending a wide range of physical phenomena. The core concept of this technique is based on transforming the differential equation into the Fredholm integral equation, then it is converted into system of linear or nonlinear algebraic equations by utilizing the collocation technique based on Chebyshev polynomials. Subsequently, we employ an iterative numerical method, such as the Newton's method, for solving the system to get the approximate solution. Error analysis is included which helps to assess the accuracy of the obtained solutions and provides insights into the reliability of the numerical results. Furthermore, we have also considered various examples to demonstrate the applicability of the collocation technique based on Chebyshev polynomials and compared with the existing results.

Keywords: green's function, chebyshev polynomials, singular and doubly sbvps, functional approximation

MSC: 65L10, 34B16, 34B27

1. Introduction

Lane-Emden-Fowler singular-type BVPs [1–3] is essential for understanding the structure, behavior, and evolution of self-gravitating systems like stars and for addressing fundamental questions in astrophysics, cosmology, various other scientific and engineering fields [4–10]. Many natural or physical processes like oxygen concentration inside spherical cells [11], shallow membrane caps [12], heat conduction in the human head [13] etc. demonstrate the crucial existence of the Lane-Emden-Fowler BVPs. Therefore we focus on studying Lane-Emden-Fowler singular-type BVPs as follows:

$$\begin{cases} \left(t^b v'(t)\right)' = t^b \phi(t,\ v(t)),\ t \in (0,\ 1), \\ v(0) = \Gamma \text{ or } v'(0) = 0,\ \gamma_1\, v(1) + \gamma_2\, v'(1) = \gamma_3, \end{cases} \tag{1}$$

where $b > 0$, $\gamma_1 > 0$, γ_2, γ_3 and Γ are real constants.

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The analytical solution of second-order singular boundary value problems (SBVPs) was only known for $b = 0$ in the early 1940s, while for $b = 1,\ 2$ the analytical solution with boundary conditions $v'(0) = 0,\ v(1) = 0$ was explored by Chambre [14] in 1952. The collocation method, the patch basis method and the finite difference approach were later proposed by Russell and Shampine [15] to solve SBVPs involving boundary conditions (BCs) $v'(0) = 0,\ v(1) = B$ and $b = 0,\ 1$ or 2. Using the finite difference method, Chawla and Katti [16] assessed a numerical solution of $(t^b v'(t))' = \phi(t,\ v(t))$, $0 < b < 1$ with BCs $v(0) = A$, $v(1) = B$ and for $b \geq 1$, Chawla et al. [17] explored this method based on uniform mesh. The approach of the spline finite difference was proposed by Iyengar and Jain [18] for solving the particular case of second order SBVPs with BCs $v(0) = c_1$ or $v'(0) = 0$ and $v(1) = c_2$ while the non-polynomial spline approach was proposed by Sakai and Usmani [19] for second order SBVPs with BCs $v'(0) = 0$, $v(1) = c_1$ for $b \geq 1$ and $v(0) = c_2$, $v(1) = c_3$ for $0 < b < 1$.

Numerous semi-numerical techniques like homotopy perturbation method [20], the optimal homotopy analysis method [21], the Adomian decomposition method with Green's function [22–25], the variational iteration method [26, 27] have been used to solve second order SBVPs. To deal with such second order SBVPs numerous collocation techniques have been used recently. For example the Haar-wavelet collocation method [28, 29], the Laguerre wavelets collocation method [30], the Haar wavelet quasi-linearization method [31, 32] and the cubic B-spline collocation method [33], singular and doubly SBVPs is solved by using collocation method [34, 35]. Despite the fact that these numerical techniques are effective in their use, identifying the numerical solution of non-linear SBVPs requires a significant amount of computational work.

The use of Chebyshev polynomials [36–38] has become a preferred choice in scientific computing due to its ability to reduce oscillations, minimize approximation errors, and exhibit stable numerical properties. Therefore, we introduce an innovative technique based on Chebyshev polynomials to solve singular and doubly SBVPs.

Structure of this paper is as follows. We develop a method by transforming the SBVPs into the Fredholm integral equation. Subsequently, we employ the collocation technique based on Chebyshev polynomial to convert the Fredholm integral equation into system of non-linear equations and then we get approximate solution by using Newton's approach. Error estimation describe the accuracy of the current technique. Various examples are also given to examine the accuracy by comparing its numerical results with the existing results of the BCM [35]. We have also included the graph for few examples. It end with a concise conclusion.

Overall, the paper presents a novel collocation technique by using Chebyshev polynomial to approximate singular and doubly SBVPs. The current technique renders a promising alternative for efficient and accurate solutions to these challenging mathematical problems.

2. The construction of proposed method

This section includes equivalent Fredholm integral equation of singular and doubly SBVPs.

2.1 *Transformation of the emden-fowler bvps into fredholm integral form*

We explore the differential equation with Dirichlet-Robin BCs as

$$\begin{cases} \left(t^b v'(t)\right)' = t^b \phi(t,\ v(t)),\ t \in (0,\ 1), \\ v(0) = \Gamma,\ \gamma_1\, v(1) + \gamma_2\, v'(1) = \gamma_3. \end{cases} \tag{2}$$

Eq. (2) is equivalent to

$$v(t) = \Gamma + \frac{(\gamma_3 - \gamma_1 \Gamma)}{\gamma_1 + \gamma_2(1-b)} t^{1-b} + \int_0^1 \kappa(t,\ \zeta)\ \zeta^b\ \phi(\zeta,\ v(\zeta))d\zeta,\ t \in (0,\ 1), \tag{3}$$

where

$$\kappa(t,\ \zeta) = \begin{cases} \dfrac{1}{1-b}\left(1 - \dfrac{\gamma_1\ \zeta^{1-b}}{\gamma_1 + \gamma_2(1-b)}\right) t^{1-b}, & t \le \zeta, \\[2ex] \dfrac{1}{1-b}\left(1 - \dfrac{\gamma_1\ t^{1-b}}{\gamma_1 + \gamma_2(1-b)}\right) \zeta^{1-b}, & \zeta \le t. \end{cases} \tag{4}$$

We explore the differential equation with Neumann-Robin BCs as

$$\begin{cases} \left(t^b v'(t)\right)' = t^b \phi(t,\ v(t)),\ t \in (0,\ 1), \\[1ex] v'(0) = 0,\ \gamma_1\ v(1) + \gamma_2\ v'(1) = \gamma_3. \end{cases} \tag{5}$$

Eq. (5) is equivalent to

$$v(t) = \frac{\gamma_3}{\gamma_1} + \int_0^1 \kappa(t,\ \zeta)\ \zeta^a\ \phi(\zeta,\ v(\zeta))d\zeta,\ t \in (0,\ 1), \tag{6}$$

where

$$\kappa(t,\ \zeta) = \begin{cases} \ln \zeta - \dfrac{\gamma_2}{\gamma_1}, & t \le \zeta,\ \text{for } b = 1, \\[2ex] \ln t - \dfrac{\gamma_2}{\gamma_1}, & \zeta \le t \end{cases} \tag{7}$$

and

$$\kappa(t,\ \zeta) = \begin{cases} \dfrac{\zeta^{1-b} - 1}{1-b} - \dfrac{\gamma_2}{\gamma_1}, & t \le \zeta,\ \text{for } b > 1, \\[2ex] \dfrac{t^{1-b} - 1}{1-b} - \dfrac{\gamma_2}{\gamma_1}, & \zeta \le t. \end{cases} \tag{8}$$

2.2 *Transformation of the doubly SBVPs into fredholm integral form*

We explore doubly SBVPs with Drichlet-Robin BCs

$$
\begin{cases}
(p(t)v'(t))' = q(t)\phi(t,\ v(t)),\ t \in (0,\ 1), \\
v(0) = \Gamma,\ \gamma_1\ v(1) + \gamma_2\ v'(1) = \gamma_3.
\end{cases} \tag{9}
$$

Its equivalent integral equation is

$$
v(t) = \Gamma + \frac{(\gamma_3 - \gamma_1\ \Gamma)}{\gamma_1\ \ell(1) + \gamma_2\ \ell'(1)}\ell(t) + \int_0^1 \kappa(t,\ \zeta)\ q(\zeta)\ \phi(\zeta,\ v(\zeta))d\zeta,\ t \in (0,\ 1), \tag{10}
$$

where

$$
\kappa(t,\ \zeta) = \begin{cases}
\ell(t) - \dfrac{\gamma_1\ l(\zeta)\ \ell(t)}{\gamma_1\ l(1) + \gamma_2\ \ell'(1)}, & t \le \zeta, \\
l(\zeta) - \dfrac{\gamma_1\ \ell(t)\ l(\zeta)}{\gamma_1\ l(1) + \gamma_2\ \ell'(1)}, & \zeta \le t
\end{cases} \tag{11}
$$

and $\ell(t) = \int_0^t \frac{1}{p(t)}dt$.

We explore doubly SBVPs with the Neumann-Robin BCs

$$
\begin{cases}
(p(t)v'(t))' = q(t)\phi(t,\ v(t)),\ t \in (0,\ 1), \\
v'(0) = 0,\ \gamma_1\ v(1) + \gamma_2\ v'(1) = \gamma_3.
\end{cases} \tag{12}
$$

Equivalently,

$$
v(t) = \frac{\gamma_3}{\gamma_1} + \int_0^1 \kappa(t,\ \zeta)\ q(\zeta)\ \phi(\zeta,\ v(\zeta))d\zeta,\ t \in (0,\ 1), \tag{13}
$$

where

$$
\kappa(t,\ \zeta) = \begin{cases}
\ell(1) - l(\zeta) + \dfrac{\gamma_2}{\gamma_1}l'(1), & t \le \zeta, \\
\ell(1) - \ell(t) + \dfrac{\gamma_2}{\gamma_1}\ell'(1), & \zeta \le t.
\end{cases} \tag{14}
$$

3. Chebyshev collocation method (CCM)

This section includes derivation of the CCM to approximate integral equations (10) and (13).

Definition 1 Shifted Chebyshev polynomials are defined on $[0,\ 1]$ therefore we introduce a new variable $s = 2t-1$ and define shifted Chebyshev polynomials in the interval $[0,\ 1]$ as

$$\begin{cases} \tau_0(t) = 1, \\ \tau_1(t) = 2t-1, \\ \tau_n(t) = 2(2t-1)\tau_{n-1}(t) - \tau_{n-2}(t) \end{cases} \tag{15}$$

which forms a complete basis and shifted Chebyshev polynomials $\tau_n(t)$ are orthogonal w.r.t weight function $w(t) = \dfrac{1}{2\sqrt{(t-t^2)}}$.

We can approximate a function $f(t) \in L^2[0,\ 1]$ by shifted Chebyshev polynomials as

$$f(t) = \sum_{r=0}^{\infty} a_r\ \tau_r(t). \tag{16}$$

For numerical purpose, we consider the first $(n+1)$ terms of the above expansion as

$$f(t) \approx \sum_{r=0}^{n} a_r\ \tau_r(t) = \mathbf{A}^{\mathbf{T}}\ \boldsymbol{\tau}(t), \tag{17}$$

where $\mathbf{A}$ and $\boldsymbol{\tau}(t)$ are column vectors of order $(n+1)$ and are defined as

$$\mathbf{A} = [a_0,\ a_1,\ \cdots,\ a_n]^T,\ \boldsymbol{\tau}(t) = [\tau_0^n(t),\ \tau_1^n(t)\cdots,\ \tau_n^n(t)]^T. \tag{18}$$

3.1 *Dirichlet-Robin BCs*

We apply the present method in equation (10)

$$v(t) = \Gamma + \frac{(\gamma_3 - \gamma_1\ \Gamma)}{\gamma_1\ \ell(1) + \gamma_2\ \ell'(1)}\ell(t) + \int_0^1 \kappa(t,\ \zeta)\ q(\zeta)\ \phi(\zeta,\ v(\zeta))d\zeta,\ t \in (0,\ 1). \tag{19}$$

We take

$$z(t) = \phi(t,\ v(t)), \tag{20}$$

in equation (19). We approximate $v(t)$ and $z(t)$ by using equation (17) as

$$v(t) \approx \mathbf{A}^{\mathbf{T}}\ \tau(t) \text{and} z(t) \approx \mathbf{B}^{\mathbf{T}}\ \tau(t), \tag{21}$$

where $\tau^{\mathbf{T}} = [b_0,\ b_1,\ b_2,\ \cdots,\ b_n]$.

Employing (20) and (21), the integral equation (19) reduces as follows

$$\mathbf{A}^{\mathbf{T}}\,\tau(t)=g(t)+\int_{0}^{1}\kappa(t,\ \zeta)\,q(\zeta)\,\mathbf{B}^{\mathbf{T}}\,\tau(\zeta)d\zeta. \tag{22}$$

It becomes

$$\mathbf{A}^{\mathbf{T}}\,\tau(t)=g(t)+\mathbf{B}^{\mathbf{T}}\,K(t), \tag{23}$$

where

$$g(t)=\Gamma+\frac{(\gamma_3-\gamma_1\,\Gamma)}{\gamma_1\,\ell(1)+\gamma_2\,\ell'(1)}\ell(t) \tag{24}$$

and

$$K(t)=\int_{0}^{1}\kappa(t,\ \zeta)\,q(\zeta)\,\tau(\zeta)d\zeta. \tag{25}$$

Substituting equation (21) in equation (20), we get

$$\mathbf{B}^{\mathbf{T}}\,\tau(t)=\phi\left(t,\ \mathbf{A}^{\mathbf{T}}\,\tau(t)\right). \tag{26}$$

To insert the collocation points $t_i=\frac{1}{2}\left(\cos\left(\frac{(2i+1)\pi}{2n}\right)+1\right)$, $i=0(1)n$ in equation (23), we have

$$\mathbf{B}^{\mathbf{T}}\,\tau(t_i)-\phi\left(t_i,\ g(t_i)+\mathbf{B}^{\mathbf{T}}\,K(t_i)\right)=0. \tag{27}$$

Unknown vector **B** can be determined by rewriting the system of equations (27) in the matrix form as

$$\varphi(\mathbf{B})=\mathbf{0}, \tag{28}$$

where **0** is the column vector of (n+1) order, and

$$\varphi(\mathbf{B})=[\varphi_0(\mathbf{B}),\ \varphi_1(\mathbf{B}),\ \cdots,\ \varphi_n(\mathbf{B})]^T$$

with $\varphi_i(\mathbf{B})=\mathbf{B}^{\mathbf{T}}\,\mathbf{P}(t_i)-\phi\left(t_i,\ g(t_i)+\mathbf{B}^{\mathbf{T}}\,K(t_i)\right)$.

We apply the Newton's method to get approximate solution of equation (28) as

$$\mathbf{B}^{[r+1]} - \mathbf{B}^{[r]} = -J^{-1}(\mathbf{B}^{[r]})\varphi(\mathbf{B}^{[r]}),\ r = 0,\ 1,\ 2,\ \cdots, \tag{29}$$

where

$$J((B))_{ml} = \partial\varphi_m(\mathbf{B})/\partial\mathbf{B}_l,\ \text{for}\ m,\ l = 0(1)n$$

and $\mathbf{B}^{[r]}$ is the rth iterative solution of (28).

In order to obtain the approximate solution of (19), we substitute the unknown co-efficient in (23) which are obtained by applying the iteration approach (29).

Note that a desired accuracy ε of Newton's method can be obtained by using the stopping criteria $\|\mathbf{B}^{[r+1]} - \mathbf{B}^{[r]}\| < \varepsilon$.

3.2 *Neumann-Robin BCs*

Consider the equation (13) as

$$v(t) = \frac{\gamma_3}{\gamma_1} + \int_0^1 \kappa(t,\ \zeta)\, q(\zeta)\, \phi(\zeta,\ v(\zeta))d\zeta,\ t \in (0,\ 1). \tag{30}$$

Using similar steps as in earlier subsection, we set the expressions from equations (20) and (21) into equation (30), we have

$$\mathbf{A}^{\mathbf{T}}\ \tau(t) = \frac{\gamma_3}{\gamma_1} + \int_0^1 \kappa(t,\ \zeta)\, q(\zeta)\, \mathbf{B}^{\mathbf{T}}\ \tau(\zeta)d\zeta \tag{31}$$

which can be further expressed as

$$\mathbf{A}^{\mathbf{T}}\ \tau(t) = \frac{\gamma_3}{\gamma_1} + \mathbf{B}^{\mathbf{T}}\ K(t), \tag{32}$$

where $K(t)$ is given by (25). Using equation (32) into equation (26), we have

$$\mathbf{B}^{\mathbf{T}}\ \tau(t) = \phi\left(t,\ \frac{\gamma_3}{\gamma_1} + \mathbf{B}^{\mathbf{T}}\ K(t)\right). \tag{33}$$

We insert the collocation points $t_i = \frac{1}{2}\left(cos\left(\frac{(2i+1)\pi}{2n}\right) + 1\right),\ i = 0(1)n$ into equation (33)

$$\mathbf{B}^{\mathbf{T}}\ \tau(t_i) = \phi\left(t_i,\ \frac{\gamma_3}{\gamma_1} + \mathbf{B}^{\mathbf{T}}\ K(t_i)\right) \tag{34}$$

and $b_0,\ b_1,\ b_2,\ \ldots,\ b_n$ are the unknown.

In order to obtain the approximate solution of (30), we substitute the unknown co-efficient in (32) which are obtained by applying Newton's method [39] in equation (34).

4. Error analysis

The CCM's error bound for solving integral equations (10) and (13) is presented in this section. For this, we take following equation as

$$v = g + \int_0^1 \kappa(t,\ \zeta)\ q(\zeta)\ \phi(\zeta,\ v(\zeta))d\zeta,\ t \in (0,\ 1). \tag{35}$$

We observe that the equations (10) and (13) are the particular cases of equation (35) when $g = \Gamma + \dfrac{\ell(t)(\gamma_3 - \gamma_1\Gamma)}{\gamma_1\ell(1) + \gamma_2\ell'(1)}$ and $g = \dfrac{\gamma_3}{\gamma_1}$ respectively.

Let the maximum norm for the Banach space $X = C[0,\ 1]$ be described as

$$\|v\| = \max_{t\in[0,\ 1]} |v(t)|. \tag{36}$$

Theorem 1 If the Chebyshev approximation function is $\tau_n(f) = \sum_{i=0}^{n} a_i \tau_i^n(x)$ of the function $f \in C[0,\ 1]$, then the sequence $\{\tau_n(f)\}$ converges uniformly to f i.e a number n exists corresponding to any given $\varepsilon > 0$ such that

$$\|\tau_n(f) - f\| < \varepsilon.$$

Proof. To get proof of this theorem see [40].

Theorem 2 Let f be bounded function and its second derivative exists in $[0,\ 1]$ then the error bound for Chebyshev approximation function is found as

$$\|\tau_n(f) - f\| \leq \frac{1}{2n} t(1-t)\|f''\|, \tag{37}$$

where $\|.\|$ denotes the maximum norm.

Proof. For proof of this theorem see [41].

Theorem 3 Consider the Banach space X with maximum norm. Let $v_n(t)$ and $v(t)$ denote the estimated and exact solutions respectively to the integral equation (35). Let the function $\phi(t,\ v(t))$ meets the state of Lipschitz condition

$$|\phi(t,\ v(t)) - \phi(t,\ v^*(t))| \leq L|v(t) - v^*(t)|. \tag{38}$$

where L is the Lipschitz constant and if

$$M = \max_{t \in [0,\ 1]} \left| \int_0^1 \kappa(t,\ \zeta)\ q(\zeta)\ d\zeta \right|, \tag{39}$$

then the error bound is approximated as

$$\|v(t) - v_n(t)\| \leq \frac{ML}{8n} \|v''\|. \tag{40}$$

Proof. Consider

$$\|v(t) - v_n(t)\| = \max_{t \in [0,\ 1]} \left| g(t) + \int_0^1 \kappa(t,\ \zeta)\ q(\zeta)\ \phi(\zeta,\ v(\zeta)) d\zeta - g(t) - \int_0^1 \kappa(t,\ \zeta)\ q(\zeta)\ \phi(\zeta,\ v_n(\zeta)) d\zeta \right|$$

$$= \max_{t \in [0,\ 1]} \left| \int_0^1 \kappa(t,\ \zeta)\ q(\zeta) \left(\phi(\zeta,\ v(\zeta)) - \phi(\zeta,\ v_n(\zeta)) \right) d\zeta \right|$$

$$\leq \max_{\zeta \in [0,\ 1]} \left| \phi(\zeta,\ v(\zeta)) - \phi(\zeta,\ v_n(\zeta)) \right| \times \max_{t \in [0,\ 1]} \left| \int_0^1 \kappa(t,\ \zeta)\ q(\zeta)\ d\zeta \right|.$$

Using equation (4.4) and (4.5) into above inequality then it becomes

$$\|v(t) - v_n(t)\| \leq ML \max_{\zeta \in [0,\ 1]} |v(\zeta) - v_n(\zeta)|. \tag{41}$$

To apply the CCM, the numerical solution of (35) is $\tau_n(v(t))$ and substituting $v_n(\zeta)$ as $\tau_n\big(v(\zeta)\big)$, the equation (41) is reduced to

$$\|v(t) - v_n(t)\| \leq ML \max_{\zeta \in [0,\ 1]} |v(\zeta) - \tau_n(v(\zeta))|. \tag{42}$$

Using equation (37) into above equation, we obtain

$$\|v(t) - v_n(t)\| \leq ML \|v - \tau_n(v)\|$$

$$\leq ML \frac{\|v''\|}{2n} \max_{\zeta \in [0,\ 1]} \big(\zeta(1 - \zeta)\big). \tag{43}$$

Hence, we have

$$\|v(t)-v_n(t)\| \leq ML\frac{\|v''\|}{8n}. \tag{44}$$

5. Numerical illustrations

To examine the accuracy of the current method, we utilized MATLAB R2015a to get an estimated solution, while we computed maximum absolute errors using both L_∞ and L_2 norms for different examples of singular and doubly SBVPs. Subsequently, we compared these results with the BCM [35] in the different tables. Graph drawn for some of the examples between the estimated solutions and the exact solutions demonstrate the behaviour of the solution. We specify L_∞ and L_2 norm errors as follows:

$$L_\infty = \max_{t\in[0,\ 1]} |v(t)-v_n(t)|,$$

and

$$L_2 = \left(\sum_{j=1}^{m} |v(t_j)-v_n(t_j)|^2\right)^{1/2},$$

where $v_n(t)$ and $v(t)$ are the estimated and exact solutions respectively.

Example 1

$$\left(t^{\frac{1}{2}}v'\right)' = t^{\frac{1}{2}}\left(\frac{e^v}{2}-e^{2v}\right),\ v(0)=\ln(2),\ v(1)=0,\ t\in(0,\ 1).$$

Equivalent integral form of the above equation is

$$v(t) = \Gamma + \frac{\gamma_3-\gamma_1\,\Gamma}{\gamma_1+\gamma_2\,(1-b)}t^{1-b} + \int_0^1 \kappa(t,\ \zeta)\,\zeta^{\frac{1}{2}}\left(\frac{1}{2}e^{v(\zeta)}-e^{2v(\zeta)}\right)d\zeta,$$

where

$$\kappa(t,\ \zeta) = \begin{cases} 2\left(1-\zeta^{\frac{1}{2}}\right)t^{\frac{1}{2}}, & t\leq\zeta, \\ 2\left(1-t^{\frac{1}{2}}\right)\zeta^{\frac{1}{2}}, & \zeta\leq t. \end{cases}$$

Here $b=\frac{1}{2},\ \gamma_1=1,\ \gamma_2=0,\ \gamma_3=0,\ \Gamma=\ln(2)$.

Exact solution is

$$v(t) = \ln\left(\frac{2}{t^2+1}\right).$$

The precise and approximate solutions which is assessed using the present method are shown in Table 1, while error estimation is given in Table 2. We observe from Table 1 and Table 2 that CCM is more accurate than the BCM.

Table 1. Exact and estimated solutions of Example 1

t	CCM $n = 3$	BCM $n = 3$	Exact sol.
0.1	0.683197	0.682056	0.683197
0.2	0.653926	0.652587	0.653926
0.3	0.606969	0.605782	0.606969
0.4	0.544727	0.543801	0.544727
0.5	0.470003	0.469224	0.470004
0.6	0.385662	0.384828	0.385662
0.7	0.294370	0.293380	0.294371
0.8	0.198450	0.197435	0.198451
0.9	0.099820	0.099133	0.099820

Table 2. The error analysis of Example 1

	L_∞		L_2	
n	CCM	BCM	CCM	BCM
3	1.09E-03	1.34E-03	2.89E-03	3.03E-03
4	1.38E-04	2.73E-04	2.73E-04	5.79E-04
5	2.04E-05	9.43E-05	3.49E-05	2.15E-04
6	2.25E-06	5.69E-06	3.77E-06	1.24E-05
7	3.24E-07	3.52E-06	5.22E-07	7.25E-06
8	1.71E-07	7.00E-07	1.83E-07	1.05E-06
9	8.17E-09	9.14E-08	1.43E-08	2.38E-07

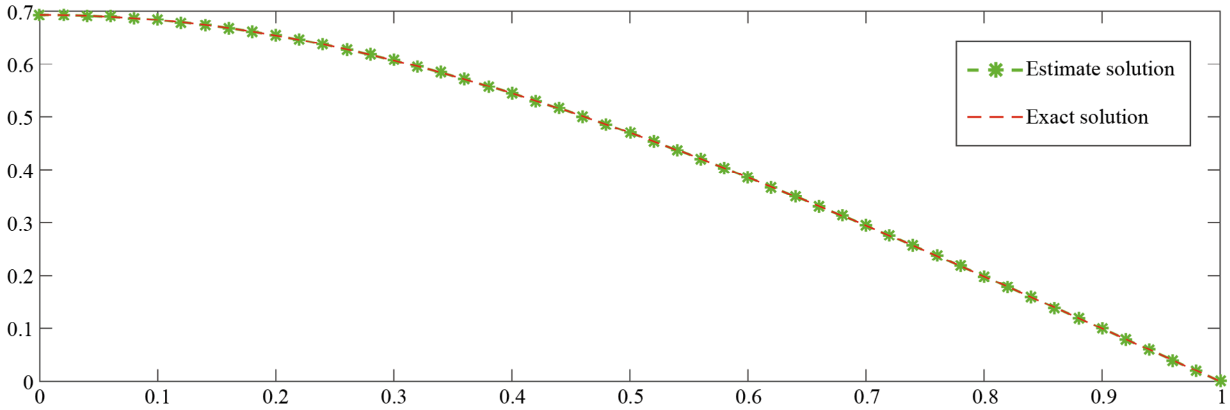

Figure 1. Comparison of estimate and exact solution of example 1

Example 2

$$\left(t^{\frac{1}{2}}v'\right)' = t^{\frac{1}{2}}\left(t^2 e^v\left(14-16t^4 e^v\right)\right),\ v(0)=\ln\left(\frac{1}{4}\right),\ v(1)=\ln\left(\frac{1}{5}\right),\ t\in(0,\ 1).$$

Its equivalent integral form is

$$v(t)=\Gamma+\frac{\gamma_3-\gamma_1\,\Gamma}{\gamma_1+\gamma_2\,(1-b)}t^{1-b}+\int_0^1\kappa(t,\ \zeta)\,\zeta^{\frac{1}{2}}\,\left(\zeta^2 e^{v(\zeta)}\left(14-16\zeta^4 e^{v(\zeta)}\right)d\zeta,\right.$$

where

$$\kappa(t,\ \zeta)=\begin{cases}2\left(1-\zeta^{\frac{1}{2}}\right)t^{\frac{1}{2}}, & t\le\zeta,\\ 2\left(1-t^{\frac{1}{2}}\right)\zeta^{\frac{1}{2}}, & \zeta\le t.\end{cases}$$

Here $b=\frac{1}{2}$, $\gamma_1=1$, $\gamma_2=0$, $\gamma_3=\ln\left(\frac{1}{5}\right)$, $\Gamma=\ln\left(\frac{1}{4}\right)$.
Exact solution is

$$v(t)=\ln\left(\frac{1}{t^4+4}\right).$$

The precise and approximate solutions which is assessed using the provided method are shown in Table 3, while error analysis is given in Table 4. We see from Table 3 and Table 4 that CCM is more accurate than the BCM.

Table 3. Exact and estimated solutions of Example 2

t	CCM $n = 3$	BCM $n = 3$	Exact sol.
0.1	-1.38631933	-1.38713067	-1.38631936
0.2	-1.38669429	-1.38754235	-1.38669428
0.3	-1.38831739	-1.38891015	-1.38831731
0.4	-1.39267386	-1.39297002	-1.39267397
0.5	-1.40179830	-1.40196436	-1.40179855
0.6	-1.41818062	-1.41844631	-1.41818055
0.7	-1.44458715	-1.44507103	-1.44458685
0.8	-1.48378386	-1.48438429	-1.48378398
0.9	-1.53817801	-1.53861008	-1.53817819
1.0	-1.60943791	-1.60943791	-1.60943791

Table 4. Errors analysis of Example 2

	L_∞		L_2	
n	CCM	BCM	CCM	BCM
3	8.04E-04	8.48E-04	1.36E-03	1.64E-03
4	5.79E-05	1.07E-04	1.18E-04	2.55E-04
5	1.58E-05	3.91E-05	3.08E-05	9.83E-05
6	2.15E-06	6.85E-06	3.71E-06	8.89E-06
7	3.02E-07	1.55E-06	4.65E-07	3.13E-06
8	1.09E-07	4.24E-07	1.77E-07	6.83E-07
9	9.19E-09	8.76E-08	1.51E-08	1.40E-07

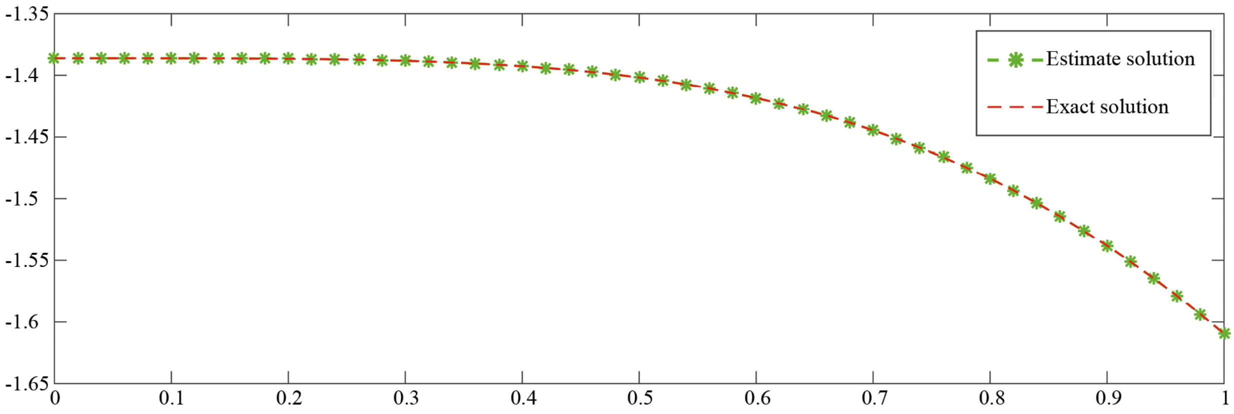

Figure 2. Comparison of estimate and exact solution of example 2

Example 3

$$\left(t^2 v'\right)' = -t^2 v^5,\ v'(0) = 0,\ v(1) = \sqrt{\frac{3}{4}},\ t \in (0,\ 1).$$

Equivalent Integral form is

$$v(t) = \frac{\gamma_3}{\gamma_1} + \int_0^1 \kappa(t,\ \zeta)\,\zeta^2\left(-v^5(\zeta)\right)d\zeta,$$

where

$$\kappa(t,\ \zeta) = \begin{cases} 1 - \dfrac{1}{\zeta}, & t \le \zeta, \\ 1 - \dfrac{1}{t}, & \zeta \le t \end{cases}$$

and $b = 2$, $\gamma_1 = 1$, $\gamma_2 = 0$, $\gamma_3 = \sqrt{\frac{3}{4}}$, $\Gamma = 0$.

Exact solution is

$$v(t) = \sqrt{\frac{3}{3 + t^2}}.$$

The precise and approximate solutions which is assessed using the provided method are shown in Table 5, while error analysis is given in Table 6. We see from Table 5 and Table 6 that CCM is more accurate rather than the BCM.

Table 5. Exact and estimated solutions of Example 3

t	CCM $n = 3$	BCM $n = 3$	Exact sol.
0.1	0.998348	0.998337	0.998337
0.2	0.993408	0.993393	0.993399
0.3	0.985336	0.985325	0.985329
0.4	0.974360	0.974361	0.974355
0.5	0.960773	0.960788	0.960769
0.6	0.944911	0.944939	0.944911
0.7	0.927139	0.927170	0.927146
0.8	0.907831	0.907855	0.907841
0.9	0.887348	0.887358	0.887357
1.0	0.866025	0.866025	0.866025

Table 6. Errors analysis of Example 3

	L_∞		L_2	
n	CCM	BCM	CCM	BCM
3	1.11E-05	2.74E-05	2.56E-05	4.50E-05
4	6.56E-06	1.40E-05	1.03E-05	2.36E-05
5	3.81E-07	7.70E-07	6.94E-07	1.16E-06
6	4.59E-08	1.29E-07	6.79E-08	1.83E-07
7	8.89E-09	3.19E-08	1.29E-08	3.66E-08
8	4.43E-09	6.95E-09	1.01E-09	1.39E-09
9	8.15E-10	8.70E-10	8.35E-10	9.52E-10

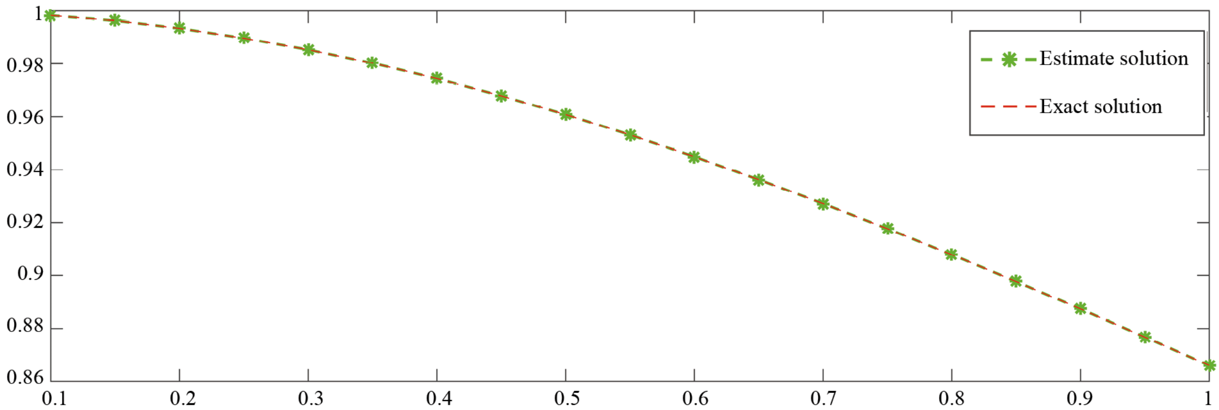

Figure 3. Comparison of estimate and exact solution of example 3

Example 4

$$\left(tv'\right)' = -te^{v},\ v'(0) = 0,\ v(1) = 0,\ t \in (0,\ 1).$$

Its equivalent integral form is

$$v(t) = \frac{\gamma_3}{\gamma_1} + \int_0^1 \kappa(t,\ \zeta)\ \zeta\ \left(-e^{v(\zeta)}\right) d\zeta,$$

where

$$\kappa(t,\ \zeta) = \begin{cases} \ln\zeta, & t \le \zeta, \\ \ln t, & \zeta \le t. \end{cases}$$

Here $b = 1$, $\gamma_1 = 1$, $\gamma_2 = 0$, $\gamma_3 = 0$, $\Gamma = 0$.
Exact solution is

$$v(t) = 2\ln\left(\frac{4-2\sqrt{2}}{(3-2\sqrt{2})(t^2+1)}\right).$$

The precise and approximate solutions which is assessed using the provided method are shown in Table 7, while error analysis is included in Table 8. It can be observe from Table 7 and Table 8 that CCM is more accurate rather than the BCM.

Table 7. Exact and estimated solutions of Example 4

t	CCM $n = 3$	BCM $n = 3$	Exact sol.
0.1	0.313265	0.313265	0.313266
0.2	0.303015	0.303019	0.303015
0.3	0.286047	0.286053	0.286047
0.4	0.262531	0.262536	0.262531
0.5	0.232696	0.232700	0.232697
0.6	0.196826	0.196829	0.196827
0.7	0.155248	0.155250	0.155248
0.8	0.108322	0.108324	0.108323
0.9	0.056438	0.056438	0.056438
1.0	0.000000	0.056438	0.000000

Table 8. Errors analysis of Example 4

	L_∞		L_2	
n	CCM	BCM	CCM	BCM
3	1.12E-05	5.55E-06	2.07E-07	9.59E-06
4	1.14E-06	3.26E-06	2.13E-06	5.77E-06
5	1.35E-08	2.41E-08	2.42E-08	4.02E-08
6	6.59E-09	2.21E-08	1.11E-08	3.70E-08
7	2.85E-10	6.80E-10	4.27E-10	1.01E-09
8	3.76E-11	1.28E-10	6.45E-11	2.13E-10
9	2.86E-12	9.89E-12	5.60E-12	1.17E-11

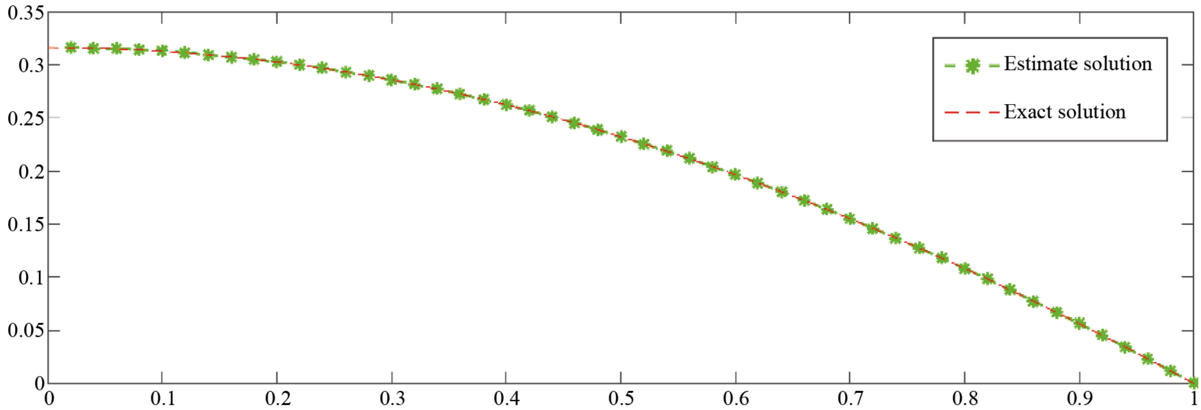

Figure 4. Comparison of estimate and exact solution of example 4

Example 5

$$\left(t^2 v'\right)' = -t^2 e^{-v},\ v'(0) = 0,\ 2v(1) + v'(1) = 0,\ t \in (0,\ 1).$$

Its integral form is

$$v(t) = \frac{\gamma_3}{\gamma_1} + \int_0^1 \kappa(t,\ \zeta)\ \zeta^2\ \left(-e^{-v(\zeta)}\right) d\zeta,$$

where $b = 2$, $\gamma_1 = 2$, $\gamma_2 = 1$, $\gamma_3 = 0$, $\Gamma = 0$ and

$$\kappa(t,\ \zeta) = \begin{cases} 1 - \dfrac{1}{\zeta}, & t \le \zeta, \\ 1 - \dfrac{1}{t}, & \zeta \le t. \end{cases}$$

We have compared absolute difference of estimated solutions $E_{45} = \|v_4 - v_5\|$ of CCM with the BCM in Table 9 which shows that the CCM is far better than the BCM.

Table 9. Approximated solutions of Example 5

	CCM			BCM		
t	v_4	v_5	E_{45}	v_4	v_5	E_{45}
0.1	0.26875701	0.26875690	1.05E-08	0.26875692	0.26875694	1.60E-08
0.2	0.26493295	0.26493282	1.31E-08	0.26493280	0.26493285	5.50E-08
0.3	0.25853993	0.25853979	1.41E-08	0.25853984	0.25853982	2.35E-08
0.4	0.24954829	0.24954819	1.05E-07	0.24954832	0.24954820	1.21E-07
0.5	0.23791594	0.23791590	4.31E-08	0.23791607	0.23791591	1.60E-07
0.6	0.22358772	0.22358771	4.73E-09	0.22358787	0.22358773	1.40E-07
0.7	0.20649451	0.20649448	2.69E-08	0.20649462	0.20649450	1.17E-07
0.8	0.18655211	0.18655201	9.49E-08	0.18655218	0.18655203	1.43E-07
0.9	0.16365983	0.16365969	1.42E-08	0.16365991	0.16365971	2.08E-07
1.0	0.13769888	0.13769875	1.27E-08	0.13769900	0.13769877	2.25E-07

Example 6

$$\left\{ \left(t^2 v'\right)' = t^2 \frac{0.76129v}{v+0.03119},\ v'(0)=0,\ 5v(1)+v'(1)=5,\ t\in(0,\ 1). \right.$$

It is equivalent to

$$v(t)=\frac{\gamma_3}{\gamma_1}+\int_0^1 \kappa(t,\ \zeta)\,\zeta^2\left(\frac{0.76129\,v(\zeta)}{v(\zeta)+0.03119}\right)d\zeta,$$

where

$$\kappa(t,\ \zeta)=\begin{cases}\left(1-\frac{1}{\zeta}\right)-\frac{1}{5}, & t\le\zeta,\\ \left(1-\frac{1}{t}\right)-\frac{1}{5}, & \zeta\le t\end{cases}$$

and $b=2$, $\gamma_1=5$, $\gamma_2=1$, $\gamma_3=5$, $\Gamma=0$.

We have compared the absolute difference of estimated solutions $E_{45}=\|v_4-v_5\|$ of CCM with the BCM in Table 10 and we can observe that the CCM perform better than the BCM.

Table 10. Estimated solutions of Example 6

	CCM			BCM		
t	v_4	v_5	E_{45}	v_4	v_5	E_{45}
0.1	0.82970609	0.82970609	8.53E-10	0.82970610	0.82970609	5.21E-09
0.2	0.83337473	0.83337473	6.07E-10	0.0.83337474	0.83337473	6.83E-09
0.3	0.83948991	0.83948991	3.37E-10	0.83948992	0.83948991	4.07E-09
0.4	0.84805278	0.84805278	1.22E-10	0.84805279	0.84805278	4.58E-10
0.5	0.85906492	0.85906492	1.06E-10	0.85906493	0.85906493	1.09E-09
0.6	0.87252832	0.87252831	7.50E-11	0.87252832	0.87252832	4.46E-10
0.7	0.88844530	0.88844530	6.95E-11	0.88844531	0.88844531	4.49E-10
0.8	0.90681854	0.90681854	2.15E-11	0.90681855	0.90681855	2.70E-10
0.9	0.92765098	0.92765098	1.82E-11	0.92765099	0.92765099	2.35E-09
1.0	0.95094579	0.95094579	1.28E-11	0.95094580	0.95094580	2.90E-09

Example 7

$$\left\{(t^b v'(t))' = t^{b-1}(te^{2v(t)} - be^{v(t)}),\ v(0) = \ln\left(\frac{1}{2}\right),\ v(1) = \ln\left(\frac{1}{3}\right),\ t \in (0,\ 1).\right.$$

It is equivalent to

$$v(t) = \Gamma + \frac{(\gamma_3 - \gamma_1\Gamma)}{\gamma_1\ell(1) + \gamma_2\ell'(1)}\ell(t) + \int_0^1 \kappa(t,\ \zeta)\ \zeta^{b-1}\ \left(\zeta e^{2v(\zeta)} - be^{v(\zeta)}\right)d\zeta,$$

where

$$\kappa(t,\ \zeta) = \begin{cases} \ell(t) - \dfrac{\ell(\zeta)\ell(t)}{\gamma_1\ell(1) + \gamma_2\ell'(1)}, & t \le \zeta, \\ \ell(\zeta) - \dfrac{\ell(t)\ell(\zeta)}{\gamma_1\ell(1) + \gamma_2\ell'(1)}, & \zeta \le t. \end{cases}$$

$$\gamma_1 = 1,\ \gamma_2 = 0,\ \gamma_3 = \ln\left(\frac{1}{3}\right),\ \Gamma = \ln\left(\frac{1}{2}\right),$$

$$p(t) = t^b,\ \ell(t) = \frac{t^{1-b}}{1-b},\ \ell(1) = \frac{1}{1-b},\ \ell'(1) = \frac{1}{p(1)}.$$

Exact solution is

$$v(t) = \ln\left(\frac{1}{t+2}\right).$$

We provided estimated solution by CCM, estimated solution by BCM and the analytic solution of Example 7 in Table 11, while error analysis is given in Table 12. We observe from the data in Tables 11 and Table 12 that the CCM performs more accurately than the BCM.

Table 11. Exact and estimated solutions of Example 7 for $b = 0.25$

x	CCM $n = 5$	BCM $n = 5$	Exact sol.
0.1	-0.741937344	-0.741936031	-0.741937345
0.2	-0.788457360	-0.788456236	-0.788457360
0.3	-0.832909121	-0.832908230	-0.832909123
0.4	-0.875468736	-0.875467976	-0.875468737
0.5	-0.916290732	-0.916290088	-0.916290732
0.6	-0.955511444	-0.955510943	-0.955511445
0.7	-0.993251772	-0.993251406	-0.993251773
0.8	-1.029619417	-1.029619152	-1.029619417
0.9	-1.064710737	-1.064710574	-1.064710737

Table 12. Errors analysis of Example 7

	L_∞		L_2	
n	CCM	BCM	CCM	BCM
3	2.85E-05	8.89E-05	4.30E-05	1.67E-04
4	2.61E-06	1.02E-05	3.21E-06	1.83E-05
5	2.02E-07	1.31E-06	2.42E-07	2.29E-06
6	1.20E-08	1.66E-07	1.85E-08	2.79E-07
7	1.11E-09	2.16E-08	1.39E-09	3.70E-08
8	8.51E-11	2.79E-09	1.27E-10	4.72E-09
9	1.21E-11	3.70E-10	1.39E-11	6.43E-10

Example 8

$$\left(t^b v'\right)' = t^{b+m-2}\left((m-bm)e^v - 4m^2 e^{2v}\right),\ v'(0) = 0,\ v(1) = \ln\left(\frac{1}{5}\right),\ t \in (0,\ 1).$$

Equivalently,

$$v(t) = \frac{\gamma_3}{\gamma_1} + \int_0^1 \kappa(t,\ \zeta)\, q(\zeta)\, \left((l-bm)e^{v(\zeta)} - 4m^2 e^{2v(\zeta)}\right) d\zeta,$$

where

$$\kappa(t,\ \zeta) = \begin{cases} l(1) - l(\zeta), & t \leq \zeta, \\ l(1) - l(t), & \zeta \leq t \end{cases}$$

$\gamma_1 = 1$, $\gamma_2 = 0$, $\gamma_3 = \ln\left(\frac{1}{5}\right)$, $\Gamma = 0$, $p(t) = t^b$, $l(t) = \dfrac{t^{(1-b)}}{1-b}$, $l(1) = \dfrac{1}{1-b}$ and $l'(1) = \dfrac{1}{p(1)}$.
Exact solution is

$$v(t) = \ln\left(\frac{1}{t^m + 4}\right).$$

We have compared estimated solution of CCM with the estimated solution of the BCM and the analytic solution of Example 8 in Table 13, while error analysis is given in Table 14. We can observe from the data in Tables 13 and Table 14 that the CCM performs more accurately than the BCM.

Table 13. Exact and estimated solutions of Example 8 for $b = 0.25$ and $m = 1.25$

t	CCM	BCM	Exact sol.
0.1	-1.400231277	-1.399003503	-1.400254990
0.2	-1.419163423	-1.418078681	-1.419184516
0.3	-1.440296721	-1.439353928	-1.440314833
0.4	-1.462801962	-1.462000686	-1.462817411
0.5	-1.486228255	-1.485567112	-1.486241095
0.6	-1.510285017	-1.509761697	-1.510295147
0.7	-1.534768071	-1.534379929	-1.534775540
0.8	-1.559526775	-1.559271039	-1.559531716
0.9	-1.584446497	-1.584320188	-1.584448947

Table 14. Errors analysis of Example 8

	L_∞		L_2	
n	CCM	BCM	CCM	BCM
3	2.85E-04	5.53E-03	5.36E-04	1.02E-02
4	1.23E-04	2.34E-03	2.35E-04	4.33E-03
5	6.49E-05	1.25E-03	1.22E-04	2.31E-03
6	3.77E-05	7.85E-04	7.05E-05	1.45E-03
7	2.37E-05	5.40E-04	4.39E-05	1.00E-03
8	1.56E-05	3.94E-04	2.89E-05	7.29E-04
9	1.08E-05	3.00E-04	1.99E-05	5.50E-04

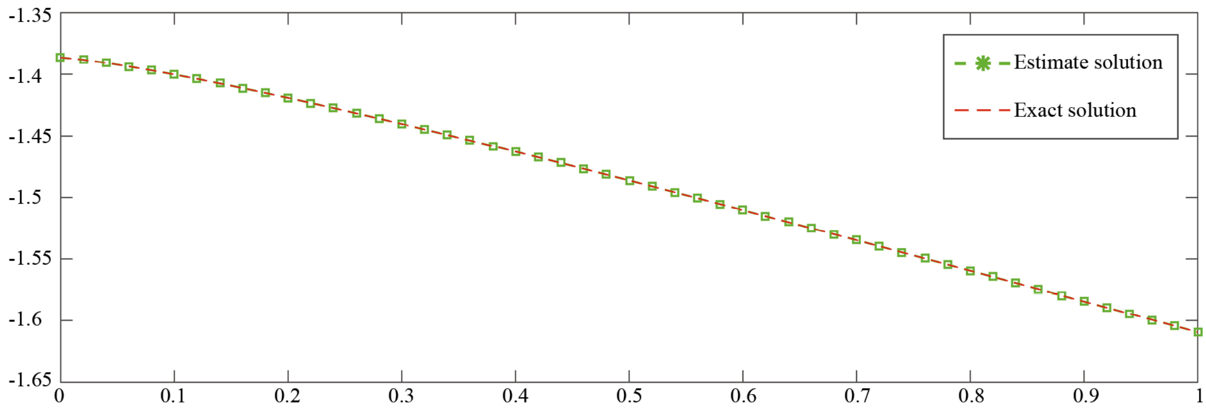

Figure 5. Comparison of estimate and exact solution of example 8

6. Conclusion

The estimated solution of the Lane-Emden-Fowler BVPs with various BCs has been carried out using the collocation technique based on Chebyshev polynomial. The Fredholm integral form of non-linear singular and doubly SBVPs have been taken into consideration to get the approximate solutions numerically. The primary advantage of the current technique is to reach the requisite level of accuracy compared to other established techniques, such as the BCM [35]. The error analysis of the method for various numerical examples using L_∞ and L_2 norms establishes the fact that the estimated solutions are quite near to the exact solutions rather than the BCM. Additionally, the graph is drawn to compare estimated solutions with exact solutions for some of the examples which show that the accuracy of the current technique is quite high.

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Conflict of interest

The authors declare there is no conflict of interest at any point with reference to research findings.

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