Contemporary Mathematics
http://ojs.wiserpub.com/index.php/CM/

Research Article

Geometric Vorticity and the Twofold Nature of Orthogonality

Ioannis Dimitriou

Research and Innovation Center, BMW Group Schleißheimerstr. 424, 80788 Munich, Germany
E-mail: ioannis.dimitriou@bmw.de

Received: 18 October 2023; **Revised:** 12 December 2023; **Accepted:** 6 February 2024

Abstract: Planar vector fields can be visualized using their tangent lines. It is shown that orthogonality between these curves and their associated orthogonal trajectories can be classified as well-ordered or irregular. Criterion for this taxonomy is the Global Curvature vector $\boldsymbol{K_G}$ (a quantity involving the local curvatures of the two sets of lines) and in particular its rotation, $\nabla \times \boldsymbol{K_G}$. The latter, which has been termed Geometric Vorticity $\boldsymbol{\Gamma}$, is an important quantity for the characterization of a two-dimensional vector field. Depending on the kinematical constraints the field is subjected to, $\boldsymbol{\Gamma}$ can either vanish (well-ordered orthogonality) or not (irregular case). The main theorem of the study asserts that every Laplacian vector field is geometrically irrotational $(\nabla \times \boldsymbol{K_G} = 0)$ and therefore well-ordered. Conversely, well-ordered orthogonal nets (which are sets of curves admitting zero geometric vorticity) can always be attributed to a Laplacian vector field. The necessary and sufficient condition for this behavior is the harmonicity of their angle function φ $(\Delta\varphi = 0)$, which is defined as the angle of incidence of the field lines. This provides a pure geometric criterion a vector field should fulfill, in order to satisfy the Laplace conditions. It connects its "appearance" with its "nature", thus allowing the experimentalist to study the violation of continuity and irrotationality in physical processes by mere observation.

Keywords: orthogonality, angle function, global curvature, geometric vorticity, laplace equation, harmonic function

MSC: 53A45, 53A04, 53A17

1. Introduction

The word "orthogonal" comes from the Greek word "ορθογώνιος". The latter derives from the words "ορθός" and "γωνία" meaning "upright" (or "correct") and "angle" respectively. Therefore, orthogonal is a term used to describe an upright angle. In mathematics and specifically in geometry the concept of orthogonality is employed to describe a certain behavior of lines: two isolated curves are said to be orthogonal if they are perpendicular at their point of intersection (Figure 1a). In mathematical terms this condition is captured by the negative (or opposite) reciprocal nature of their slopes. In particular, assuming the curves have no parallel slopes to the axis, the orthogonality condition at their point of intersection reads as:

$$m_1 \equiv -\frac{1}{m_2} \quad \text{(or equivalently } m_1 m_2 = -1\text{)}$$

DOI: https://doi.org/10.37256/cm.5320243794

where m_1 and m_2 denote the slopes of the two curves. Taking into consideration that a slope m of a curve is the tangent function of its angle of inclination φ, the last equation can be equally written as:

$$\tan\varphi_1 \tan\varphi_2 = -1 \quad \left(\text{where obviously } \varphi_2 - \varphi_1 = \frac{\pi}{2}\right).$$

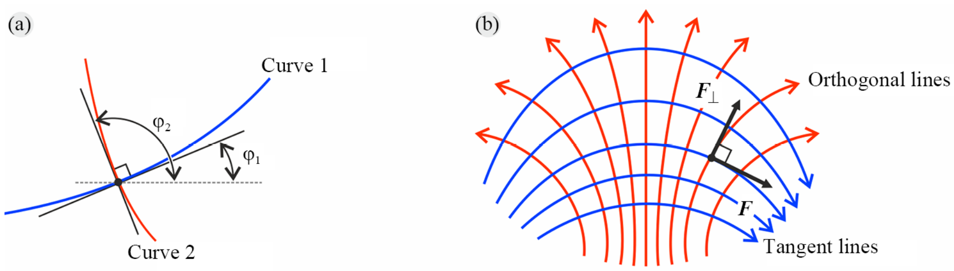

Figure 1. Orthogonality between two isolated curves (a) and two vector fields (b)

The concept of orthogonality can be generalized to vector fields as well. In this case we are not interested in the orthogonality of just two isolated lines but rather two sets of lines as shown in Figure 1b. The blue colored set of lines can be thought of as the field lines of a vector field $\boldsymbol{F}$ while the red set corresponds to a field that is perpendicular to the initial one. For any planar vector field $\boldsymbol{F}$ with components F_x and F_y in the Cartesian coordinate system, there is a new vector field $\boldsymbol{F}_\perp = (-F_y,\ F_x)$ that is vertical to $\boldsymbol{F}$. The two fields are said to be orthogonal, or perpendicular, if and only if their inner product vanishes identically [1]:

$$\boldsymbol{F} \cdot \boldsymbol{F}_\perp = 0. \tag{1}$$

At each point in their domains, the vector fields are pairwise orthogonal and consequently so are their associated tangent lines (often called integral curves, field lines, lines of forces or streamlines), which are defined as the curves $\boldsymbol{x}(t)$ satisfying the following differential equation (in case of the vector field $\boldsymbol{F}$):

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{F}(\boldsymbol{x}),$$

where the variable t is used to parameterize the plane curves.

Yet, no matter how clear and definite equation (1) regarding the mutual orthogonality of planar fields appears to be, careful examination of the geometric properties of the lines involved reveals further information on the very nature of the field they belong to. As we are about to show, orthogonality can be further classified as "well-ordered" or "irregular", depending on the value of the sum of the rate of change of the streamline curvature (K_S), with respect to the streamline arclength s, and the rate of change of the orthogonal line curvature (K_N), with respect to its arclength n $\left(\dfrac{\partial K_S}{\partial s} + \dfrac{\partial K_N}{\partial n}\right)$. This behavior can be mathematically captured by a geometric quantity, the so-called Global Curvature vector $(\boldsymbol{K_G})$ and in particular its rotation $\boldsymbol{\Gamma}\,(\boldsymbol{\Gamma} = \nabla \times \boldsymbol{K_G})$, a quantity that has been given the name Geometric Vorticity. More specifically, in the most general case, that is when the field $\boldsymbol{F}$ is not constrained by kinematical conditions such as irrotationality or incompressibility, geometric vorticity could take either positive or negative values. Accordingly, the orthogonality of the lines between $\boldsymbol{F}$ and $\boldsymbol{F}_\perp$ will be considered as "irregular". If the vector field $\boldsymbol{F}$ on the other hand is both solenoidal $(\nabla \cdot \boldsymbol{F} = 0)$ and irrotational $(\nabla \times \boldsymbol{F} = \boldsymbol{0})$, geometric vorticity must vanish everywhere in the field. This leaves a specific footprint on the pattern of the lines involved, a certain "signature" which not only can be quantified but also observed experimentally. In this special case the orthogonality between the streamlines of $\boldsymbol{F}$ and $\boldsymbol{F}_{\perp}$ will be considered as "well-ordered".

In section 5, it will be shown that being well-ordered is intimately related to the harmonicity of the angle of incidence φ, which is the angle formed between the tangent lines of $\boldsymbol{F}$ and the x-axis. It is well known that scalar functions, which have continuous second partial derivatives and satisfy Laplace's equation are called harmonic functions. They occur in several applications in engineering and physical sciences (ranging from robotics [2] to quantum mechanics [3]) and they depict a certain notion of "stability", whenever one point in the interior of a planar region is influenced in terms of its values on the boundary [4]. Exactly this behavior is captured by the scalar φ when Laplacian vector fields are considered and explains why the term "ordered" was adopted to describe them in the first place. It does not only offer a geometric interpretation of Laplacian vector fields, but also provides a link between observation of a certain phenomenon and the underlying physics, thus making it valuable from an experimental point of view. In section 6 we offer a summary of the main results of the study along with some use cases to demonstrate their applications. Finally, a brief outlook on interesting possibilities for future work on this subject is presented.

At this point we would like to mention that the majority of the calculations in this manuscript is based on the streamline, or intrinsic coordinate system, which employs the arc-lengths s and n along the stream and orthogonal lines, as independent variables. It aligns with both the field-lines and their normal trajectories, thus forming a right-handed orthogonal curvilinear system, defined by the unit tangent vector $\boldsymbol{t}$ and the unit vector $\boldsymbol{n}$ normal to it, as depicted in Figure 2. The streamlines correspond to the lines of constant n-values, while the orthogonal trajectories correspond to the lines of constant s-values. The $(s,\ n)$ system has been chosen on purpose because it is built upon the geometry of the field. Consequently, it incorporates its geometric footprint, something which is going to be beneficial in our further analysis, significantly facilitating our efforts in performing mathematical calculations.

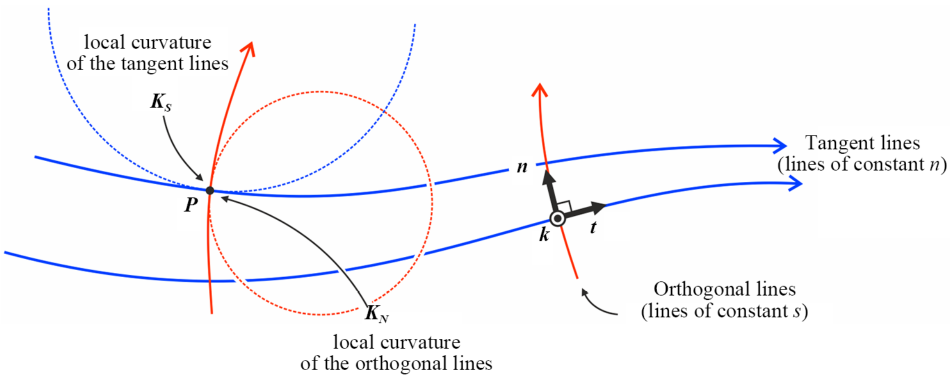

Figure 2. Schematic representation of the streamline coordinate system and the local curvatures K_S and K_N corresponding to the tangent and orthogonal trajectories of a planar vector field

In the following section, the definition of global curvature $\boldsymbol{K_G}$, which builds the foundation for our further study will be presented (for tracing its origin and exact evolution one would have to chronologically follow some of the studies presented in the list of references [5–7]). The need for its introduction will later become apparent, thus justifying our foresighted decision for its conception in 2007 [5].

2. The global curvature vector

Curvature is literally a measure of how much a line "bends" at each point. Leonhard Euler defined curvature K as the ratio

$$K \equiv \frac{d\varphi}{ds} \quad (\text{Definition of curvature}) \tag{2}$$

which is the change in the angle of inclination (or "angle function") φ of a curve divided by the change in arc-length in an infinitely small location (Figure 3a). Obviously, a large change of the angle in a short distance will produce a large curvature and vice versa. Therefore, a straight line has zero curvature, while a circle has a constant one that is inversely proportional to its radius. Newton eventually introduced the idea of curvature radius (Figure 3b), as the radius of the largest possible circle (the so called "osculating circle"), which is tangent to a curve on its concave side and it is inversely proportional to its curvature [8]:

$$R = \frac{1}{K} \quad (\text{Definition of radius of curvature}).$$

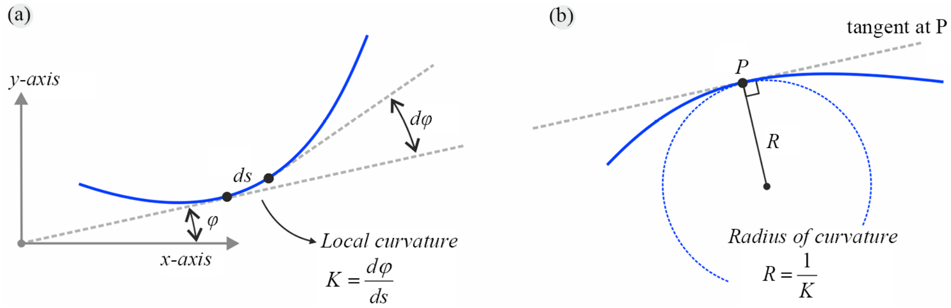

Figure 3. Definition of the angle function φ, curvature K and radius of curvature R

Now consider a set of curves that correspond to the tangent lines of a planar vector field. One could uniquely define a perpendicular set of curves to which we will refer to as the orthogonal trajectories of the initial vector field. Each individual point of the vector field is defined by the intersection of two distinct curves, one tangent line and one orthogonal trajectory (see for example point "P" in Figure 2). One could therefore assign two numbers at every point of the field representing the local curvatures K_S and K_N of these two lines (the scalar function K_S, also called the curvature function of the field, gives the local curvature of the streamlines at every point in the plane). Eventually, these curvature functions, can be combined to form the global curvature Vector $\boldsymbol{K_G}$ defined as follows:

Definition 1 The global curvature vector $\boldsymbol{K_G}$ is defined in the streamline coordinate system as,

$$\boldsymbol{K_G} \equiv (-K_N,\ K_S). \tag{3}$$

Its tangent component is the negative of the local curvature of the orthogonal line K_N, while its vertical component equals to the streamline curvature K_S.

For both components of $\boldsymbol{K_G}$ the signed curvature is implied and therefore one should take into account its correct sign, depending on the flow topology being investigated. Following the right-hand rule, the sign is positive for an anti-clockwise sense of rotation, during a hypothetical motion of a point particle along the curve. For the vector field topology at "P" shown in Figure 4, K_S and K_N are negative and positive respectively, which explains the specific direction of $\boldsymbol{K_G}$.

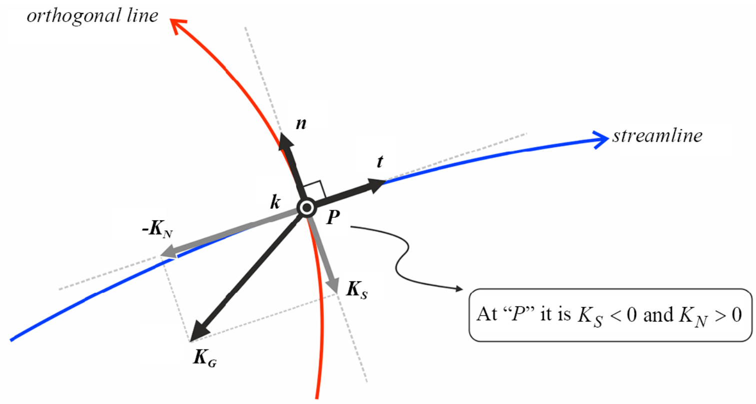

Figure 4. Schematic representation of the global curvature vector $\boldsymbol{K_G}$

As already mentioned in the introduction the notion of "*global curvature*" (and eventually that of "*geometric vorticity*") is crucial in our further enquiry. Based on this, some holistic results regarding the nature of a vector field can be directly deduced. One of them reveals that there are two types of orthogonality as long as a pair of mutually orthogonal lines is involved. But before developing the corresponding theorem, we would like to present two methods for the computation of the curvature functions K_S and K_N, even when an explicit analytical expression of the associated field trajectories is unknown.

3. Computation of the curvature functions corresponding to a planar vector field

3.1 *Vector field geometry based on the components of the vector field*

We will first show how the streamline curvature at every point can be found, as long as the vector field components are known. For the rest of the treatise a unique description of a vector field $\boldsymbol{F}$ in terms of its angle function φ will be assumed. This scalar along with the two vector components F_x and F_y are supposed to be piecewise analytic with defined and continuous partial derivatives.

Lemma 1 The curvature function K_S of the tangent lines of a vector field $\boldsymbol{F}$ is equal to the rotation (*Curl* operator $\nabla\times$) of its normalized vector $\boldsymbol{t}$:

$$K_S = (\nabla \times \boldsymbol{t}) \cdot \boldsymbol{k}$$

where $\boldsymbol{k}$ is the unit vector perpendicular to the planar field (refer to Figure 4).

Proof. The total differential of the angle function $\varphi(x,\ y)$ that describes $\boldsymbol{F}$ can be written in the Cartesian coordinate system as:

$$d\varphi = \frac{\partial \varphi}{\partial x}dx + \frac{\partial \varphi}{\partial y}dy.$$

Division of the last equation with the line element the line element ds leads to:

$$\frac{d\varphi}{ds} = \frac{\partial \varphi}{\partial x}\frac{dx}{ds} + \frac{\partial \varphi}{\partial y}\frac{dy}{ds} \overset{(2)}{\Leftrightarrow}$$
$$K_S = \frac{\partial \varphi}{\partial x}\frac{dx}{ds} + \frac{\partial \varphi}{\partial y}\frac{dy}{ds}, \tag{4}$$

where K_S denotes the streamline curvature. Taking into consideration the following two relations

$$\sin\varphi = \frac{dy}{ds} \quad \text{and} \quad \cos\varphi = \frac{dx}{ds}, \tag{5}$$

which are valid everywhere in the field, equation (4) updates to:

$$K_S = \frac{\partial}{\partial x}(\sin\varphi) - \frac{\partial}{\partial y}(\cos\varphi). \tag{6}$$

On the other hand, the unit tangent vector $\boldsymbol{t}$ can be written in terms of the angle function φ as:

$$\boldsymbol{t} = (\cos\varphi,\ \sin\varphi).$$

By applying the *Curl* vector operator to $\boldsymbol{t}$ we obtain (we view here $\boldsymbol{t}$ as a vector in R^3 with a zero component in the vertical direction $\boldsymbol{k}$):

$$\nabla \times \boldsymbol{t} = \left(\frac{\partial}{\partial x}(\sin\varphi) - \frac{\partial}{\partial y}(\cos\varphi) \right) \boldsymbol{k}.$$

Comparison between the last equation and equation (6) shows that the curvature of the vector field streamlines in every point is equal to the algebraic value of the rotation of the corresponding unit vector:

$$K_S = (\nabla \times \boldsymbol{t}) \cdot \boldsymbol{k}. \tag{7}$$

The above method for calculating K_S was first developed in 2007 and is referred to as the "*Method of rotation*" [5]. Obviously, if the vector field components are known, one could work out the components of its normalized vector, to feed equation (7) for the computation of the streamline curvature everywhere in the field. The key in succeeding our goal was to study the field as a whole. Introducing the angle function φ was crucial. It is this scalar which, when combined with the very definition of curvature, leads to an analytical solution for the field curvature even if the field streamlines are entirely unknown. We now proceed with the application of Lemma 1 to the set of lines that are orthogonal to the generator field $\boldsymbol{F}$.

Lemma 2 The curvature function K_N corresponding to the orthogonal set of lines of a vector field $\boldsymbol{F}$ is equal to the divergence of its normalized vector $\boldsymbol{t}$:

$$K_N = \nabla \cdot \boldsymbol{t}.$$

Proof. Because of the orthogonality condition, the unit tangent vector $\boldsymbol{n}$ of the orthogonal vector field $\boldsymbol{F}_\perp$ has the following components:

$$\boldsymbol{n} = (-\sin\varphi,\ \cos\varphi).$$

According to the rotation method presented earlier, the curvature function K_N will be given from the following equation:

$$\begin{aligned}
&K_N = (\nabla \times \boldsymbol{n}) \cdot \boldsymbol{k} \Leftrightarrow \\
&K_N = \left(\frac{\partial}{\partial x}(\cos\varphi) - \frac{\partial}{\partial y}(-\sin\varphi)\right) \Leftrightarrow \\
&K_N = \left(\frac{\partial}{\partial x}(\cos\varphi) + \frac{\partial}{\partial y}(\sin\varphi)\right) \Leftrightarrow \\
&K_N = \nabla \cdot \boldsymbol{t}.
\end{aligned} \tag{8}$$

A very interesting aspect associated with Lemmas 1 and 2 is that the two basic operators of vector calculus imply a strong geometric character when acting on its normalized field. Furthermore, equations (7) and (8) do not rely upon parametrizations of the curves they describe.

3.2 *Vector field geometry based on the angle function*

We would now like to present the second method, which allows the computation of streamline curvature based on the knowledge of the angle function φ.

Lemma 3 The curvature function K_S of a vector field $\boldsymbol{F}$ is equal to the directional derivative of its angle function, calculated with respect to the direction of its tangent lines:

$$K_S = \nabla\varphi \cdot \boldsymbol{t}.$$

Proof. The directional derivative of φ in the field direction is accordingly:

$$\nabla\varphi \cdot \boldsymbol{t} = \left(\frac{\partial\varphi}{\partial s}\boldsymbol{t} + \frac{\partial\varphi}{\partial n}\boldsymbol{n}\right) \cdot \boldsymbol{t} = \frac{\partial\varphi}{\partial s}.$$

But since the change in the angle $d\varphi$ of the tangent vector over the change in arc-length ds in an infinitely small location equals the streamline curvature K_S at that point (equation 2),

$$K_S \equiv \frac{\partial \varphi}{\partial s} \tag{9}$$

equation (7) updates to:

$$K_S = \nabla\varphi \cdot \boldsymbol{t}. \tag{10}$$

We will refer to the equation above as the "*Directional Derivative Method*" (for further details please refer to [5]), in order to distinguish it from the "*rotation method*" presented at the beginning of this section (equation 7). Based on Lemma 3, it will now be shown how it is possible to compute the curvature function corresponding to the orthogonal trajectories of the field $\boldsymbol{F}$.

Lemma 4 The curvature function K_N for the set of lines that are orthogonal to the field's tangent lines is given by the directional derivative of the angle function, calculated at right angles (anticlockwise) to the direction of the generator vector field:

$$K_N = \nabla\varphi \cdot \boldsymbol{n}$$

Proof. If φ is the angle function of the generator vector field $\boldsymbol{F}$, the following scalar

$$\vartheta = \varphi + \frac{\pi}{2}$$

represents the angle function of a vector field $\boldsymbol{F}_\perp$ that is perpendicular to $\boldsymbol{F}$ (Figure 5).

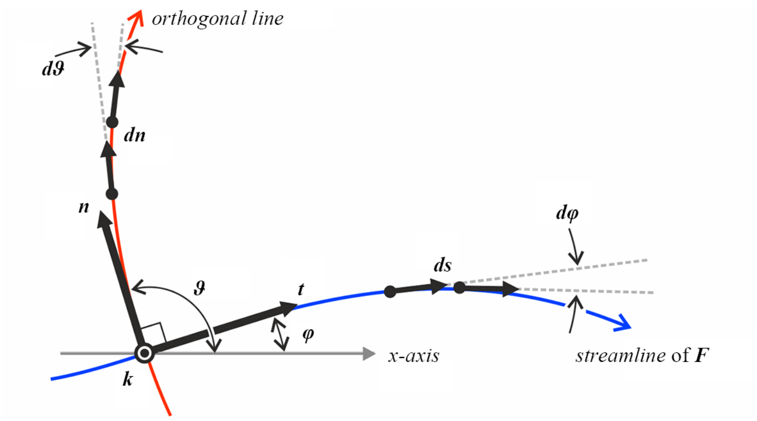

Figure 5. Graphical description of the angle functions corresponding to a 2D vector field $\boldsymbol{F}$

Taking into account that the unit tangent vector of the new vector field is now $\boldsymbol{n}$, the corresponding curvature function K_N can be obtained by making direct use of the directional derivative method. According to equation (10):

$$
\begin{aligned}
&K_N = \nabla \vartheta \cdot \boldsymbol{n} = \nabla \left(\varphi + \frac{\pi}{2} \right) \cdot \boldsymbol{n} \Leftrightarrow \\
&K_N = \nabla \varphi \cdot \boldsymbol{n}
\end{aligned}
\tag{11}
$$

and the Lemma has been proven. The factor $\frac{\partial \varphi}{\partial n}$, that is the rate of change of the angle function φ along the $\boldsymbol{n}$-direction, represents the curvature of the orthogonal trajectories, since:

$$
\begin{aligned}
&K_N = \left(\frac{\partial \varphi}{\partial s} \boldsymbol{t} + \frac{\partial \varphi}{\partial n} \boldsymbol{n} \right) \cdot \boldsymbol{n} \Leftrightarrow \\
&K_N = \frac{\partial \varphi}{\partial n}.
\end{aligned}
\tag{12}
$$

Lemma 5 Global curvature $\boldsymbol{K_G}$ is an incompressible vector field:

$$
\nabla \cdot \boldsymbol{K_G} = 0
$$

Proof. In virtue of equations (9) and (12) the definition of $\boldsymbol{K_G}$ (equation 3) can be rewritten as:

$$
\boldsymbol{K_G} = \left(-\frac{\partial \varphi}{\partial n}, \frac{\partial \varphi}{\partial s} \right).
$$

The *Divergence* of $\boldsymbol{K_G}$ can then be calculated in the intrinsic coordinate system as:

$$
\nabla \cdot \boldsymbol{K_G} = \nabla \cdot \left(-\frac{\partial \varphi}{\partial n}, \frac{\partial \varphi}{\partial s} \right) = -\frac{\partial}{\partial s} \left(\frac{\partial \varphi}{\partial n} \right) + \frac{\partial}{\partial n} \left(\frac{\partial \varphi}{\partial s} \right).
$$

After assuming sufficient smoothness, to permit changing the order of differentiation, the above equation leads to the incompressibility of $\boldsymbol{K_G}$:

$$
\nabla \cdot \boldsymbol{K_G} = 0. \tag{13}
$$

For practical purposes (for example, in order to plot K_S and K_N when a vector field is given), both equation (10) and (11) can be analyzed in the Cartesian coordinate system. For the vector field $\boldsymbol{F}$ with an angle of incidence φ the curvature function of its tangent lines is given by:

$$
\begin{aligned}
K_S &= \left(\frac{\partial \varphi}{\partial x}\boldsymbol{i} + \frac{\partial \varphi}{\partial y}\boldsymbol{j}\right) \cdot (\cos\varphi \boldsymbol{i} + \sin\varphi \boldsymbol{j}) \Leftrightarrow \\
K_S &= \frac{\partial \varphi}{\partial x}\cos\varphi + \frac{\partial \varphi}{\partial y}\sin\varphi,
\end{aligned} \tag{14}
$$

while for the orthogonal trajectories, equation (11) gives:

$$
\begin{aligned}
K_N &= \left(\frac{\partial \varphi}{\partial x}\boldsymbol{i} + \frac{\partial \varphi}{\partial y}\boldsymbol{j}\right) \cdot (-\sin\varphi \boldsymbol{i} + \cos\varphi \boldsymbol{j}) \Leftrightarrow \\
K_N &= -\frac{\partial \varphi}{\partial x}\sin\varphi + \frac{\partial \varphi}{\partial y}\cos\varphi.
\end{aligned} \tag{15}
$$

In Figure 6 several plots are depicted, all corresponding to the following vector field $\boldsymbol{v}$:

$$
\boldsymbol{v} = (v_x,\ v_y) = \left(\frac{x^4 + y^4 - x^2 + y^2 + 2x^2y^2}{(x^2+y^2)^2},\ \frac{-2xy}{(x^2+y^2)^2}\right)
$$

representing the velocity distribution of an incompressible ($\nabla \cdot \boldsymbol{v} = 0$) and at the same time inviscid ($\nabla \times \boldsymbol{v} = \boldsymbol{0}$) uniform flow over a circular cylinder with unit radius [9]. Such vector fields, satisfying incompressibility ($\nabla \cdot \boldsymbol{F} = 0$) and irrotationality ($\nabla \times \boldsymbol{F} = \boldsymbol{0}$) conditions are often called potential or Laplacian vector fields. In that specific case the corresponding orthogonal trajectories are usually called potential or equipotential lines (instead of orthogonal). The first plot (top left, Figure 6a) shows the flow streamlines (blue colored lines) and their orthogonal trajectories (red colored lines). The next diagram displays the iso-lines of the corresponding angle of incidence φ (6b), computed by using the equation below:

$$
\varphi = \tan^{-1}\left(\frac{v_y}{v_x}\right) = \tan^{-1}\left(\frac{-2xy}{x^4 + y^4 - x^2 + y^2 + 2x^2y^2}\right).
$$

The other two diagrams depict the iso-lines of the curvature functions K_S and K_N (6c and 6d), which were generated with the aid of equations (14) and (15) respectively.

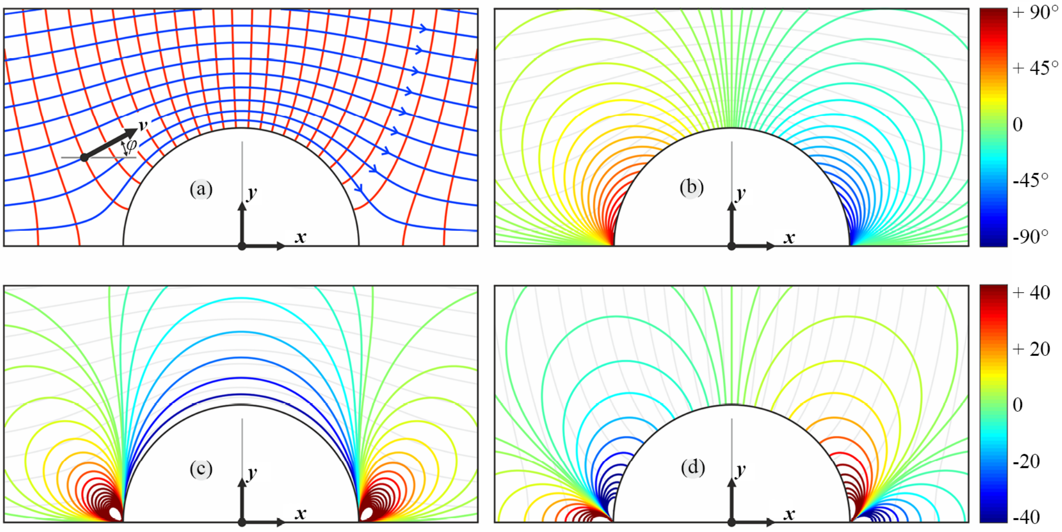

Figure 6. Stream and potential line plot (a), angle function plot (b) and streamline curvature plots (c for K_S and d for K_N) for the velocity vector field of an inviscid flow around a circular cylinder

4. The concept of geometric vorticity

Definition 2 Let $\boldsymbol{F}$ be a continuously differentiable vector field. The *Curl* of the corresponding global curvature Vector $\boldsymbol{K_G}$, to which we will refer by the symbol $\boldsymbol{\Gamma}$, constitutes the so-called geometric vorticity field of $\boldsymbol{F}$:

$$\boldsymbol{\Gamma} \equiv \nabla \times \boldsymbol{K_G} \quad \text{(definition of geometric vorticity)} . \tag{16}$$

In two dimensions $\boldsymbol{\Gamma}$ has a single component parallel to $\boldsymbol{k}$ (which is the unit vector perpendicular to the planar field), that is

$$\boldsymbol{\Gamma} = (\mathrm{Curl}_z \boldsymbol{K_G}) \cdot \boldsymbol{k} \equiv \gamma \boldsymbol{k}, \tag{17}$$

where γ is defined as the scalar geometric vorticity. The term "vorticity" is obviously borrowed from the field of fluid mechanics where the quantity $\nabla \times \boldsymbol{v}$ represents the well-established vorticity vector of the flow velocity field $\boldsymbol{v}$. As a direct consequence of the definition above we can prove the following Lemma:

Lemma 6 The scalar geometric vorticity γ of any plane vector field equals to the sum of the rate of change of the streamline curvature with respect to s and the rate of change of the orthogonal line curvature with respect to n . In other words:

$$\gamma = \frac{\partial K_S}{\partial s} + \frac{\partial K_N}{\partial n}.$$

Proof. In the natural coordinate system, we have:

$$
\begin{aligned}
&\mathbf{\Gamma} = \nabla \times (-K_N \boldsymbol{t} + K_S \boldsymbol{n}) = -K_N \nabla \times \boldsymbol{t} - \nabla K_N \times \boldsymbol{t} + K_S \nabla \times \boldsymbol{n} + \nabla K_S \times \boldsymbol{n} \overset{(7)}{\Leftrightarrow} \\
&\mathbf{\Gamma} = -K_N K_S \boldsymbol{k} - \left(\frac{\partial K_N}{\partial s}\boldsymbol{t} + \frac{\partial K_N}{\partial n}\boldsymbol{n}\right) \times \boldsymbol{t} + K_S \nabla \times \boldsymbol{n} + \left(\frac{\partial K_S}{\partial s}\boldsymbol{t} + \frac{\partial K_S}{\partial n}\boldsymbol{n}\right) \times \boldsymbol{n}.
\end{aligned}
\tag{18}
$$

But according to equation (8) we have:

$$
\nabla \times \boldsymbol{n} = (\nabla \cdot \boldsymbol{t})\boldsymbol{k} = K_N \boldsymbol{k}
$$

and equation (18) updates to:

$$
\begin{aligned}
&\mathbf{\Gamma} = -K_N K_S \boldsymbol{k} - \left(\frac{\partial K_N}{\partial s}\boldsymbol{t} + \frac{\partial K_N}{\partial n}\boldsymbol{n}\right) \times \boldsymbol{t} + K_S K_N \boldsymbol{k} + \left(\frac{\partial K_S}{\partial s}\boldsymbol{t} + \frac{\partial K_S}{\partial n}\boldsymbol{n}\right) \times \boldsymbol{n} \Leftrightarrow \\
&\mathbf{\Gamma} = -\left(\frac{\partial K_N}{\partial n}\right)\boldsymbol{n} \times \boldsymbol{t} + \left(\frac{\partial K_S}{\partial s}\right)\boldsymbol{t} \times \boldsymbol{n} \Leftrightarrow \\
&\mathbf{\Gamma} = \left(\frac{\partial K_S}{\partial s} + \frac{\partial K_N}{\partial n}\right)\boldsymbol{k} \Leftrightarrow \\
&\mathbf{\Gamma} \cdot \boldsymbol{k} = \frac{\partial K_S}{\partial s} + \frac{\partial K_N}{\partial n}.
\end{aligned}
$$

In virtue of the definition (17) we finally obtain the following formula for the scalar geometric vorticity:

$$
\gamma = \frac{\partial K_S}{\partial s} + \frac{\partial K_N}{\partial n}. \tag{19}
$$

Moreover, it is possible to link γ to the angle function of the vector field. The following theorem reveals this connection:

Theorem 1 The Laplacian of the angle function φ of any vector field $\boldsymbol{F}$ equals its geometric vorticity γ

$$
\Delta\varphi = \gamma.
$$

Proof. The mathematical expression of the Laplace operator in the intrinsic coordinate system incorporates the global curvature vector and is given by the following equation (the exact derivation can be found in [10]):

$$
\Delta = -\boldsymbol{K_G} \cdot \nabla + \frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial n^2}.
$$

Its application on the scalar function φ gives:

$$\Delta\varphi = -\boldsymbol{K_G}\cdot\nabla\varphi + \frac{\partial^2\varphi}{\partial s^2} + \frac{\partial^2\varphi}{\partial n^2} \Leftrightarrow$$

$$\Delta\varphi = -(-K_N, K_S)\cdot\nabla\varphi + \frac{\partial^2\varphi}{\partial s^2} + \frac{\partial^2\varphi}{\partial n^2} \overset{(9)\&(12)}{\Longleftrightarrow}$$

$$\Delta\varphi = -\left(-\frac{\partial\varphi}{\partial n}, \frac{\partial\varphi}{\partial s}\right)\cdot\left(\frac{\partial\varphi}{\partial s}, \frac{\partial\varphi}{\partial n}\right) + \frac{\partial^2\varphi}{\partial s^2} + \frac{\partial^2\varphi}{\partial n^2} = 0 + \frac{\partial^2\varphi}{\partial s^2} + \frac{\partial^2\varphi}{\partial n^2} \Leftrightarrow \tag{20}$$

$$\Delta\varphi = \frac{\partial}{\partial s}\left(\frac{\partial\varphi}{\partial s}\right) + \frac{\partial}{\partial n}\left(\frac{\partial\varphi}{\partial n}\right) \Leftrightarrow$$

$$\Delta\varphi = \frac{\partial K_S}{\partial s} + \frac{\partial K_N}{\partial n}.$$

Comparison between equations (19) and (20) yields the following Poisson equation:

$$\Delta\varphi = \gamma. \tag{21}$$

Equations (19) and (21) have a general character, since they can be applied to all types of fields. In the following section though we will see how the divergence-free and irrotationality conditions, which are often encountered in the fields of electrodynamics and fluid mechanics, expose an interesting geometric feature for the vector field to which these constraints were imposed to.

5. Laplacian vector fields and well-ordered orthogonality

In this section we restrict our focus to the study of vector fields that are *incompressible* (*solenoidal*) and *conservative* at the same time. Such fields, which are constrained through both incompressibility ($\nabla\cdot\boldsymbol{F} = 0$) and irrotationality ($\nabla\times\boldsymbol{F} = \boldsymbol{0}$) conditions are termed "*Laplacian*". It can be shown that their geometry is such so that $\boldsymbol{K_G}$ apart from incompressible ($\nabla\cdot\boldsymbol{K_G} = 0$) is also an irrotational ($\nabla\times\boldsymbol{K}_G = \boldsymbol{0}$) vector field.

Theorem 2 The global curvature $\boldsymbol{K_G}$ corresponding to a planar Laplacian vector field $\boldsymbol{F}$ is a Laplacian field itself.

Proof. In general, the *Divergence* of a plane vector field $\boldsymbol{F}$ can be developed as:

$$\nabla\cdot\boldsymbol{F} = F(\nabla\cdot\boldsymbol{t}) + \nabla F\cdot\boldsymbol{t} \overset{(8)}{\Leftrightarrow}$$

$$\nabla\cdot\boldsymbol{F} = FK_N + \left(\frac{\partial F}{\partial s}\boldsymbol{t} + \frac{\partial F}{\partial n}\boldsymbol{n}\right)\cdot\boldsymbol{t} \Leftrightarrow \tag{22}$$

$$\nabla\cdot\boldsymbol{F} = FK_N + \frac{\partial F}{\partial s}.$$

If $\boldsymbol{F}$ is solenoidal though ($\nabla \cdot \boldsymbol{F} = 0$) the last equation updates to:

$$K_N = -\frac{1}{F}\frac{\partial F}{\partial s}.$$

Furthermore the Rotation of a plane vector field can be developed as:

$$\begin{aligned}
&\nabla \times \boldsymbol{F} = \nabla F \times \boldsymbol{t} + F \nabla \times \boldsymbol{t} \overset{(7)}{\Leftrightarrow} \\
&\nabla \times \boldsymbol{F} = \nabla F \times \boldsymbol{t} + F\left(K_S \boldsymbol{k}\right) \Leftrightarrow \\
&\nabla \times \boldsymbol{F} = F K_S \boldsymbol{k} + \left(\frac{\partial F}{\partial s}\boldsymbol{t} + \frac{\partial F}{\partial n}\boldsymbol{n}\right) \times \boldsymbol{t} \Leftrightarrow \\
&\nabla \times \boldsymbol{F} = \left(F K_S - \frac{\partial F}{\partial n}\right)\boldsymbol{k} \Leftrightarrow \\
&|\nabla \times \boldsymbol{F}| = F K_S - \frac{\partial F}{\partial n}
\end{aligned} \tag{23}$$

Because the examined field is planar, its Rotation has a single component parallel to $\boldsymbol{k}$: $\nabla \times \boldsymbol{F} = |\nabla \times \boldsymbol{F}|\boldsymbol{k}$. If $\boldsymbol{F}$is conservative ($\nabla \times \boldsymbol{F} = \boldsymbol{0}$), equation (23) simplifies to:

$$K_S = \frac{1}{F}\frac{\partial F}{\partial n}.$$

Summarizing our results, the two kinematical constraints imply that:

$$\begin{cases} K_N = -\dfrac{1}{F}\dfrac{\partial F}{\partial s} \\ \\ K_S = \dfrac{1}{F}\dfrac{\partial F}{\partial n} \end{cases} \Longleftrightarrow \begin{cases} -K_N = \dfrac{\partial}{\partial s}(\ln F) \\ \\ K_S = \dfrac{\partial}{\partial n}(\ln F) \end{cases} \tag{24}$$

which means that $\boldsymbol{K_G}$ can be written as the gradient of a single scalar (potential):

$$\boldsymbol{K_G} = \nabla(\ln F).$$

Since the *Curl* of the gradient of any scalar field (which is continuously twice-differentiable) is always zero, it is:

$$\nabla \times \boldsymbol{K}_G = \boldsymbol{0}. \tag{25}$$

Together with the solenoidal character of $\boldsymbol{K_G}$ demonstrated earlier (please refer to lemma 5) it can be concluded that the global curvature of a Laplacian vector field is Laplacian as well.

We are about to reveal that the two families of curves (tangent and potential lines), corresponding to $\boldsymbol{F}$, exhibit a special geometric regularity. Specifically, equation (19) considerably simplifies as the next theorem shows:

Theorem 3 (Geometric Conservation Law of Laplacian flows) The geometric vorticity γ of any planar Laplacian field vanishes identically ($\gamma = 0$), while its angle function φ is harmonic and thus satisfies Laplace's equation ($\Delta\varphi = 0$).

Proof. The vanishing of geometric vorticity follows immediately from theorem 2 and specifically equation (25). Yet, there is another way to prove that, by calculating the derivatives of equations (24) with respect to n and s respectively. It is then:

$$\begin{cases} \dfrac{\partial K_N}{dn} = -\dfrac{\partial^2(\ln F)}{\partial n \partial s} \\ \\ \dfrac{\partial K_S}{ds} = \dfrac{\partial^2(\ln F)}{\partial s \partial n} \end{cases}.$$

After assuming sufficient smoothness to permit changing the order of differentiation, addition of the last two equations leads to an interesting relation between the two curvatures

$$\frac{\partial K_S}{\partial s} + \frac{\partial K_N}{\partial n} = 0. \tag{26}$$

Comparison between equations (19) and (26) directly yields

$$\gamma = 0. \tag{27}$$

Therefore, the sum of the rate of change of the streamline curvature, with respect to the streamline arc-length and the rate of change of the potential line curvature, with respect to the potential line arc-length, equals zero. Or shortly, geometric vorticity vanishes identically. For this reason, γ can be seen as a geometric invariant, while equation (27) can be characterized as a geometric conservation law that Laplacian vector fields must fulfill.

There is even an alternative expression for the aforementioned law, which instead of the two curvature functions K_S and K_N employs just one scalar, namely the angle function φ of the vector field. In the most general case, φ satisfies Poisson's equation. But according to theorem 2, if the field is Laplacian the corresponding global curvature Vector $\boldsymbol{K_G}$ is irrotational ($\gamma = 0$) and equation (21) immediately simplifies to:

$$\Delta\varphi = 0. \tag{28}$$

Concluding, if $\boldsymbol{F}$ represents a Laplacian vector field, its geometry must be such that the scalar φ is a harmonic function.

Definition 3 From now on we would like to refer to orthogonal sets of lines as well-ordered, if these satisfy either of equations (27) or (28). Analogously, a planar vector field satisfying equations (27) or (28) will be named a well-ordered field.

It follows to wonder whether the opposite argument holds as well. Specifically, does the vanishing of geometric vorticity automatically secure the existence of a vector field which is Laplacian, and which could act as the global curvature vector $\boldsymbol{K_G}$? Even more, does the harmonicity of φ imply the "Laplacian nature" of the vector field $\boldsymbol{F}$, to which $\boldsymbol{K_G}$ is ascribed to? Before answering the above raised questions it is necessary to rewrite the Cauchy-Riemann (C-R) equations met in complex analysis (which together with some continuity and differentiability criteria, form a necessary and sufficient condition for a complex function to be holomorphic), using streamline, instead of Cartesian coordinates. This is presented in the following Lemma:

Lemma 7 In the streamline coordinate system, the Cauchy-Riemann equations take the following form:

$$\begin{cases} \dfrac{\partial P}{\partial s} = \dfrac{\partial Q}{\partial n} \\ \\ \dfrac{\partial P}{\partial n} = -\dfrac{\partial Q}{\partial s} \end{cases}.$$

where P and Q are scalar functions representing the real and imaginary parts of a complex function.

Proof. Firstly, it is assumed that the partial derivatives of both scalar functions over x and y exist, are continuous and that in the domain of interest there are no critical points for which the angle function of the level-curves is 90° (in such case the local slope cannot be defined). In addition they satisfy the C-R equations, which in cartesian coordinates $(x,\ y)$ are given by the following system of partial differential equations:

$$\begin{cases} \dfrac{\partial P}{\partial x} = \dfrac{\partial Q}{\partial y} \\ \\ \dfrac{\partial P}{\partial y} = -\dfrac{\partial Q}{\partial x} \end{cases}. \tag{29}$$

The two scalars P and Q can be expressed in terms of the streamline coordinates $(s,\ n)$ as well. By using the chain rule, their partial derivatives over s and n, can be expressed as functions of x and y. For the first scalar it is:

$$\begin{aligned} &\frac{\partial P}{\partial s} = \frac{\partial P}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial P}{\partial y}\frac{\partial y}{\partial s} \Leftrightarrow \\ &\frac{\partial P}{\partial s} = \frac{\partial P}{\partial x}\cos\varphi + \frac{\partial P}{\partial y}\sin\varphi \Leftrightarrow \\ &\frac{\partial P}{\partial x} = \frac{1}{\cos\varphi}\frac{\partial P}{\partial s} - \frac{\partial P}{\partial y}\tan\varphi. \end{aligned} \tag{30}$$

Similarly, the term $\dfrac{\partial P}{\partial n}$ can be expressed as:

$$\frac{\partial P}{\partial n}=\frac{\partial P}{\partial x}\frac{\partial x}{\partial n}+\frac{\partial P}{\partial y}\frac{\partial y}{\partial n}\Leftrightarrow$$

$$\frac{\partial P}{\partial n}=\frac{\partial P}{\partial x}(-\sin\varphi)+\frac{\partial P}{\partial y}\cos\varphi\Leftrightarrow \tag{31}$$

$$\frac{\partial P}{\partial y}=\frac{1}{\cos\varphi}\frac{\partial P}{\partial n}+\frac{\partial P}{\partial x}\tan\varphi.$$

Analogously, following equations are obtained for the scalar Q:

$$\frac{\partial Q}{\partial x}=\frac{1}{\cos\varphi}\frac{\partial Q}{\partial s}-\frac{\partial Q}{\partial y}\tan\varphi, \tag{32}$$

$$\frac{\partial Q}{\partial y}=\frac{1}{\cos\varphi}\frac{\partial Q}{\partial n}+\frac{\partial Q}{\partial x}\tan\varphi. \tag{33}$$

By substituting equations (30)-(33) into equations (29), it is:

$$\begin{cases}\dfrac{1}{\cos\varphi}\dfrac{\partial P}{\partial s}-\dfrac{\partial P}{\partial y}\tan\varphi=\dfrac{1}{\cos\varphi}\dfrac{\partial Q}{\partial n}+\dfrac{\partial Q}{\partial x}\tan\varphi\\[2ex] \dfrac{1}{\cos\varphi}\dfrac{\partial P}{\partial n}+\dfrac{\partial P}{\partial x}\tan\varphi=-\dfrac{1}{\cos\varphi}\dfrac{\partial Q}{\partial s}+\dfrac{\partial Q}{\partial y}\tan\varphi\end{cases}\Leftrightarrow$$

$$\begin{cases}\dfrac{1}{\cos\varphi}\dfrac{\partial P}{\partial s}=\dfrac{1}{\cos\varphi}\dfrac{\partial Q}{\partial n}+\left(\dfrac{\partial Q}{\partial x}+\dfrac{\partial P}{\partial y}\right)\tan\varphi\\[2ex] \dfrac{1}{\cos\varphi}\dfrac{\partial P}{\partial n}=-\dfrac{1}{\cos\varphi}\dfrac{\partial Q}{\partial s}+\left(\dfrac{\partial Q}{\partial y}-\dfrac{\partial P}{\partial x}\right)\tan\varphi\end{cases}\Leftrightarrow \tag{34}$$

$$\begin{cases}\dfrac{\partial P}{\partial s}=\dfrac{\partial Q}{\partial n}\\[2ex] \dfrac{\partial P}{\partial n}=-\dfrac{\partial Q}{\partial s}\end{cases}.$$

The latter represents an alternative expression of the C-R equations in the streamline coordinate system. The following theorem can then be proven:

Theorem 4 For every well-ordered net of orthogonal lines ($\Delta\varphi=0$), there exists a unique vector field $\boldsymbol{K_G}$ which is Laplacian and which describes the net.

Proof. From complex analysis, it is known that a harmonic function P ($\Delta P=0$) always admits a conjugate function Q ($\Delta Q=0$), which is unique up to a constant. Furthermore, every harmonic function is the real part of a complex differentiable function and thus, together with its conjugate, satisfy the C-R equations [11]. In the present context, the harmonicity of the angle function φ ($\Delta\varphi = 0$) implies that there is a harmonic conjugate N ($\Delta N = 0$), uniquely defined up to an additive constant. Moreover, the scalars φ and N are respectively the real and imaginary parts of a holomorphic function $f(z)$ of z the complex variable :

$$f(z) = \varphi + iN,$$

such as their first-order partial derivatives satisfy equations (34):

$$\begin{cases} \dfrac{\partial \varphi}{\partial s} = \dfrac{\partial N}{\partial n} \\ \\ \dfrac{\partial \varphi}{\partial n} = -\dfrac{\partial N}{\partial s} \end{cases} \overset{(9)\&(12)}{\Longleftrightarrow} \begin{cases} K_S = \dfrac{\partial N}{\partial n} \\ \\ K_N = -\dfrac{\partial N}{\partial s} \end{cases}. \tag{35}$$

As a result, the global curvature vector that could be assigned to the well-ordered net is given by the following equation:

$$\boldsymbol{K_G} = (-K_N,\, K_S) \overset{(35)}{\Longleftrightarrow} \boldsymbol{K_G} = \left(\frac{\partial N}{\partial s},\, \frac{\partial N}{\partial n}\right) \Leftrightarrow \boldsymbol{K_G} = \nabla N. \tag{36}$$

It shows that the scalar N, the so-called "geometric potential", serves as a potential function of the geometry of the field and therefore $\boldsymbol{K_G}$ is irrotational ($\nabla \times \boldsymbol{K_G} = \mathbf{0}$). In addition, the divergence of $\boldsymbol{K_G}$ can be written as,

$$\nabla \cdot \boldsymbol{K_G} = \nabla \cdot \nabla N.$$

Because N is a harmonic conjugate, the right-hand side of the equation above vanishes and $\boldsymbol{K_G}$ is incompressible ($\nabla \cdot \boldsymbol{K_G} = 0$). Consequently, $\boldsymbol{K_G}$ is a Laplacian vector field.

It is noteworthy that equation (36) immediately follows from equation (16). The assumption that the angle function is harmonic, automatically implies that geometric vorticity vanishes ($\nabla \times \boldsymbol{K_G} = \mathbf{0}$) and therefore the global curvature vector admits a scalar potential N, such $\boldsymbol{K_G} = \nabla N$, which is in fact equation (36). This way the proof of theorem 4 would have been much shorter and there would be no need for proving Lemma 7 in advance. Nevertheless, it was intentionally decided to follow the "complex analysis path", in order to reveal the uniqueness of $\boldsymbol{K_G}$ as well as the geometric relation between the angle function and the geometric potential. Since they represent the real and imaginary parts of a complex holomorphic function, φ and N are related as having orthogonal trajectories, that is their isolines cross at right angles. The same applies for their gradients ($\nabla\varphi \cdot \nabla N = 0$). In this regard $f(z) = \varphi + iN$ would be identified as the geometric complex potential. The fact that the C-R conditions hold for both φ and N, is a necessary and sufficient condition for the function $f(z)$ to be analytic. This in turn implies that it is also differentiable with a finite limit, which is independent of the direction of differentiation.

So far, we have established the existence of a unique expression for $\boldsymbol{K_G}$ when a harmonic angle function φ is given. Since $\boldsymbol{K_G}$ is Laplacian, one might think that it could represent the global curvature vector of a Laplacian vector field $\boldsymbol{F}$ (the converse argument of theorem 2). This is indeed the case as the next theorem shows.

Theorem 5 Well-ordered orthogonality ($\Delta\varphi = 0$) can always be attributed to the geometry of a Laplacian vector field $\boldsymbol{F}$.

Proof. It is assumed that the angle function φ of a well-ordered net corresponds to a non-vanishing vector field $\boldsymbol{F}$, in the sense that the angle of incidence of the unit tangent vector $\boldsymbol{t}$ coincides everywhere with the slope of the lines of one of the two sets. In virtue of equations (22) and (23), $\boldsymbol{K_G}$ can be developed as:

$$\begin{aligned}
&\boldsymbol{K_G} = (-K_N,\ K_S) \Longleftrightarrow \\
&\boldsymbol{K_G} = \left(\frac{1}{F}\frac{\partial F}{\partial s} - \frac{\nabla \cdot \boldsymbol{F}}{F},\ \frac{1}{F}\frac{\partial F}{\partial n} + \frac{|\nabla \times \boldsymbol{F}|}{F}\right) \Leftrightarrow \\
&\boldsymbol{K_G} = \left(\frac{\partial \ln F}{\partial s},\ \frac{\partial \ln F}{\partial s}\right) + \frac{1}{F}(\nabla \cdot \boldsymbol{F},\ |\nabla \times \boldsymbol{F}|) \Leftrightarrow \\
&\boldsymbol{K_G} = \nabla \ln F + \frac{1}{F}(\nabla \cdot \boldsymbol{F},\ |\nabla \times \boldsymbol{F}|)
\end{aligned} \tag{37}$$

This is the most general expression of $\boldsymbol{K_G}$, because up to this point, no kinematical restrictions have been imposed to $\boldsymbol{F}$. Theorem 4 states that the harmonicity of φ implies the existence of a well-defined curvature vector $\boldsymbol{K_G}$, which is irrotational ($\boldsymbol{K_G} = \nabla N$) with components that are intimately related to the geometry of the net. Considering that if a potential function exists, it is uniquely defined up to an additive constant, comparison between equations (36) and (37) leads to the following results:

$$\begin{cases} N = \ln F + c \\ \nabla \cdot \boldsymbol{F} = 0 \\ \nabla \times \boldsymbol{F} = \boldsymbol{0} \end{cases} \Longleftrightarrow \begin{cases} F = Ce^N \\ \nabla \cdot \boldsymbol{F} = 0 \\ \nabla \times \boldsymbol{F} = \boldsymbol{0} \end{cases} \quad \text{(where } c \text{ and } C \text{ are some constants).} \tag{38}$$

Hence, one possible solution could emerge from the equality between (36) and (37), which demands that the vector field $\boldsymbol{F}$ is solenoidal ($\nabla \cdot \boldsymbol{F} = 0$) and conservative ($\nabla \times \boldsymbol{F} = \boldsymbol{0}$) at the same time. The direction of $\boldsymbol{F}$ is per definition identical to one of the line-sets of the net. Its magnitude can be derived with the aid of the geometric potential and appropriate boundary conditions.

Recalling Helmholtz theorem [12], a vector field can be uniquely reconstructed, if its divergence and curl are known functions. In our case they are both prescribed, so if in addition the field vanishes at infinity, it can be uniquely specified. Nevertheless, even if certain convergence properties at spatial infinity [13] make $\boldsymbol{F}$ unique, theorem 5 does not assert the uniqueness of a solution in general. In fact, there is at least one non-Laplacian vector field that fulfills the harmonicity of φ as well. This can be qualitatively demonstrated by imaging a Laplacian vector field $\boldsymbol{F}$ and its normalized one, $\boldsymbol{t} = \dfrac{\boldsymbol{F}}{|\boldsymbol{F}|}$. They both have identical tangent and orthogonal lines. Although they are described by the same angle function, a simple calculation shows that they cannot be Laplacian fields simultaneously. Thus, there is at least one additional, non-Laplacian solution that is consistent with the given geometry. Its multiplication with any constant number would lead to a new, geometrically equivalent vector field. We can conclude that there is an infinite number of fields with the same geometric footprint. This is a manifestation of the fact that streamlines alone are not conclusive on the magnitude of the vector elements that created them. Under these considerations we can formulate the following corollary:

Corollary 1 If the scalar geometric vorticity of a smooth non-zero vector field $\boldsymbol{F}$ vanishes identically ($\gamma = 0$), then it is possible to scale $\boldsymbol{F}$ by a positive-valued function into a Laplacian vector field.

Proof. If the geometric vorticity of $\boldsymbol{F}$ is zero everywhere, then Proposition 1 of [14] guarantees that in the neighborhood of every point, there exists an analytic function

$$f(z) = U + iV$$

with the property that the gradient field ∇U is a positive multiple of $\boldsymbol{F}$. Since U is the real part of an analytic function, U must be harmonic, while its gradient ∇U is a Laplacian function. Hence it is possible to scale $\boldsymbol{F}$ into a Laplacian vector field within the neighborhood.

The harmonicity of φ does not provide a necessary and sufficient condition for the Laplacian character of $\boldsymbol{F}$. A well-ordered net does not guarantee this, but at least it provides (via theorem 5 and corollary 1) the possibility to construct a Laplacian field that perfectly corresponds to the picture of the net. However it is certain that an orthogonal net with non-zero geometric vorticity ($\gamma \neq 0$, or equivalently $\Delta\varphi \neq 0$) can never be assigned to the tangent and potential lines of a Laplacian field. In such case, global curvature cannot be written as the gradient of a scalar ($\boldsymbol{K}_G \neq \nabla N$) and the arguments employed in theorem 5 will no longer hold.

6. Discussion and concluding remarks

After extensive research in the existing literature, we believe that equation (26) was first introduced by Bivens [14]. The author posed in 1992 the question "when does a set of orthogonal lines (also termed "net") possess a complex potential?". He studied the geometric aspects of imaginary valued analytic functions and in the context of complex analysis he showed a criterion for the existence of such a potential in terms of the arc-length derivatives of the plane curvature functions for the curves in the net (which in effect is equation 26). Accordingly, "one should be able to make an informed decision as to whether or not a net is isothermal (such special orthogonal nets corresponding to analytic functions were termed "isothermal") based solely upon an accurate picture of the net". He illustrated this by considering the orthogonal net of ellipses and parabolas shown in Figure 7.

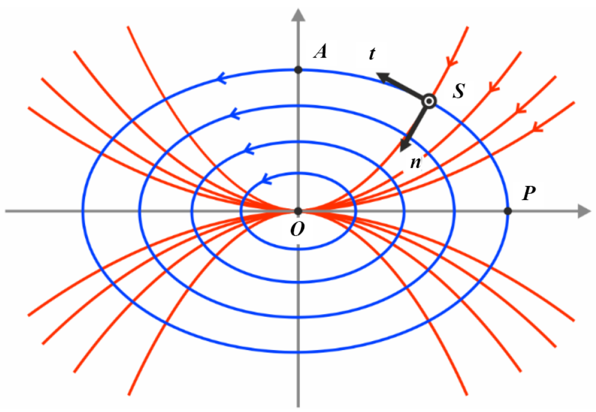

Figure 7. Demonstration of the non-vanishing geometric vorticity for the orthogonal net of ellipses $\left(x^2 + 2y^2 = C\right)$ and parabolas $\left(y = Cx^2\right)$ [14]

The curvature at point P is greater than in A and assuming a counterclockwise orientation along the ellipse (which here represents a tangent line) there will be a point in between, for example S, for which the factor $\dfrac{\partial K_S}{\partial s}$ is negative. However, for the parabola (which here represents the orthogonal trajectory) passing through , the signed curvature decreases towards the origin for a clockwise sense of rotation, implying that the factor $\dfrac{\partial K_N}{\partial n}$ will be negative as well. Therefore, the sum of the respective arc-length derivatives has to be negative and the net cannot be isothermal.

In 2009 [6], unaware of Biven's work and while studying sets of streamlines that are directly attributed to potential flow fields, we rediscovered equation (26). Later, in 2017 [7], it was generalized in three dimensions, while in the same year [10] it took its final form, based on the notion of geometric vorticity (equation 27). The latter offers a criterion to categorize orthogonal nets and to draw conclusions on the properties of the vector fields which they could represent. There are two types of orthogonality depending on whether γ is zero or not, in other words a "well-ordered" or an "irregular" one respectively. We would like to illustrate this by depicting the nets of three different vector fields: the "uniform" field consisting of parallel straight lines (Figure 8a), the "source" field having radial lines originating from a common point (Figure 8b) and a theoretical field the streamlines of which are straight lines emanating tangentially from a circle (Figure 8c). Although in all three nets the streamlines are characterized by the same curvature ($K_S = 0$), not all of them are well-ordered (geometrically irrotational). It can be easily verified that the sets of lines shown in Figure 8a and 8b fulfill equation (26). In the first case the corresponding curvature functions K_S and K_N vanish, while in the second example, both sets of curves have constant curvatures (zero for the streamlines and $1/\gamma$ for the orthogonal trajectories). Consequently, their derivatives with respect to their line elements vanish as well, automatically satisfying the geometric conservation law. On the other hand, the third field obviously does not fulfill this condition. Its tangent lines have zero curvature and hence the rate of change of with respect to vanishes. Nevertheless, its orthogonal set of lines, which consists of involutes of a circle (these lines are traced out by the end of an imaginary thread, tautly unwound from a stationary circular spool). shows a decreasing curvature outward from the center and consequently the factor $\dfrac{\partial K_N}{\partial n}$ takes negative values. Therefore, the net depicted in Figure 8c cannot correspond to a Laplacian vector field.

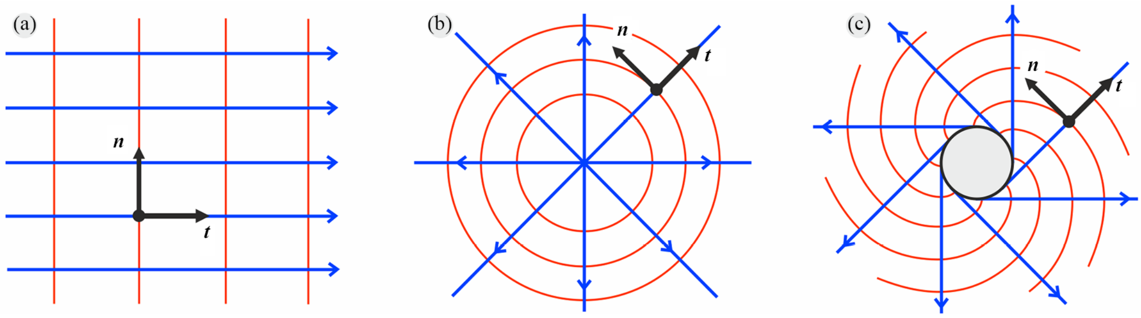

Figure 8. Illustration of three sets of mutually perpendicular lines with identical streamline curvature but different geometric vorticities

Whenever a set of field lines together with the associated orthogonal trajectories fulfill either of equations (26), (27) or (28), it is always possible to assign a simultaneously incompressible and irrotational vector field to it. Inversely, if these equations are not satisfied, then the field is non-continuous, or it is rotational or both at the same time. Kinematical conditions such as irrotationality and incompressibility leave a distinctive footprint in the field geometry. This gives us the means to tell fields apart and to deduce the nature of some observable physical processes. Imagine for example having a two-dimensional flow visualization. The visible streamlines could be digitalized, in order to evaluate their geometric vorticity. The angle function of the velocity field can easily be established at every point. It is now numerically possible to test its harmonicity. Violation of the geometric conservation law would identify the areas where friction plays an important role. On the contrary if the Laplacian of φ vanishes at some flow regions, one could almost be certain that the flow there is incompressible and free of vorticity.

For the sake of clarity, the obtained results of our study could be combined and summarized as follows: the streamlines and equipotential lines of any Laplacian vector field $\boldsymbol{F}$ (with $F = |\boldsymbol{F}|$) always form a well-ordered net. This condition is mathematically described by either of the following equivalent statements:

-The global curvature vector $\boldsymbol{K_G}$ is a Laplacian field,

-The angle function φ is harmonic, $\Delta\varphi = 0$,

-The geometric vorticity γ vanishes, $\gamma = 0$(or $\nabla \times \boldsymbol{K}_G = \mathbf{0}$),

-The global curvature vector $\boldsymbol{K_G}$ admits a geometric potential N, such as $\boldsymbol{K_G} = \nabla N$,

-The geometric potential N is a harmonic function, $\Delta N = 0$(where $N = \ln F + c$).

While every Laplacian vector field $\boldsymbol{F}$ always has a well-ordered appearance, the converse argument is not automatically satisfied and therefore each one of the statements above provides a necessary but not a sufficient condition for the existence of a Laplacian field. Fortunately, however, theorem 5 and corollary 1 tell us that a Laplacian vector field can always be assigned to the geometry of a well-ordered orthogonal net. Inversely, the statement that a net of orthogonal lines which is not well-ordered ($\Delta\varphi \neq 0$) can never be attributed to the geometry of a Laplacian vector field $\boldsymbol{F}$, is always true.

The geometric conditions obtained in the preceding analysis, associated with well-ordered nets, are graphically presented in the following diagram (Figure 9). Arrows coming together to the same "box", denote the simultaneous possession of the "features" which they come from. For instance, a planar Laplacian field is a well-ordered field, it is geometrically irrotational and exhibits a harmonic angle function. On the other hand, a well-ordered field is not necessarily Laplacian, but at least it can be scaled to a Laplacian vector field (this property is depicted by the white arrows).

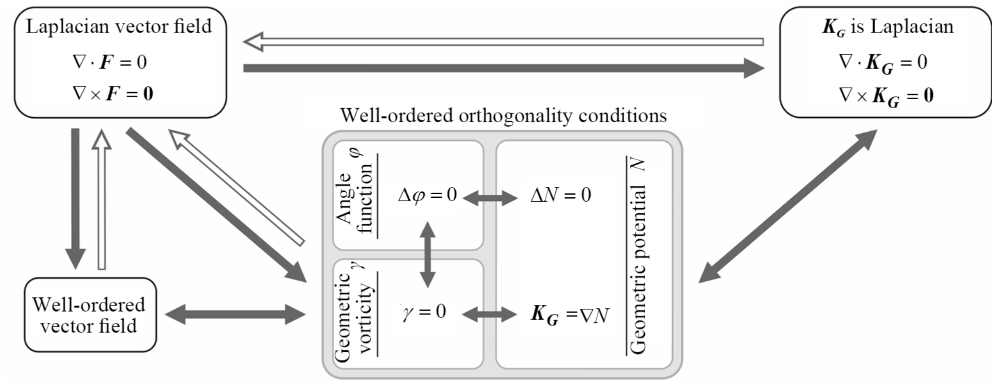

Figure 9. Geometric conditions describing well-ordered nets, or equivalently, Laplacian and well-ordered vector fields

The validity of the core conditions highlighted in Figure 9, can be demonstrated in some trivial examples met in fluid mechanics. We recall that the velocity field of a uniform flow over a circular cylinder (with unit radius) is both incompressible and irrotational (Laplacian field). In polar coordinates $(r,\ \vartheta)$ it is given by the following equation [15]

$$\boldsymbol{v} = (v_r,\ v_\vartheta) = \left(v_\infty\left(1 - \frac{1}{r^2}\right)\cos\vartheta,\ -v_\infty\left(1 + \frac{1}{r^2}\right)\sin\vartheta\right),$$

while the corresponding angle function reads as

$$\varphi = \tan^{-1}\left(\frac{\sin 2\vartheta}{\cos 2\theta - r^2}\right).$$

The first and second derivatives of φ over r and ϑ can be calculated. It is:

$$\frac{\partial \varphi}{\partial r} = \frac{2r\sin 2\vartheta}{1+r^4-2r^2\cos 2\vartheta}$$

$$\frac{\partial^2 \varphi}{\partial r^2} = 2\frac{\sin 2\vartheta - 3r^4\sin 2\vartheta + 2r^2\sin 2\vartheta\cos 2\vartheta}{\left(1+r^4-2r^2\cos 2\vartheta\right)^2}$$

$$\frac{\partial \varphi}{\partial \vartheta} = \frac{2-2r^2\cos 2\vartheta}{1+r^4-2r^2\cos 2\vartheta}$$

$$\frac{\partial^2 \varphi}{\partial \vartheta^2} = 2\frac{2r^6\sin 2\vartheta - 2r^2\sin 2\vartheta}{\left(1+r^4-2r^2\cos 2\vartheta\right)^2}.$$

Substituting the last equations into the Laplacian expression of φ in the polar coordinate system

$$\Delta\varphi = \frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\vartheta^2}$$

and after carrying some trivial computations it can easily be verified that $\Delta\varphi = 0$. The angle function satisfies Laplace equation and is therefore harmonic as expected.

The spiral vortex is another example of an elementary flow field that is Laplacian. For this reason, the flow admits a geometric potential N, which is connected to its velocity magnitude $\left(v = \frac{1}{r}\right)$, via the formula:

$$N = \ln\left(\frac{1}{r}\right) + c.$$

The Laplacian of N can be computed as:

$$\begin{aligned}\Delta N =& \frac{\partial^2 N}{\partial r^2} + \frac{1}{r}\frac{\partial N}{\partial r} + \frac{1}{r^2}\frac{\partial^2 N}{\partial\vartheta^2}\\ =& \frac{\partial}{\partial r}\left(\frac{\partial N}{\partial r}\right) + \frac{1}{r}\frac{\partial N}{\partial r} + 0 = \frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\right) + \frac{1}{r}r\frac{\partial}{\partial r}\left(\frac{1}{r}\right) = -\frac{\partial}{\partial r}\left(r\frac{1}{r^2}\right) - \frac{1}{r^2} = 0,\end{aligned}$$

which demonstrates that the geometry associated with the spiral vortex flow is well-ordered.

Now we consider the following vector field $\boldsymbol{F} = (x,\, y)$, the streamlines of which are rays originating from a common point (origin) while the orthogonal curves are circles centered at the origin with a counterclockwise orientation. The associated curvature functions are:

$$\begin{cases} K_S = 0 \\ \\ K_N = \dfrac{1}{\sqrt{x^2+y^2}}. \end{cases}$$

Since K_N is constant along an orthogonal curve it follows immediately that geometric vorticity vanishes identically ($\gamma = 0$). Furthermore, it can be verified that the angle function $\left(\varphi = \tan^{-1}(y/x)\right)$ is harmonic. It is:

$$\frac{\partial \varphi}{\partial x} = \frac{-y}{x^2+y^2} \quad \text{and} \quad \frac{\partial \varphi}{\partial y} = \frac{x}{x^2+y^2}.$$

Subsequently, the second derivatives of φ over x and y can be calculated:

$$\frac{\partial^2 \varphi}{\partial x^2} = \frac{2xy}{(x^2+y^2)^2} \quad \text{and} \quad \frac{\partial^2 \varphi}{\partial y^2} = \frac{-2yx}{(x^2+y^2)^2}.$$

Adding the last two equations together leads to:

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0$$

as expected. Although, $\boldsymbol{F}$ is a well-ordered vector field, it is not a Laplacian one, since it has a non-zero divergence $(\nabla \cdot \boldsymbol{F} = 2)$. Nevertheless, based on theorem 5, $\boldsymbol{F}$ can be scaled by a positive-valued function to form a Laplacian field. In our case a straightforward verification shows that the field

$$\boldsymbol{F}_L = \frac{1}{x^2+y^2}\boldsymbol{F} = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right),$$

is indeed incompressible and irrotational.

At last, it is important to mention that the application of the findings is not limited only to line sets corresponding to vector fields. In theory one could equally derive the same results by just studying sets of curves in an analytic manner. The reason why we ascribed a vector field to them for their description was to overcome the problem of curvature computations, which typically rely upon parametrizations of these curves. By doing so and by using tools of vector calculus and some basic results of differential geometry we managed to extract the geometric information needed for our investigation (see for example equations 7 and 8), even without prior knowledge of their analytic expressions. This fact does not restrict the validity of the findings, which can be applied to any sets of curves, not necessarily belonging to vector fields.

7. Outlook and future perspectives

Orthogonal sets of plane curves are uniquely defined by the opposite reciprocal nature of their slopes. Because of this unambiguous definition one would suppose that orthogonality is an "αυτοτελής" concept, meaning it is self-sufficient and does not demand any further analysis. Yet, as revealed in this manuscript, the nature of orthogonality is twofold and depends on whether or not geometric vorticity is zero. The vanishing of geometric vorticity has been demonstrated in the past in the case of ideal flows [16]. It would be very interesting though to investigate whether the geometric regularity imprinted in the flow streamlines outside the boundary layer developed on the surface of an airfoil profile (such as the one considered in [7], left diagram in Figure 6), could also be found in other "lines" or "patterns", not necessarily related to fluids. For example, the set of curves that corresponds to the streamlines of the spiral vortex flow is fascinatingly encountered in nature and art (Figure 10), which encourages us to search for further examples. In each individual pattern (the sunflower's core in the middle or the central panel from a tessellated floor of a roman villa on the right), two mutually orthogonal sets of spiral lines can be distinguished. The geometric vorticity for each pair is zero everywhere ($\gamma = 0$), which is equivalent to the irrotationality of the associated global curvature vector. Based on these examples it can be argued, that if the geometric conservation law is occasionally favored in nature, beauty is perhaps not solely manifested through symmetry, but also through the vanishing of geometric vorticity. This new parameter might provide an additional criterion for what is perceived to be aesthetically pleasing to the eye.

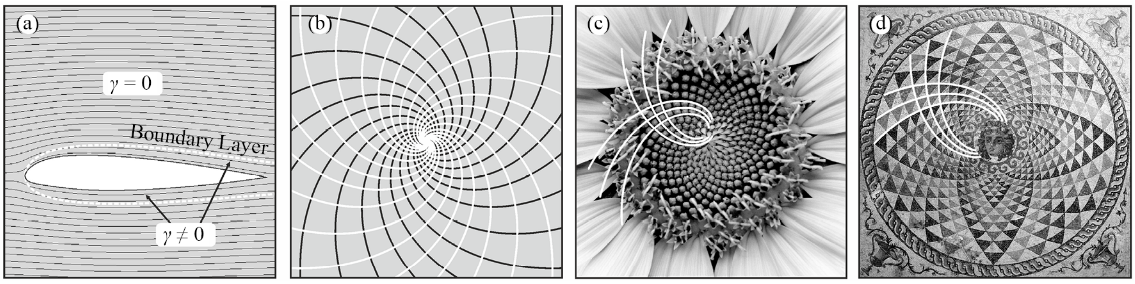

Figure 10. Demonstration of the vanishing of geometric vorticity in nature and art

In this study orthogonality was categorized in terms of the vanishing of geometric vorticity. It would however be interesting to further investigate orthogonal pairs of curves admitting either positive or negative values of γ. Moreover, the global curvature vector along with its derivative concept (geometric vorticity) were defined in two dimensions. Yet three dimensional fields are of equal if not of more interest, due to the vast number of real-life examples involving them. Such fields could encompass more than two, mutually vertical, sets of lines. Hence, research on their geometric properties might reveal additional information regarding their nature and the constraints they are subjected to. As a next step it would be logical to extend and generalize the notion of global curvature in three dimensions. This task would demand taking the concept of torsion into account as well, thus increasing the complexity of the problem. As a consolation, the taxonomy of orthogonality in three, or even more dimensions could be richer than the "binary" character that plane perpendicular curves possess.

7.1 *Table of variables*

The variables found throughout the document are presented in the following list.

7.1.1 *Geometrical variables*

$(x,\ y)$: Cartesian coordinate system
$(r,\ \vartheta)$: Polar coordinate system
$(s,\ n)$: Streamline coordinate system
s: Arc length along the streamline
n: Arc length along the orthogonal trajectory
K_S: Streamline curvature
K_N: Curvature of the orthogonal trajectories
$\boldsymbol{K_G}$: Global Curvature Vector

N: Curvature Potential
$\boldsymbol{\Gamma}$: Geometric Vorticity
γ: Scalar Geometric Vorticity
$\boldsymbol{t}$: Unit tangent vector
$\boldsymbol{n}$: Unit vector tangent to the orthogonal trajectories
$\boldsymbol{k}$: Unit vector perpendicular to the plane of motion
φ: Angle function (denotes the angle between the tangent of a streamline and the horizontal x-axis)
ϑ: Angle between the unit normal vector $\boldsymbol{n}$ and the horizontal x-axis
m_1 (or m_2): slope of a plane curve

7.1.2 *Physical variables*

$\boldsymbol{v}$: Velocity vector (vector velocity)
v: Velocity magnitude (scalar velocity)
v_∞: Free stream velocity
Φ: Scalar velocity potential
ζ: Scalar vorticity
p: Static pressure

7.1.3 *Miscellaneous variables*

$\boldsymbol{F}$: Vector field with two spatial dimensions
F_x: the x-component of $\boldsymbol{F}$ in Cartesian coordinates
F_y: the y-component of $\boldsymbol{F}$ in Cartesian coordinates
F: the magnitude of the vector $\boldsymbol{F}$
$\boldsymbol{F}_\perp$: Normal vector to $\boldsymbol{F}$
P, Q: Scalar functions representing the real and imaginary parts of a complex function respectively
U, V: Scalar functions representing the real and imaginary parts of a complex function respectively
$f(z)$: Geometric complex potential (holomorphic function) of the complex variable z
c, C: Constants

Acknowledgement

This work is dedicated to the most memorable teachers who I have had the privilege of meeting. In chronological order: Xeni and Vasileios Dimitriou, Foteini Papakogkou, Li Lykoudi, Perikles Daskalakis, Dimitris Maxouris, Thanasis Skufas, Manolis Papoulias, Dimitris Zazas, Manolarakis, Panayiotis Stavrinos and Nikos Voglis. Their inspiring, entertaining and motivating personalities always captivated me, shaped the person who I have become and quickly made me realize that for real happiness all you need is a pencil and a piece of paper.

Conflict of interest

The author declares there is no conflict of interest at any point with reference to research findings.

References

[1] Marsden J, Tromba A. *Vector Calculus*. 6th ed. New York: W. H. Freeman and Company Publishers; 2011.

[2] Connolly CI, Grupen RA. The application of harmonic functions to robotics. *Journal of Robotic Systems*. 1993; 10(7): 931-946.

[3] Alsoboh A, Amourah A, Darus M, Rudder CA. Studying the harmonic functions associated with quantum calculus. *Mathematics*. 2023; 11(10): 2220.

[4] Needham T. The geometry of harmonic functions. *Mathematics Magazine*. 1994; 67(2): 92-108.

[5] Dimitriou I. On the geometry of a steady two-dimensional potential flow and its physics. *Journal of Applied Mathematics and Physics*. 2007; 58(1): 100-120.

[6] Dimitriou I. Introducing a geometric potential theory for two-dimensional steady flows. *Journal of Engineering Mathematics*. 2009; 63(1): 1-15.

[7] Dimitriou I. Geometrical interpretations of continuous and complex-lamellar steady flows. *European Journal of Mechanics-B/Fluids*. 2017; 61(1): 86-99.

[8] Struik DJ. *Lectures on Classical Differential Geometry*. New York: Dover Publications; 1988.

[9] Fox RW, McDonald AT. *Introduction to Fluid Mechanics Fourth Edition*. New Jersey: John Wiley and Sons; 1994.

[10] Dimitriou I. Planar incompressible navier-stokes and euler equations: A geometric approach. *Physics of Fluids*. 2017; 29(11): 117101.

[11] Brown J. W., Churchill R. V. *Complex Variables and Applications*. 9th ed. New York: Mc Graw Hill; 2014.

[12] Martensen E. *Potentialtheorie BG*. Stuttgart: Teubner Stuttgart; 1968. p.246-254.

[13] Stewart AM. Does the helmholtz theorem of vector decomposition apply to the wave fields of electromagnetic radiation? *Physical Scripts*. 2014; 89(6): 065502.

[14] Bivens CI. When do orthogonal families of curves possess a complex potential? *Mathematics Magazine*. 1992; 65(4): 226-235.

[15] Kuethe AM, Chow C-Y. *Foundations of Aerodynamics: Bases of Aerodynamic Design, Fifth Edition*. New Jersey: John Wiley and Sons; 1998.

[16] Dimitriou I, Rodríguez JÁ. Quantitative analysis of two-dimensional flow visualizations using the geometric potential method. *Journal of Engineering Mathematics*. 2016; 98(1): 145-161.