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Research Article

On Topological Structures on Γ-*BCK*-Algebras

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Abstract: In this paper, we first study topological structures on Γ-*BCK*-algebras and obtain some of its properties. Next, we introduce the notion of quotient Γ-*BCK*-algebra by ideals and investigate some of topological properties on a quotient Γ-*BCK*-algebra. Finally, we define a quotient Γ-*BCK*-algebras by dual ideals of a Γ-*BCK*-algebra and give uniform structures on quotient Γ-*BCK*-algebras.

***Keywords*:** Γ-*BCK*-algebra, topological Γ-*BCK*-algebra, quotient Γ-*BCK*-algebra by Γ-ideals, topological quotient Γ-*BCK*-algebra, quotient Γ-*BCK*-algebra by dual Γ-ideals, uniform structure

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1. Introduction

In 1978, Iséki and Tanaka [1] introduced the concept of *BCK*-algebras as a generalization of *I*-algebras proposed by Imai and Iséki [2]. After that time, many researchers introduced and studied some proper subclasses of *BCK*-algebras, for example, *BCI*-algebras [3], *BCH*-algebras [4, 5], *BH*-algebras [6], *QS*-algebras [7] and *Q*-algebras [8]. In particular, Iséki [9] and Meng [10] investigated properties of ideals in *BCK*-algebras. Meng [11, 12] introduced the notions of commutative ideals and dual ideals in *BCK*-algebras and dealt with some of their properties respectively. Moreover, Jun and Roh [13], Lee and Ryu [14], and Roudabri and Torkzadeh [15] applied *BCK*-algebras to topology respectively. Mohammed et al. [16] studied topological structures on *BCK*-algebras. Also, Jun et al. [17] and Hasankhani et al. [18] and Ahn and Kwon [19] dealt with topological structures on *BCI*-algebras.

Recently, Saeid et al. [20] introduced the concept of Γ-*BCK*-algebras and investigated some of its properties. By modifying a Γ-*BCK*-algebra proposed by Saeid et al., Shi et al. [21] redefined a Γ-*BCK*-algebra and discussed its various properties. We think it is necessary to study Γ-*BCK*-algebras from the perspective of a topological group.

The purpose of our research is to study topological structures on Γ-*BCK*-algebras and the quotient Γ-*BCK*-algebras as a preliminary step to research on topological groups. The layout of this paper is as follows. In section 2, we recall some definitions need in next section. In section 3, we define a topological Γ-*BCK*-algebra and obtain some of its examples and properties. In section 4, we define a quotient Γ-*BCK*-algebra by ideals of a Γ-*BCK*-algebra and study some of topological

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properties on a quotient Γ-*BCK*-algebra. Finally, we introduce the notion of quotient Γ-*BCK*-algebras by dual ideals of a Γ-*BCK*-algebra and give uniform structures on quotient Γ-*BCK*-algebras.

2. Preliminaries

We recall some definitions needed in next sections.

Definition 2.1 [1, 3] Let X be a nonempty set with a constant 0 and a binary operation $*$. Consider the following axioms: for any $x, y, z \in X$,

(A_1) $[(x*y)*(x*z)]*(z*y)=0$,

(A_2) $[x*(x*y)]*y=0$,

(A_3) $x*x=0$,

(A_4) $x*y=0$ and $y*x=0$ imply $x=y$,

(A_5) $0*x=0$.

Then X is called a:

(i) *BCI-algebra*, if it satisfies axioms (A_1)-(A_4),

(ii) *BCK-algebra*, if it satisfies axioms (A_1)-(A_5).

In *BCI*-algebra or *BCK*-algebra X, we define a binary operation $\leq$ on X as follows: for any $x, y \in X$,

$$x \leq y \text{ if and only if } x*y=0.$$

Definition 2.2 [22] Let X and Γ be two nonempty sets. Then X is called a Γ-*semigroup*, if there is a mapping $f: X\times\Gamma\times X \to X$, denoted by $f(x, \alpha, y) = x\alpha y$ for each $(x, \alpha, y) \in X\times\Gamma\times X$, such that it satisfies the following condition: for any $x, y, z \in X$ and any $\alpha, \beta \in \Gamma$,

$$x\alpha(y\beta z) = (x\alpha y)\beta z. \tag{1}$$

Definition 2.3 [21] Let X be a set with a constant 0 and let Γ be a nonempty set. Then X is called a Γ-*BCK-algebra*, if there is a mapping $f: X\times\Gamma\times X \to X$, denoted by $f(x, \alpha, y) = x\alpha y$ for each $(x, \alpha, y) \in X\times\Gamma\times X$, satisfying the following axioms: for any $x, y, z \in X$ and $\alpha, \beta \in \Gamma$,

(ΓA_1) $[(x\alpha y)\beta(x\alpha z)]\beta(z\alpha y)=0$,

(ΓA_2) $[x\alpha(x\beta y)]\alpha y=0$,

(ΓA_3) if $x\alpha y=0=y\alpha x$, then $x=y$,

(ΓA_4) $x\alpha x=0$,

(ΓA_5) $0\alpha x=0$.

For a Γ-*BCK*-algebra X and a fixed $\alpha \in \Gamma$, we define the operation $*: X\times X \to X$ as follows: for any $x, y \in X$,

$$x*y = x\alpha y.$$

Then it is clear $(X, *, 0)$ is a *BCK*-algebra and is denoted by X_α.

Definition 2.4 [21] A Γ-*BCK*-algebra X is said to be *positive implicative*, if it satisfies the following axiom: for any $x, y, z \in X$ and any $\alpha, \beta \in \Gamma$,

$$(x\alpha z)\beta(y\alpha z) = (x\alpha y)\beta z. \quad (2)$$

Definition 2.5 [20] A Γ-BCK-algebra X is said to be *commutative*, if it satisfies the following axiom: for any $x,\ y \in X$ and any $\alpha,\ \beta \in \Gamma$,

$$y\alpha(y\beta x) = x\alpha(x\beta y). \quad (3)$$

Definition 2.6 [21] Let X be a Γ-BCK-algebra. Then X is said to be *implicative*, if it satisfies the following condition:

$$x = x\alpha(y\beta x) \text{ for any } x,\ y \in X \text{ and any } \alpha,\ \beta \in \Gamma. \quad (4)$$

Definition 2.7 [20] Let X be a Γ-BCK-algebra and let A be a nonempty subset of X. Then A is called a Γ-*BCK-subalgebra* of X, if it satisfies the following condition:

$$x\alpha y \in A \text{ for any } x,\ y \in A \text{ and for each } \alpha \in \Gamma. \quad (5)$$

Definition 2.8 [20] Let X be Γ-BCK-algebra and let I be a nonempty set of X. Then I is called a Γ-*ideal* of X, if it satisfies the following conditions: for any $x,\ y \in X$ and each $\alpha \in \Gamma$,

(ΓI_1) $0 \in I$,

(ΓI_2) if $x\alpha y \in I$ and $y \in I$, then $x \in I$.

An ideal I is said to be *proper*, if $I \neq X$. It is obvious that X and $\{0\}$ are ideals of X. In particular, X is called a *trivial* Γ-*ideal* of X.

3. Topological structures on Γ-BCK-algebras

We first recall some terms and notations related to a general topology (see [23, 24]). For a subset A of a topological space $(X,\ \tau)$, we denote the closure and the interior of A as $cl_\tau(A)$, $cl(A)$ or $\bar{A}$ and $int_\tau(A)$, $int(A)$ or A°. A subfamily $\mathscr{B}$ of τ is called a *base* for τ, if for each $U \in \tau$ either $U = \emptyset$ or there is $\mathscr{B}' \subset \mathscr{B}$ such that $U = \bigcup \mathscr{B}'$. A subset A of X is called a *neighborhood* of $x \in X$, denoted by $N(x)$, if there is $U \in \tau$ such that $x \in U \subset A$. We denote the set of all neighborhoods of x as $\mathscr{N}_\tau(x)$ or $\mathscr{N}(x)$ and $\mathscr{N}(x)$ is called the *neighborhood filter* of $x \in X$. See [23, 24] for the definitions of a discrete space and a Hausdorff space.

We next introduce a definition of *topological BCK-algebra* (briefly, $TBCK$-algebra) and some examples of $TBCK$-algebras .

Definition 3.1 [21] Let X be a set with a constant 0 and let Γ be a nonempty set. Then X is called a Γ-*BCK-algebra*, if there is a mapping $f: X \times \Gamma \times X \to X$, denoted by $f(x,\ \alpha,\ y) = x\alpha y$ for each $(x,\ \alpha,\ y) \in X \times \Gamma \times X$, satisfying the following axioms: for any $x,\ y,\ z \in X$ and $\alpha,\ \beta \in \Gamma$,

(ΓA_1) $[(x\alpha y)\beta(x\alpha z)]\beta(z\alpha y) = 0$,

(ΓA_2) $[x\alpha(x\beta y)]\alpha y = 0$,

(ΓA_3) if $x\alpha y = 0 = y\alpha x$, then $x = y$,

(ΓA_4) $x\alpha x = 0$,

(ΓA_5) $0\alpha x = 0$.

For a Γ-BCK-algebra X and a fixed $\alpha \in \Gamma$, we define the operation $* : X \times X \to X$ as follows: for any $x,\ y \in X$,

$$x * y = x\alpha y.$$

Then it is clear $(X,\ *,\ 0)$ is a BCK-algebra and is denoted by X_α.

Definition 3.2 [14] Let X be a BCK-algebra and let τ be a topology on X. Then X is called a *topological BCK-algebra* (briefly, $TBCK$-algebra), if $* : (X \times X,\ \tau \times \tau) \to (X,\ \tau)$ is continuous, i.e., for any $x,\ y \in X$ and each $W \in N(x * y)$ there are $U \in N(x)$ and $V \in N(y)$ such that $U * V \subset W$, where $U * V = \{x * y \in X : x \in U,\ y \in V\}$.

Definition 3.3 Let X be a Γ-BCK-algebra and let τ be a topology on X. Then X is called a *topological* Γ-*BCK-algebra* (briefly, $T\Gamma$-BCK-algebra), if the mapping $f : (X,\ \tau) \times \Gamma \times (X,\ \tau) \to (X,\ \tau)$ is continuous at each $(x,\ \alpha,\ y) \in X \times \Gamma \times X$, i.e., for each $\alpha \in \Gamma$, any $x,\ y \in X$ and each $W \in N(x\alpha y)$ there are $U \in N(x)$ and $V \in N(y)$ such that $U\alpha V \subset W$, where $U\alpha V \subset W = \{x\alpha y : x \in U,\ y \in V\}$.

The above definition, it is obvious that if X is a $T\Gamma$-BCK-algebra, then X_α is a $TBCK$-algebra for each $\alpha \in \Gamma$. However, the converse is not true in general (see Example 3.4 (3)).

Example 3.4 (1) Let $\Gamma = \{\alpha,\ \beta\}$ and let $X = \{0,\ 1,\ 2,\ 3,\ 4\} \cdots$ be the Γ-BCK-algebra given in Example 3.20 (3) in [21], having the the ternary operation be defined as the following Table 1:

Table 1. The ternary operation 1

	α					β				
	0	1	2	3	4	0	1	2	3	4
0	0	0	0	0	0	0	0	0	0	0
1	1	0	1	0	0	1	0	1	0	0
2	2	2	0	0	0	2	0	0	1	0
3	3	1	0	0	0	3	1	0	0	0
4	4	4	4	4	0	4	4	4	4	0

Consider the topology τ on X given by:

$$\tau = \{\emptyset,\ \{4\},\ \{0,\ 1,\ 2,\ 3\},\ X\}.$$

Then we can easily see that $(X,\ \tau)$ is a $T\Gamma$-BCK-algebra. Moreover, X_α and X_β are $TBCK$-algebras.

(2) For a set $\Gamma = \{\alpha,\ \beta,\ \gamma\}$, let $X = \{0,\ 1,\ 2,\ 3\}$ be the Γ-BCK-algebra given in Example 3.2 (3) in [21] with the ternary operation defined by the Table 2:

Table 2. The ternary operation 2

	α				β				γ			
	0	1	2	3	0	1	2	3	0	1	2	3
0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	0	0	0	1	0	0	0	1	0	0	1
2	2	2	0	0	2	2	0	0	2	3	0	2
3	3	2	0	0	3	3	0	0	3	3	0	0

Consider a topology τ on X given by:

$$\tau = \{\emptyset,\ \{0\},\ \{0,\ 1\},\ \{0,\ 2,\ 3\},\ X\}.$$

Then we can check that $(X,\ \tau)$ is a $T\Gamma$-BCK-algebra.
(3) Let $X = \{0,\ 1,\ 2,\ 3\}$ and let $*$ and $*'$ be the binary operations on X given by the following Table 3:

Table 3. The binary operation 3

	$*$				$*'$			
	0	1	2	3	0	1	2	3
0	0	0	0	0	0	0	0	0
1	1	0	0	1	1	0	0	1
2	2	2	0	2	2	1	0	2
3	3	3	3	0	3	3	3	0

Then clearly, $(X,\ *,\ 0)$ and $(X,\ *',\ 0)$ are BCK-algebras. Consider the topology τ on X given by:

$$\tau = \{\emptyset,\ \{2\},\ \{3\},\ \{0,\ 1\},\ \{2,\ 3\},\ \{0,\ 1,\ 2\},\ \{0,\ 1,\ 3\},\ X\}.$$

Then we can easily see that $(X,\ *,\ \tau)$ and $(X,\ *',\ \tau)$ are $TBCK$-algebras. Now let $\alpha = *$ and $\beta = *'$ and consider the following ternary operation defined by the Table 4:

Table 4. The ternary operation 3

	α				β			
	0	1	2	3	0	1	2	3
0	0	0	0	0	0	0	0	0
1	1	0	0	1	1	0	0	1
2	2	2	0	2	2	1	0	2
3	3	3	3	0	3	3	3	0

Then $[2\alpha(2\beta 1)]\alpha 1 = 2 \neq 0$. Thus X is not a Γ-BCK-algebra.

Proposition 3.5 Let X be a $T\Gamma$-BCK-algebra. If $\{0\}$ is open in X, then X is discrete.

Proof. Let $x \in X$ and let $\alpha \in \Gamma$. Then clearly, $x\alpha x = 0 \in \{0\} \in N(0)$. Thus there are $U,\ V \in N(x)$ such that $U\alpha V = \{0\}$. Let $W = U \cap V$. Then $W\alpha W \subset U\alpha V = \{0\}$. Thus $W\alpha W = \{0\}$. Since $x \in U \cap V$, $x \in W$. So $W = \{x\}$ and W is open in X. Hence X is discrete. □

The following is an immediate consequence of Proposition 3.5.

Corollary 3.6 (See Proposition 2.2, [14]) Let X be a $T\Gamma$-BCK-algebra. If $\{0\}$ is open in X_α for each $\alpha \in \Gamma$, then X_α is discrete.

Theorem 3.7 Let X be a $T\Gamma$-BCK-algebra. Then $\{0\}$ is closed in X if and only if X is Hausdorff.

Proof. Suppose $\{0\}$ is closed in X, let $x,\ y \in X$ such that $x \neq y$ and let $\alpha \in \Gamma$. Then $x\alpha y \neq 0$ or $y\alpha x \neq 0$, say $x\alpha y \neq 0$. Since $\{0\}$ is closed in X and $x\alpha y \neq 0$, $\{0\}^c$ is open in X and $x\alpha y \in \{0\}^c$. Thus $\{0\}^c \in N(x\alpha y)$. Since X be a

$T\Gamma$-BCK-algebra, by Definition 3.3, there are $U \in N(x)$ and $V \in N(y)$ such that $U\alpha V \subset \{0\}^c$. So $U \cap V = \emptyset$. Hence X is Hausdorff.

Conversely, suppose X is Haousdorff and let $x \in \{0\}^c$. Then $x \neq 0$. By the hypothesis, there are $U \in N(x)$ and $V \in N(0)$ such that $U \cap V = \emptyset$. Thus $0 \notin U$. So $U \subset \{0\}^c$. Hence $\{0\}^c$ is open in X. Therefore $\{0\}$ is closed in X. □

The following is an immediate consequence of Theorem 3.7.

Corollary 3.8 (See Proposition 2.3, [14]) Let X be a $T\Gamma$-BCK-algebra. If $\{0\}$ is closed in X_α for each $\alpha \in \Gamma$, then X_α is Hausdorff.

Proposition 3.9 Let X be a $T\Gamma$-BCK-algebra and let A be open in X. If A is a Γ-BCK-subalgebra of X, then A is a $T\Gamma$-BCK-algebra.

Proof. Let τ be the topology on X and let τ_A be the subspace topology on A with respect to τ. Let $x,\ y \in A$ and let $\alpha \in \Gamma$. Since A is a Γ-BCK-subalgebra of X, $x\alpha y \in A$. Let $W_A \in N_{\tau_A}(x\alpha y)$, where $N_{\tau_A}(x\alpha y)$ denotes the neighborhood of $x\alpha y$ in the subspace $(A,\ \tau_A$ of $(X,\ \tau)$. Then there is $W \in N(x\alpha y)$ such that $W_A = A \cap W$. Since X is a $T\Gamma$-BCK-algebra, there are $U \in N(x)$ and $V \in N(y)$ such that $U\alpha V \subset W$. Thus $U_A = A \cap U \in N_{\tau_A}(x)$ and $V_A = A \cap V \in N_{\tau_A}(x)$. It is clear that

$$U_A\alpha V_A = (A \cap U)\alpha(A \cap V) \subset W \text{ and } U_A\alpha V_A \subset A.$$

So $U_A\alpha V_A \subset A \cap W = W_A$. Hence A is a $T\Gamma$-BCK-algebra. □

Proposition 3.10 Let X be a $T\Gamma$-BCK-algebra and let I be open in X. If I is a Γ-ideal of X, then I is closed in X.

Proof. Let $x \in I^c$ and let $\alpha \in \Gamma$. Since $x\alpha x = 0 \in I$ and I is open, $I \in N(0)$. Since X is a $T\Gamma$-BCK-algebra, there is $U \in N(x)$ such that $U\alpha U \subset I$. Assume that $U \not\subset I^c$. Then there is $y \in X$ such that $y \in U \cap I$. It is obvious that $z\alpha y \in U\alpha U \subset I$ for each $z \in U$. Since I is a Γ-ideal of X and $y \in I$, $z \in I$. Thus $U \subset I$. This is a contradiction. So $U \subset I^c$, i.e., I^c is open in X. Hence I is closed in X. □

Proposition 3.11 Let X be a $T\Gamma$-BCK-algebra and let I be a Γ-ideal of X. If $0 \in int(I)$, then I is open in X.

Proof. Let $x \in I$ and let $\alpha \in \Gamma$. Since $0 \in int(I)$ and $x\alpha x = 0 \in I$, there is $W \in N(0) = N(x\alpha x)$ such that $W \subset I$. Since X is a $T\Gamma$-BCK-algebra, by Definition 3.3, there are $U,\ V \in N(x)$ such that $U\alpha V \subset W \subset I$. It is clear that $y\alpha x \in U\alpha V \subset I$ for each $y \in U$. Since I is a Γ-ideal of X and $x \in I$, $y \in I$. Then $y \in I$. Thus $U \subset I$. So I is open in X. □

In Proposition 3.11, when $0 \neq x \in int(I)$, I need not open in X (see Example 3.12).

Example 3.12 For a set $\Gamma = \{\alpha,\ \beta\}$, let $X = \{0,\ 1,\ 2,\ 3\}$ be a Γ-BCK-algebra with the ternary operation be defined by the Table 5:

Table 5. The ternary operation 4

	α				β			
	0	1	2	3	0	1	2	3
0	0	0	0	0	0	0	0	0
1	1	0	0	1	1	0	0	2
2	2	2	0	2	2	2	0	2
3	3	3	0	0	3	3	0	0

Then $I = \{0,\ 3\}$ is a Γ-ideal of X. Consider a topology τ on X given by:

$$\tau = \{\emptyset,\ \{2\},\ \{3\},\ \{0,\ 1\},\ \{2,\ 3\},\ \{0,\ 1,\ 3\},\ X\}.$$

Then clearly, $3 \in int(I)$. But $I \notin \tau$. Now consider another topology $\tau^{'}$ on X given by:

$$\tau^{'} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{0, 3\}, \{1, 2\}, \{0, 1, 3\}, X\}.$$

Then we can easily check that I is closed in X. Thus we can see that Proposition 3.10 holds.

Proposition 3.13 Let X be a $T\Gamma$-BCK-algebra. Then $\bigcap N(0) = \{0\}$ and thus $\bigcap \mathscr{N}(0) = \{0\}$.

Proof. Assume that $0 \neq x \notin \bigcap N(0)$. Then clearly, there is $U \in N(0)$ such that $o \in U$ but $x \notin U$. Thus $x \notin \bigcap N(0)$. This is a contradiction. So $\bigcap N(0) = \{0\}$. □

Proposition 3.14 Let (X, τ) be a $T\Gamma$-BCK-algebra and let $\mathscr{B}_1$, $\mathscr{B}_2$ be the families of subsets of X given by:

$$\mathscr{B}_1 = \{x\alpha U : x \in X, \alpha \in \Gamma, U \in \mathscr{N}(0)\}, \mathscr{B}_2 = \{U\alpha x : x \in X, \alpha \in \Gamma, U \in \mathscr{N}(0)\},$$

where $x\alpha U = \{x\alpha u : u \in U\}$ and $U\alpha x = \{u\alpha x : u \in U\}$. Then $\mathscr{B}_1$ and $\mathscr{B}_2$ are bases for τ.

Proof. Let $x \in X$. Since $0 \in U \in \mathscr{N}(0)$, $x\alpha 0 = x$. Then $\bigcup \mathscr{B}_1 = X$. Suppose $B_1, B_2 \in \mathscr{B}_1$ and $z \in B_1 \cap B_2$. Then there are $U_1, U_2 \in \mathscr{N}(0)$ such that $B_1 = x\alpha U_1$, $B_2 = x\alpha U_2$ and $B_1 \cap B_2 = x\alpha(U_1 \cap U_2)$. Since $z \in B_1 \cap B_2$, there is $y \in U_1 \cap U_2$. Since $U_1, U_2 \in N(0)$, $U_1 \cap U_2 \in N(0)$. So there is $V \in \mathscr{N}(0)$ such that $y \in V \subset U_1 \cap U_2$. Hence $z = x\alpha y \in x\alpha V \in \mathscr{B}_1$. Therefore $\mathscr{B}_1$ is a base for τ. Similarly, we can prove that $\mathscr{B}_2$ is a base for τ. □

To give a filter base on X generating a topology on a Γ-BCK-algebra, let us define a subset $U(a)$ of X generated by each $a \in X$ and each $U \in P(X)$ as follows:

$$U(a) = \{x \in X : x\alpha a \in U, a\alpha x \in U, \alpha \in \Gamma\}.$$

Proposition 3.15 Let X be a Γ-BCK-algebra. Suppose $\mathscr{B}$ is a filter base on X satisfying the following condition:

(1) for each $u \in U \in \mathscr{B}$ there is $B \in \mathscr{B}$ such that $B(u) \subset U$,

(2) for each $u \in U \in \mathscr{B}$ and each $\alpha \in \Gamma$ if $x\alpha u = 0$, then $x \in U$,

(3) for each $U \in \mathscr{B}$ there is $B \in \mathscr{B}$ such that $B(b) \subset U$ for each $b \in B$.

Then there is a unique topology τ on X such that $\mathscr{B} = \mathscr{N}_\tau(0)$ and (X, τ) is a $T\Gamma$-BCK-algebra.

Proof. Let $\tau = \{O \in P(X)$: for each $a \in O$ there is $B \in \mathscr{B}$ such that $B(a) \subset O\}$.

Claim 1: τ is a topology on X. By the definition of τ, it is clear that $X, \emptyset \in \tau$. Suppose $\{O_\alpha\}_\alpha \in \Lambda \subset \tau$ and let $a \in \bigcup_{\alpha\in\Lambda} O_\alpha$, where Λ is a index set. Then there is $\alpha \in \Lambda$ such that $a \in O_\alpha$. Thus there is $B \in \mathscr{B}$ such that $B(a) \subset O_\alpha \subset \bigcup_{\alpha\in\Lambda} O_\alpha$. So $\bigcup_{\alpha\in\Lambda} O_\alpha \in \tau$. Now suppose $O_1, O_2 \in \tau$ and let $a \in O_1 \cap O_2$. Then there are $B_1, B_2 \in \mathscr{B}$ such that $B_1(a) \subset O_1$ and $B_2(a) \subset O_2$. Since $\mathscr{B}$ is a filter base on X, there is $B \in \mathscr{B}$ such that $B \subset B_1 \cap B_2$. On the other hand, we get

$$B(a) \subset (B_1 \cap B_2)(a) \subset B_1(a) \cap B_2(a) \subset O_1 \cap O_2.$$

Thus $O_1 \cap O_2 \in \tau$. So τ is a topology on X.

Claim 2: $B(a) \in \tau$. Let $x \in B(a)$. Then $x\alpha a$, $a\alpha x \in B$ for each $\alpha \in \Gamma$. Thus by the condition (1), there are $B_1, B_2 \in \mathscr{B}$ such that $B_1(x\alpha a) \subset B$ and $B_2(a\alpha x) \subset B$. Since $\mathscr{B}$ is a filter base on X, there is $U \in \mathscr{B}$ such that $U \in B_1 \cap B_2$. Let $x\alpha y$, $y\alpha x \in U$, i.e., $y \in U(x)$. By Proposition 3.6 (2) in [21] , we have

$$(x\alpha a)\beta(y\alpha a) \leq x\alpha y, \ (y\alpha a)\beta(x\alpha y) \leq y\alpha x.$$

Then $[(x\alpha a)\beta(y\alpha a)]\beta(x\alpha y) = 0,\ [(y\alpha a)\beta(x\alpha y)]\beta(y\alpha x) = 0$. By the condition (2), $x\alpha y,\ y\alpha x \in U$. Thus we get

$$y\alpha a \in U(x\alpha a) \subset B_1(x\alpha a) \subset B.$$

So $y\alpha a \in B$. Similarly, we can show that $a\alpha y \in U$. Hence $y \in U(a)$, i.e., $U(x) \subset B(a)$. Therefore $B(a) \in \tau$.

Claim 3: $\mathscr{B} = \mathscr{N}_\tau(0)$. Let $A \in \mathscr{B}$ and let $x \in A$. Since X is a Γ-BCK-algebra, by the axiom (ΓA_5), $0\alpha x = 0$. By the condition (2), $0 \in A$. By the condition (1), there is $B \in \mathscr{B}$ such that $B(0) \subset A$. Then by Claim 2, $B(0) \in \tau$. Thus $A \in N_\tau(0)$. So $\mathscr{B} \subset N_\tau(0)$. Hence by the condition (3), $\mathscr{B} \subset \mathscr{N}_\tau(0)$. It can be easily proved that $\mathscr{N}_\tau(0) \subset \mathscr{B}$. Therefore $\mathscr{B} = \mathscr{N}_\tau(0)$.

Claim 4: A mapping $f : (X,\ \tau) \times \Gamma \times (X,\ \tau) \to (X,\ \tau)$ is continuous at each $(x,\ \alpha,\ y) \in X \times \Gamma \times X$. Let $x,\ y \in X$, let $\alpha \in \Gamma$ and let $W \in N_\tau(x\alpha y)$. Since $x\alpha y \in W$, by the condition (1), there is $W' \in \mathscr{B}$ such that $W'(x\alpha y) \subset W$. Since $W' \in \mathscr{B}$, by the condition (3), there is $B \in \mathscr{B}$ such that $B(b) \subset W'$ for each $b \in W'$. Let $U = B(x),\ V = B(y)$ and let $u \in U,\ v \in V$. Then we have

$$\begin{aligned}
[(x\alpha y)\beta(u\alpha v)]\beta(x\alpha u) =& [(x\alpha u)\beta(x\alpha u)]\beta(u\alpha v) \text{ [By Proposition 3.5, [21]]}\\
=& [x\alpha(x\alpha u)\beta y]\beta(u\alpha v)\\
\leq& (u\alpha y)\beta(u\alpha v)\\
& \text{[By Proposition 3.3 and Proposition 3.6, [21]]}\\
\leq& v\alpha y. \text{ [By Proposition 3.3, [21]]}
\end{aligned}$$

Thus $([(x\alpha y)\beta(u\alpha v)]\beta(x\alpha u))\beta(v\alpha y) = 0$. Since $v\alpha y \in B$, by the condition (2), $[(x\alpha y)\beta(u\alpha v)]\beta(x\alpha u) \in B$. Similarly, we have $(x\alpha u)\beta[(x\alpha y)\beta(u\alpha v)] \in B$. So we get

$$(x\alpha y)\beta(u\alpha v) \in B(x\alpha u) \subset W', \text{ i.e., } (x\alpha y)\beta(u\alpha v) \in W'.$$

Similarly, $(u\alpha v)\beta(x\alpha y) \in W'$. Hence we have

$$u\alpha v \in W'(x\alpha y), \text{ i.e., } U\alpha V = B(x)\alpha B(y) \subset W'(x\alpha y) \subset W.$$

Therefore f is continuous. The proof of uniqueness for τ is easy. This completes the proof. □

Example 3.16 (1) Let X be the Γ-BCK-algebra and let $\mathscr{I}$ be the collection of all ideals of X. Let $x \in I \in \mathscr{I}$. Then clearly, $I(x) \subset I$. Thus $\mathscr{I}$ satisfies the conditions (1) and (3) in Proposition 3.15. Let $y \in I \in \mathscr{I}$ and suppose $x\alpha y = 0$. Then $x\alpha y = 0 \in I$. Thus $x \in I$. So $\mathscr{I}$ satisfies the condition (2) in Proposition 3.5. So $\mathscr{I}$ forms a filter base of X satisfying all the conditions in Proposition 3.15. Hence $(X,\ \tau)$ is a $(X,\ \tau)$ is a $T\Gamma$-BCK-algebra, where τ is the topology on X generated by $\mathscr{I}$.

(2) (See Example 3.14 (2), [21]) Let $\Gamma = \{\alpha,\ \beta\}$ and let $X = \{0,\ 1,\ 2,\ 3\}$ be the Γ-BCK-algebra with the the ternary operation be defined as the following Table 6:

Table 6. The ternary operation 5

	α				β			
	0	1	2	3	0	1	2	3
0	0	0	0	0	0	0	0	0
1	1	0	1	1	1	0	1	1
2	2	2	0	0	2	3	0	0
3	3	3	3	0	3	3	3	0

Consider the family $\mathscr{B}$ of subsets of X given by:

$$\mathscr{B} = \{\{0,\ 1\},\ \{0,\ 2\},\ \{0,\ 3\},\ \{0,\ 1,\ 2\},\ \{0,\ 1,\ 3\},\ \{0,\ 2,\ 3\}\}.$$

Then we can easily check that $\mathscr{B}$ is a filter base on X. Moreover, we have

$$\{0,\ 1\}(0) = \{0,\ 1\}(1) = \{0,\ 1\},\ \{0,\ 1\}(2) = \{2\},\ \{0,\ 1\}(3) = \{3\},$$

$$\{0,\ 2\}(0) = \{0,\ 2\}(2) = \{0,\ 2\},\ \{0,\ 2\}(1) = \{1\},\ \{0,\ 2\}(3) = \{3\},$$

$$\{0,\ 3\}(0) = \{0,\ 3\}(3) = \{0,\ 3\},\ \{0,\ 3\}(1) = \{1\},\ \{0,\ 3\}(2) = \{2\},$$

$$\{0,\ 1,\ 2\}(0) = \{0,\ 1,\ 2\}(1) = \{0,\ 1,\ 2\}(2) = \{0,\ 1,\ 2\},\ \{0,\ 1,\ 2\} = \{3\},$$

$$\{0,\ 1,\ 3\}(0) = \{0,\ 1,\ 3\}(1) = \{0,\ 1,\ 3\}(3) = \{0,\ 1,\ 3\},\ \{0,\ 1,\ 3\}(2) = \{2\},$$

$$\{0,\ 2,\ 3\}(0) = \{0,\ 2,\ 3\}(2) = \{0,\ 2,\ 3\}(3) = \{0,\ 2,\ 3\},\ \{0,\ 2,\ 3\}(1) = \{1\}.$$

Thus $\mathscr{B}$ is a filter base on X satisfying al the conditions in Proposition 3.15. So the topology τ on X generated by $\mathscr{B}$ is given as follows:

$$\tau = \{\emptyset,\ \{0,\ 1\},\ \{0,\ 2\},\ \{0,\ 3\},\ \{0,\ 1,\ 2\},\ \{0,\ 1,\ 3\},\ \{0,\ 2,\ 3\},\ X\}.$$

Hence $(X,\ \tau)$ is a $T\Gamma$-BCK-algebra.

Lemma 3.17 Let X be a Γ-BCK-algebra and let τ be the topology on X generated by $\mathscr{B}$, where $\mathscr{B}$ is a filter base on satisfying all the conditions in Proposition 3.15. Then for each $B \in \mathscr{B}$ and each $a \in X$,

(1) $B(a) \in N_\tau(a)$,

(2) $B(A) = \bigcup_{a \in A} B(a) \in N_\tau(A)$ for each $A \in P(X)$.

Proof. The proof is straightforward. □

Proposition 3.18 Let X be a Γ-BCK-algebra and let τ be the topology on X generated by $\mathscr{B}$, where $\mathscr{B}$ is a filter base on X satisfying all the conditions in Proposition 3.15. Then for each $B \in \mathscr{B}$, $cl_\tau(A) = \bigcap_{B \in \mathscr{B}} B(A)$.

Proof. Let $x \in cl_\tau(A)$ and let $B \in \mathscr{B}$. By Lemma 3.17 (1), $B(x) \in N_\tau(x)$. Then $B(x) \cap A \neq \emptyset$. Thus there is $a \in A$ such that $a\alpha x,\ x\alpha a \in B$ for each $\alpha \in \Gamma$. So $x \in B(a) \subset B(A)$, i.e., $x \in \bigcap_{B \in \mathscr{B}} B(A)$. Hence $cl_\tau(A) \subset \bigcap_{B \in \mathscr{B}} B(A)$. Conversely, let $x \in \bigcap_{B \in \mathscr{B}} B(A)$. Then $x \in U(A)$ for each $U \in \mathscr{B}$. Thus there is $a \in A$ such that $x \in B(a)$, i.e., $x\alpha a,\ a\alpha x \in B$ for each $\alpha \in \Gamma$. So $a \in B(x)$, i.e., $B(x) \cap A \neq \emptyset$. Hence $x \in cl_\tau(A)$, i.e., $\bigcap_{B \in \mathscr{B}} B(A) \subset cl_\tau(A)$. Therefore $cl_\tau(A) = \bigcap_{B \in \mathscr{B}} B(A)$. □

4. Quotient Γ-*BCK*-algebras by ideals

We introduce the concept of quotient Γ-*BCK*-algebras by ideals of Γ-*BCK*-algebra and study some of its properties. Next, we deal with topological structures on quotient Γ-*BCK*-algebras.

Proposition 4.1 Let X be a Γ-*BCK*-algebra and let $I \in \mathscr{I}(X)$, where $\mathscr{I}(X)$ is the set of all Γ-ideals of X. Define a relation $\backsim^I$ on X as follows: for any $x,\ y \in X$ and each $\alpha \in \Gamma$,

$$x \backsim^I y \text{ if and only if } x\alpha y,\ y\alpha x \in I.$$

Then $\backsim^I$ is a congruence relation on X, i.e., it satisfies the following conditions: for any $x,\ y,\ z \in X$ and each $\alpha \in \Gamma$,
(1) $x \backsim^I x$, i.e., $\backsim^I$ is reflexive,
(2) if $x \backsim^I y$, then $y \backsim^I x$, i.e., $\backsim^I$ is symmetric,
(3) if $x \backsim^I y$ and $y \backsim^I z$, then $x \backsim^I z$, i.e., $\backsim^I$ is transitive,
(4) if $x \backsim^I u$ and $y \backsim^I v$, then $x\alpha y \backsim^I u\alpha v$.

Proof. (1) Since $0 \in I$, by the axiom (ΓA_4), $x\alpha x = 0 \in I$ for each $x \in X$ and each $\alpha \in \Gamma$. Then $x \backsim^I x$. Thus $\backsim^I$ is reflexive.

(2) The proof is easy.

(3) Suppose $x \backsim^I y$ and $y \backsim^I z$, and let $\alpha,\ \beta \in \Gamma$. Then $x\alpha y,\ y\alpha z \in I$. Moreover, by the axiom (ΓA_1),

$$[(x\alpha z)\beta(x\alpha y)]\beta(y\alpha z) = 0.$$

Thus by Corollary 4.9 in [21], $x\alpha z \in I$. Similarly, we get $z\alpha x \in I$. So $x \backsim^I z$. Hence $\backsim^I$ is transitive.

(4) Suppose $x \backsim^I u$ and $y \backsim^I v$, and let $\alpha,\ \beta \in \Gamma$. Then clearly, we get

$$x\alpha u,\ u\alpha x,\ y\alpha v,\ v\alpha y \in I.$$

Furthermore, by Theorem 3.3 in [21], we have

$$(x\alpha y)\beta(x\alpha v) \leq v\alpha y \text{ and } (x\alpha v)\beta(x\alpha y) \leq y\alpha v \in I.$$

By Proposition 4.5 in [21], we get

$$(x\alpha y)\beta(x\alpha v),\ (x\alpha v)\beta(x\alpha y) \in I.$$

Thus $x\alpha y \backsim^I x\alpha v$. On the other hand, by Proposition 3.6 in [21],

$$(x\alpha v)\beta(u\alpha v) \leq x\alpha u \in I \text{ and } (u\alpha v)\beta(x\alpha v) \leq u\alpha x \in I.$$

So $x\alpha v \backsim^I u\alpha v$. Hence by the condition (3), $x\alpha y \backsim^I u\alpha v$. □

For a congruence relation $\backsim^I$ on a Γ-BCK-algebra X and each $x \in X$, a subset $I[x]$ of X defined by

$$I[x] = \{y \in X : x \backsim^I y\} = \{y \in X : x\alpha y,\ y\alpha x \in I \text{ for each } \alpha \in \Gamma\}$$

is called the *congruence class* in X determined by x with respect to $\backsim^I$. The set of all congruence classes in X is denoted by X/I.

Proposition 4.2 Let X be a Γ-BCK-algebra, $I \in \mathscr{I}(X)$ and let $\backsim^I$ be a congruence relation on X. We define a mapping $f : X/I \times \Gamma \times X/I \to X/I$ as follows: for each $(I[x],\ \alpha,\ I[y]) \in X/I \times \Gamma \times X/I$,

$$f(I[x],\ \alpha,\ I[y]) = I[x]\alpha I[y] = I[x\alpha y].$$

Then X/I is a Γ-BCK-algebra such that $I[0] = I$. In this case, X/I is called the *quotient* Γ-*BCK-algebra of* X *by* I.

Proof. It is obvious that f is well-defined.

Let $x \in I$ and let $\alpha \in \Gamma$. By the axiom (ΓA_5) and Proposition 3.6 in [21], $0\alpha x \in I$ and $x\alpha 0 = x \in I$. Then $x \backsim^I 0$, i.e., $x \in I[0]$. Thus $I \subset I[0]$. Conversely, let $x \in I[0]$ and let $\alpha \in \Gamma$. Then $x = x\alpha 0 \in I$. Thus $I[0] \subset I$. So $I = I[0]$.

Let $x,\ y,\ z \in X$ and let $\alpha,\ \beta \in \Gamma$. Then by the axiom ($\Gamma A_1$), we get

$$[(x\alpha y)\beta(x\alpha z)]\beta(z\alpha y) = 0$$

Thus we have

$$[(I[x]\alpha I[y])\beta I[x]\alpha I[z])]\beta(I[z]\alpha I[y]) = I[[(x\alpha y)\beta(x\alpha z)]\beta(z\alpha y)] = I[0].$$

So the axiom (ΓA_1) holds.

Let $x,\ y \in X$ and let $\alpha,\ \beta \in \Gamma$. Then by the axiom ($\Gamma A_2$), $[x\alpha(x\beta y)]\alpha y = 0$. Thus we have

$$[I[x]\alpha(I[x]\beta I[y])]\alpha I[y] = I[[x\alpha(x\beta y)]\alpha y] = I[0].$$

So the axiom (ΓA_2) holds.

It can be easily proved that the axioms (ΓA_3) and (ΓA_5).

Now suppose $I[x]\alpha I[y] = I[y]\alpha I[x] = I[0]$ for any $x,\ y \in X$ and each $\alpha \in \Gamma$. Then clearly, $x\alpha y \backsim^I 0$ and $y\alpha x \backsim^I 0$. Thus $x\alpha y,\ y\alpha x \in I$. So $I[x] = I[y]$. Hence the axiom (ΓA_4) holds. Therefore X/I is a Γ-BCK-algebra with the zero element $I[0] = I$. This completes the proof. □

Consider the Γ-BCK-algebra X and the Γ-ideal $I = \{0,\ 3\}$ of X in Example 3.12. Then we can easily obtain the quotient Γ-BCK-algebra X/I with the following ternary operation (Table 7):

Table 7. The ternary operation on the quotient set

	α			β		
	$I[0]$	$I[1]$	$I[2]$	$I[0]$	$I[1]$	$I[2]$
$I[0]$	$I[0]$	$I[0]$	$I[0]$	$I[0]$	$I[0]$	$I[0]$
$I[1]$	$I[1]$	$I[0]$	$I[1]$	$I[1]$	$I[0]$	$I[1]$
$I[2]$	$I[2]$	$I[2]$	$I[0]$	$I[2]$	$I[2]$	$I[0]$

We can define a partial ordering $\leq$ on X/I as follows: for any $x,\ y \in X$ and each $\alpha \in \Gamma$,

$$I[x] \leq I[y] \ \text{ if and only if } \ I[x]\alpha I[y] = I[0] = I.$$

Then from the proof process of Proposition 4.2, we have the following.

Proposition 4.3 Let X be a Γ-BCK-algebra and let X/I be the quotient Γ-BCK-algebra of X by $I \in \mathscr{I}(X)$. Then the followings hold: for any $x,\ y,\ z \in X$ and any $\alpha,\ \beta \in \Gamma$,

(1) $(I[x]\alpha I[y])\beta(I[x]\alpha I[z]) \leq I[z]\alpha I[y]$,

(2) $I[x]\alpha(I[x]\beta I[y]) \leq I[y]$,

(3) if $I[x]leqI[y]$ and $I[y] \leq I[x]$, then $I[x] = I[y]$,

(4) $I[x] \leq I[x]$,

(5) $I \leq I[x]$.

Lemma 4.4 Let X be a Γ-BCK-algebra and let $I,\ J \in \mathscr{I}(X)$ such that $I \subset J$. Then

(1) I is an ideal of the subalgebra J,

(2) J/I is the quotient algebra of J by I and J/I is an ideal of X/I.

Proof. (1) The proof is straightforward from Proposition 4.10 in [21] and Definition 2.8.

(2) From (1) and Proposition 4.3, it is obvious that J/I is the quotient algebra of J by I. Let $J/I = \{I_J[x] : x \in J\}$ and let $y \in I_J[x]$ for each $x \in J$ and each $y \in X$. Then $y \backsim^I x$. Thus $y\alpha x \in I$ for each $\alpha \in \Gamma$. So $y\alpha x \in J$ for each $\alpha \in \Gamma$. Since J is an ideal of X and $x \in J$, $y \in J$. Hence $I_J[x] \in X/I$. Therefore $J/I \subset X/I$.

Since $I \subset J$, $I[0] = I \in J/I$. For any $I[x],\ I[y] \in X/I$ and each $\alpha \in \Gamma$, suppose $I[x]\alpha I[y] \in J/I$ and $I[y] \in J/I$. Then $I[x\alpha y] = I[x]\alpha I[y] \in J/I$. Thus $x\alpha y \in J$ and $y \in J$. Since J is an ideal of X, $x \in J$. So $I[x] \in J/I$. Hence J/I is an ideal of X/I. □

Lemma 4.5 If J^* is an ideal of X/I, then $J = \bigcup_{x \in J^*} I[x]$ is an ideal of X and $I \subset J$.

Proof. It is clear that $I = I[0] \in J^*$. Then $0 \in J$ and $I \subset J$. For any $x,\ y \in X$ and each $\alpha \in \Gamma$, suppose $x\alpha y \in J$ and $y \in J$. Then $I[x]\alpha I[y] = I[x\alpha y] \in J^*$ and $I[y] \in J^*$. Since J^* is an ideal of X/I, $I[x] \in J^*$. Thus $x \in J$. So J is an ideal of X. □

Proposition 4.6 If I is an ideal of a Γ-BCK-algebra X, then there is a bijection from $\mathscr{I}(X,\ I)$ to $\mathscr{I}(X/I)$ where $\mathscr{I}(X,\ I)$ is the set of all ideals containing I of X.

Proof. We define the mapping $f : \mathscr{I}(X,\ I) \rightarrow \mathscr{I}(X/I)$ as follows:

$$f(J) = J/I \text{ for each } J \in \mathscr{I}(X,\ I).$$

From Lemmas 4.4 and 4.5, f is well-defined and surjective. Let $A,\ B \in \mathscr{I}(X,\ I)$ such that $A \neq B$. Then there is $x \in X$ such that $x \in A - B$ or $x \in B - A$, say $x \in B - A$. Assume that $f(A) = f(B)$. Then $I[x] \in f(A) \cap f(B)$. Thus there is $y \in A$ such that $I[x] = I[y]$. So $x \backsim^I y$, i.e., $x\alpha y,\ y\alpha x \in I$. Since $I \subset A$, $x\alpha y \in A$. Since A is an ideal of X and $y \in A$, $x \in A$. This is a contradiction to $x \notin A$. Hence $A = B$, i.e., f is injective. Therefore f is bijective. □

Remark 4.7 We define the mapping $\pi : X \rightarrow X/I$ as follows:

$$\pi(x) = I[x] \text{ for each } x \in X.$$

Then we can easily check that π is a surjective homomorphism. In this case, π is called the *natural homomorphism*.

From Remark 4.7, Lemma 4.5 can be restated as follows.

Proposition 4.8 Let X be a Γ-BCK-algebra and let $\pi : X \to X/I$ be the natural homomorphism, where I is an ideal of X. If A is an ideal of X/I, then $\pi^{-1}(A)$ is an ideal of X and $I \subset \pi^{-1}(A)$.

Now we deal with characterizations of quotient Γ-BCK-algebra by commutative [resp. positive implicative and implicative] ideals.

Theorem 4.9 Let X be a Γ-BCK-algebra and let $I \in \mathscr{I}(X)$. Then I is commutative if and only if X/I is a commutative Γ-BCK-algebra.

Proof. Suppose I is commutative. To prove that X/I is commutative, the following identity holds: for any $x,\ y \in X$ and any $\alpha,\ \beta \in \Gamma$,

$$I[y]\alpha(I[y]\beta I[x]) = I[x]\alpha(I[x]\beta(I[y]\alpha(I[y]\beta I[x]))). \tag{6}$$

Let $x,\ y \in X$ and let $\alpha,\ \beta \in \Gamma$ and let $u = y\alpha(y\beta x)$. Since $I \in \mathscr{I}(X)$, by the axiom (ΓA_2), $u\beta x = 0 \in I$. Since I is commutative, by Theorem 4.31 in [21], $u\beta(x\alpha(x\beta u)) \in I$. Then $I[u]\beta(I[x]\alpha(I[x]\beta I[u])) = I[u\beta(x\alpha(x\beta u))] = I = I[0]$, i.e.,

$$I[u] \le I[x]\alpha(I[x]\beta I[u]).$$

Thus we have

$$I[y]\alpha(I[y]\beta I[x]) = I[y\alpha(y\beta x)] = I[u] \le I[x]\alpha(I[x]\beta I[u]).$$

It is clear that $I[x]\alpha(I[x]\beta I[y]) \le I[u]$. So we get

$$I[y]\alpha(I[y]\beta I[x]) = I[x]\alpha(I[x]\beta I[u]) = I[x]\alpha(I[x]\beta(I[y]\alpha(I[y]\beta I[x]))).$$

Hence (6) holds. Therefore by Theorem 3.18 in [21], X/I is commutative.

Conversely, suppose X/I is commutative. By Theorem 4.18 in [21], $\{I[0]\}$ is commutative. Suppose $x\alpha y \in I$ for any $x,\ y \in X$ and each $\alpha \in \Gamma$. Then we have

$$I[x]\alpha I[y] = I[x\alpha y] = I[0] \in \{I[0]\}.$$

Since $\{I[0]\}$ is commutative, by Theorem 4.31 in [21], we get

$$I[x\alpha(y\beta(y\alpha x))] = I[x]\alpha(I[y]\beta(I[y]\alpha I[x])) \in \{I[0]\} \text{ for each } \beta \in \Gamma.$$

Thus $I[x\alpha(y\beta(y\alpha x))] = I[0]$. So $x\alpha(y\beta(y\alpha x)) \in I$. Hence by Theorem 4.31 in [21], I is commutative. $\square$

Theorem 4.10 Let X be a Γ-BCK-algebra and let $I \in \mathscr{I}(X)$. Then I is positive implicative if and only if X/I is a positive implicative Γ-BCK-algebra.

Proof. Suppose I is positive implicative and for any $c,\ y,\ z \in X$ and any $\alpha,\ \beta \in \Gamma$, let $u = (x\beta y)\alpha z$. Then by Proposition 3.5 in [21] and the axiom (ΓA_3), we get

$$((x\beta u)\alpha y)\beta z = ((x\beta y)\alpha z)\beta u = 0 \in I.$$

By Theorem 4.16 in [21], we have

$$((x\beta u)\alpha z)\beta(y\beta z) \in I.$$

On the other hand, by Proposition 3.5 in [21], we get

$$(((x\beta u)\alpha z)\beta(y\beta z)) = (x\beta z)\alpha(y\beta z))\beta((x\beta y)\alpha z).$$

Thus $((x\beta z)\alpha(y\beta z))\beta((x\beta y)\alpha z) \in I$. So we have

$$((I[x]\beta I[z])\alpha(I[y]\beta I[z]))\beta((I[x]\beta I[y])\alpha I[z]) = I[((x\beta z)\alpha(y\beta z))\beta((x\beta y)\alpha z)] = I[0].$$

It can be easily see that the following identity holds:

$$((I[x]\beta I[y])\alpha I[z])\beta((I[x]\beta I[z])\alpha(I[y]\beta I[z])) = I[0].$$

By the axiom (ΓA_3), we get

$$(I[x]\beta I[z])\alpha(I[y]\beta I[z]) = (I[x]\beta I[y])\alpha I[z].$$

Hence X/I is positive implicative.

Conversely, suppose X/I is positive implicative. By Theorem 4.18 in [21], $\{I[0]\}$ is a positive implicative ideal of X/I. Suppose $(x\alpha y)\beta z \in I$ for any $x,\ y,\ z \in X$ and any $\alpha,\ \beta \in \Gamma$. Then we have

$$(I[x]\alpha I[y])\beta I[z] = I[(x\alpha y)\beta z] = I[0] \in \{I[0]\}.$$

Then by Theorem 4.16 in [21], we get

$$(I[x]\alpha I[z])\beta(I[y]\alpha I[z]) \in \{I[0]\}.$$

Thus $I[(x\alpha z)\beta(y\alpha z)] = (I[x]\alpha I[z])\beta(I[y]\alpha I[z]) = I[0]$. So $(x\alpha z)\beta(y\alpha z) \in I$. Hence I is positive implicative. □

Theorem 4.11 Let X be a Γ-BCK-algebra and let $I \in \mathscr{I}(X)$. Then I is implicative if and only if X/I is an implicative Γ-BCK-algebra.

Proof. The proof follows from Theorem 3.21, Proposition 4.26 in [21] and Theorems 4.9, 4.10. □

See section 5 for the definition of a bounded Γ-BCK-algebra.

Proposition 4.12 Let X be a bounded Γ-BCK-algebra with the greatest element 1 and let $I \in \mathscr{I}(X)$. Then X/I is bounded with the greatest element $I[1]$.

Proof. Let $x \in X$ and let $\alpha \in \Gamma$. Then $I[x]\alpha I[1] = I[x\alpha 1] = I[0]$. Thus $I[x] \leq I[1]$. So $I[1]$ is the greatest element of X/I. □

Now we will discuss topological structures on quotient Γ-BCK-algebras. We denote subsets of X/I as $\dot{A}$, $\dot{B}$, $\dot{C}$, etc. and $\dot{\emptyset} = \emptyset$, $\dot{X} = X/I$.

Proposition 4.13 Let $(X,\ \tau)$ be a $T\Gamma$-BCK-algebra, $I \in \mathscr{I}(X)$ and let $\pi : X \to X/I$ be the natural homomorphism. We define a collection τ_π of subsets of X/I as follows:

$$\tau_\pi = \{\dot{A} \in P(X/I) : \pi^{-1}(\dot{A}) \in \tau\},$$

where $\dot{A} = \{I[a] : a \in A\}$ for some $A \subset X$. Then

(1) τ_π is a topology on X/I,

(2) $\pi : (X,\ \tau) \to (X/I,\ \tau_\pi)$ is continuous, open and closed,

(3) τ_π is the finest topology on X/I with respect to which π is continuous,

(4) $(X/I,\ \tau_\pi)$ is a $T\Gamma$-BCK-algebra.

In this case, τ_π is called the *quotient topology on X/I induced by* π and $(X/I,\ \tau_\pi)$ is called a *quotient $T\Gamma$-BCK-algebra* and π is called a *quotient mapping*.

Proof. (1) It is clear that $\pi^{-1}(\dot{\emptyset}) = \pi^{-1}(\emptyset) = \emptyset$, $\pi^{-1}(\dot{\emptyset}) = \pi^{-1}(X/I) = X \in \tau$. Then $\emptyset \in \tau_\pi$. Suppose $\dot{A}$, $\dot{B} \in \tau_\pi$. Then $\pi^{-1}(\dot{A})$, $\pi^{-1}(\dot{B}) \in \tau$. Thus $\pi^{-1}(\dot{A} \cap \dot{B}) = \pi^{-1}(\dot{A}) \cap \pi^{-1}(\dot{B}) \in \tau$. So $\dot{A} \cap \dot{B} \in \tau_\pi$. Now suppose $(\dot{A}_j)_{j\in J} \subset \tau_\pi$, where J denotes an index set. Then clearly, $(\pi^{-1}(\dot{A}_j))_{j\in J} \subset \tau$. Thus $\pi^{-1}(\bigcup_{j\in J} \dot{A}_j) = (\bigcup_{j\in J} \pi^{-1}(\dot{A}_j) \in \tau$. So $\bigcup_{j\in J} \dot{A}_j \in \tau_\pi$. Hence τ_π is a topology on X/I.

(2) The proof is straightforward.

(3) Let δ be any topology on X/I such that $\pi : (X,\ \tau) \to (X/I,\ \delta)$ is continuous and let $\dot{V} \in \delta$. Then $\pi^{-1}(\dot{V}) \in \tau$. Thus $\dot{V} \in \tau_\pi$. So $\delta \subset \tau_\pi$. Hence τ_π is the finest topology on X/I which π is continuous.

(4) We prove that a mapping $f : (X/I,\ \tau_\pi) \times \Gamma \times (X/I,\ \tau_\pi) \to (X/I,\ \tau_\pi)$ is continuous at each $(I[a],\ \alpha,\ I[b]) \in X/I \times \Gamma \times X/I$. Let $a,\ b \in X$, let $\alpha \in \Gamma$ and let $\dot{W} \in N_{\tau_\pi}(I[a]\alpha I[b]) = N_{\tau_\pi}(I[a\alpha b])$. Then there is $\dot{W}' \in \tau_\pi$ such that $I[a\alpha b] \in \dot{W}' \subset \dot{W}$. Thus $a\alpha b = \pi^{-1}(I[a\alpha b]) \in \pi^{-1}(\dot{W}') \subset \pi^{-1}(\dot{W})$. Since $\pi^{-1}(\dot{W}') \in \tau$, $\pi^{-1}(\dot{W}) \in N_\tau(a\alpha b)$. Since $(X,\ \tau)$ is a $T\Gamma$-BCK-algebra, by Definition 3.3, there are $U' \in N_\tau(a)$ and $V' \in N_\tau(b)$ such that $U'\alpha V' \subset \pi^{-1}(\dot{W})$. Since π is surjective, $\dot{U} = \pi(U') \in \pi(N(a))$ and $\dot{V} = \pi(V') \in \pi(N(b))$ such that $\dot{U}\alpha\dot{V} \subset \dot{W}$. By (2), $\dot{U}$, $\dot{V} \in \tau_\pi$. Note that $N_{\tau_\pi}(I[a]) = \pi(N_\tau(a))$ and $N_{\tau_\pi}(I[b]) = \pi(N_\tau(b))$. Thus $\dot{U} \in N_{\tau_\pi}(I[a])$, $\dot{V} \in N_{\tau_\pi}(I[b])$ and $\dot{U}\alpha\dot{V} \subset \dot{W}$. So f is continuous. Hence $(X/I,\ \tau_\pi)$ is a $T\Gamma$-BCK-algebra. □

Proposition 4.14 Let $(X,\ \tau)$ be a $T\Gamma$-BCK-algebra, $I \in \mathscr{I}(X)$ and let $\pi : X \to X/I$ be the natural homomorphism. If $\{I\}$ is open in $(X/I,\ \tau_\pi)$, then X/I is discrete.

Proof. Suppose $\{I\}$ is open in $(X/I,\ \tau_\pi)$, let $a \in X$ and let $\alpha \in \Gamma$. Then clearly, $I[a]\alpha I[a] = I[a\alpha a] = I[0] = I$. Thus by the hypothesis, $\{I\} \in N_{\tau_\pi}(I[a]\alpha I[a])$. So there are $\dot{U}$, $\dot{V} \in N_{\tau_\pi}(I[a])$ such that $\dot{U}\alpha\dot{V} \subset \{I\}$, i.e., $\dot{U}\alpha\dot{V} = \{I\}$. Let $\dot{W} = \dot{U} \cap \dot{V}$. Then $\dot{W}\alpha\dot{W} = \{I\}$. Thus $\dot{W} = \{I[a]\}$, i.e., $\{I[a]\}$ is open in X/I. So X/I is discrete. □

The following is an immediate consequence of Propositions 4.13 and 4.14.

Corollary 4.15 Let $(X,\ \tau)$ be a $T\Gamma$-BCK-algebra, $I \in \mathscr{I}(X)$ and let $\pi : X \to X/I$ be the natural homomorphism. If $\{0\}$ is open in X, then X/I is discrete.

Proposition 4.16 Let $(X,\ \tau)$ be a $T\Gamma$-BCK-algebra, $I \in \mathscr{I}(X)$ and let $\pi : X \to X/I$ be the natural homomorphism. If $(X/I,\ \tau_\pi)$ is a T_1-space, then $\{0\}$ is closed in X.

Proof. Suppose X/I is a T_1-space and let $a \in X$. Then clearly, $\{I[a]\}$ is closed in X/I. In particular, $\{I[0]\}$ is closed in X/I. Since $\pi : (X,\ \tau) \to (X/I,\ \tau_\pi)$ is continuous, $\{0\} = \{\pi^{-1}(I[0])\}$ is closed in X. □

Theorem 4.17 Let $(X,\ \tau)$ be a $T\Gamma$-BCK-algebra, $I \in \mathscr{I}(X)$ and let $\pi : X \to X/I$ be the natural homomorphism. Then $\{I\}$ is closed in $(X/I,\ \tau_\pi)$ if and only if X/I is Hausdorff.

Proof. Suppose $\{I\}$ is closed in $(X/I,\ \tau_\pi)$, let $a,\ b \in X$ such that $a \neq b$ and let $\alpha \in \Gamma$. Then clearly, $I[a] \neq I[b]$, i.e., $I[a]\alpha I[b] \neq I$ or $I[b]\alpha I[a] \neq I$, say $I[a]\alpha I[b] \neq I$. By the hypothesis, $\{I\}^c \in \tau_\pi$, i.e., $\{I\}^c \in N_{\tau_\pi}(I[a]\alpha I[b])$. Thus there are $\dot{U} \in N_{\tau_\pi}(I[a])$ and $\dot{V} \in N_{\tau_\pi}(I[b])$ such that $\dot{U}\alpha\dot{V} \subset \{I\}^c$. So $\dot{U} \cap \dot{V} = \dot{\emptyset}$. Hence X/I is Hausdorff.

Suppose X/I is Hausdorff and let $a \in \{I\}^c$. Then $I[a] \neq I$. By the hypothesis, there are $\dot{U} \in N_{\tau_\pi}(I[a])$ and $\dot{V} \in N_{\tau_\pi}(I)$ such that $\dot{U} \cap \dot{V} = \dot{\emptyset}$. Thus $I \notin \dot{U}$. So $\dot{U} \subset \{I\}^c$, i.e., $\{I\}^c$ is open in X/I. Hence $\{I\}$ is closed in X/I. □

The following is an immediate consequence of Proposition 4.13 and Theorem 4.17.

Corollary 4.18 Let $(X,\ \tau)$ be a $T\Gamma$-BCK-algebra, $I \in \mathscr{I}(X)$ and let $\pi : X \to X/I$ be the natural homomorphism. Then $\{0\}$ is closed in X if and only if X/I is Hausdorff.

Proposition 4.19 Let $(X,\ \tau)$ be a $T\Gamma$-BCK-algebra, $I \in \mathscr{I}(X)$ and let $\pi : X \to X/I$ be the natural homomorphism. If $\dot{A}$ is an ideal of X/I and $I \in int_{\tau_\pi}(\dot{A})$, then $\dot{A}$ is open in X/I.

Proof. Let $I[a] \in \dot{A}$ and let $\alpha \in \Gamma$. Then $I[a]\alpha I[a] = I$. Since $I \in int_{\tau_\pi}(\dot{A})$, there is $\dot{W} \in N_{\tau_\pi}(I) = N_{\tau_\pi}(I[a]\alpha I[a])$ such that $\dot{W} \subset \dot{A}$. Since τ_π is a topology on X/I, by Definition 3.3, there are $\dot{U},\ \dot{V} \in N_{\tau_\pi}(I[a])$ such that $\dot{U}\alpha\dot{V} \subset \dot{W} \subset \dot{A}$. On the other hand, $I[b]\alpha I[a] \in \dot{U}\alpha\dot{V} \subset \dot{A}$ for each $I[b] \in \dot{U}$. Since $\dot{A}$ is an ideal of X/I, $I[b] \in \dot{A}$. So $I[a] \in U \subset \dot{A}$. Hence $\dot{A}$ is open in X/I. □

Lemma 4.20 Let X be a Γ-BCK-algebra, $I \in \mathscr{I}(X)$ and let $\pi : X \to X/I$ be the natural homomorphism. If A is an ideal of X, then $\pi(A)$ is an ideal of X/I.

Proof. Suppose $I[a]\alpha I[b] \in \pi(A)$ and $I[b] \in \pi(A)$ for any $a \in X$ and each $\alpha \in \Gamma$. Then there are $c,\ d \in A$ such that $[I(a\alpha b] = I[a]\alpha I[b] = I[c]$ and $I[b] = I[d]$. Since $c \sim^I a\alpha b$ and $d \sim^I b$, $a\alpha b \in A$ and $b \in A$. Thus by the hypothesis, $a \in A$. So $I[a] = \pi(a) \in \pi(A)$. Hence $\pi(A)$ is an ideal of X/I. □

The following is an immediate consequence of Propositions 4.13, 4.19 and Lemma 4.20.

Corollary 4.21 Let $(X,\ \tau)$ be a $T\Gamma$-BCK-algebra, $I \in \mathscr{I}(X)$ and let $\pi : X \to X/I$ be the natural homomorphism. If A is an ideal of X and $I \in int_\tau(\pi(A))$, then $\pi(A)$ is open in X/I.

Proposition 4.22 Let $(X,\ \tau)$ be a $T\Gamma$-BCK-algebra, $I \in \mathscr{I}(X)$ and let $\pi : X \to X/I$ be the natural homomorphism. If $\dot{A}$ is an ideal of X/I and is open in X/I, then $\dot{A}$ is closed in X/I.

Proof. Suppose $\dot{A}$ is an ideal of X/I and is open in X/I and let $I[x] \in \dot{A}^c$, $\alpha \in \Gamma$. Since $I[x]\alpha I[x] = I$ and $\dot{A}$ is open in X/I, $\dot{A} \in N_{\tau_\pi}(I) = N_{\tau_\pi}(I[x\alpha x])$. Since $(X/I, \tau_\pi)$ is a $T\Gamma$-BCK-algebra, there is $\dot{U} \in N_{\tau_\pi}(I[x])$ such that $\dot{U}\alpha\dot{U} \subset \dot{A}$. Assume that $\dot{U} \not\subset \dot{A}^c$. Then there is $I[y] \in X/I$ such that $I[y] \in \dot{U} \cap \dot{A}$. It is clear that $I[z\alpha y] = I[z]\alpha I[y] \in \dot{U}\alpha\dot{U} \subset \dot{A}$ for each $I[z] \in \dot{U}$. Since $I[y] \in \dot{A}$ and $\dot{A}$ is an ideal of X/I, $I[z] \in \dot{A}$. Thus $\dot{U} \subset \dot{A}$. This is a contradiction. So $\dot{U} \subset \dot{A}^c$. Hence $\dot{A}^c$ is open in X/I, i.e., $\dot{A}$ is closed in X/I. □

The following is an immediate consequence of Propositions 4.13 and 4.22.

Corollary 4.23 Let $(X,\ \tau)$ be a $T\Gamma$-BCK-algebra, $I \in \mathscr{I}(X)$ and let $\pi : X \to X/I$ be the natural homomorphism. If A is an ideal of X and is open in X, then $\pi(A)$ is closed in X/I.

Proposition 4.24 Let $(X,\ \tau)$ be a $T\Gamma$-BCK-algebra, $I \in \mathscr{I}(X)$ and let $\pi : X \to X/I$ be the natural homomorphism. If $(X/I,\ \tau_\pi)$ is Hausdorff, then $\bigcap_{\dot{U} \in N_{\tau_\pi}(I)} \dot{U} = \{I\}$. Moreover, $\bigcap_{\dot{U} \in \mathscr{N}_{\tau_\pi}(I)} \dot{U} = \{I\}$.

Proof. Assume that $I \neq I[x] \in \bigcap_{\dot{U} \in N_{\tau_\pi}(I)} \dot{U}$. Then there is $\dot{V} \in N_{\tau_\pi}(I)$ such that $I[x] \notin \dot{V}$. Thus $I[x] \notin \bigcap_{\dot{U} \in N_{\tau_\pi}(I)} \dot{U}$. This is a contradiction. So $\bigcap_{\dot{U} \in N_{\tau_\pi}(I)} \dot{U} = \{I\}$.

Lemma 4.25 Let $(X,\ \tau)$ be a $T\Gamma$-BCK-algebra, $I \in \mathscr{I}(X)$ and let $\pi : X \to X/I$ be the natural homomorphism. Then $\pi(\mathscr{N}_\tau(0) = \mathscr{N}_{\tau_\pi}(I)$.

Proof. From Proposition 4.13, it is obvious that $\pi(N_\tau(0)) = N_{\tau_\pi}(I)$. Let $\dot{V} \in N_{\tau_\pi}(I)$. Then clearly, $\pi^{-1}(\dot{V}) \in N_\tau(0)$. Thus there is $U \in N_\tau(0)$ such that $U \subset \pi^{-1}(\dot{V})$. So $\pi(U) \subset \pi(\pi^{-1}(\dot{V})) = \dot{V}$. This completes the proof. □

Lemma 4.26 Let $(X,\ \tau)$ be a $T\Gamma$-BCK-algebra, $I \in \mathscr{I}(X)$ and let $\pi : X \to X/I$ be the natural homomorphism. Then If X is Hausdorff, then $(X/I,\ \tau_\pi)$ is Hausdorff.

Proof. The proof is straightforward. □

The following is an immediate consequence of Propositions 4.13, 4.24 and Lemmas 4.25, 4.26.

Proposition 4.27 Let (X, τ) be a $T\Gamma$-BCK-algebra, $I \in \mathscr{I}(X)$ and let $\pi : X \to X/I$ be the natural homomorphism. If X is Hausdorff, then $\bigcap_{\dot{U} \in \mathscr{N}_{\tau_\pi}(I)} \dot{U} = \{I\}$.

5. Quotient Γ-*BCK*-algebras by dual Γ-ideals

First of all, we introduce the concepts of dual ideals of a Γ-BCK-algebra X and quotient Γ-BCK-algebras by dual Γ-ideals, and study some of their properties. Next, we discuss uniform structures on a quotient Γ-BCK-algebra.

Definition 5.1 A Γ-BCK-algebra X is said to be *bounded*, if there is $1 \in X$ (called the *unit* of X) such that $x\alpha 1 = 0$, i.e., $x \leq 1$ for each $x \in X$ and each $\alpha \in \Gamma$. In a bounded Γ-BCK-algebra X, we denote $1\alpha x$ for each $x \in X$ and each $\alpha \in \Gamma$ by Nx.

Example 5.2 Let $\Gamma = \{\alpha, \beta\}$ and let $X = \{0, a, b, c, d, 1\}$ be the Γ-BCK-algebra with the ternary operation be defined by the following Table 8:

Table 8. The ternary operation 6

	α						β					
	0	a	b	c	d	1	0	a	b	c	d	1
0	0	0	0	0	0	0	0	0	0	0	0	0
a	a	0	a	a	0	0	a	0	a	a	0	0
b	b	b	0	0	0	0	b	b	0	0	0	0
c	c	c	b	0	b	0	c	c	0	0	b	0
d	d	b	a	a	0	0	d	c	a	a	0	0
1	1	c	d	a	b	0	1	c	d	a	b	0

Then we can easily check that X is bounded with the unit 1.

Proposition 5.3 In a bounded Γ-BCK-algebra X with the unit 1, the followings hold: for any $x, y \in X$ and each $\alpha, \beta \in \Gamma$,

(1) $N1 = 0$, $N0 = 1$,

(2) $NNx \leq x$,

(3) $Nx\alpha Ny \leq y\beta x$,

(4) if $y \leq x$, then $Nx \leq Ny$,

(5) $Nx\alpha y = Ny\alpha x$,

(6) $NNNx = Nx$.

Proof. (1) The proofs are obvious.

(2) By Proposition 3.7 in [21] and the axiom (ΓA_4), we have

$$NNx\beta x = N(1\alpha x)\beta x = [1\alpha(1\alpha x)]\beta x = (1\alpha x)\beta(1\alpha x) = 0.$$

Then $NNx \leq x$.

(3) The proof follows from Theorem 3.3, Proposition 3.6 in [21], and axiom (ΓA_4).

(4) The proof follows from Proposition 3.4 in [21].

(5) We can easily prove that $(Nx\alpha y)\beta(Ny\alpha x) = 0$, $(Ny\alpha x)\beta(Nx\alpha y) = 0$. Then by the axiom ($\Gamma A_3$), $Nx\alpha y = Ny\alpha x$.

(6) The proof is easy. □

For a Γ-BCK-algebra X and any $x, y \in X$, let us $x \wedge y$ define as follows:

$$x \wedge y = y\alpha(y\beta x), \text{ for each } \alpha, \beta \in \Gamma.$$

Then we obtain the following.

Proposition 5.4 In a bounded Γ-BCK-algebra X with the unit 1, the followings hold: for any $x \in X$,

(1) $1 \wedge x = x$,

(2) $x \wedge 1 = NNx$.

Definition 5.5 Let X be a bounded Γ-BCK-algebra. Then $x \in X$ is called an *involution*, if $NNx = x$.

We denote the set of all involutions of X by $IV(X)$. Then clearly, $0,\ 1 \in IV(X)$. Thus $IV(X) \neq \emptyset$.

The following is an immediate consequence of Definition 5.5.

Proposition 5.6 Let X be a bounded Γ-BCK-algebra. Then for any $x,\ y \in IV(X)$ and each $\alpha \in \Gamma$,

$$x\alpha Ny = y\alpha Nx.$$

Definition 5.7 Let X be a bounded Γ-BCK-algebra and let D be a nonempty subset of X. Then D is called a *dual* Γ-*ideal* of X, if it satisfies the following conditions: for any $x,\ y \in X$ and any $\alpha \in \Gamma$,

(ΓD_1) $1 \in D$,

(ΓD_2) if $N(Nx\alpha Ny) \in D$ and $y \in D$, then $x \in D$.

We denote the set of all dual Γ-ideals of X by $\mathscr{DI}(X)$. Then it is clear that $\{1\},\ X \in \mathscr{DI}(X)$. Furthermore, if $D_1,\ D_2 \in \mathscr{DI}(X)$, then $D_1 \cap D_2 \in \mathscr{DI}(X)$.

Proposition 5.8 Let X be a bounded Γ-BCK-algebra, let $D \in \mathscr{DI}(X)$ and let $x,\ y \in X$. If $y \in D$ and $Nx \leq Ny$, then $x \in D$.

Proof. The proof follows from hypothesis , Proposition 5.3, and condition (ΓD_2) of Definition 5.7. □

The following is an immediate consequence of Propositions 5.8 and 5.3.

Corollary 5.9 Let X be a bounded Γ-BCK-algebra, let $D \in \mathscr{DI}(X)$ and let $x,\ y \in X$. If $y \leq x$ and $y \in D$, then $x \in D$.

Theorem 5.10 Let X be a bounded commutative Γ-BCK-algebra and let $\emptyset \neq D \subset X$. Then $D \in \mathscr{DI}(X)$ if and only if $ND \in \mathscr{I}(X)$, where $ND = \{Nd : d \in D\}$.

Proof. Suppose $D \in \mathscr{DI}(X)$. Then clearly, $1 \in D$. Thus by Proposition 5.3 (1), $0 = N1 \in ND$. Now suppose $u\alpha v \in ND$ and $v \in ND$ for each $\alpha \in \Gamma$. Then there are $x,\ y \in D$ such that $u\alpha v = Ny$ and $v = Nx$. By commutativity of X and Lemma 3.16 in [21], we get

$$N(NNu\alpha Nx) = N(u\alpha Nx) = NNy = y \in D.$$

Note that $Nu \in D$. Thus Lemma 3.16 in [21], $u = NNu \in ND$. So $ND \in \mathscr{I}(X)$.

Conversely, suppose $I \in \mathscr{I}(X)$. Then we can prove similarly that $NI \in \mathscr{DI}(X)$. □

Note that if X is not commutative, then Theorem 5.10 need not hold (see Example 5.11).

Example 5.11 Let $\Gamma = \{\alpha,\ \beta\}$ and let $X = \{0,\ a,\ b,\ c,\ 1\}$ be the Γ-BCK-algebra with the ternary operation be defined by the following Table 9:

Table 9. The ternary operation 7

	α					β				
	0	a	b	c	1	0	a	b	c	1
0	0	0	0	0	0	0	0	0	0	0
a	a	0	0	0	0	a	0	0	0	0
b	b	a	0	a	0	b	a	0	0	0
c	c	c	c	0	0	c	c	c	0	0
1	1	c	c	a	0	1	c	c	a	0

Then clearly, X is non-commutative bounded with the unit 1. Consider the subset $D = \{c,\ 1\}$ of X. Then we can easily see that D is a dual ideal of X but $ND = \{Nc,\ N1\} = \{0,\ a\}$ is not an ideal of X.

Definition 5.12 (See [12]) Let X be a bounded Γ-BCK-algebra and let A be a nonempty subset of X. Then the intersection of all dual ideals containing A is called the *dual* Γ*-ideal of* X *generated by* A and denoted by $[A)$.

It is obvious that $[\emptyset) = \{1\}$. Moreover, if $A \in \mathscr{DI}(X)$, then $[A) = A$. In fact, $[A)$ is the least dual ideal of X containing A.

Lemma 5.13 Let X be a bounded Γ-BCK-algebra and let A be a nonempty subset of X. Then for each $\alpha \in \Gamma$,

$$[A) = \{x \in X : \exists\, a_1,\ a_2,\ \cdots,\ a_n \in A \text{ such that } (\cdots(Nx\alpha Na_1)\alpha\cdots)\alpha Na_n = 0\}.$$

Proof. The proof is almost same the proof of Theorem 4.2 in [25]. □

Definition 5.14 (See [26]) Let X be a lower Γ-BCK-semilattice and let F be a nonempty subset of X. Then F is called a Γ*-BCK-filter* of X, if it satisfies the following conditions: for any $x,\ y \in X$,

(ΓF_1) if $x \in F$ and $x \le y$, then $y \in F$,

(ΓF_2) if $x,\ y \in F$, then $x \wedge y \in F$, where $x \wedge y = max\{x,\ y\}$.

Example 5.15 In Example 5.11, D is a Γ-BCK-filter of X.

In a a bounded commutative Γ-BCK-algebra X, it is obvious that $N(Nx\alpha Ny) = N(y\alpha x)$ for any $x,\ y \in X$ and each $\alpha \in \Gamma$. Then we obtain the following characterization of dual ideals.

Theorem 5.16 Let X be a bounded commutative Γ-BCK-algebra and let $\emptyset \neq D \subset X$. Then $D \in \mathscr{DI}(X)$ if and only if it satisfies the conditions (ΓF_1) and for any $x,\ y \in X$ and each $\alpha \in \Gamma$,

$$N(y\alpha x) \in D \text{ and } y \in D \text{ imply } x \in D. \tag{7}$$

Proof. The proof is straightforward. □

Lemma 5.17 Let X be a bounded Γ-BCK-algebra and let $D \in \mathscr{DI}(X)$. Define a relation $\backsim^D$ on X as follows: for any $x,\ y \in X$ and each $\alpha \in \Gamma$,

$$x \backsim^D y \text{ if and only if } Nx\alpha y,\ Ny\alpha x \in D.$$

Then $\backsim^D$ is a congruence relation on X, i.e., it satisfies the following conditions: for any $x,\ y,\ z \in X$ and each $\alpha \in \Gamma$,

(1) $x \backsim^D x$, i.e., $\backsim^D$ is reflexive,

(2) if $x \backsim^D y$, then $y \backsim^D x$, i.e., $\backsim^D$ is symmetric,

(3) if $x \backsim^D y$ and $y \backsim^D z$, then $x \backsim^D z$, i.e., $\backsim^D$ is transitive,

(4) if $x \backsim^D u$ and $y \backsim^D v$, then $x\alpha y \backsim^D u\alpha v$.
Proof. (1) From the axiom (ΓA_4) and Proposition 5.3, $N(x\alpha x) \in D$. Then $x \backsim^D x$.
(2) The proof follows from the definition of $\backsim^D$.
(3) Suppose $x \backsim^D y$ and $y \backsim^D z$, and let $\alpha,\ \beta \in \Gamma$. Then clearly, we have

$$Nx\alpha y,\ Ny\alpha x,\ Ny\alpha z,\ Nz\alpha y \in D.$$

On the other hand, by Theorem 3.3 (1) in [21] and Proposition 5.3 (4), we get

$$Nz\alpha y \leq N(x\alpha y)\beta(x\alpha z)$$

$$\leq N[(Nx\alpha z)\beta(Nx\alpha y)].$$

Since $Nz\alpha y \in D$, by Corollary 5.9, $N[(Nx\alpha z)\beta(Nx\alpha y)] \in D$. Since $D \in \mathscr{D}\mathscr{I}(X)$ and $Nx\alpha y \in D$, $Nx\alpha z \in D$. Similarly, we have $Nz\alpha x \in D$. Thus $x \backsim^D z$. So $\backsim^D$ is transitive.
(4) Suppose $x \backsim^D u$ and $y \backsim^D v$ and let $\alpha,\ \beta \in \Gamma$. Then we have

$$Nx\alpha u,\ Nu\alpha x,\ Ny\alpha v,\ Nv\alpha y \in D.$$

By calculations, we easily obtain the following inequalities:

$$Nv\alpha y \leq N(x\alpha y)\beta(x\alpha v),\ Ny\alpha v \leq N(x\alpha v)\beta(x\alpha y).$$

Since $Nv\alpha y,\ Ny\alpha v \in D$, by Corollary 5.9, we get

$$N(x\alpha y)\beta(x\alpha v),\ N(x\alpha v)\beta(x\alpha y) \in D.$$

Thus $x\alpha y \backsim^D x\alpha v$. Also, we obtain the following inequalities:

$$Nx\alpha u \leq N(x\alpha v)\beta(u\alpha v),\ Nu\alpha x \leq N(u\alpha v)\beta(x\alpha v).$$

Since $Nx\alpha u,\ Nu\alpha x \in D$, by Corollary 5.9,

$$N(x\alpha v)\beta(u\alpha v),\ N(u\alpha v)\beta(x\alpha v) \in D.$$

So $x\alpha v \backsim^D u\alpha v$. Hence by the transitivity, $x\alpha y \backsim^D u\alpha v$. Therefore $\backsim^D$ is a congruence relation on X. □
For a congruence relation $\backsim^D$ on a bounded Γ-BCK-algebra X and each $x \in X$, a subset $D[x]$ of X defined by

$$D[x] = \{y \in X : x \backsim^D y\}$$

is called the *congruence class* in X determined by x. The set of all congruence classes in X denoted by X/D.

Proposition 5.18 Let X be a bounded Γ-BCK-algebra, $D \in \mathscr{D}\mathscr{I}(X)$ and let $\backsim^D$ be a congruence relation on X. We define a mapping $f : X/D \times \Gamma \times X/D \to X/D$ as follows: for each $(D[x],\ \alpha,\ D[y]) \in X/D \times \Gamma \times X/D$,

$$f(D[x],\ \alpha,\ D[y]) = D[x]\alpha D[y] = D[x\alpha y].$$

Then X/D is a bounded Γ-BCK-algebra with $D[0]$ and $D[1]$ as the zero element and the unit respectively. In this case, X/D is called the *Γ-BCK-algebra by D.*

Proof. It is obvious that f is well-defined.

Let $x,\ y,\ z \in X$ and let $\alpha,\ \beta \in \Gamma$. Then by the axiom ($\Gamma A_1$), we get

$$((x\alpha y)\beta(x\alpha z))\beta(z\alpha y) = 0.$$

Thus we have

$$[(D[x]\alpha D[y])\beta D[x]\alpha D[z])]\beta(D[z]\alpha D[y]) = D[((x\alpha y)\beta(x\alpha z))\beta(z\alpha y)] = D[0].$$

So the axiom (ΓA_1) holds.

Let $x,\ y \in X$ and let $\alpha,\ \beta \in \Gamma$. Then by the axiom ($\Gamma A_2$), $(x\alpha(x\beta y))\alpha y = 0$. Thus we have

$$(D[x]\alpha(D[x]\beta D[y]))\alpha D[y] = D[(x\alpha(x\beta y))\alpha y] = D[0].$$

So the axiom (ΓA_2) holds.

It can be easily proved that the axioms (ΓA_3) and (ΓA_5).

Now suppose $D[x]\alpha D[y] = D[y]\alpha D[x] = D[0]$ for any $x,\ y \in X$ and each $\alpha \in \Gamma$. Then clearly, $x\alpha y \backsim^D 0$ and $y\alpha x \backsim^D 0$. Thus $Nx\alpha y,\ Ny\alpha x \in D$. So $D[x] = D[y]$. Hence the axiom (ΓA_4) holds. Therefore X/D is a Γ-BCK-algebra.

Finally, we prove that $D[1]$ is the unit of X/D. Since 1 is the unit of X, $x\alpha 1 = 0$ for each $x \in X$ and each $\alpha \in \Gamma$. Then we get

$$D[x]\alpha D[1] = D[x\alpha 1] = D[0].$$

Thus $D[1]$ is the unit of X/D. This completes the proof. □

We can define a partial ordering $\leq$ on X/D as follows: for any $x,\ y \in X$ and each $\alpha \in \Gamma$,

$$D[x] \leq D[y] \text{ if and only if } D[x]\alpha D[y] = D[0].$$

Then it is obvious that $D[x] \leq D[1]$ for each $x \in X$.

Proposition 5.19 Let X be a bounded Γ-BCK-algebra, $D \in \mathscr{DI}(X)$ and let $\backsim^D$ be a congruence relation on X. If X is commutative [resp. positive implicative, implicative], then so is X/D.

Proof. The proof is straightforward. □

In order to discuss a relationship between $\mathscr{DI}(X)$ and $\mathscr{DI}(X/D)$, we first deal with some properties.

Lemma 5.20 Let X be a bounded commutative Γ-BCK-algebra, $D \in \mathscr{DI}(X)$ and let $\backsim^D$ be a congruence relation on X. Then $D[1] = D$.

Proof. Let $x \in D[1]$. Then clearly, $x \backsim^D 1$, i.e., $N1\alpha x \in D$ for each $\alpha \in \Gamma$. Since X is commutative and $x \leq 1$, by Theorem 3.17 in [21], $x = 1\beta(1\alpha x) = N1\alpha x$. Thus $x \in D$. So $D[1] \subset D$. Conversely, let $x \in D$. Then clearly, $N1\alpha x = x \in D$ and $Dx\alpha 1 \in D$ for each $\alpha \in \Gamma$. Thus $x \backsim^D 1$. So $x \in D[1]$, i.e., $D \subset D[1]$. Hence $D[1] = D$. □

Lemma 5.21 Let X be a bounded commutative Γ-BCK-algebra, $D,\ D^* \in \mathscr{DI}(X)$ and let $D \subset D^*$. Then $D^{**} = \{D[x] : x \in D^*\} \in \mathscr{DI}(X/D)$.

Proof. It is clear that $1 \in D^*$. Then by the definition of D^{**}, $D[1] \in D^{**}$. Suppose the followings hold: for any $x,\ y \in X$ and any $\alpha,\ \beta \in \Gamma$,

$$D[1]\beta(D[y]\alpha D[x]),\ D[y] \in D^{**}.$$

It is obvious that $D[1]\beta(D[y]\alpha D[x]) = D[1\beta(y\alpha x)] = D[Ny\alpha x]$. Since $D[1]\beta(D[y]\alpha D[x]) \in D^{**}$, $D[Ny\alpha x] \in D^{**}$. Then $Ny\alpha x \in D^*$. By the hypothesis and the definition of D^{**}, $y \in D^*$. Thus by Theorem 5.16, $x \in D^*$. So $D[x] \in D^{**}$. Hence $D^{**} \in \mathscr{DI}(X)$. □

Proposition 5.22 Let X be a bounded commutative Γ-BCK-algebra, $D,\ D^* \in \mathscr{DI}(X)$, let $D \subset D^*$ and let $D[x] \in X/D$. If $D[x] \cap D^* \neq \emptyset$, then $D[x] \subset D^*$.

Proof. Suppose $D[x] \cap D^* \neq \emptyset$. Then there is $y \in D[x] \cap D^*$. Thus $D[x] = D[y]$ and $y \in D^*$. Moreover, $N(y\alpha z) \in D$ for each $z \in D[x] = D[y]$ and each $\alpha \in \Gamma$. Since $D \subset D^*$, $N(y\alpha z) \in D^*$. So by Theorem 5.16, $z \in D^*$. Hence $D[x] \subset D^*$. □

From Proposition 5.22, we obtain the subset D^*/D of X/D defined by

$$D^*/D = \{D[x] \in X/D : x \in D^*\}.$$

Lemma 5.23 Let X be a bounded commutative Γ-BCK-algebra, $D \in \mathscr{DI}(X)$ and let $D^{**} \in \mathscr{DI}(X/D)$. Then the followings hold:

$$D^* = \bigcup\{D[x] \in X/D : D[x] \in D^{**}\} \in \mathscr{DI}(X) \text{ and } D \subset D^*.$$

Proof. Since $D^{**} \in \mathscr{DI}(X/D)$, by Lemma 5.21, $D = D[1] \in D^{**}$. Then by the definition of D^*, it is obvious that $D \subset D^*$ and $1 \in D^*$. Now suppose $Ny\alpha x \in D^*$ and $y \in D^*$ for any $x,\ y \in X$ and each $\alpha \in \Gamma$. Then by the definition of D^*,

$$D[Ny\alpha x] = D[1\beta(y\alpha x)] = D[1]\beta(D[y]\alpha D[x]) \in D^{**} \text{ for each } \beta \in \Gamma \text{ and } D[y] \in D^{**}.$$

Thus by Theorem 5.16, $D[x] \in D^{**}$. So $x \in D^*$. Hence $D^* \in \mathscr{DI}(X)$. □

Let $\mathscr{DI}(X,\ D)$ denote the set of all dual ideals of X containing D. Then we have the following.

Proposition 5.24 Let X be a bounded commutative Γ-BCK-algebra and let $D \in \mathscr{DI}(X)$. Then there is a bijection from $\mathscr{DI}(X,\ D)$ to $\mathscr{DI}(X/D)$.

Proof. Let $h : \mathscr{DI}(X,\ D) \to \mathscr{DI}(X/D)$ be the mapping defined as follows:

$$h(D^*) = D^*/D \text{ for each } D^* \in \mathscr{D}\mathscr{I}(X,\ D).$$

From Lemmas 5.20, 5.21 and 5.23, h is surjective. Assume that there are $A \neq B \in \mathscr{D}\mathscr{I}(X,\ D)$ such that $h(A) = h(B)$. Then there is $x \in A - B$ or $x \in B - A$, say $x \in B - A$. Thus $D[x] \in h(B)$. Since $h(A) = h(B)$, $D[x] \in h(A)$. So there is $y \in A$ such that $D[y] = D[x]$, i.e., $x \backsim^D y$. Hence for each $\alpha \in \Gamma$, we have

$$Nx\alpha y,\ Ny\alpha x \in D.$$

Since $D \subset A$, we get

$$Nx\alpha y,\ Ny\alpha x \in A.$$

Since $y \subset A$ and A is an dual ideal of X, $x \in A$. This contradicts to $x \notin A$. Therefore h is injective. This completes the proof. □

Now we will deal with uniform structures on a Γ-*BCK*-algebra. First of all, let us consider the following notations: for a nonempty set X and any subsets $U,\ V$ of $X \times X$,

(i) $U \circ V = \{(x,\ y) \in X \times X :$ there is $z \in X$ such that $(x,\ z) \in U,\ (z,\ y) \in V\}$,

(ii) $U^{-1} = \{(x,\ y) \in X \times X : (y,\ x) \in U\}$,

(iii) $\triangle = \{(x,\ x) \in X \times X : x \in X\}$.

Definition 5.25 [23] Let X be a nonempty set and let $\mathscr{U}$ be a set of subsets of $X \times X$. Then $\mathscr{U}$ is called a *uniform structure* or *uniformity* on X, if it satisfies the following conditions:

(U_1) if $U \in \mathscr{U}$ and $U \subset V \subset X \times X$, then $V \in \mathscr{U}$,

(U_2) if $U,\ V \in \mathscr{U}$, then $U \cap V \in \mathscr{U}$,

(U_3) $\triangle \subset U$ for each $U \in \mathscr{U}$,

(U_4) if $U \in \mathscr{U}$, then $U^{-1} \in \mathscr{U}$,

(U_5) for each $U \in \mathscr{U}$ there is $V \in \mathscr{U}$ such that $V \circ V \subset U$.

In this case, each member of $\mathscr{U}$ is called an *entourage* of the uniformity on X by $\mathscr{U}$ and $(X,\ \mathscr{U})$ is called a *uniform space*.

Proposition 5.26 Let X be a bounded Γ-*BCK*-algebra and for each $D \in \mathscr{D}\mathscr{I}(X)$, let U_D be the subset of $X \times X$ defined as follows: for each $\alpha \in \Gamma$,

$$U_D = \{(x,\ y) \in X \times X : Nx\alpha y,\ Ny\alpha x \in D\} = \{(x,\ y) \in X \times X : x \backsim^D y\}.$$

Then $\mathscr{U}^* = \{U_D : D \in \mathscr{D}\mathscr{I}(X)\}$ satisfies the conditions (U_2)-(U_5).

Proof. Suppose $U_D,\ U_E \in \mathscr{U}^*$, where $D,\ E \in \mathscr{D}\mathscr{I}(X)$. Let $(x,\ y) \in U_D \cap U_E$. Then clearly, $(x,\ y) \in U_D$ and $(x,\ y) \in U_E$. By the definitions of U_D and U_E, we have the followings: for each $\alpha \in \Gamma$,

$$Nx\alpha y,\ Ny\alpha x \in D \text{ and } Nx\alpha y,\ Ny\alpha x \in E.$$

Thus $Nx\alpha y,\ Ny\alpha x \in D\cap E$. Since $D\cap E \in \mathscr{DI}(X)$, $(x,\ y)\in U_{D\cap E}$. So $U_D \cap U_E \subset U_{D\cap E}$. Similarly, we can prove that $U_{D\cap E} \subset U_D \cap U_E$. Hence $U_D \cap U_E = U_{D\cap E}$. Therefore $\mathscr{U}^*$ satisfies the condition (U_2).

Let $U_D \in \mathscr{U}^*$ and let $(x,\ y)\in \triangle$, where $D \in \mathscr{DI}(X)$. It is obvious that $Nx\alpha x = N0 = 1 \in D$ for each $\alpha \in \Gamma$. Then $(x,\ y)\in U_D$. Thus $\mathscr{U}^*$ satisfies the condition (U_3).

Suppose $U_D \in \mathscr{U}^*$ and let $(x,\ y)\in U_D$, where $D \in \mathscr{DI}(X)$. By the definition of U_D, $x \backsim^D y$. By Lemma 5.17, $y \backsim^D x$. Then $(y,\ x)\in U_D$. Thus $(x,\ y)\in U_D^{-1}$. So $U_D \subset U_D^{-1}$. Similarly, $U_D^{-1} \subset U_D$. Hence $U_D = U_D^{-1}$, i.e., $U_D^{-1} \in \mathscr{U}^*$. Therefore $\mathscr{U}^*$ satisfies the condition (U_4).

Let $U_D \in \mathscr{U}^*$ and let $A = (D_j)_{j\in J}$ be a family of dual ideals of X contained in D, where J denotes an index set. Then clearly, $A \neq \emptyset$, i.e., $\bigcup_{j\in J} D_j \neq \emptyset$. Let $I = [\bigcup_{j\in J} D_j)$. Since $I \in \mathscr{DI}(X)$, $U_I \in \mathscr{U}^*$. Now let $(x,\ y)\in U_I \circ U_I$. Then there is $z \in X$ such that $(x,\ z),\ (z,\ y)\in U_I$. By Lemma 5.17, $(x,\ y)\in U_I$, i.e., for each $\alpha \in \Gamma$,

$$Nx\alpha y,\ Ny\alpha x \in I.$$

Since $\bigcup_{j\in J} D_j \subset D$ and I is the least dual ideal of X containing $\bigcup_{j\in J} D_j$, $I \subset D$. Thus $Nx\alpha y,\ Ny\alpha x \in D$. So $(x,\ y)\in U_D$. Hence $U_I \circ U_I \subset U_D$. Therefore $\mathscr{U}^*$ satisfies the condition (U_5). This completes the proof. □

Proposition 5.27 Let X be a bounded Γ-BCK-algebra and let $\mathscr{U}^*$ be the class given in Proposition 5.26. Then $\mathscr{U} = \{U \subset X\times X : \exists\, U_D \in \mathscr{U}^* \text{ such that } U \supset U_D\}$ is a uniform structure on X and thus $(X,\ \mathscr{U})$ is uniform space.

Proof. From Proposition 5.26, we can obtain our result. □

For each $x\in X$ and each $U \in \mathscr{U}$, let us a subset $U[x]$ of X and a set $\mathscr{B}(x)$ of subsets of X defined as follows:

$$U[x] = \{y \in X : (x,\ y)\in U\} \text{ and } \mathscr{B}(x) = \{U[x] : U \in \mathscr{U}\}.$$

Then we have the following consequence.

Proposition 5.28 Let X be a bounded Γ-BCK-algebra and let $\mathscr{U}$ be the class given in Proposition 5.27. Then there is a unique topology τ on X such that for each $x\in X$, $\mathscr{B}(x)$ is the neighborhood filter of x in $(X,\ \tau)$, i.e., $\mathscr{B}(x) = N_\tau(x)$. In fact, $\tau = \{V \subset X : \forall x \in V,\ \exists U \in \mathscr{U} \text{ such that } U[x]\subset V\}$ and $(X,\ \tau)$ is a bounded $T\Gamma$-BCK-algebra.

In this case, τ is called the *topology on X induced by $\mathscr{U}$*.

Proof. We prove $\mathscr{B}(x)$ satisfies the conditions (N_1)-(N_4) of the neighborhood filter of $x \in X$. The proofs of the conditions (N_1), (N_2) and (N_3) follow immediately from fact that $\mathscr{U}$ satisfies the conditions (U_1), (U_2) and (U_3). Let $V \in \mathscr{U}$. By (U_5), there is $W \in \mathscr{U}$ such that $W\circ W \subset V$. Let $(x,\ z)\in circW$. Then there is $y\in X$ such that $(x,\ y),\ (y,\ x)\in W$. Since $W\circ W\subset V$, $(x,\ z)\in V$. Thus $W[y]\subset V[x]$ for each $y\in W[x]$. So $V[x]\in \mathscr{B}(x)$ for each $y \in W[x]$. Hence $\mathscr{B}(x)$ satisfies all the conditions of the neighborhood filter of $x\in X$. Moreover, it can be easily proved that $\mathscr{B}(x) = N_\tau(x)$. This completes the proof. □

6. Conclusions

We obtained some of topological properties on Γ-BCK-algebras. We proposed the concept of quotient Γ-BCK-algebras by ideals and discussed topological structures on quotient Γ-BCK-algebras. Also, we defined a quotient Γ-BCK-algebra by dual ideals of a bounded Γ-BCK-algebra and gave a uniform structure on quotient Γ-BCK-algebras by dual ideals. In the future, We will study the definitions and properties of various types of Γ-algebras. Furthermore, we will investigate the topological structures on Γ-algebras and the quotient Γ-algebra by ideals in the sense of Khalaf and Ahmed [27].

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Conflict of interest

The authors declare no competing financial interest.

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