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Research Article

The Relationship between Model Spaces and Invariant Subspaces of the Unilateral Shift Operator

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Abstract: Let u be an inner function and K_u^2 be the model space. For inner function v, the space vH^2 is the invariant subspace of the unilateral shift operator on H^2. In this article, the relationship between model spaces K_u^2 and invariant subspaces vH^2 of the unilateral shift operator is discussed from perspectives of the Toeplitz kernels $\ker T_{\overline{u}v}$ $(v \neq u)$, the spectrum of u and v, the left invertible property of $T_{\overline{u}v}$, the minimal isometric dilation and the completeness problem. We obtain that the Toeplitz operator T_u on H^2 is a minimal isometric dilation of A_u^v defined on the model space K_v^2 if and only if $K_u^2 \cap vH^2 = \{0\}$. Moreover, $K_u^2 \cap vH^2 = \{0\}$ when $(\sigma(u) \cap \mathbb{T}) \subset (\sigma(v) \cap \mathbb{T})$.

***Keywords*:** model spaces, invariant subspaces, Toeplitz kernels, minimal isometric dilations

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1. Introduction

Let $\mathbb{D} = \{\lambda \in \mathbb{C}: |\lambda| < 1\}$ denote the open unit disk in the complex plane $\mathbb{C}$ and $\mathbb{T}$ denote the unit circle. Ues L^2 to denote the Hilbert space with the inner product

$$\|h\|^2 = \int_{\mathbb{T}} |h(t)|^2 dm(t),\ h \in L^2,$$

and it is finite, where dm is the Lebesgue measure on $\mathbb{T}$. The Banach space of all functions in L^2 essentially bounded on $\mathbb{T}$ is denoted L^∞. Hardy spaces H^2 consist of all analytic functions f on $\mathbb{D}$ having square-summable Taylor coefficients at 0. The space H^∞ consists of all bounded holomorphic functions in $\mathbb{D}$ with the norm

$$\|h\|_\infty = \sup_{z \in \mathbb{D}} |h(z)|.$$

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An analytic function u is called an inner function if $|u| = 1$ a.e. on $\mathbb{T}$. Hilbert spaces of the form $K_u^2 = H^2 \ominus uH^2$ called mode spaces, which are proper nontrivial invariant subspaces of the backward shift S^*, given by

$$(S^*g)(z) = \frac{g(z) - g(0)}{z},$$

on H^2.

Toeplitz operators are compressions to H^2 of multiplication operators on L^2. For $f \in L^\infty$, a Toeplitz operator T_f is defined on H^2 by

$$T_f h = P(fh), \ h \in H^2,$$

where P denotes the orthogonal projection from L^2 onto H^2. The function f called the symbol of T_f.

Truncated Toeplitz operators A_f^u are compressions of Toeplitz operators T_f to K_u^2. For $f \in L^\infty$, A_f^u induced by f and u is defined by

$$A_f^u \phi = P_u(f\phi), \ \phi \in K_u^2,$$

where P_u denotes the orthogonal projection from L^2 onto K_u^2. Clearly, $(A_f^u)^* = A_{\overline{f}}^u$. More additional detail of truncated Toeplitz operators can be found in the paper initiated by Sarason [1].

Based on the view of operator theory, truncated Toeplitz operators represent the scalar version of the Sz.-Nagy and Foias general theory of contractions in a Hilbert space [2]. In paper [3], authors proved that $I - (A_\varphi^u)^* A_\varphi^u$ has a finite rank n if and only if the symbol φ is a finite Blaschke product B_n of degree n, where u is a nontrivial inner function and K_u^2 is an infinite dimension model space, and $\varphi \in K_u^2 \cap L^\infty$ with $\|\varphi\|_\infty \leq 1$ such that $I - (A_\varphi^u)^* A_\varphi^u \neq 0$ and the symbol of A_φ^u is unique. From the proof of this conclusion, we find that if $I - A_\varphi^u (A_\varphi^u)^*$ has a finite rank on K_u^2, then

$$K_\varphi^2 \cap uK_\varphi^2 = \{0\}.$$

Since

$$H^2 = K_\varphi^2 \oplus \varphi H^2 \text{ and } K_\varphi^2 \perp u\varphi H^2,$$

we get that

$$K_\varphi^2 \cap uH^2 = K_\varphi^2 \cap u(K_\varphi^2 \oplus \varphi H^2) = K_\varphi^2 \cap (uK_\varphi^2 \oplus u\varphi H^2) = \{0\}.$$

It follows that $K_\varphi^2 \cap uH^2 = \{0\}$ when $I - A_\varphi^u (A_\varphi^u)^*$ has a finite rank on K_u^2. Therefore, we find that the relationship between model spaces K_u^2 and invariant subspaces vH^2 of the unilateral shift operator is a very interesting problem.

Although $K_u^2 \perp uH^2$, but the relationship between K_u^2 and vH^2 $(v \neq u)$ is complicated. The relationship between K_u^2 and vH^2 is related to invariant subspaces of truncated Toeplitz operators (see Remark 1 in the Subsection 2.1), kernels spaces of Toeplitz operators on Hardy space (see Lemma 1 in the Subsection 2.1), the minimum isometric dilation of truncated Toeplitz operators (see Theorem 2 in the Subsection 2.4) and the completeness problem which means that a sequence $\{k_\lambda\}_{\lambda\in\wedge}$ of kernel functions for Hardy spaces forms a Riesz basis for K_u^2 (see Lemma 7 in the Subsection 2.5). From the above related perspectives, we are devoted to using properties and the structure of inner functions to describe the intersection of K_u^2 and vH^2, and discuss which properties of inner functions affect the intersection of K_u^2 and vH^2.

For inner functions u, v with $v \neq u$, in our paper, we discuss the relationship between K_u^2 and vH^2 by kernels spaces of Toeplitz operators, the minimum isometric dilation of Toeplitz operators and the completeness problem. That is, use the following equivalence relations to discuss the relationship between K_u^2 and vH^2.

1. $K_u^2 \cap vH^2 = \{0\}$.
2. $\ker T_{\bar{u}v} = \{0\}$.
3. $K_\wedge = K_u^2$, where $\wedge \subset \mathbb{D}$ means a Blaschke sequence and a infinite Blaschke product v with the zero set $\wedge$. Use $K_\wedge$ to denote the space spanned by

$$\left\{k_\lambda = \frac{1}{1-\overline{\lambda}z}, \lambda \in \wedge\right\}.$$

4. $\overline{\operatorname{ran} T_{\bar{v}u}} = H^2$. In fact, $(\ker T_{\bar{u}v})^\perp = \overline{\operatorname{ran} T_{\bar{v}u}}$.
5. The minimal isometric dilation of A_u^v defined on K_v^2 is a Toeplitz operator T_u defined on H^2.

Combining the above research ideas, this paper is organized as follows. In Section 2, we study the relationship between K_u^2 and vH^2 and consider the intersection of K_u^2 and vH^2 in particular. In Subsection 2.1, we obtain that $K_u^2 \cap vH^2 = v\ker T_{\bar{u}v}$; In Subsection 2.2, we know that $K_u^2 \cap vH^2 = \{0\}$ when $(\sigma(u)\cap\mathbb{T}) \subset (\sigma(v)\cap\mathbb{T})$; In Subsection 2.3, use the left invertible property of $T_{\bar{u}v}$ to discuss the relationship between K_u^2 and vH^2; In Subsection 2.4, the Toeplitz operator T_u on H^2 is a minimal isometric dilation of A_u^v defined on K_v^2 if and only if $K_u^2 \cap vH^2 = \{0\}$; In Subsection 2.5, use the completeness problem to study the relationship between K_u^2 and vH^2.

2. The relationship between K_u^2 and vH^2

For kernel spaces of Toeplitz operators, Coburn's Theorem (see Proposition 7.24 in [4]) claimed that either $\ker T_g = \{0\}$ or $\ker T_g^* = \{0\}$ for $g \in L^\infty$. Whenever $\ker T_g \neq \{0\}$, it will switch to some description about kernel spaces of Toeplitz operators. For the structure of Toeplitz kernels, consider a special case of Toeplitz operators T_f with symbols of the form $f = \overline{\theta}$, where θ is some inner function. Then clearly, $\ker T_{\overline{\theta}} = K_\theta^2$. Does there exist some analogous characterization for the kernels of Toeplitz operators with general symbols? Hayashi's results [5] play a crucial role. Whenever $\ker T_g \neq \{0\}$, then $\ker T_g = \varphi K_\eta^2$, where φ is an outer function and η is an inner function with $\eta(0) = 0$, and further, multiplication by φ acts isometrically on $\ker T_g$. Not hard to get that the Toeplitz kernel is nearly S^*-invariant by $S^*T_gS = T_g$. Hitt [6] showed some description about nearly S^*-invariant subspaces. Later, Sarason [7] gave some new proof of Hitt's theorem by the de Branges-Rovnyak spaces. More research details of Toeplitz kernels can refer to [8].

As we have seen, the class of kernel spaces of Toeplitz operators, which includes the class of model spaces, can itself be described in terms of model spaces. Moreover, model spaces and Toeplitz kernels have a number of important connections. Some classical results about the connection between Toeplitz kernels and model spaces are given in [8].

Above introductions, we can discuss the relationship between K_u^2 and vH^2 with the help of the Toeplitz kernel $\ker T_{\bar{u}v}$, and what kind of information on the relationship between K_u^2 and vH^2 can be deduced from the kernels of Toeplitz operators.

2.1 *In terms of the Toeplitz kernel* $\ker T_{\bar{u}v}$

As is known to all that K_u^2 has a natural conjugation C, antiunitary, involution operator, defined by

$$Cg = \overline{zg}u,\ g \in K_u^2.$$

Using the conjugation, model spaces have a very important property and it is frequently used in subsequent proofs, that is,

$$K_u^2 = H^2 \cap u\overline{zH^2}.$$

For the sake of completeness of the article, we give the following proof of possible well-known conclusion.

Lemma 1 Let u, v be inner functions, then

$$\ker T_{\overline{u}v} = \{g \in K_u^2 \colon\ vg \in K_u^2\}. \tag{1}$$

Moreover, $\ker T_{\overline{u}v} \neq \{0\}$ if and only if $vH^2 \cap K_u^2 \neq \{0\}$.

Proof. Setting $\mathscr{G} = \{g \in K_u^2 \colon\ vg \in K_u^2\}$. For a nonzero function $\varphi \in \mathscr{G} \subset K_u^2$, because K_u^2 has a conjugation C, there is $\psi \in K_u^2$ such that

$$v\varphi = u\overline{z\psi}.$$

Then

$$T_{\overline{u}v}\varphi = P(\overline{u}v\varphi) = P(\overline{u}u\overline{z\psi}) = P(\overline{z\psi}) = 0.$$

It follows that

$$\varphi \in \ker T_{\overline{u}v} \text{ and } \mathscr{G} \subseteq \ker T_{\overline{u}v}.$$

For a nonzero function $g \in \ker\, T_{\overline{u}v}$, we have that $T_{\overline{u}v}g = P(\overline{u}vg) = 0$. There is $h \in H^2$ such that

$$\overline{u}vg = \overline{zh}. \tag{2}$$

From this, we can obtain that $vg = u\overline{zh} \in H^2$. In terms of

$$K_u^2 = u\overline{zH^2} \cap H^2, \tag{3}$$

we know that $vg \in K_u^2$. By the equality (2), we get that $g = u\overline{zhv} \in H^2$. Then, by the equality (3), $g \in K_u^2$ and $g \in \mathscr{G}$. Thus $\ker T_{\overline{u}v} \subseteq \mathscr{G}$.

Suppose that $vH^2 \cap K_u^2 \neq \{0\}$. There is $0 \neq \phi \in H^2$ such that

$$v\phi \in K_u^2.$$

Because K_u^2 carries a conjugation, we can find a nonzero function $f \in K_u^2$ such that

$$v\phi = uz\overline{f},$$

and

$$T_{\overline{u}v}\phi = P(\overline{u}v\phi) = P(\overline{u}uz\overline{f}) = 0.$$

Therefore,

$$\phi \in \ker T_{\overline{u}v} \text{ and } \ker T_{\overline{u}v} \neq \{0\}.$$

By the equatity (1), we can easy to get that $vH^2 \cap K_u^2 \neq \{0\}$ when $\ker T_{\overline{u}v} \neq \{0\}$. The proof is completed. □

Proposition 1 Let u, v be inner functions satisfied $\ker T_{\overline{u}v} \neq 0$, then

$$K_u^2 \cap vH^2 = v\ker T_{\overline{u}v}.$$

Proof. For any nonzero $f \in K_u^2 \cap vH^2$, there exists a function $g \in H^2$ such that $f = vg$. This implies that $\overline{v}f \in H^2$. Since $K_u^2 = CK_u^2 = uz\overline{K_u^2}$ and $f \in K_u^2$, we can find a function $\eta \in K_u^2$ such that $f = uz\overline{\eta}$. Then

$$T_{\overline{u}v}(\overline{v}f) = P(\overline{u}v\overline{v}f) = P(\overline{u}f) = P(\overline{u}uz\overline{\eta}) = 0.$$

Thus

$$K_u^2 \cap vH^2 \subseteq v\ker T_{\overline{u}v}.$$

For any nonzero $h \in \ker T_{\overline{u}v}$, by the Lemma 1, we know that $vh = \psi \in K_u^2$. It follows that

$$\psi \in K_u^2 \cap vH^2 \text{ and } h = \overline{v}\psi \in \overline{v}(K_u^2 \cap vH^2).$$

Thus

$$v\ker T_{\overline{u}v} \subseteq K_u^2 \cap vH^2.$$

The proof is completed. □

The greatest common divisor of inner functions θ_1 and θ_2 is denoted by $GCD(\theta_1, \theta_2)$, it's unique if it's different by a constant multiple.

Lemma 2 [Lemma 3.6 in [9]] Let u, θ be non-constant inner functions having a nontrivial greatest common divisor, denoted by $GCD(u, \theta) = v$ and $u = vu_1$, then

$$K_u^2 \cap \theta H^2 \subseteq vK_{u_1}^2,$$

where v, u_1 are inner functions.

Remark 1 When $GCD(u, \theta) = \theta$, it is not difficult to find that $u = \theta u_1$ and

$$K_u^2 \cap \theta H^2 = \theta K_{u_1}^2.$$

It is an invariant subspace of the truncated Toeplitz operator A_z^u defined on K_u^2.

Corollary 1 If u, θ are nontrivial inner functions with $GCD(u, \theta) = \eta$ and $u = \eta u_1$, where η is a nontrivial inner function, then

$$\ker T_{\overline{u}\theta} \subseteq \overline{\theta}\eta K_{u_1}^2. \tag{4}$$

Proof. By Proposition 1 and Lemma 2, we know that $\ker T_{\overline{u}\theta} \neq 0$ and

$$\ker T_{\overline{u}\theta} = \overline{\theta}(K_u^2 \cap \theta H^2) \subseteq \overline{\theta}\eta K_{u_1}^2.$$

□

Remark 2 When $GCD(u, \theta) = \theta$, the equal sign of formula (4) holds. In fact, by $GCD(u, \theta) = \theta$ and $u = u_1\theta$, we get that $\ker T_{\overline{u_1}} = \ker T_{\overline{u}\theta}$. Since $\ker T_{\overline{u_1}} = K_{u_1}^2$, we have that

$$\ker T_{\overline{u_1}} = \ker T_{\overline{u}\theta} = K_{u_1}^2 = \overline{\theta}\theta K_{u_1}^2.$$

2.2 *In terms of the spectrum of inner functions u and v*

For a inner function $u = B_\wedge S_\mu$, where $B_\wedge$ is a Blaschke product having the zero set $\wedge$ and S_μ is a singular inner function with corresponding singular measure μ, then the spectrum of u is the set

$$\sigma(u) = \wedge^- \cup \text{supp } \mu.$$

Use $\wedge^-$ to denote the closure of the zero set of u and supp μ to denote the support set of singular measure μ about S_μ. More details reference section 6.2 in [10].

Lemma 3 [Proposition 6.9 in [10]] Each ϕ in K_u^2 has an analytic continuation across $\widehat{\mathbb{C}} \setminus \{\frac{1}{\overline{w}}: w \in \sigma(u)\}$, where $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

Lemma 4 [Theorem 6.1 and 6.2 in [11]] The inner function u can be analytically continued to $\mathbb{T} \setminus \sigma(u)$.

Proposition 2 Let v and u be inner functions with $v \neq u$. If $K_u^2 \cap vH^2 \neq 0$, then

$$(\sigma(v)\cap\mathbb{T})\subset(\sigma(u)\cap\mathbb{T}).$$

Proof. Since $K_u^2\cap vH^2\neq 0$, by Lemma 1, we know that $\ker T_{\overline{u}v}\neq 0$. There is $0\neq h\in\ker T_{\overline{u}v}$ such that

$$T_{\overline{u}v}h=0.$$

Then $P(\overline{u}vh)=0$. There is a nonzero function $g\in H^2$ such that $\overline{u}vh=\overline{zg}$ and

$$vh=u\overline{zg}.$$

Since $K_u^2=H^2\bigcap u\overline{zH^2}$, we obtain that $vh\in K_u^2$. By Lemma 3, we have that vh has an analytic continuation across $\widehat{\mathbb{C}}\setminus\{\frac{1}{\overline{z}}: z\in\sigma(u)\}$. The Lemma 4 implies that v can be analytically continued to $\mathbb{T}\setminus\sigma(v)$. Thus, $(\sigma(v)\cap\mathbb{T})\subset(\sigma(u)\cap\mathbb{T})$. □

Corollary 2 Let v and u be inner functions with $v\neq u$. If $(\sigma(u)\cap\mathbb{T})\subset(\sigma(v)\cap\mathbb{T})$, then $K_u^2\cap vH^2=\{0\}$.

Proof. Suppose that $K_u^2\cap vH^2\neq\{0\}$. By Lemma 1,we know that $\ker T_{\overline{u}v}\neq 0$, and

$$\ker T_{\overline{u}v}=\{f\in K_u^2: vf\in K_u^2\}.$$

Then vf has the same analytic continuation across $\mathbb{T}\setminus\sigma(u)$. Since v has the analytic continuation across $\mathbb{T}\setminus\sigma(v)$, we get that $(\sigma(v)\cap\mathbb{T})\subset(\sigma(u)\cap\mathbb{T})$. It is the contradiction. Thus we prove that $K_u^2\cap vH^2=\{0\}$ if $(\sigma(u)\cap\mathbb{T})\subset(\sigma(v)\cap\mathbb{T})$. □

2.3 *In terms of the left invertible property of* $T_{\overline{u}v}$

We use $\mathrm{dist}(\varphi, H^\infty)$ to denote the distance between the function φ and the set H^∞. That is,

$$\mathrm{dist}(\varphi, H^\infty)=\inf_{\phi\in H^\infty}\|\varphi-\phi\|_\infty.$$

Lemma 5 [Theorem 7.30 in [4]] If φ is a unimodular in L^∞, then the operator T_φ is left invertible if and only if $\mathrm{dist}(\varphi, H^\infty)<1$.

Theorem 1 Let v, u be inner functions with $v\neq u$. If $K_u^2\cap vH^2\neq\{0\}$, then

$$\frac{1}{2}\leq\sup_{z\in\mathbb{D}}|\mathrm{Im}u(z)|<1,\ \frac{1}{2}\leq\sup_{z\in\mathbb{D}}|\mathrm{Re}u(z)|<1 \text{ and } \frac{\sqrt{3}}{2}\leq\sup_{z\in\mathbb{D}}|\mathrm{Re}u(z)-1|\leq 2,$$

where $\mathrm{Re}u(z)$ denotes the real part of $u(z)$ and $Imu(z)$ denotes the imaginary part of $u(z)$.

Proof. Since $K_u^2\cap vH^2\neq 0$, by Lemma 1, we get that $\ker T_{\overline{u}v}\neq 0$, and $T_{\overline{u}v}$ must be not left invertible. Then, by Lemma 5,

$$\operatorname{dist}(\overline{u}v, H^\infty) \geq 1.$$

That is,

$$\inf_{h\in H^\infty} \|\overline{u}v - h\|_\infty \geq 1.$$

It follows that

$$\|\overline{u}v - h\|_\infty \geq 1 \text{ for any } h \in H^\infty. \tag{5}$$

So $\|\overline{u}v - uv\|_\infty \geq 1$. That is,

$$\sup_{z\in\mathbb{D}} |(\overline{u}v)(z) - (uv)(z)| \geq 1. \tag{6}$$

Since B is an inner function and $|v(z)| < 1$, by (6), we have that

$$\sup_{z\in\mathbb{D}} |\overline{u}(z) - u(z)| \geq 1. \tag{7}$$

Setting $u(z) = \mathrm{Re}u(z) + iImu(z)$. The inequality (7) follows that $\sup\limits_{z\in\mathbb{D}} |Imu(z)| \geq \frac{1}{2}$. Since $|Imu(z)| < 1$, we prove that $\frac{1}{2} \leq \sup\limits_{z\in\mathbb{D}} |Imu(z)| < 1$.

By (5), we also obtain that

$$1 \leq \|\overline{u}v + uv\|_\infty = \sup_{z\in\mathbb{D}} |(\overline{u}v)(z) + (uv)(z)| = \sup_{z\in\mathbb{D}} |v(z)||\overline{u}(z) + u(z)|.$$

Since v is an inner function and $|v(z)| < 1$, we get that $\sup\limits_{z\in\mathbb{D}} |\overline{u}(z) + u(z)| \geq 1$. Then

$$1 \leq \sup_{z\in\mathbb{D}} |\overline{u}(z) + u(z)| = \sup_{z\in\mathbb{D}} |\mathrm{Re}u(z) - iImu(z) + \mathrm{Re}u(z) + iImu(z)|.$$

Thus $\sup\limits_{z\in\mathbb{D}} |\mathrm{Re}u(z)| \geq \frac{1}{2}$. Since $|\mathrm{Re}u(z)| < 1$, we prove that $\frac{1}{2} \leq \sup\limits_{z\in\mathbb{D}} |\mathrm{Re}u(z)| < 1$.

By (5), we know that

$$1 \leq \|\overline{u}v - v\|_\infty = \sup_{z\in\mathbb{D}} |(\overline{u}v)(z) - v(z)| = \sup_{z\in\mathbb{D}} |v(z)||\overline{u}(z) - 1|.$$

Since v is an inner function and $|v(z)| < 1$, we have that $\sup_{z\in\mathbb{D}} |\overline{u}(z) - 1| \geq 1$. Then

$$\sup_{z\in\mathbb{D}} |\overline{u}(z) - 1|^2 \geq 1.$$

That is,

$$\begin{aligned} 1 \leq \sup_{z\in\mathbb{D}} |\overline{u}(z) - 1|^2 &= \sup_{z\in\mathbb{D}} |\mathrm{Re}u(z) - iImu(z) - 1|^2 \\ &= \sup_{z\in\mathbb{D}} [(\mathrm{Re}u(z) - 1)^2 + (Imu(z))^2] \\ &= \sup_{z\in\mathbb{D}} (\mathrm{Re}u(z) - 1)^2 + \sup_{z\in\mathbb{D}} (Imu(z))^2. \end{aligned}$$

By $\sup_{z\in\mathbb{D}} |Imu(z)| \geq \frac{1}{2}$, we know that

$$\sup_{z\in\mathbb{D}} (\mathrm{Re}u(z) - 1)^2 \geq \frac{3}{4}.$$

Thus $\frac{\sqrt{3}}{2} \leq \sup_{z\in\mathbb{D}} |\mathrm{Re}u(z) - 1| \leq 2$. The proof is completed. □

2.4 *In terms of the minimal isometric dilation*

Let $\mathscr{H}$ be a Hilbert space. The set of all bounded linear operators on $\mathscr{H}$ is denoted $\mathscr{L}(\mathscr{H})$. Use $P_{\mathscr{H}}$ to represent the orthogonal projection onto $\mathscr{H}$. For an operator $A \in \mathscr{L}(\mathscr{H})$, an isometric dilation of A is an isometric operator $T \in \mathscr{L}(\mathscr{K})$, with $\mathscr{K} \supset \mathscr{H}$, such that

$$P_{\mathscr{H}} T^n|_{\mathscr{H}} = A^n$$

for any $n \in \mathbb{N}$. If

$$A = P_{\mathscr{H}} T|_{\mathscr{H}} \text{ and } T\mathscr{H}^{\perp} \subset \mathscr{H}^{\perp},$$

then T is a dilation. A minimal isometric dilation $T \in \mathscr{L}(\mathscr{K})$ means that

$$\mathscr{K} = \bigvee_{n=0}^{\infty} T^n \mathscr{H}.$$

It's uniquely defined that modulo a unitary isomorphism commuting with the dilations. More details can refer to the book [2].

Lemma 6 [Proposition 3.18 in [10]] When $\{u_k\}_{k\geq 1}$ may be a finite sequence of inner function such that $u = \prod_{k\geq 1} u_k$ exists, then

$$K_u^2 = K_{u_1}^2 \oplus \bigoplus_{n\geq 2} \left(\prod_{k=1}^{n-1} u_k\right) K_{u_n}^2.$$

Moreover, if u, θ are inner functions, then

$$K_{u\theta}^2 = K_u^2 \oplus uK_\theta^2.$$

Theorem 2 Let v, u be two inner functions. Then the following are equivalent.
1. The operator T_u on H^2 is a minimal isometric dilation of A_u^v defined on the model space K_v^2.
2. $K_u^2 \cap vH^2 = \{0\}$.
3. $\ker T_{\overline{u}v} = \{0\}$.

Proof. (2) $\Leftrightarrow$ (3) See Lemma 1.

(1) $\Rightarrow$ (3). Suppose that $\ker T_{\overline{u}v} \neq \{0\}$. There exists $0 \neq f \in H^2$ such that $T_{\overline{u}v} f = 0$. Then we can find $0 \neq \eta \in H^2$ such that

$$vf = u\overline{z\eta}.$$

By $K_u^2 = H^2 \cap u\overline{zH^2}$, we know that $vf = u\overline{z\eta} \in K_u^2$. Then

$$vf = u\overline{z\eta} \in K_u^2 \cap vH^2.$$

It follows that

$$\langle vf, u^k g\rangle = 0, \quad \text{for } g \in K_v^2,\ k = 0, 1, 2, \cdots,$$

which implies

$$H^2 \neq \bigvee_{k=0}^{\infty} T_{u^k} K_v^2$$

and hence T_u on H^2 is not a minimal isometric dilation of A_u^v. It is the contradiction. Thus $\ker T_{\overline{u}v} = \{0\}$.

(3) $\Rightarrow$ (1) (see [12] [Theorem 4.1]). For the sake of completeness, we provide a proof.

Claim: For each positive integer n,

$$K_v^2 + uK_v^2 + \cdots + u^n K_v^2 = K_{u^n v}^2. \tag{8}$$

For $n = 0$, equality (8) is right. Suppose that it is right up to $n-1$, it is left then to prove that

$$K^2_{u^{n-1}v} + u^n K^2_v = K^2_{u^n v}. \tag{9}$$

For any $g + u^n h \in K^2_{u^{n-1}v} + u^n K^2_v,$ where $g \in K^2_{u^{n-1}v}$ and $h \in K^2_v$, for any $\phi \in H^2$, we get that

$$\langle g + u^n h,\, u^n v\phi \rangle = \langle g,\, u^n v\phi \rangle + \langle h,\, v\phi \rangle = 0.$$

Thus

$$K^2_{u^{n-1}v} + u^n K^2_v \subseteq K^2_{u^n v}.$$

On the other hand, by Lemma 6, we have

$$K^2_{u^n v} = K^2_{u^{n-1}v} \oplus u^{n-1} v K^2_u = u^n K^2_v \oplus K^2_{u^n}.$$

Suppose that $f \in K^2_{u^n v}$ orthogonal with $K^2_{u^{n-1}v}$ as well as to $u^n K^2_v$. It obtains that

$$f \in (vu^{n-1} K^2_u) \cap K^2_{u^n}.$$

Thus, there is $g \in K^2_u$ such that $f = vu^{n-1} g$, and also $f \perp u^n H^2$, which means

$$vg \perp uH^2 \text{ or } vg \in K^2_u.$$

It follows that

$$0 = T_{\overline{u}}(vg) = T_{v\overline{u}} g.$$

By $\ker T_{\overline{u}v} = \{0\},$ we know that $g = 0$. Thus $f = 0$ and we prove the equality (9).
Since

$$\left(\bigvee_n K^2_{u^n v} \right)^{\perp} = \bigcap_n u^n v H^2 = \{0\},$$

it follows that

$$H^2 = \bigvee_n T_u^n K_v^2.$$

Therefore T_u is a minimal isometric dilation of A_u^v defined on K_v^2. □

2.5 *In terms of the completeness problem*

Let $\{z_n\}_{n\geq 1}$ be a sequence made up of elements of $\mathbb{D}\setminus\{0\}$, repeated based on multiplicity, and satisfies

$$\sum_{n=1}^{\infty}(1-|z_n|)<\infty,$$

then we call it a Blaschke sequence. If $\wedge\subset\mathbb{D}$ is a sequence, use $K_\wedge$ to denote the space spanned by $\{k_\lambda=\frac{1}{1-\bar{\lambda}z},\ \lambda\in\wedge\}$.

Lemma 7 [Proposition 9.1 in [10]] Let $\wedge\subset\mathbb{D}$ be a Blaschke sequence and B be a Blaschke product with the zero set $\wedge$, then, for any inner function u, $K_\wedge\neq K_u^2$ if and only if

$$\ker T_{\bar{u}B}\neq 0.$$

Proposition 3 For inner function u and Blaschke sequence $\wedge\subset\mathbb{D}$, if B is a Blaschke product with the zero set $\wedge$ having a cluster point in $\mathbb{T}\setminus\ \sigma(u)$, then

$$K_u^2\cap BH^2=\{0\}.$$

Proof.

When $\wedge$ has a cluster point in $\mathbb{T}\setminus\ \sigma(u)$, we claim that $K_\wedge=K_u^2$. Suppose that $K_\wedge\neq K_u^2$. There exists $0\neq\psi\in K_u^2$ such that $\psi\perp K_\wedge$. Then

$$\langle\psi,k_\lambda\rangle=\psi(\lambda)=0 \text{ for any } \lambda\in\wedge.$$

That is, the function ψ vanishes on $\wedge$. Since $\psi\in K_u^2$, we can obtain that ψ can be analytically continued to $\mathbb{T}\setminus\ \sigma(u)$. Since ψ vanishes on $\wedge$ which has a cluster point in $\mathbb{T}\setminus\ \sigma(u)$, the analytic continuation of the nonzero function ψ vanishes on $\wedge$ having a cluster point in $\mathbb{T}\setminus\ \sigma(u)$. It is a contradiction. Thus $K_\wedge=K_u^2$. By Lemmas 1 and 7, we prove that $K_u^2\cap BH^2=\{0\}$. □

Remark 3 If K_u^2 does not contain nonzero functions that vanish on a zero set of the Blaschke product B, then $K_u^2\cap BH^2$ must be trivial.

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Conflict of interest

The authors declare no competing financial interest.

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