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Research Article

A Certain q-Sălăgean Differential Operator and Its Applications to Subclasses of Analytic and Bi-Univalent Functions Involving (p, q)-Chebyshev Polynomial

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Abstract: In the present investigation, we make use of the q- analogue of the Sălăgean differential operator and introduce a new subclass of analytic and bi-univalent functions $S_{\Sigma}^{\eta,\,\mu}$ involving the $(\mathscr{P},\ \mathscr{Q})$-Chebyshev polynomials. Furthermore, we derive coefficient inequalities and obtain the Fekete-Szegö problem for this new function class $f \in S_{\Sigma}^{\eta,\,\mu}$ of functions.

***Keywords*:** bi-univalent functions, $(\mathscr{P}, \mathscr{Q})$-chebyshev polynomials, sălăgean differential operator

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1. Introduction and motivation

Let $\mathscr{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

analytic in the open unit disk $\mathscr{U}$. Also we let $\mathscr{S}$ denote the class of all functions in $\mathscr{A}$ which are univalent in $\mathscr{U}$. The well known example in this class is the Koebe function $k(z)$, defined by

$$k(z) = \frac{z}{(z-1)^2} = z + \sum_{n=2}^{\infty} n z^n.$$

The Bieberbach conjecture about the coefficient of the univalent functions in the unit disk was formulated by Bieberbach [1] in the year 1916. The conjecture states that for every function $f \in S$ given by (1), we have $|a_n| \leq n$,

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for every n. Strictly inequality holds for all n unless f is the Koebe function or one of its rotation. For many years, this conjecture remained as a challenge to mathematicians. After the proof of $|a_3| \leq 3$ by Lowner in 1923, Fekete-Szegö [2] surprised the mathematicians with the complicated inequality

$$|a_3 - \mu a_2^2| \leq 1 + 2\exp\left(\frac{-2\mu}{1-\mu}\right),$$

which holds good for all values $0 \leq \mu \leq 1$. Note that this inequality region was thoroughly investigated by Schaefer and Spencer [3].

For a class functions in $\mathscr{A}$ and a real (or more generally complex) number μ, the Fekete-Szegö problem is all about finding the best possible constant $C(\mu)$ so that $\left|a_3 - \mu a_2^2\right| \leq C(\mu)$ for every function in $\mathscr{A}$.

It is well known that every function $f \in S$ has a function f^{-1}, defined by

$$f^{-1}[f(z)] = z;\ (z \in \mathscr{U}),$$

and

$$f[f^{-1}(w)] = w;\ \ (|w| < r_0(f);\ r_0 f \geq \frac{1}{4}).$$

In fact, the inverse function f^{-1} is given by

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)\, w^4 + \cdots. \tag{2}$$

Several authors worked on Chebyshev polynomial expansion to find coefficient estimates for bi-univalent functions defined in the open unit disk. Still it attracts more attention on researches in this field. Most recent studies of Altınkaya and S. Yalçın [4] motivated us to define the new class of Sakaguchi type function subordinate to $(\mathscr{P},\ \mathscr{Q})$-Chebyshev polynomials.

Let f and g be analytic in the open unit desk $\mathscr{U}$. The function f is subordinate to g written as $f \prec g$ in $\mathscr{U}$, if there exist a function w analytic in $\mathscr{U}$ with $w(0) = 0$ and $|w(z)| < 1;\ (z \in \mathscr{U})$ such that $f(z) = g(w(z)),\ (z \in \mathscr{U})$.

For any integer $n \geq 2$ and $0 < \mathscr{Q} < \mathscr{P} \leq 1$, the $(\mathscr{P},\ \mathscr{Q})-$Chebyshev polynomials of the second kind is defined by the following recurrence relations:

$$U_n(x,\ s,\ \mathscr{P},\ \mathscr{Q}) = (\mathscr{P}^n + \mathscr{Q}^n)xU_{n-1}(x,\ s,\ \mathscr{P},\ \mathscr{Q}) + (\mathscr{P}\mathscr{Q})^{n-1} sU_{n-2}(x,\ s,\ \mathscr{P},\ \mathscr{Q}),$$

with the initial values $U_0(x,\ s,\ \mathscr{P},\ \mathscr{Q}) = 1$ and $U_1(x,\ s,\ \mathscr{P},\ \mathscr{Q}) = (\mathscr{P}\mathscr{Q})x$ and s is a variable. In slight view of this recurrence relation, A list of some special cases of the $(\mathscr{P},\ \mathscr{Q})$-Chebyshev polynomials of second kind can be defined see [4]. These polynomials defined recursively over the integers share numerous interesting properties and have been extensively studied. They have been also found to be topics of interest in many different areas of pure and applied science. The generating function of the $(\mathscr{P},\ \mathscr{Q})$-Chebyshev polynomials of the second kind is as follows:

$$G_{\mathscr{P},\ \mathscr{Q}}(z) = \frac{1}{1 - x\mathscr{P}z\eta - x\mathscr{Q}z\eta_{\mathscr{Q}} - s\mathscr{P}\mathscr{Q}z^2\eta_{\mathscr{P},\ \mathscr{Q}}} = \sum_{n=1}^{\infty} U_n(x,\ s,\ \mathscr{P},\ \mathscr{Q})z^n,$$

where $\eta_{\mathscr{Q}} f(z) = f(\mathscr{Q} z)$ are known as Fibonacci operator introduced and studied by [5]. Similarly, the operator $\eta_{\mathscr{P},\, \mathscr{Q}} f(z) = f(\mathscr{P}\mathscr{Q} z)$ was defined in [6].

Now let's review some basic definitions and concepts of the q -calculus, that are helpful in our research. Throughout the paper, we asume that $0 < q < 1$ and

$$\mathbb{N} := \{1,\ 2,\ 3,\ ...\} = \mathbb{N}_0 \backslash \{0\},\ \ \mathbb{N}_0 := \{0,\ 1,\ 2,\ 3,\ ...\}.$$

Definition (See [7, 8]) Let $q \in (0,\ 1)$ and define the q-number $[\lambda]_q$ by

$$[\lambda]_q = \begin{cases} \frac{1-q^\lambda}{1-q}, & \lambda \in \mathbb{C}, \\ \sum_{k=0}^{n-1} q^k = 1 + q + q^2 + ... + q^{n-1}, & \lambda = n \in \mathbb{N}. \end{cases}$$

Definition (See [7, 8]) Let $q \in (0,1)$ and define the q-factorial $[n]_q!$ by

$$[n]_q! = \begin{cases} 1, & n = 0, \\ \prod_{k=1}^{n-1} [k]_q, & n \in \mathbb{N}. \end{cases}$$

Definition (See [7, 8]) In terms of q-numbers, the Jackson q-exponential function e_q^ω is defined by

$$e_q^\omega = \sum_{n=0}^{\infty} \frac{\omega^n}{[n]_q!}.$$

Definition (See [7, 8]) The q-difference operator denoted as $D_q f(z)$ is defined by

$$D_q f(z) = \frac{f(z) - f(qz)}{z(1-q)}, \quad (f \in \mathscr{A},\ z \in \mathscr{U} - \{0\}),$$

and $D_q f(0) = f'(0)$, where $q \in (0,\ 1)$. It can be easily seen that $D_q f(z) \to f'(z)$ as $q \to 1^-$. If $f(z)$ is of the form (1), a simple computation yields

$$D_q f(z) = 1 + \sum_{n=2}^{\infty} \frac{1-q^n}{1-q} a_n z^n, \quad (z \in \mathscr{U}). \tag{3}$$

Definition The q-analogue of Sălăgean differential operator (see [9]) $R_q^m f(z) : \mathscr{A} \to \mathscr{A}$ for $m \in N$, is formed as follows.

$$
\begin{aligned}
lR_q^0 f(z) &= f(z) \\
R_q^1 f(z) &= z(D_q f(z)) \\
&\vdots \\
R_q^m f(z) &= R_q^1\left(R_q^{m-1} f(z)\right).
\end{aligned}
$$

The so-called q-polynomials are a significant and fascinating group of special functions, specifically orthogonal polynomials. They can be found in various disciplines of the natural sciences, such as coding theory, discrete mathematics (graph theory and combinatorics), Eulerian series, elliptic functions, theta functions, continuous fractions, and so on (see [10, 11]), and algebras and quantum groups (see [12–14]). In Srivastava's recently-published survey-cum-expository review article [15], one can find an introductory overview of some important and potential useful developments concerning the Bessel polynomials and the q-Bessel polynomials, as well as a number of other orthogonal polynomials, orthogonal q-polynomials, hypergeometric polynomials, the q-hypergeometric polynomials, and so on.

2. Coefficient bounds and fekete-szegö inequality

In this section, we define the bi-univalent function class $S_{\Sigma}^{\eta,\,\mu}$ associated with the $(\mathscr{P},\,\mathscr{Q})$-Chebyshev polynomials. Then we will derive the $(\mathscr{P},\,\mathscr{Q})$-Chebyshev polynomial bounds for the initial coefficients and determine Fekete-Szegö functional for $f \in S_{\Sigma}^{\eta,\,\mu}$.

Definition A function $f(z) \in \mathscr{A}$ is said to be in the class $S_{\Sigma}^{\eta,\,\mu}$ satisfies the differential inequality if and only if

$$
\left[\frac{(1-\lambda)\, z\,(R_q^m f(z))^{(q)} + \lambda\, z(R_q^{m+k} f(z))^{(q)}}{(1-\lambda)R_q^m f(z) + \lambda R_q^{m+k} f(z)}\right] \prec G_{\mathscr{P},\,\mathscr{Q}}(z), \tag{4}
$$

and

$$
\left[\frac{(1-\lambda)\, z\,(R_q^m g(z))^{(q)} + \lambda\, z(R_q^{m+k} g(z))^{(q)}}{(1-\lambda)R_q^m g(z) + \lambda R_q^{m+k} g(z)}\right] \prec G_{\mathscr{P},\,\mathscr{Q}}(w), \tag{5}
$$

$$
(z \in \mathscr{U};\ m,\ k \in M_0;\ \lambda \geq 0)
$$

where (q) denotes the q-derivative of f as defined in (3) and $g = f^{-1}$.

Theorem Let the function f given by (1) be in the class $S_{\Sigma}^{\eta,\,\mu}$, then

$$|a_2| \leq \frac{(\mathscr{P}+\mathscr{Q})x\sqrt{((\mathscr{P}+\mathscr{Q})x)}}{\mathscr{D}(\mathscr{P}+\mathscr{Q})^2x^2 - 2q\mathscr{C}\left((\mathscr{P}^2+\mathscr{Q}^2)(\mathscr{P}+\mathscr{Q})^2x^2+\mathscr{P}\mathscr{Q}s\right)}, \tag{6}$$

$$|a_3| \leq \frac{(\mathscr{P}+\mathscr{Q})x}{B}\left(1+\left(\frac{\mathscr{C}+\mathscr{D}}{\mathscr{C}}\right)(\mathscr{P}+\mathscr{Q})x\right), \tag{7}$$

where

$$\mathscr{B} = q(1+q)(1+q+q^2)^m(1-\lambda+\lambda(1+q+q^2)^k),$$

$$\mathscr{C} = q(1+q)^{2m}(1-\lambda+\lambda(1+q)^k)^2,$$

$$\mathscr{D} = (1+q)^{2m}q(1+q)(1-\lambda+\lambda(1+q+q^2)).$$

Proof. Suppose that $f \in S_{\Sigma}^{\eta,\,\mu}$. Thus from (4) and (5), we can write

$$\left[\frac{(1-\lambda)\,z(R_q^m f(z))^{(q)}+\lambda\,z(R_q^{m+k} f(z))^{(q)}}{(1-\lambda)R_q^m f(z)+\lambda R_q^{m+k} f(z)}\right] = G_{\mathscr{P},\,\mathscr{Q}}(z), \tag{8}$$

and for the inverse map

$$\left[\frac{(1-\lambda)\,z(R_q^m g(z))^{(q)}+\lambda\,z(R_q^{m+k} g(z))^{(q)}}{(1-\lambda)R_q^m g(z)+\lambda R_q^{m+k} g(z)}\right] = G_{\mathscr{P},\,\mathscr{Q}}(w), \tag{9}$$

for some analytic functions $\phi,\ \varphi$ such that $\phi(0)=\varphi(0)=0$ and $|\phi(z)|<1,\ |\varphi(w)|<1$ for all $z,\ w \in \mathscr{U}$. It is fairly well known that if

$$|\phi(z)| = \left|c_1z+c_2z^2+c_3z^3+\cdots\right| < 1,\ (z \in \mathscr{U}),$$

$$|\varphi(z)| = \left|d_1w+d_2w^2+d_3w^3+\cdots\right| < 1,\ (w \in \mathscr{U}),$$

and it is well known that

$$|c_n| \leq 1,\ |d_n| \leq 1,\ \ n \in \mathscr{N}. \tag{10}$$

From the equalities (8) and (9), we obtain that

$$\left[\frac{(1-\lambda)\,z\,(R_q^m f(z))^{(q)}+\lambda\,z(R_q^{m+k} f(z))^{(q)}}{(1-\lambda)R_q^m f(z)+\lambda R_q^{m+k} f(z)}\right]=U_0(x,\ s,\ \mathscr{P},\ \mathscr{Q})+U_1(x,\ s,\ \mathscr{P},\ \mathscr{Q})\phi(z)$$
$$+U_2(x,\ s,\ \mathscr{P},\ \mathscr{Q})\phi^2(z)+\cdots, \tag{11}$$

and for the inverse map

$$\left[\frac{(1-\lambda)\,z\,(R_q^m g(z))^{(q)}+\lambda\,z(R_q^{m+k} g(z))^{(q)}}{(1-\lambda)R_q^m g(z)+\lambda R_q^{m+k} g(z)}\right]=U_0(x,\ s,\ \mathscr{P},\ \mathscr{Q})+U_1(x,\ s,\ \mathscr{P},\ \mathscr{Q})\phi(w)$$
$$+U_2(x,\ s,\ \mathscr{P},\ \mathscr{Q})\phi^2(w)+\cdots. \tag{12}$$

Thus we write,

$$\left[\frac{(1-\lambda)\,z\,(R_q^m f(z))^{(q)}+\lambda\,z(R_q^{m+k} f(z))^{(q)}}{(1-\lambda)R_q^m f(z)+\lambda R_q^{m+k} f(z)}\right]=1+U_1(x,\ s,\ \mathscr{P},\ \mathscr{Q})c_1 z$$
$$+[U_1(x,\ s,\ \mathscr{P},\ \mathscr{Q})c_2+U_2(x,\ s,\ \mathscr{P},\ \mathscr{Q})c_1^2]z^2+\cdots, \tag{13}$$

and for the inverse map

$$\left[\frac{(1-\lambda)\,z\,(R_q^m g(z))^{(q)}+\lambda\,z(R_q^{m+k} g(z))^{(q)}}{(1-\lambda)R_q^m g(z)+\lambda R_q^{m+k} g(z)}\right]=1+U_1(x,\ s,\ \mathscr{P},\ \mathscr{Q})d_1 w$$
$$+[U_1(x,\ s,\ \mathscr{P},\ \mathscr{Q})d_2+U_2(x,\ s,\ \mathscr{P},\ \mathscr{Q})d_1^2]\,w^2+\cdots. \tag{14}$$

Now, equating the coefficients in (13) and (14), we obtain

$$q(1+q)^m(1-\lambda+\lambda(1+q)^k)a_2=U_1(x,\ s,\ \mathscr{P},\ \mathscr{Q})c_1, \tag{15}$$

$$(1+q)(1+q+q^2)^m(1-\lambda+\lambda(1+q+q^2)^k)a_3-q(1+q)^{2m}$$

$$(1-\lambda+\lambda(1+q)^k)^2a_2^2=U_1(x,\ s,\ \mathscr{P},\ \mathscr{Q})c_2+U_2(x,\ s,\ \mathscr{P},\ \mathscr{Q})c_1^2, \tag{16}$$

and

$$-q(1+q)^m(1-\lambda+\lambda(1+q)^k)a_2=U_1(x,\ s,\ \mathscr{P},\ \mathscr{Q})d_1, \tag{17}$$

$$2(1+q)^{2m}q(1+q)(1-\lambda+\lambda(1+q+q^2))a_2^2-q(1+q)(1+q+q^2)^m$$

$$(1-\lambda+\lambda(1+q+q^2)^k)a_3+q(1+q)^{2m}(1-\lambda+\lambda(1+q)^k)^2a_2^2 \tag{18}$$

$$=U_1(x,\ s,\ \mathscr{P},\ \mathscr{Q})d_2+U_2(x,\ s,\ \mathscr{P},\ \mathscr{Q})d_1^2.$$

From (15) and (17), it is clear that

$$c_1=-d_1. \tag{19}$$

Also squaring and adding of (15) and (17),

$$\frac{2q^2(1+q)^{2m}(1-\lambda+\lambda(1+q)^k)^2}{U_1^2(x,\ s,\ \mathscr{P},\ \mathscr{Q})}a_2^2=c_1^2+d_1^2. \tag{20}$$

Now, by adding (16) and (18), we get

$$\begin{aligned}2(1+q)^{2m}q(1+q)(1-\lambda+\lambda(1+q+q^2))a_2^2=U_1(x,\ s,\ \mathscr{P},\ \mathscr{Q})(c_2+d_2)+\\ U_2(x,\ s,\ \mathscr{P},\ \mathscr{Q})(c_1^2+d_1^2).\end{aligned} \tag{21}$$

Making use of (20) in (21)

$$a_2^2=\frac{U_1^3(x,\ s,\ \mathscr{P},\ \mathscr{Q})(c_2+d_2)}{A_{q,1}}, \tag{22}$$

where

$$A_{q,1}=2(1+q)^{2m}q(1+q)(1-\lambda+\lambda(1+q+q^2))U_1^2(x,\ s,\ \mathscr{P},\ \mathscr{Q})$$

$$-2q^2(1+q)^{2m}(1-\lambda+\lambda(1+q)^k)^2U_2(x,\ s,\ \mathscr{P},\ \mathscr{Q}).$$

From (19) and (21) together with (10), we obtained that

$$|a_2|\leq\frac{(\mathscr{P}+\mathscr{Q})x\sqrt{((\mathscr{P}+\mathscr{Q})x)}}{A_{q,\ 2}} \tag{23}$$

where

$$A_{q,\,2} = 2(1+q)^{2m}q(1+q)(1-\lambda+\lambda(1+q+q^2))(\mathscr{P}+\mathscr{Q})^2x^2$$

$$-2q^2(1+q)^{2m}(1-\lambda+\lambda(1+q)^k)^2\left((\mathscr{P}^2+\mathscr{Q}^2)(\mathscr{P}+\mathscr{Q})^2x^2+\mathscr{P}\mathscr{Q}s\right)$$

In order to estimates the bound on $|a_3|$, we subtract (18) from (16) and we get

$$\begin{aligned}&2q(1+q)(1+q+q^2)^m(1-\lambda+\lambda(1+q+q^2)^k)a_3\\&-\left(2q(1+q)^{2m}(1-\lambda+\lambda(1+q)^k)^2+2(1+q)^{2m}q(1+q)(1-\lambda+\lambda(1+q+q^2))\right)a_2^2\\=&U_1(x,\ s,\ \mathscr{P},\ \mathscr{Q})(c_2-d_2)+U_2(x,\ s,\ \mathscr{P},\ \mathscr{Q})(c_1^2-d_1^2).\end{aligned} \tag{24}$$

Then Eqs. (19), (20) and (22), become

$$a_3 = \frac{U_1(x,\ s,\ \mathscr{P},\ \mathscr{Q})(c_2-d_2)}{2\mathscr{B}} + \left(\frac{\mathscr{C}+\mathscr{D}}{2\mathscr{B}\mathscr{C}}\right)(c_1^2+d_1^2)U_1^2(x,\ s,\ \mathscr{P},\ \mathscr{Q}), \tag{25}$$

where

$$\begin{aligned}&\mathscr{B} = q(1+q)(1+q+q^2)^m(1-\lambda+\lambda(1+q+q^2)^k),\\&\mathscr{C} = q(1+q)^{2m}(1-\lambda+\lambda(1+q)^k)^2,\\&\mathscr{D} = (1+q)^{2m}q(1+q)(1-\lambda+\lambda(1+q+q^2)).\end{aligned} \tag{26}$$

Therefore, from (10), we find that

$$|a_3| \le \frac{(\mathscr{P}+\mathscr{Q})x}{B}\left(1+\left(\frac{\mathscr{C}+\mathscr{D}}{\mathscr{C}}\right)(\mathscr{P}+\mathscr{Q})x\right). \tag{27}$$

This completes the proof of the Theorem. □

In the next theorem, we present the Fekete-Szegö inequality for the family $S_{\Sigma}^{\eta,\mu}$.

Theorem For $0 \le \lambda \le 1$ and $x, \mu \in \mathbb{R}$, let $f \in \mathscr{A}$ be in the family $S_{\Sigma}^{\eta,\mu}$. Then

$$\left|a_3-\mu a_2^2\right| \le \begin{cases} \dfrac{(\mathscr{P}+\mathscr{Q})x}{\mathscr{B}}, & 0 \le |y(\mu)| \le \dfrac{1}{B} \\ \\ (\mathscr{P}+\mathscr{Q})x\,|y(\mu)|, & |y(\mu)| \le \dfrac{1}{\mathscr{B}}. \end{cases}$$

Proof. For any real μ,

$$a_3 - \mu a_2^2 = (1-\mu)a_2^2 + a_3 - a_2^2$$

$$= (1-\mu)\frac{U_1^3(x,\ s,\ \mathscr{P},\ \mathscr{Q})(c_2+d_2)}{A_{q,1}} + \frac{U_1(x,\ s,\ \mathscr{P},\ \mathscr{Q})(c_2-d_2)}{2\mathscr{B}} \tag{28}$$

$$= \frac{U_1(x,\ s,\ \mathscr{P},\ \mathscr{Q})}{2}\left[\left(y(\mu)+\frac{1}{2\mathscr{B}}\right)c_2 + \left(y(\mu)-\frac{1}{2\mathscr{B}}\right)d_2\right],$$

where

$$y(\mu) = \frac{U_1(x,\ s,\ \mathscr{P}, \mathscr{Q})^2(1-\mu)}{\mathscr{D}U_1(x,\ s,\ \mathscr{P},\ \mathscr{Q})^2 + q\mathscr{C}U_2(x,\ s,\ \mathscr{P},\ \mathscr{Q})}, \tag{29}$$

$$\mathscr{B} = q(1+q)(1+q+q^2)^m(1-\lambda+\lambda(1+q+q^2)^k),$$

$$A_{q,1} = 2(1+q)^{2m}q(1+q)(1-\lambda+\lambda(1+q+q^2))U_1^2(x,\ s,\ \mathscr{P},\ \mathscr{Q})$$

$$-2q^2(1+q)^{2m}(1-\lambda+\lambda(1+q)^k)^2U_2(x,\ s,\ \mathscr{P},\ \mathscr{Q}).$$

Hence, we have reached the desired assertion of the Theorem (6),

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} c\dfrac{(\mathscr{P}+\mathscr{Q})x}{\mathscr{B}}, & 0 \leq |y(\mu)| \leq \dfrac{1}{B} \\ \\ (\mathscr{P}+\mathscr{Q})x\,|y(\mu)|, & |y(\mu)| \leq \dfrac{1}{\mathscr{B}}. \end{cases}$$

This completes the proof of the Theorem. □

3. Conclusion

In this paper, we introduced and analysed the bi-univalent function class $S_{\Sigma}^{\eta,\mu}$ associated with the $(\mathscr{P},\ \mathscr{Q})$-Chebyshev polynomials using Sălăgean differential operator. As a result, we obtained the second and third Taylor-Maclaurin coefficients of this class of functions. These results improve the previous estimates, according to the recent studies.

Conflict of interest

Authors declare there is no conflict of interest at any point with reference to research findings.

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