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Research Article

Extended Higher Order Iterative Method for Nonlinear Equations and its Convergence Analysis in Banach Spaces

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Abstract: In this article, a novel higher order iterative method for solving nonlinear equations is developed. The new iterative method obtained from fifth order Newton-Özban method attains eighth order of convergence by adding a single step with only one additional function evaluation. The method is extended to Banach spaces and its local as well as semi-local convergence analysis is done under generalized continuity conditions. The existence and uniqueness results of solution are also provided along with radii of convergence balls. From the numerical experiments, it can be inferred that the proposed method is more accurate and effective in high precision computations than existing eighth order methods. The computation of error analysis and norm of functions demonstrate that proposed method takes a lead over the considered methods.

***Keywords*:** nonlinear equations, Newton's method, order of convergence, error analysis, banach space, convergence

MSC: 65H05, 65G99, 47J25, 49M15

1. Introduction

In numerical analysis, higher order iterative methods have acquired foremost significance for solving nonlinear equations that arise in numerous branches of science and technology [1, 2]. Various researchers have developed a plethora of iterative methods [3-13] for solving nonlinear equations given in the form

$$f(x) = 0, \tag{1}$$

where $f : \mathcal{D} \subset \mathbb{R} \to \mathbb{R}$ is a continuously differentiable nonlinear function and $\mathcal{D}$ is an open interval. Most widely used iterative method is quadratically convergent Newton's method given by

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \quad k = 0, 1, 2, \dots. \tag{2}$$

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Numerous applications in fields of chemical speciation, transportation, chemical engineering, electron theory, the geometric theory of relativistic string, queuing models etc. also give rise to innumerable such equations. But most of the time the transformed nonlinear equations can not be solved using analytical approach. Thus, to find the numerical solution of such equations, iterative methods are taken into consideration. To have an efficient approximation and more accuracy in finding the solution of nonlinear equations of the form (1), current trend is to develop higher order iterative methods. Such methods are of utmost importance as a number of applications in multidisciplinary areas need faster convergence. But maintenance of an equilibrium between the convergence order and operational cost is another important issue at the same time. In order to improve the convergence of Newton's method, various higher-order methods have been proposed by researchers worldwide.

Neta and Johnson [14] has developed eighth order iterative method (NJ8) which is given for $k = 0, 1, 2, \ldots$ as:

$$y_k = x_k - \frac{f(x_k)}{f'(x_k)},$$

$$z_k = x_k - \frac{f(x_k)}{\frac{1}{6}f'(x_k) + \frac{1}{6}f'(y_k) + \frac{2}{3}f'(\eta_k)},$$

$$x_{k+1} = z_k - \frac{f(z_k)}{f'(x_k)}\frac{f'(x_k) + f'(y_k) - f'(\eta_k)}{2f'(y_k) - f'(\eta_k)}, \tag{3}$$

where $\eta_k = x_k - \frac{1}{8}\frac{f(x_k)}{f'(x_k)} - \frac{3}{8}\frac{f(x_k)}{f(y_k)}$.

An eighth order iterative method (Cordero Torregrosa Vassileva Method (CTVM)) has been developed by Cordero et al. [15] given for $k = 0, 1, 2, \ldots$ as:

$$y_k = x_k - \frac{f(x_k)}{f'(x_k)},$$

$$t_k = x_k - \frac{f(x_k) - f(y_k)}{f(x_k) - 2f(y_k)}\frac{f(x_k)}{f'(x_k)},$$

$$z_k = t_k - \frac{f(t_k)}{f'(x_k)}\left(\frac{f(x_k) - f(y_k)}{f(x_k) - 2f(y_k)} + \frac{1}{2}\frac{f(t_k)}{f(y_k) - 2f(t_k)}\right)^2,$$

$$x_{k+1} = z_k - \frac{3(\beta_2 + \beta_3)(z_k - t_k)}{\beta_1(z_k - t_k) + \beta_2(y_k - x_k) + \beta_3(t_k - x_k)}, \tag{4}$$

where $\beta_j \in \mathbb{R} (j = 1, 2, 3)$ and $\beta_2 + \beta_3 \neq 0$.

Dzunić and Petković [16] has developed Ostrowski's type iterative method (Džunić Petković Method (DPM)) given as:

$$y_k = x_k - \frac{f(x_k)}{f'(x_k)},$$

$$t_k = y_k - \frac{f(x_k)}{f(x_k) - 2f(y_k)} \frac{f(y_k)}{f'(x_k)},$$

$$x_{k+1} = t_k - \frac{f(t_k)}{f'(x_k)\left[1 - 2\frac{f(y_k)}{f(x_k)} - \left(\frac{f(y_k)}{f(x_k)}\right)^2\right]\left[1 - \frac{f(t_k)}{f(y_k)}\right]\left[1 - 2\frac{f(t_k)}{f(x_k)}\right]}. \tag{5}$$

A weighted Newton eighth order method (Sharma Sharma Kalra Method (SSKM)) has been developed by Sharma et al. [17] given $k = 0, 1, 2, \ldots$ as:

$$y_k = x_k - \frac{f(x_k)}{f'(x_k)},$$

$$t_k = y_k - \frac{f^2(x_k) + 10f(x_k)f(y_k) + 16f^2(y_k)}{f^2(x_k) + 8f(x_k)f(y_k) + f^2(y_k)} \frac{f(y_k)}{f'(x_k)},$$

$$x_{k+1} = t_k - \left(1 + 2\frac{f(y_k)}{f(x_k)} + 4\frac{f(t_k)}{f(x_k)} + \frac{\frac{f(t_k)}{f(y_k)}}{1 - \frac{f(t_k)}{f(y_k)}}\right) \frac{f(t_k)}{f'(x_k)}. \tag{6}$$

Thukral proposed an eighth order iterative method [18] given for $k = 0, 1, 2, \ldots$ as:

$$y_k = x_k - \frac{f(x_k)}{f'(x_k)},$$

$$t_k = x_k - \frac{f^2(x_k) + f^2(y_k)}{f'(x_k)(f(x_k) - f(y_k))},$$

$$x_{k+1} = t_k - \left(\left(\frac{1 + \mu_k^2}{1 + \mu_k}\right)^2 - 2\mu_k^2 - 6\mu_k^3 + \frac{f(t_k)}{f(y_k)} + 4\frac{f(t_k)}{f(y_k)}\right) \frac{f(t_k)}{f'(x_k)}, \tag{7}$$

where $\mu_k = \frac{f(y_k)}{f(x_k)}$.

Motivated by ongoing research in this direction, the aim of the present work is to propose an eighth order iterative methods to solve nonlinear equations. This is done by adding one step to fifth order iterative method developed by Grau-Sánchez et al. [19]. The order of the proposed method is enhanced from five to eight by only one additional functional evaluation which is the driving force behind the present work. Further, the motivation leads to extension of the proposed method to Banach spaces where the technique of majorizing sequences is utilized to analyze its local and semi-local convergence [20-26]. Various nonlinear equations are solved and comparison results are presented which indicate better performance of the proposed method over the existing ones.

(Q_1) It is worth noticing that all the aforementioned methods (3)-(7) are defined on the real line. There are common limitations in the aforementioned works related to the usage of Taylor series to show the convergence of these methods. Moreover, this approach requires assumptions on the existence of higher order derivatives that do not appear on these methods. Let us consider the motivational example. Define the function $f : D := [-1.4, 1.4] \to \mathbb{R}$ by

$$f(t) = m_1 t^2 \log t + m_2 t^5 + m_3 t^4, \text{ if } t \neq 0, \text{ and}$$

$$= 0, \text{ if } t = 0, \text{ where } m_1 \in \mathbb{R} - \{0\}, \quad m_2, m_3 \in \mathbb{R}$$

and satisfy $m_2 + m_3 = 0$. Then, clearly $t_* = 1 \in D$, and $f(t_*) = 0$. However, $f'''(t)$ is not continuous at $t_* = 0 \in D$. Hence, the results involving methods (3)-(7) cannot guarantee their convergence because all of them require existence of f''' and even higher. However, these methods may converge. Furthermore, there are other limitations:

(Q_2) There are no computable a priori estimates on $\|x_k - x_*\|$. That is we do not know in advance how many iterates should be computed to reach a predecided error tolerance.

(Q_3) The choice of the initial point is hard (i.e. a "shot in dark"). This problem exists, since no radius of convergence is found for these methods.

(Q_4) No isolation of the solution results are given either.

In particular, these concerns exist for our method studied in Section 1. The limitations (Q_1)-(Q_4) restrict the usage of these methods. The novelty of this method is that we address all these concerns positively in Section 3, where the method of Section 2 is extended in the setting of Banach spaces. Notice also that the technique developed in Section 3 is very general. We simply use it on an eighth convergence order method. However, the same technique can be used to extend the applicability of other methods using inverses of linear operators. Hence, this is the motivation and the novelty of this paper. This is the direction of our future research.

The contents of the paper are summarized as under. Section 2 includes the establishment of the eighthorder method and its convergence analysis is discussed. In section 3, the proposed method is extended to Banach spaces and its local and semi-local convergence analysis is provided. In Section 4, numerical examples are figured out to ascertain the theoretical postulates for comparing the proposed methods with the current methods. Section 5 contains the concluding remarks.

2. Development of eighth order method

In this section, we propose a new iterative method for solving the nonlinear equation of the form (1) from fifth-order Newton-Özban composition given by Grau-Sánchez et al. [19]. This method is given as follows:

$$y_k = x_k - f'(x_k)^{-1} f(x_k),$$

$$z_k = x_k - \frac{1}{2}\left(f'(x_k)^{-1} + f'(y_k)^{-1}\right) f(x_k),$$

$$x_{k+1} = z_k - f'(y_k)^{-1} f(z_k). \tag{8}$$

Extension of fifth order method (8) to obtain an eighth order iterative method is done by adding a step in the following manner:

$$y_k = x_k - f'(x_k)^{-1} f(x_k),$$

$$z_k = x_k - \frac{1}{2}\left(f'(x_k)^{-1} + f'(y_k)^{-1}\right) f(x_k),$$

$$w_k = z_k - f'(y_k)^{-1} f(z_k),$$

$$x_{k+1} = w_k - \frac{1}{2}\left(1 + f'(x_k)f'(y_k)^{-1} f'(x_k)f'(y_k)^{-1}\right) f'(x_k)^{-1} f(w_k). \tag{9}$$

where, k = 0, 1, 2, ... and the initial approximation x_0 is suitably chosen. The foremost aim of our study is to develop a novel and efficient eighth-order iterative method. The convergence analysis of the eighth-order method (9) is established in the next theorem.

Theorem 2.1 Let $f : \mathcal{D} \subset \mathbb{R} \to \mathbb{R}$ be a sufficiently differentiable function in an open interval $\mathcal{D}$ and x_0 is a close approximation to its simple root $x_* \in \mathcal{D}$. The iterative method (9) satisfies the following error equation:

$$e_{k+1} = 2\left(2a_2^5 a_3 - a_2^3 a_3^2\right) e_k^8 + O(e_k^9), \tag{10}$$

where $a_k = \left(\frac{f^k(x_*)}{k!\,f'(x_*)}\right)$, for k = 2, 3, ...

Proof. Let $e_k = x_k - x_*$ be the error in k^{th} iterate. Applying Taylor expansion of $f(x_k)$ and $f'(x_k)$ about x_*, we get

$$f(x_k) = f'(x_*)(e_k + a_2 e_k^2 + a_3 e_k^3 + a_4 e_k^4 + a_5 e_k^5 + a_6 e_k^6 + O(e_k^7), \tag{11}$$

$$f'(x_k) = f'(x_*)(1 + 2a_2 e_k + 3a_3 e_k^2 + 4a_4 e_k^3 + 5a_5 e_k^4 + 6a_6 e_k^5 + O(e_k^6). \tag{12}$$

Substituting (11) and (12) in first substep of (9), we get

$$y_k = x_* + a_2 e_k^2 + 2(a_3 - a_2^2)e_k^3 + (4a_2^3 - 7a_2 a_3 + 3a_4)e_k^4 - 2(3a_3^2 + 4a_2^4 + 5a_2 a_4 - 10a_2^2 a_3 - 2a_5)e_k^5$$

$$+(16a_2^5 + 33a_2 a_3^2 - 52a_2^3 a_3 + 28a_2^2 a_4 - 17a_3 a_4 - 13a_2 a_5 + 5a_6)e_k^6 + O(e_k^7).$$

Again, using Taylor expansion about x_* gives,

$$f(y_k) = f'(x_*)\Big[a_2^2 e_k^2 - (a_3 + 2a_2^2)e_k^3 + (3a_4 + 5a_2^3 - 7a_2 a_3)e_k^4$$

$$-(-24a_2^2 a_3 + 12a_2^4 + 10a_2 a_4 + 4a_5 + 6a_3^2)e_k^5$$

$$+(28a_2^5 - 73a_2^3 a_3 + 34a_2^2 a_4 + 37a_2 a_3^2 - 17a_3 a_4 - 13a_2 a_5 + 5a_6)e_k^6 + O(e_k^7)\Big], \tag{13}$$

and

$$f'(y_k) = f'(x_*)\Big[1 + 2a_2^2 e_k^2 - 4a_2(-a_3 + a_2^2)e_k^3 + a_2(8a_2^3 - 11a_2 a_3 + 6a_4)e_k^4$$

$$-4a_2(4a_2^4 + 5a_2 a_4 - 7a_2^2 a_3 - 2a_5)e_k^5$$

$$+2(30a_2^3 a_4 + 6a_3^3 - 34a_2^4 a_3 + 16a_2^5 - 8a_2 a_3 a_4 + 5a_2 a_6 - 13a_2^2 a_5)e_k^6 + O(e_k^7)\Big]. \tag{14}$$

Substituting (11), (12) and (14) in second substep of (9) renders

$$z_k = x_* + \frac{1}{2}\Big[a_3e_k^3 + (2a_2^3 - 3a_2a_3 + 2a_4)e_k^4 + (-8a_2^4 + 15a_2^2a_3 - 6a_3^2 - 4a_2a_4 + 3a_5)e_k^5$$

$$+(20a_2^5 - 55a_2^3a_3 + 37a_2a_3^2 + 16a_2^2a_4 - 17a_3a_4 - 5a_2a_5 + 4a_6)e_k^6 + O(e_k^7)\Big]. \tag{15}$$

Expanding $f(z_k)$ about x_* using Taylor expansion, we get

$$f(z_k) = f'(x_*)\left[\frac{1}{2}a_3e_k^3 + \left(a_2^3 - \frac{3}{2}a_2a_3 + a_4\right)e_k^4 + \left(4a_2^4 - \frac{15}{2}a_2^2a_3 + 3a_3^2 + 2a_2a_4 - \frac{3}{2}a_5\right)e_k^5\right.$$

$$\left. + \left(10a_2^5 - \frac{55}{2}a_2^3a_3 + \frac{75}{4}a_2a_3^3 + 8a_2^2a_4 - \frac{17}{2}a_3a_4 - \frac{5}{2}a_2a_5 + 2a_6\right)e_k^6 + O(e_k^7)\right], \tag{16}$$

Substituting (14) and (16) in the second last substep of (9), we obtain

$$w_k = x_* + a_2^2a_3e_k^5 + \frac{1}{4}(8a_2^5 - 20a_2^3a_3 + 7a_2a_3^2 + 8a_2^2a_4)e_k^6$$

$$+(-12a_2^6 + 26a_2^4a_3 - 16a_2^2a_3^2 - 8a_2^3a_4 + 6a_2a_3a_4 + 3a_2^2a_5)e_k^7$$

$$+\frac{1}{4}(156a_2^7 - 444a_2^5a_3 + 369a_2^3a_3^2 - 36a_2a_3^3 + 128a_2^4a_4 - 200a_2^2a_3a_4$$

$$+20a_2a_4^2 - 44a_2^3a_5 + 34a_2a_3a_5 + 16a_2^2a_6)e_k^8 + O(e_k^9). \tag{17}$$

Expanding $f(w_k)$ about x_* using Taylor expansion, we get

$$f(w_k) = f'(x_*)\Big[a_2^2a_3 + \frac{1}{4}a_2(8a_2^4 - 20a_2^2a_3 + 7a_3^2 + 8a_2a_4)e_k^6$$

$$+a_2(-12a_2^5 + 26a_2^3a_3 - 16a_2a_3^2 - 8a_2^2a_4 + 6a_3a_4 + 3a_2a_5)e_k^7$$

$$+\frac{1}{4}a_2(156a_2^6 - 444a_2^4a_3 - 36a_3^3 + 128a_2^3a_4 + 20a_4^2 + a_2^2(369a_3^2 - 44a_5)$$

$$+34a_3a_5 - 8a_2(25a_3a_4 - 2a_6))e_k^8 + O(e_k^9)\Big], \tag{18}$$

Substituting (12), (14) and (18) in the last substep of (9), we obtain

$$e_{k+1} = 2\left(2a_2^5a_3 - a2^3\,a3^2\right)e_k^8 + O(e_k^9).$$

Thus, the proof is completed. □

3. An extension

There are certain limitations with the local convergence analysis of the previous section.

(L_1) The analysis is provided only for nonlinear equations defined on the real line.

(L_2) The Taylor series expansion technique requires the existence of derivatives such as $f^{(j)}$, $j = 2, 3, ..., 7$ which are not present on the method.

Let us consider the function $f : [-2, 1.5] \to \mathbb{R}$ defined by $f(t) = t^3 \ln t + \sqrt{2}t^5 - \sqrt{2}t^4$ for $t \neq 0$ and $f(t) = 0$ for $t = 0$. It follows by this definition that f''' is unbounded on the interval $[-2, 1.5]$, since f''' is not continuous at $t = 0$. Notice also that $t_* = 1$ solves the equation $f(t) = 0$. Therefore, the results of the previous section cannot guarantee the convergence of the method to t_*. But the method converges to t_* say for $t_0 = 0.9$.

(L_3) There are no error estimates on the distances $\|x_k - x_*\|$ which can be computed a priori. Hence, we do not know in advance how many iterations should be carried out to obtain a desired error tolerance.

(L_4) There are no results on uniqueness of the solution x_*.

(L_5) The more interesting semi-local convergence of the method is not given.

The limitations (L_1)-(L_5) constitute the motivation for writing this section. These problems are positively addressed as follows:

(L_1)$'$ The convergence analysis is carried out for Banach space valued operators.

(L_2)$'$ The convergence conditions involve only f' which is the only derivative appearing on the method.

(L_3)$'$ Upper bounds on the error distances $\|x_k - x_*\|$ are provided which can be computed in advance. Therefore, we do know the number of iterations required to achieve a certain error tolerance.

(L_4)$'$ A certain neighborhood of x_* is determined with only one solution.

(L_5)$'$ The semi-local convergence the method is developed based on majorizing sequences [22]. Both type of analyses rely on the concept of generalized continuity of the derivative [20, 22].

In order to achieve the extensions (L_1)$'$-(L_5)$'$ the method has to be rewritten in a Banach space as follows:

For $\varphi : D \subset B_1 \to B_2$, where the letters B_1 and B_2 denote Banach spaces, D an open and convex set, and φ a continuously differentiable operator in the Fréchet sense. Then, we approximate a solution $x_* \in D$ of the equation

$$\varphi(x) = 0 \tag{19}$$

using the extension of the method for $x_0 \in D$ and each $k = 0, 1, 2, ...$ by

$$y_k = x_k - \varphi'(x_k)^{-1}\varphi(x_k),$$

$$z_k = x_k - \frac{1}{2}\left(\varphi'(x_k)^{-1} + \varphi'(y_k)^{-1}\right)\varphi(x_k),$$

$$w_k = z_k - \varphi'(y_k)^{-1}\varphi(z_k),$$

$$x_{k+1} = w_k - \frac{1}{2}\left(I + \varphi'(x_k)\varphi'(y_k)^{-1}\varphi'(x_k)\varphi'(y_k)^{-1}\right)\varphi'(x_k)^{-1}\varphi(w_k). \tag{20}$$

Clearly, if $B_1 = B_2 = \mathbb{R}$ and $\varphi = f$, the method (20) reduces to method (9) for solving the equation $f(t) = 0$.

Next, we first study the local convergence analysis of the method (20) based on conditions for $T = [0, +\infty)$:

(H_1) There exists a continuous as well as nondecreasing function (continuous and nondecreasing functions (CNF)) $P_0 : T \to \mathbb{R}$ so that the equation $P_0(t) - 1 = 0$ has a smallest solution (SS) $R_0 \in T - \{0\}$. Set $T_0 = [0, R_0)$.

There exist CNF $P : T_0 \to \mathbb{R}$, $q_1 : T_0 \to \mathbb{R}$ so that the equation $q_1(t) - 1 = 0$ has an SS $r_1 \in T_0 - \{0\}$, with

$$q_1(t) = \frac{\int_0^1 P((1-\theta)t)d\theta}{1-P_0(t)}.$$

The equation $P_0(q_1(t)t) - 1 = 0$ has a SS $R_1 \in T_0 - \{0\}$. Set $T_1 = [0, R_1)$.
Define the function

$$\overline{P}(t) = \begin{cases} P((1+q_1(t))t) \\ \text{or} \\ P_0(t) + P_0(q_1(t)t). \end{cases}$$

The equation $q_2(t) - 1 = 0$ has an SS $r_2 \in T_1 - \{0\}$, where $q_2 : T_1 \to \mathbb{R}$ is defined by

$$q_2(t) = \frac{\int_0^1 P((1-\theta)t)d\theta}{1-P_0(t)}$$

$$+\frac{\overline{P}(t)(1+\int_0^1 P_0(\theta t)d\theta)}{(1-P_0(t))(1-P_0(q_1(t)t))}.$$

The equation $P_0(q_2(t)t) - 1 = 0$ has an SS $R_2 \in T_1 - \{0\}$. Set $T_2 = [0, R_2)$.
Define the function

$$\overline{\overline{P}}(t) = \begin{cases} P((q_1(t)+q_2(t))t) \\ \text{or} \\ P_0(q_1(t)t) + P_0(q_2(t)t). \end{cases}$$

The equation $q_3(t) - 1 = 0$ has an SS $r_3 \in T_2 - \{0\}$, where

$$q_3(t) = \left[\frac{\int_0^1 P((1-\theta)q_2(t)t)d\theta}{1-P_0(q_2(t)t)}\right.$$

$$\left.+\frac{\overline{\overline{P}}(t)(1+\int_0^1 P_0(\theta q_2(t)t)d\theta)}{(1-P_0(q_1(t)t))(1-P_0(q_2(t)t))}\right]q_2(t).$$

The equation $P_0(q_3(t)t) - 1 = 0$ has an SS $R_3 \in T_2 - \{0\}$. Set $T_3 = [0, R_3)$.
Define the function

$$\overline{\overline{\overline{P}}}(t) = \begin{cases} P((1+q_3(t))t) \\ \text{or} \\ P_0(t) + P_0(q_3(t)t). \end{cases}$$

The equation $q_4(t) - 1 = 0$ has an SS $r_4 \in T_3 - \{0\}$, where

$$q_4(t) = \left[\frac{\int_0^1 P((1-\theta)q_3(t)t)d\theta}{1-P_0(q_3(t)t)} \right.$$

$$+\frac{\bar{\bar{\bar{P}}}(t)(1+\int_0^1 P_0(\theta q_3(t)t)d\theta)}{(1-P_0(t))(1-P_0(q_3(t)t))}$$

$$\left. +\frac{1}{2(1-P_0(t))(1-P_0(q_1(t)t))}\bar{P}(t)\left(2+\frac{\bar{P}(t)}{1-P_0(q_1(t)t)}\right)\left(1+\int_0^1 P_0(\theta q_3(t)t)d\theta\right)\right] q_3(t).$$

Define the parameter

$$r_* = \min\{r_i\},\ i = 1,2,3,4. \tag{21}$$

The parameter r_* is shown to be a radius of convergence for the method (20) in Theorem 3.1. However, some more conditions are needed. Let $\mathcal{S}(x, \rho)$, $\mathcal{S}[x, \rho]$ denote respectively, the open and closed balls in B_1 with center $x \in B_1$ and of radius $\rho > 0$.

(H_2) There exists an invertible linear operator M so that for each $x \in D$

$$\left\|M^{-1}(\varphi'(x)-M)\right\| \le P_0\left(\|x-x_*\|\right).$$

Set $\mathcal{S}_0 = \mathcal{S}(x_*, R_0) \cap D$.

(H_3) For each $x, y \in \mathcal{S}_0$

$$\left\|M^{-1}(\varphi'(y)-\varphi'(x))\right\| \le P\left(\|y-x\|\right).$$

and

(H_4)

$$\mathcal{S}[x_*, r] \subset D.$$

The conditions (H_1)-(H_4) are sufficient for the local convergence analysis of the method (20).

Theorem 3.1 Suppose that the conditions (H_1)-(H_4) hold. Then, for $x_0 \in \mathcal{S}(x_*, r_*) - \{x_*\}$, the sequence $\{x_k\}$ generated by the method (20) exists in $\mathcal{S}(x_*, r_*)$, stays in $\mathcal{S}(x_*, r_*)$ and is convergent to x_* so that for each $k = 0, 1, 2, ...$

$$\|y_k - x_*\| \le q_1\left(\|x_k - x_*\|\right)\|x_k - x_*\| \le \|x_k - x_*\| < r_*, \tag{22}$$

$$\|z_k - x_*\| \le q_2\left(\|x_k - x_*\|\right)\|x_k - x_*\| \le \|x_k - x_*\|, \tag{23}$$

$$\|w_k - x_*\| \le q_3\left(\|x_k - x_*\|\right)\|x_k - x_*\| \le \|x_k - x_*\|, \tag{24}$$

and

$$\|x_{k+1}-x_*\| \le q_4\left(\|x_k-x_*\|\right)\|x_k-x_*\| \le \|x_k-x_*\|, \tag{25}$$

where the functions q_i are as given previously, and the radius r_* is as defined in (21).

Proof. Mathematical induction is employed to validate the assertions (22)-(25). Let $u \in \mathcal{S}(x_*, r_*)$. Then, the condition (H_2) and (21) give

$$\left\|M^{-1}(\varphi'(u)-M)\right\| \le P_0\left(\|u-x_*\|\right) \le P_0(r_*) < 1. \tag{26}$$

The estimate (26) and the celebrated Banach Lemma on linear invertible operators imply the invertability of $\varphi'(u)$ and

$$\left\|\varphi'(u)^{-1}M\right\| \le \frac{1}{1-P_0\left(\|u-x_*\|\right)}. \tag{27}$$

If $u = x_0$, then by the first substep of the method (20) the iterate y_0 exists, since $x_0 \in \mathcal{S}(x_*, r_*) - \{x_*\}$. Then, we can write

$$y_0 - x_* = x_0 - x_* - \varphi'(x_0)^{-1}\varphi(x_0)$$

$$= \int_0^1 \varphi'(x_0)^{-1}(\varphi'(x_* + \theta(x_0 - x_*)) - \varphi'(x_0))d\theta(x_0 - x_*). \tag{28}$$

It follows by (H_4), (21), (27) (for $u = x_0$) and (28) that

$$\|y_0 - x_*\| \le \frac{\int_0^1 P\left((1-\theta)\|x_0-x_*\|\right)d\theta\|x_0-x_*\|}{1-P_0\left(\|x_0-x_*\|\right)}$$

$$\le q_1\left(\|x_0-x_*\|\right)\|x_0-x_*\| \le \|x_0-x_*\| < r_*, \tag{29}$$

showing (22) if $k = 0$, $y_0 \in \mathcal{S}(x_*, r_*)$, and (27) for $u = y_0$. Hence, the iterate z_0 exists, and

$$z_0 - x_* = x_0 - x_* - \varphi(x_0)^{-1}\varphi(x_0) - \frac{1}{2}(\varphi'(y_0)^{-1} - \varphi'(x_0)^{-1})\varphi(x_0)$$

$$= x_0 - x_* - \varphi(x_0)^{-1}\varphi(x_0) - \frac{1}{2}\varphi'(y_0)^{-1}(\varphi'(x_0) - \varphi'(y_0))\varphi'(x_0)^{-1}\varphi(x_0). \tag{30}$$

Then, as in (29), we get

$$\|z_0 - x_*\| \le \Bigg[\frac{\int_0^1 P\left((1-\theta)\|x_0-x_*\|\right)d\theta}{1-P_0\left(\|x_0-x_*\|\right)}$$

$$+\frac{\overline{P}_0\left(1+\int_0^1 P_0(\theta\|x_0-x_*\|)\right)d\theta}{\left(1-P_0\left(\|x_0-x_*\|\right)\right)\left(1-P_0\left(\|y_0-x_*\|\right)\right)}\Bigg]\|x_0-x_*\|$$

$$\leq q_2\left(\|x_0-x_*\|\right)\|x_0-x_*\|\leq\|x_0-x_*\|, \tag{31}$$

where we also used

$$\left\|M^{-1}(\varphi'(x_0)-\varphi'(y_0))\right\|\leq P\left(\|x_0-y_0\|\right)$$

$$\leq P\left(\|x_0-x_*\|+\|y_0-x_*\|\right)\leq\overline{P}_0$$

or

$$\left\|M^{-1}(\varphi'(x_0)-\varphi'(y_0))\right\|\leq\left\|M^{-1}(\varphi'(x_0)-M)\right\|$$

$$+\left\|M^{-1}(\varphi'(y_0)-M)\right\|\leq\overline{P}_0,$$

and

$$\varphi(x_0)=\varphi(x_0)-\varphi(x_*)$$

$$=\int_0^1\varphi'(x_*+\theta(x_0-x_*))d\theta(x_0-x_*)$$

$$\left\|M^{-1}\varphi(x_0)\right\|\leq\left\|M^{-1}\left(\int_0^1\varphi'(x_*+\theta(x_0-x_*))d\theta-M+M\right)\right\|\|x_0-x_*\|$$

$$\leq\left(1+\int_0^1 P_0\left(\theta\|x_0-x_*\|\right)d\theta\right)\|x_0-x_*\|.$$

Thus, the iterate $z_0\in\mathcal{S}(x_*, r_*)$, the assertion (23) holds if $k=0$, the iterate w_0 exists by the third substep of the method (20), and

$$w_0-x_*=z_0-x_*-\varphi'(z_0)^{-1}\varphi(z_0)+(\varphi'(z_0)^{-1}-\varphi'(y_0)^{-1})\varphi(z_0), \tag{32}$$

leading similarly to (31) to

$$\|w_0-x_*\|\leq\Bigg[\frac{\int_0^1 P\left((1-\theta)\|z_0-x_*\|\right)d\theta}{1-P_0\left(\|z_0-x_*\|\right)}$$

$$+\frac{\overline{\overline{P}}_0\left(1+\int_0^1 P_0\left(\theta\|z_0-x_*\|\right)\right)d\theta}{\left(1-P_0\left(\|y_0-x_*\|\right)\right)\left(1-P_0\left(\|z_0-x_*\|\right)\right)}\Bigg]\|z_0-x_*\|$$

$$\leq q_3\left(\|x_0-x_*\|\right)\|x_0-x_*\|\leq\|x_0-x_*\|. \tag{33}$$

Hence, the assertion (25) is valid if $k = 0$, the iterate $w_0 \in \mathcal{S}(x_*, r_*)$, x_1 exists by the fourth step of the method (20), and

$$x_1-x_*=w_0-x_*-\varphi'(w_0)^{-1}\varphi(w_0)+(\varphi'(w_0)^{-1}-\varphi'(x_0)^{-1})\varphi(w_0)$$

$$+\frac{1}{2}\left(I-\varphi'(x_0)^{-1}\varphi'(y_0)\varphi'(x_0)^{-1}\varphi'(y_0)\right)\varphi'(x_0)^{-1}\varphi'(w_0),$$

$$=w_0-x_*-\varphi'(w_0)^{-1}\varphi(w_0)+\varphi'(w_0)^{-1}(\varphi'(x_0)-\varphi'(w_0))\varphi'(x_0)^{-1}\varphi(w_0)$$

$$-\frac{1}{2}(\varphi(y_0)\varphi'(x_0)^{-1}-I)\left(2I+(\varphi'(y_0)\varphi'(x_0)^{-1}-I)\right)\varphi'(x_0)^{-1}\varphi'(w_0), \tag{34}$$

so

$$\|x_1-x_*\|\leq\Bigg[\frac{\int_0^1 P\left((1-\theta)\|w_0-x_*\|\right)d\theta}{1-P_0\left(\|w_0-x_*\|\right)}$$

$$+\frac{\overline{\overline{\overline{P}}}_0\left(1+\int_0^1 P_0(\theta\|w_0-x_*\|)\right)d\theta}{\left(1-P_0\left(\|x_0-x_*\|\right)\right)\left(1-P_0\left(\|w_0-x_*\|\right)\right)}$$

$$\frac{\overline{P}_0}{2\left(1-P_0\left(\|x_0-x_*\|\right)\right)\left(1-P_0\left(\|y_0-x_*\|\right)\right)}\left(2+\frac{\overline{P}_0}{1-P_0\left(\|y_0-x_*\|\right)}\right)$$

$$\times\left(1+\int_0^1 P_0\left(\theta\|w_0-x_*\|\right)d\theta\right)\Bigg]\|w_0-x_*\|$$

$$\leq q_4\left(\|x_0-x_*\|\right)\|x_0-x_*\|\leq\|x_0-x_*\|. \tag{35}$$

So, the assertion (25) is valid for $k = 0$, and the iterate $x_1 \in \mathcal{S}(x_*, r_*)$. The induction for the assertions (22)-(25) holds for $k = 0$, and the iterates $y_0, z_0, w_0, x_1 \in \mathcal{S}(x_*, r_*)$. But the calculations can be repeated if x_k, y_k, z_k, w_k, x_{k+1} replace respectively x_0, y_0, z_0, w_0, x_1. Thus, the induction for the assertions (22)-(25) holds for each $k = 0, 1, 2, \ldots$. Then, from the estimate

$$\|x_{k+1} - x_*\| \leq \lambda \|x_k - x_*\| \leq r_* \tag{36}$$

for $\lambda = q_4(\| x_0 - x_* \|) \in [0,1)$, implies that $\lim_{k\to+\infty} x_k = x_*$ and the iterate $x_{k+1} \in \mathcal{S}(x_*, r_*)$. □

Next, a region is determined inside which x_* is the only solution of the equation $\varphi(x) = 0$.

Proposition 3.2 Suppose:

The condition (H_2) holds on $\mathcal{S}(x_*, r_5)$ for some $r_5 > 0$ and there exits $r_6 \geq r_5$ so that

$$\int_0^1 P_0(\theta r_6) d\theta < 1. \tag{37}$$

Set $\mathcal{S}_1 = \mathcal{S}[x_*, r_6] \cap D$.

Then, the only solution of the equation (19) in the region $\mathcal{S}_1$ is x_*.

Proof. $\bar{x} \in \mathcal{S}_1$ with $\varphi(\bar{x}) = 0$. Consider the linear operator $\mathcal{Q} = \int_0^1 \varphi'(x_* + \theta(\bar{x} - x_*)) d\theta$. It follows by (H_2) and (37) in turn that

$$\left\| M^{-1}(\mathcal{Q} - M) \right\| \leq \int_0^1 P_0\left(\theta \|\bar{x} - x_*\|\right) d\theta$$

$$\leq \int_0^1 P_0(\theta r_6) d\theta < 1,$$

which implies the invertability of the operator $\mathcal{Q}$. Then, from the identity

$$\bar{x} - x_* = \mathcal{Q}(\varphi(\bar{x}) - \varphi(x_*)) = \mathcal{Q}^{-1}(0) = 0,$$

we conclude $\bar{x} = x_*$. □

Next, the semilocal analysis of the method (20) is developed in an analogous way to the local case but the role of x_* and the "P" functions is exchanged by x_0 and the "h" functions as follows:

(A_1) There exists a (CNF) $\psi_0 : T \to \mathbb{R}$ so that the equation $\psi_0(t) - 1 = 0$ has an SS $\rho_0 \in T - \{0\}$. Set $T_4 = [0, \rho_0)$.

There exists a function $\psi : T_4 \to \mathbb{R}$. Define the sequences $\{a_k\}$, $\{b_k\}$, $\{c_k\}$ and $\{d_k\}$ for $a_0 = 0$, some $b_0 \geq 0$ and each $k = 0, 1, 2, \ldots$ by

$$\bar{\psi}_k = \begin{cases} \psi(b_k - a_k) \\ \text{or} \\ \psi_0(a_k) + \psi_0(b_k), \end{cases}$$

$$c_k = b_k + \frac{\bar{\psi}_k (b_k - a_k)}{1 - \psi_0(b_k)},$$

$$\lambda_k = \left(1 + \int_0^1 \psi_0(b_k + \theta(c_k - b_k)) d\theta\right)(c_k - b_k)$$

$$+ \int_0^1 \psi((1-\theta)(b_k - a_k)) d\theta (b_k - a_k),$$

$$d_k = c_k + \frac{\lambda_k}{1 - \psi_0(b_k)},$$

$$\mu_k = \left(1 + \int_0^1 \psi_0(c_k + \theta(d_k - c_k))d\theta\right)(d_k - c_k) + \lambda_k,$$

$$a_{k+1} = d_k + \frac{1}{2}\left(2 + \frac{2\bar{\psi}_k}{1 - \psi_0(b_k)} + \left(\frac{\bar{\psi}_k}{1 - \psi_0(b_k)}\right)^2\right)\frac{\mu_k}{1 - \psi_0(a_k)},$$

$$e_{k+1} = \int_0^1 \psi((1-\theta)(a_{k+1} - a_k))d\theta(a_{k+1} - a_k) + (1 + \psi_0(a_k))(a_{k+1} - b_k), \tag{38}$$

and

$$b_{k+1} = a_{k+1} + \frac{e_{k+1}}{1 - \psi_0(a_{k+1})}.$$

The sequence $\{a_k\}$ is shown to be majorizing for $\{x_k\}$ in Theorem 3.3. But first a convergence condition for it is needed.

(A_2) There exists $\rho \in [0, \rho_*)$ so that for each $k = 0, 1, 2, ...$

$$\psi_0(a_k) < 1,\ \psi_0(b_k) < 1, \text{ and } a_k < \rho.$$

The conditions and (38) imply that $0 \leq a_k \leq b_k \leq c_k \leq d_k \leq a_{k+1} < \rho$, and that there exists $a_* \in [0, \rho]$ so that $\lim_{k \to +\infty} a_k = a_*$.

(A_3) There exists invertible operator M so that for each $x \in D$

$$\left\|M^{-1}(\varphi'(x) - M)\right\| \leq \psi_0\left(\|x - x_0\|\right).$$

It follows that $\|M^{-1}(\varphi'(x_0) - M)\| \leq \varphi_0(0) < 1$. Thus, $\varphi_0(x_0)^{-1}$ exists, and when we can choose $b_0 \geq \|\varphi'(x_0)^{-1}\varphi(x_0)\|$.

Set $\mathcal{S}_2 = \mathcal{S}(x_0, \rho_0) \cap D$.

(A_4)

$$\left\|M^{-1}(\varphi'(y) - \varphi'(x))\right\| \leq \psi\left(\|y - x\|\right)$$

for each $x, y \in \mathcal{S}_2$.

(A_5)

$$\mathcal{S}[x_0, a_*] \subset D.$$

The semi-local convergence analysis of the method (20) is developed in next result.

Theorem 3.3 Suppose that the conditions (A_1)-(A_5) hold. Then, the sequence $\{x_k\}$ produced by the method (20) exists in $\mathcal{S}(x_0, a_*)$, stays in $\mathcal{S}(x_0, a_*)$ for each $k = 0, 1, 2$ ... and is convergent to some $x_* \in \mathcal{S}[x_0, a_*]$ solving the equation (19), and so that

$$\|x_* - x_k\| \leq a_* - a_k. \tag{39}$$

Proof. As in Theorem 3.1 mathematical induction shall establish the assertions for each $k = 0, 1, 2 \ldots$

$$\|y_k - x_k\| \leq b_k - a_k, \tag{40}$$

$$\|z_k - y_k\| \leq c_k - b_k, \tag{41}$$

$$\|w_k - z_k\| \leq d_k - c_k \tag{42}$$

and

$$\|x_{k+1} - w_k\| \leq a_{k+1} - b_k. \tag{43}$$

The assertion (40) holds by the definition of a_0, b_0 and the first substep of the method (20), since $\|y_0 - x_0\| = \|\varphi'(x_0)^{-1}\varphi(x_0)\| \leq b_0 = b_0 - a_0 < a_*$. We also have $y_0 \in \mathcal{S}(x_0, a_*)$. Then, by subtracting the first from second substep of the method (20), we get as in local case

$$z_k - y_k = -\frac{1}{2}\varphi'(y_k)^{-1}(\varphi'(x_k) - \varphi'(y_k))\varphi'(x_k)^{-1}\varphi(x_k)$$

$$= \frac{1}{2}\varphi'(y_k)^{-1}(\varphi'(x_k) - \varphi'(y_k))(y_k - x_k),$$

so

$$\|z_k - y_k\| \leq \frac{1}{2}\frac{\bar{\psi}_k \|y_k - x_k\|}{1 - \psi_0\left(\|y_k - x_0\|\right)}$$

$$\leq \frac{\bar{\psi}_k (b_k - a_k)}{1 - \psi_0(b_k)} = c_k - b_k,$$

$$\|z_k - x_0\| \leq \|z_k - y_k\| + \|y_k - x_0\| \leq c_k - b_k + b_k - a_0 = c_k < a_*,$$

$$\varphi(z_k) = \varphi(z_k) - \varphi(y_k) + \varphi(z_k),$$

$$\left\|M^{-1}\varphi(z_k)\right\| \leq \left(1 + \int_0^1 \psi_0\left(\|y_k - x_0\| + \theta\|z_k - y_k\|\right)d\theta\right)\|z_k - y_k\|$$

$$+ \int_0^1 \psi\left((1-\theta)\|y_k - x_k\|\right)d\theta \|y_k - x_k\|$$

$$\leq \left(1 + \int_0^1 \psi_0(b_k + \theta(c_k - b_k))d\theta\right)(c_k - b_k)$$

$$+\int_0^1 \psi((1-\theta)(b_k - a_k))d\theta(b_k - a_k) = \lambda_k,$$

$$w_k - z_k = -\varphi'(y_k)^{-1}\varphi(z_k)$$

$$= -\varphi'(y_k)^{-1}(\varphi(z_k) - \varphi(y_k) + \varphi(y_k)),$$

$$\|w_k - z_k\| \le \|\varphi'(y_k)^{-1}M\|\|M^{-1}\varphi(z_k)\|$$

$$\le \frac{\lambda_k}{1-\psi_0\left(\|y_k - x_0\|\right)} \le \frac{\lambda_k}{1-\psi_0(b_k)} = d_k - c_k,$$

$$\|w_k - x_0\| \le \|w_k - z_k\| + \|z_k - x_0\|$$

$$\le d_k - c_k + c_k - a_0 = d_k < a_*,$$

$$x_{k+1} - w_k = -\frac{1}{2}(2I + 2(\varphi'(y_k)^{-1}\varphi'(x_k) - I)$$

$$+(\varphi'(y_k)^{-1}\varphi'(x_k) - I)^2)\varphi'(x_k)^{-1}\varphi(w_k).$$

But

$$\varphi(w_k) = \varphi(w_k) - \varphi(z_k) + \varphi(z_k),$$

$$\|M^{-1}\varphi(w_k)\| \le \left(1 + \int_0^1 \psi_0(c_k + \theta(d_k - c_k))d\theta\right)(d_k - c_k) + \lambda_k = \mu_k,$$

$$\|x_{k+1} - w_k\| = \frac{1}{2}\left(2 + \frac{2\bar{\psi}_k}{1-\psi_0(b_k)} + \left(\frac{\bar{\psi}_k}{1-\psi_0(b_k)}\right)^2\right)\frac{\mu_k}{1-\psi_0(a_k)}$$

$$= a_{k+1} - d_k,$$

$$\|x_{k+1} - x_0\| \le \|x_{k+1} - w_k\| + \|w_k - x_0\|$$

$$\le a_{k+1} - d_k + d_k - a_0 = a_{k+1} < a_*,$$

$$\varphi(x_{k+1}) = \varphi(x_{k+1}) - \varphi(x_k) - \varphi'(x_k)(x_{k+1} - x_k) + \varphi'(x_k)(x_{k+1} - y_k),$$

$$\|M^{-1}\varphi(x_{k+1})\| \le \int_0^1 \psi\left((1-\theta)\|x_{k+1} - x_k\|\right)d\theta\|x_{k+1} - x_k\|$$

$$+(1+\psi_0\left(\|x_k-x_0\|\right)\|x_{k+1}-y_k\|$$

$$\leq \int_0^1 \psi((1-\theta)(a_{k+1}-a_k))d\theta(a_{k+1}-a_k)$$

$$+(1+\psi_0(a_k))(a_{k+1}-b_k)=e_{k+1}, \tag{44}$$

$$\|y_{k+1}-x_{k+1}\| \leq \|\varphi(x_{k+1})^{-1}M\|\|M^{-1}\varphi(x_{k+1})\|$$

$$\leq \frac{e_{k+1}}{1-\psi_0(a_{k+1})}=b_{k+1}-a_{k+1}$$

and

$$\|y_{k+1}-x_0\| \leq \|y_{k+1}-x_{k+1}\|+\|x_{k+1}-x_0\|$$

$$\leq b_{k+1}-a_{k+1}+a_{k+1}-a_0=b_{k+1}<a_*.$$

Hence, the induction for the assertions (40)-(43) is completed, and $x_k, y_k, z_k, w_k \in \mathcal{S}(x_*, a_*)$. Moreover, we have

$$\|x_{k+1}-x_k\| \leq a_{k+1}-a_k. \tag{45}$$

Thus, the sequence $\{x_k\}$ is Cauchy in the Banach space B_1, and as such it is convergent to some $x_* \in \mathcal{S}[x_0, a_*]$. By letting $k \to +\infty$ in (44), we deduce $\varphi(x_*) = 0$.

Furthermore, by (45)

$$\|x_{k+m}-x_k\| \leq a_{k+m}-a_k \tag{46}$$

for each $m = 1, 2, ...$.

By letting $m \to +\infty$ in (46), we conclude that (39) holds. □

The uniqueness result result for a solution of the equation (19) follows.

Proposition 3.4 Suppose:

There exists a solution $\tilde{x} \in \mathcal{S}(x_0, \rho_3)$ of the equation (19). The condition (A_3) holds in the ball $\mathcal{S}(x_0, \rho_3)$, and there exists $\rho_4 \geq \rho_3$, so that

$$\int_0^1 \psi(\theta\rho_3+(1-\theta)\rho_4)d\theta<1. \tag{47}$$

Set $\mathcal{S}_3 = \mathcal{S}(x_0, \rho_4) \cap D$.

Then, the only solution of the equation (19) in the region $\mathcal{S}_3$ is $\tilde{x}$.

Proof. Let $\bar{x} \in \mathcal{S}_3$ with $\varphi(\bar{x})=0$. Define the operator $Q_1=\int_0^1 \varphi'(\tilde{x}+\theta(\bar{x}-\tilde{x}))d\theta$. Then, by (A_3) and (47), we get

$$\|M^{-1}(Q_1-M)\| \leq \int_0^1 \psi_0\left(\theta\|\tilde{x}-x_0\|+(1-\theta)\|\bar{x}-x_0\|\right)d\theta$$

$$\leq \int_0^1 \psi_0(\theta\rho_3 + (1-\theta)\rho_4)d\theta < 1.$$

Thus, we conclude again as in the local case that $\tilde{x} = \bar{x}$. □

Remark 3.5 (i) If all conditions of Theorem 3.3 hold, then take $\rho_3 = a_*$ and $x_* = \tilde{x}$.

(ii) A popular choice but not the most flexible is $M = \varphi'(x_*)$ for the local, and $M = \varphi'(x_0)$ for the semi-local case.

4. Numerical applications

This section comprises of several numerical examples to show the efficiency of our proposed method for approximating the solution by comparing with existing methods. The comparison is made with eighth order methods given by Neta and Johnson (NJ8), Cordero et al. (CVTM) (taking $\beta_1 = 0$, $\beta_2 = 1$, $\beta_3 = 0$), Dzunić and Petković (DPM), Sharma et al. (SSKM), Thukral (TM) and existing fifth order method by Grau-Sánchez et al. (GSM) (8). In Table 1, the considered test functions with initial approximation and the corresponding root are displayed. To compare the computational performance, the number of iteration indexes (k), norms of the functions ($|f(x_k)|$), error between two consecutive iterates $\|x_{k+1} - x_k\|$ and computational order of convergence given by formula [27]:

$$\rho = \frac{\ln\|f(x_k)/f(x_{k-1})\|}{\ln\|f(x_{k-1})/f(x_{k-2})\|}$$

are mentioned. Further, the theoretical results proved in Section 3 are verified and numerical experiments are performed in Banach spaces in Examples (4.2) and (4.3) but on the real line in Example (4.1).

Table 1. Test functions

$f(x)$	x_0	Root (α)
$f(x) = \sin^2 x - x^2 - 1$	3	1.404491648215341
$f_2(x) = e^{x^2+7x-30} - 1$	3.3	3.000000000000000
$f_3(x) = x^2 - e^x - 3x + 2$	3	0.257530285439861
$f_4(x) = e^{-x^2+x+2} - \cos(x+1) + x^3 + 1$	-2	−1.000000000000000
$f_5(x) = x^2 e^x - \sin(x) + x$	-2	−1.499393096901409
$f_6(x) = \log(x^2 + x + 2) - x + 1$	2	4.152590736757158
$f_7(x) = e^{-x} + \cos x$	-0.5	1.365230013414097
$f_8(x) = \arcsin(x^2 - 1) - x/2 + 1$	1	0.5948109683983692

Example 4.1 Set $D = B_1 = B_2 = \mathbb{R}$. Then, the method (9) is applied on the test functions given in Table 1.

The numerical experiments are stimulated by using Mathematica 8 on Intel(R) Core(TM) i5 - 8250U mCPU @ 1.60 GHz 1.80 GHz with 8 GB of RAM running on the Windows 10 Pro version 2017. The comparison results for all considered examples for $|x_{k+1} - x_k|$, $|f(x_k)|$ and ρ are displayed in Tables 2-9 up to the third iteration. For every method, the stopping criterion used is $|x_{k+1} - x_k| + |f(x_k)| < 10^{-100}$. It can be observed that proposed method has higher accuracy in numerical values of approximations to the root than the existing methods in all the considered examples.

Table 2. Comparison of the performances of methods for f_1

Methods	k	$\lvert x_{k+1} - x_k \rvert$	$\lvert f(x_{k+1}) \rvert$	ρ
	1	1.61e - 000	2.62e - 004	
M8	2	1.06e - 002	3.81e - 017	7.9991981
	3	1.54e - 017	7.39e - 136	
	1	1.60e - 005	1.01e - 002	
NJ8	2	4.10e - 003	1.60e - 017	8.0005908
	3	6.44e - 017	5.74e - 136	
	1	1.60e - 000	2.12e - 002	
CVTM	2	8.61e - 003	2.57e - 017	8.0013750
	3	1.04e - 017	1.08e - 136	
	1	1.58e - 000	2.99e - 002	
DPM	2	1.19e - 002	1.81e - 016	7.9990926
	3	7.30e - 017	3.64e - 130	
	1	1.57e - 000	5.20e - 002	
SSKM	2	2.06e - 002	7.19e - 012	7.9686093
	3	1.62e - 383	3.47e - 384	
	1	1.57e - 000	6.84e - 002	
TM	2	2.70e - 002	5.30e - 012	7.9905882
	3	2.13e - 012	1.01e - 092	
	1	1.63e - 000	9.27e - 002	
GSM	2	3.85e - 002	8.87e - 009	4.9850176
	3	3.57e - 009	7.80e - 044	

Table 3. Comparison of the performances of methods for f_2

Methods	k	$\lvert x_{k+1} - x_k \rvert$	$\lvert f(x_{k+1}) \rvert$	ρ
	1	1.90e - 001	3.19e - 000	
M8	2	1.07e - 001	3.17e - 002	6.49495
	3	2.40e - 003	1.42e - 014	
	1	1.78e - 001	3.89e - 000	
NJ8	2	1.15e - 001	8.11e - 002	6.00338
	3	5.99e - 003	5.67e - 011	
	1	2.20e - 001	1.81e - 000	
CVTM	2	7.89e - 002	1.03e - 003	6.99268
	3	7.94e - 005	7.10e - 014	
	1	2.40e - 001	1.17e - 000	
DPM	2	5.93e - 002	8.13e - 006	6.99909
	3	6.25e - 007	1.56e - 014	
	1	1.67e - 001	4.73e - 000	
SSKM	2	1.12e - 001	1.86e - 001	6.96860
	3	1.31e - 002	1.17e - 008	
	1	1.71e - 001	4.37e - 000	
TM	2	1.18e - 001	1.37e - 001	5.20899
	3	9.93e - 003	1.47e - 008	
	1	1.62e - 001	5.11e - 000	
GSM	2	1.20e - 001	2.59e - 001	4.39737
	3	1.77e - 002	2.77e - 005	

Table 4. Comparison of the performances of methods for f_3

Methods	k	$\lvert x_{k+1} - x_k \rvert$	$\lvert f(x_{k+1}) \rvert$	ρ
	1	2.73e - 000	4.32e - 002	
M8	2	1.14e - 002	4.07e - 021	7.9998647
	3	1.08e - 021	2.51e - 173	
	1	2.62e - 000	4.75e - 001	
NJ8	2	1.27e - 001	5.06e - 013	8.0044300
	3	1.34e - 013	6.73e - 109	
	1	2.67e - 000	2.82e - 001	
CVTM	2	7.52e - 002	1.16e - 013	8.0009526
	3	3.06e - 014	8.46e - 113	
	1	2.48e - 000	9.80e - 001	
DPM	2	2.65e - 001	7.79e - 010	7.9754496
	3	2.06e - 010	1.77e - 082	
	1	2.54e - 000	7.65e - 001	
SSKM	2	2.06e - 001	1.24e - 010	8.0559699
	3	3.27e - 011	1.43e - 089	
	1	2.62e - 000	4.69e - 001	
TM	2	1.25e - 001	2.69e - 012	7.9921705
	3	7.11e - 013	3.52e - 102	
	1	2.45e - 000	1.07e - 000	
GSM	2	2.90e - 001	3.20e - 006	4.9852520
	3	8.46e - 007	8.18e - 034	

Table 5. Comparison of the performances of methods for f_4

Methods	k	$\lvert x_{k+1} - x_k \rvert$	$\lvert f(x_{k+1}) \rvert$	ρ
	1	1.00e - 000	2.12e - 003	
M8	2	3.53e - 004	2.00e - 030	8.0000412
	3	3.34e - 021	1.29e - 246	
	1	1.00e - 000	9.91e - 003	
NJ8	2	1.65e - 003	2.20e - 030	7.9999631
	3	3.66e - 034	1.28e - 245	
	1	9.96e - 001	2.40e - 002	
CVTM	2	4.00e - 003	6.64e - 021	7.9996822
	3	1.11e - 021	2.31e - 169	
	1	9.96e - 001	2.65e - 002	
DPM	2	4.41e - 003	3.39e - 021	7.9989672
	3	5.65e - 022	2.55e - 172	
	1	1.00e - 000	2.59e - 002	
SSKM	2	4.31e - 002	3.76e - 021	8.0005490
	3	6.26e - 022	7.37e - 172	
	1	1.00e - 000	1.61e - 002	
TM	2	2.68e - 003	2.63e - 023	8.0033069
	3	4.38e - 034	1.15e - 189	
	1	9.98e - 000	1.49e - 002	
GSM	2	2.48e - 003	6.45e - 015	4.9852520
	3	1.07e - 015	9.95e - 077	

Table 6. Comparison of the performances of methods for f_5

Methods	k	$\lvert x_{k+1} - x_k \rvert$	$\lvert f(x_{k+1}) \rvert$	ρ
	1	5.01e - 001	7.71e - 005	
M8	2	1.01e - 004	4.50e - 034	7.9998908
	3	5.92e - 034	6.11e - 268	
	1	5.00e - 001	2.10e - 004	
NJ8	2	2.76e - 004	6.41e - 029	7.9999626
	3	8.43e - 029	4.81e - 225	
	1	5.00e - 001	1.39e - 004	
CVTM	2	1.82e - 004	1.36e - 030	7.9999744
	3	1.78e - 030	1.13e - 238	
	1	5.00e - 001	1.59e - 004	
DPM	2	2.09e - 004	2.94e - 030	7.9999787
	3	3.87e - 030	4.10e - 236	
	1	4.99e - 001	9.43e - 004	
SSKM	2	1.24e - 003	2.12e - 022	7.9986387
	3	2.78e - 022	1.47e - 171	
	1	4.99e - 001	9.27e - 004	
TM	2	1.22e - 003	1.22e - 022	7.9996808
	3	1.60e - 022	1.10e - 173	
	1	4.99e - 001	8.69e - 004	
GSM	2	1.14e - 003	2.37e - 017	4.9973865
	3	3.11e - 017	3.86e - 085	

Table 7. Comparison of the performances of methods for f_6

Methods	k	$\lvert x_{k+1} - x_k \rvert$	$\lvert f(x_{k+1}) \rvert$	ρ
	1	2.15e - 000	3.34e - 004	
M8	2	5.536e - 004	2.18e - 036	8.0000107
	3	3.61e - 036	7.03e - 283	
	1	2.15e - 000	1.99e - 004	
NJ8	2	3.30e - 004	1.96e - 036	7.9999939
	3	3.25e - 036	1.73e - 282	
	1	2.15e - 000	1.00e - 003	
CVTM	2	1.66e - 003	1.79e - 030	7.9999585
	3	2.98e - 030	1.90e - 244	
	1	2.15e - 000	4.46e - 004	
DPM	2	7.14e - 004	5.49e - 033	7.9999850
	3	9.12e - 033	2.91e - 264	
	1	1.47e - 000	3.92e - 001	
SSKM	2	6.80e - 001	2.68e - 007	7.9999972
	3	4.46e - 007	1.20e - 059	
	1	2.17e - 000	1.09e - 002	
TM	2	1.80e - 002	5.23e - 021	7.9992396
	3	8.69e - 021	1.60e - 167	
	1	2.16e - 000	4.68e - 003	
GSM	2	7.78e - 003	4.08e - 016	4.9998024
	3	6.78e - 016	2.07e - 081	

Table 8. Comparison of the performances of methods for f_7

Methods	k	$\lvert x_{k+1}-x_k\rvert$	$\lvert f(x_{k+1})\rvert$	ρ
	1	2.25e - 000	2.36e - 005	
M8	2	2.04e - 005	4.45e - 042	7.9999989
	3	3.84e - 042	7.02e - 336	
	1	2.25e - 000	2.07e - 004	
NJ8	2	1.79e - 004	8.90e - 035	7.9999903
	3	7.68e - 035	1.03e - 277	
	1	2.25e - 000	8.46e - 005	
CVTM	2	7.30e - 005	1.76e - 037	8.0000003
	3	1.51e - 037	6.02e - 299	
	1	2.25e - 000	1.53e - 004	
DPM	2	1.32e - 004	2.65e - 035	8.0000097
	3	2.29e - 035	2.10e - 281	
	1	2.25e - 000	1.43e - 004	
SSKM	2	1.23e - 004	2.87e - 035	7.9999989
	3	2.48e - 035	7.75e - 281	
	1	2.25e - 000	2.05e - 004	
TM	2	1.77e - 004	2.15e - 033	8.0000172
	3	1.86e - 033	3.14e - 265	
	1	2.25e - 000	1.05e - 003	
GSM	2	9.08e - 004	1.89e - 018	4.9999421
	3	1.63e - 018	3.49e - 092	

Table 9. Comparison of the performances of methods for f_8

Methods	k	$\lvert x_{k+1}-x_k\rvert$	$\lvert f(x_{k+1})\rvert$	ρ
	1	4.05e - 001	2.42e - 006	
M8	2	2.29e - 006	7.59e - 049	7.9999999
	3	7.17e - 049	7.05e - 389	
	1	4.05e - 001	1.20e - 005	
NJ8	2	1.13e - 005	3.50e - 043	8.0000008
	3	3.31e - 043	1.89e - 343	
	1	4.05e - 001	1.12e - 005	
CVTM	2	1.06e - 005	4.72e - 043	8.0000004
	3	4.46e - 043	4.68e - 342	
	1	4.05e - 001	6.44e - 006	
DPM	2	6.08e - 006	1.44e - 045	8.0000003
	3	1.36e - 045	8.83e - 363	
	1	4.05e - 001	1.85e - 005	
SSKM	2	1.75e - 005	7.97e - 041	7.9999989
	3	7.53e - 041	9.41e - 324	
	1	4.05e - 001	5.22e - 006	
TM	2	4.93e - 006	2.76e - 046	8.0000002
	3	2.61e - 046	1.69e - 368	
	1	4.05e - 001	4.76e - 004	
GSM	2	4.49e - 004	3.36e - 019	5.0000229
	3	3.17e - 019	5.91e - 095	

The next two examples demonstrate the extension of proposed method in Banach spaces presented in Section 3.

Example 4.2 Let $B_1 = B_2 = \mathbb{R}^3$ and $D = \mathcal{S}(0, 1)$. Define the operator φ on D for $s = (s_1, s_2, s_3)^T$ by

$$\varphi(s) = \left(e^{s_1} - 1, \frac{e-1}{2} s_2^2 + s_2, s_3\right)^T. \tag{48}$$

This definition implies that

$$\varphi'(s) = \begin{pmatrix} e^{s_1} & 0 & 0 \\ 0 & (e-1)s_2 + 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Clearly, $s_* = (0, 0, 0)^T$ solves the equation $\varphi(s) = 0$ and $\varphi'(s_*) = I$. Then, for $M = \varphi'(s_*)$ the conditions (H_1)-(H_3) hold, if we take $P_0(t) = (e - 1)t$, and $P(t) = e^{\frac{1}{e-1}} t$. The radius of convergence r_* for the method (20) using the formula (21) is given by $r_* = 0.159254648631897 \cdots < 1$. Hence, the condition (H_4) also holds. Therefore, all the conditions of Theorem 3.1 hold.

Example 4.3 Let $B_1 = B_2 = \mathbb{R}^3$ and $D = \mathcal{S}(0, 1)$. Define the operator φ on D by

$$\varphi(u)(x) = u(x) - 8\int_0^1 x\theta u(\theta)^3 d\theta. \tag{49}$$

By this definition, we calculate the derivative φ' to be

$$\varphi'(u(l))(x) = l(x) - 24\int_0^1 x\theta u(\theta)^2 l(\lambda) d\lambda.$$

Clearly, $x_* = 0$ solves the equation $\varphi(x) = 0$, so $\varphi'(x_*) = I$. Then, for $M = \varphi'(x_*)$ the conditions (H_1)-(H_3) hold if we take $P_0(t) = 12t$ and $P(t) = 24t$. The radius of convergence for the method (20) using again formula (21) comes out to be $r_* = 0.022092906375511003 \cdots < 1$. Therefore, all the conditions of Theorem 3.1 hold.

Thus, the novelty, applicability and theoretical results of the current work are corroborated by numerical experiments.

5. Conclusion

The construction of higher order iterative methods is at topmost importance in numerical analysis now a days due to its numerous applications in various fields. The present study consists of designing of a new eighth order method which is obtained from a fifth order by method by adding one step for which only one additional function evaluation is required. The method is extended to Banach spaces and is analyzed with local and semi-local convergence using generalized conditions. Based on the numerical findings, it is clear that the proposed method performs the best in terms of accuracy in error and norm of the function in all the considered examples as compared to other existing methods. As already noted in the introduction the methodology developed in Section 3 is applicable on other methods. This is the direction of our research which shall establish our approach further. We start with this paper. That will be an improvement of our work.

Conflict of interest

The authors declare that they do not have conflict of interests.

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