Contemporary Mathematics
http://ojs.wiserpub.com/index.php/CM/

UNIVERSAL WISER PUBLISHER

Research Article

# Conjugacy Class Graph of Some Non-Abelian Groups

**Chinmayee Kumar[*], Kuntala Patra**

Department of Mathematics, Gauhati University, India
E-mail: chinmayeekumarc@gmail.com

**Received:** 6 November 2023; **Revised:** 25 December 2023; **Accepted:** 8 January 2024

**Abstract:** The conjugacy class graph of a group $G$ is a graph whose vertices are the non-central conjugacy classes of $G$ and two vertices are adjacent if their cardinalities are not co-prime. In this paper, conjugacy class graphs of $D_n$, $Q_{4n}$, $S_n$ are studied. These graphs are found to be either complete graphs or union of complete graphs. Conjugacy classes of $D_n \times D_m$ are calculated and the results obtained are used to determine the structure of conjugacy class graphs of $D_n \times D_m$, for odd and even values of $m$ and $n$. Conjugacy class graphs of $D_n$ are non-planar for $n = 8$ and $n \geq 11$. They are non-hyperenergetic for all $n$ and hypoenergetic only for $n = 3, 5$. Also, line graphs of these graphs are regular and eulerian for $n \equiv 1 \ (mod\ 2)$ and $n \equiv 0 \ (mod\ 4)$. The conjugacy class graphs of $Q_{4n}$ are non-planar for $n = 4$ and $n \geq 6$. These graphs are non-hyperenergetic as well as non-hypoenergetic. The line graphs are eulerian for even values of $n$. It is conjectured that conjugacy class graph of $S_n$ is non-planar for $n \geq 5$.

***Keywords*:** non-abelian group, conjugacy class graph, line graph, energy (of a graph)

**MSC:** 05C10, 05C25, 05C50

## 1. Introduction

Conjugacy classes of a non-abelian group have been studied extensively since years for determining the structure of a group. Two elements $a, b \in G$ of a group $G$ are said to be conjugate if there exists an element $c \in G$ such that $b = c^{-1}ac$. The set of all elements conjugate to $a$ is called the conjugacy class of $a$ and is denoted by $Cl(a)$. The conjugacy relation is an equivalence relation that partitions $G$ into disjoint equivalence classes, called the conjugacy classes. The study of conjugacy classes of non-abelian groups is fundamental in determining their structure as they influence various parameters like coloring of graph, domination number and metric dimension among others. Conjugacy classes of a group can be used to show the existence of isomorphism between two groups, if it exists. The concept of attaching a graph to a group have proved to be useful in exploring the structural properties of the group. Algebraic methods are applied to solve problem about graphs or the other way round. In 1990, one such graph called the *conjugacy class graph* $\Gamma_g^{cl}$ was introduced by Bertram et al. [1]. The conjugacy class graph of a group $G$, denoted by $\Gamma_g^{cl}(G)$, is a graph whose vertices are the non-central conjugacy classes of $G$ and two vertices $a$, $b$ are adjacent if $gcd(a, b)$ is greater than one. In 2017, Zulkarnain [2] discussed the conjugate graphs of finite $p$-groups. Sarmin et al. [3] obtained some graph properties of conjugate graphs and conjugacy class graphs of metacyclic 2-groups of order atmost 32. Zamri and Sarmin [4] found the conjugate graphs and generalized conjugacy class graphs of metacyclic 3-groups and metacyclic 5-group. Hanan


DOI: https://doi.org/10.37256/cm.5120243875

et al. [5] introduced the non-conjugate graph associated with finite groups and contructed the resolving polynomial for generalized quaternion groups.

This paper is divided into three sections. The first section is the introductory section where we give a brief definition of conjugacy classes and conjugacy class graph. In the second section, we state some basic definitions and results that have been referred to for our study. The third section involves our main results that have been further divided into four sections. Throughout the paper, the graphs referred to are undirected and simple graphs.

## 2. Preliminaries

In this section, we state some basic definitions and results from algebra and graph theory that have been used in establishing our main results in the subsequent sections. One of the common graph structure that appears in most of our results is the complete graph $K_n$ and hence few results have been stated in connection to $K_n$.

Consider a simple graph $G(V, E)$. Let $v_0$, $e_0$, $v_1$, $e_1$, ..., $v_{k-1}$, $e_{k-1}$, $v_k$ be an alternating sequence of edges and vertices where $v_i's$ are distinct vertices and $e_i$ is an edge connecting vertices $v_i$ and $v_{i+1}$. The number of edges that connects two vertices is called the length of the path between them. The graph $G$ is said to be *connected* if there exists a path between any two distinct vertices in the graph; otherwise $G$ is said to be *disconnected*. A connected graph is said to be *eulerian* if it contains an eulerian circuit, i.e., it contains a trail that visits every edge exactly once and starts and ends on the same vertex. A connected graph is eulerian if and only if every vertex has even degree. A connected graph that can be embedded in the plane, i.e., it can be drawn without any edges crossing each other is called a *planar graph*. A graph is a planar graph if and only if it contains no subgraph isomorphic to $K_{3,3}$ or $K_5$. Given a graph $G$, the line graph $L(G)$ of $G$ is a graph whose vertices are the edges of $G$, with two vertices of $L(G)$ adjacent whenever the corresponding edges of $G$ are adjacent. Thus the line graph $L(G)$ represents the adjacencies between the edges of $G$. Regularity of graphs is a fundamental concept in graph theory as such graphs tend to be well-connected and often presents a high degree of symmetry. A graph is said to be regular if all its vertices have the same degree. A graph $G$ is said to be $k$*-regular* if the degree of each vertex in $G$ is $k$. Apart from graph theory, regular graphs find applications in various fields like network topology, cryptography, game theory, coding theory.

Spectral graph theory is another fundamental study concerned with graph parameters in relationship to eigenvalues of matrices associated to a graph, such as the adjacency matrix. The *adjacency matrix* $A(X)$ of a directed graph $X$ is the integer matrix with rows and columns indexed by the vertices of $X$, such that the $uv$-entry of $A(X)$ is equal to the number of edges from $u$ to $v$ which is usually 0 or 1. The set of all the eigen values of a matrix alongwith their multiplicities is called the *spectrum* of the given matrix. The eigen values of the complete graph $K_n$ are $n - 1$ and $-1$ with multiplicities 1 and $n - 1$ respectively.

The concept of *energy* of a graph was first introduced by Gutman in 1978 [6]. The *energy* of a graph is defined as follows.

Let $G$ be a graph with $n$ vertices and let $\lambda_1$, $\lambda_2$, ..., $\lambda_n$ be the eigen values of its adjacency matrix. Then, the *energy* $E(G)$ of the graph $G$ is defined as :

$$E(G) = \sum_{i=1}^{n} |\lambda_i|$$

Gutman conjectured that among all graphs with $n$ vertices, the complete graph $K_n$ has the maximum energy. This was however disproved by Walikar et al. [7] and a new concept of *hyperenergetic graphs* was defined. A graph $G$ with $n$ vertices is *hyperenergetic* if the energy is greater than that of the complete graph $K_n$, i.e., $E(G) > 2(n - 1)$. Gutman proved that there are no hyperenergetic graphs on less than 8 vertices. The graph $G$ is said to be *hypoenergetic* if its energy is less than the order of the graph, i.e., $E(G) < n$. Gutman proved that if a graph $G$ is regular with non-zero degree then $G$ is hypoenergetic. Two graphs $G_1$ and $G_2$ are said to be equienergetic if $E(G_1) = E(G_2)$, provided both $G_1$ and $G_2$ have the same number of vertices.

## 3. Main results

This section is divided into four subsections. The first section includes results on structure of conjugacy class graphs of the groups $D_n$, $Q_{4n}$ and $S_n$. In the second section, we find out the structure of conjugacy class graphs of $D_n \times D_m$, $D_n \times D_n$ and $D_m \times D_m$, where $n$ is odd and $m$ is even. In the third section regularity of line graphs of the conjugacy class graphs of $D_n$ and $Q_{4n}$ are discussed. Lastly, we investigate the nature of conjugacy class graphs of $D_n$ and $Q_{4n}$ in terms of their graph energies.

### 3.1 *Conjugacy class graph of $D_n$, $Q_{4n}$ and $S_n$*

**Definition 3.1.1** The dihedral group $D_n(n \geq 3)$ of order $2n$ is the group of symmetries of a regular $n$-sided polygon with the group presentation

$$D_{2n} = < a, b : a^2 = b^n = e, aba = b^{-1} >$$

The properties of Dihedral group $D_n$ depend on whether $n$ is even or odd.

We now proceed to find out the conjugacy classes of $D_n$ through the following lemma.

**Lemma 3.1.1** The number of conjugacy classes for $D_n(n\ odd)$ is $\frac{n+3}{2}$ and the conjugacy classes are given by

$$\{e = b^n\}, \{a, ab, ab^2, ab^3, \ldots.., ab^{n-2}, ab^{n-1}\}, \{b, b^{n-1}\}, \{b^2, b^{n-2}\}, \ldots.., \left\{b^{\frac{n-1}{2}}, b^{\frac{n+1}{2}}\right\}$$

and the number of conjugacy classes for $D_n(n\ even)$ is $\frac{n+6}{2}$ and the conjugacy classes are given by

$$\{e = b^n\}, \{a, ab^2, ab^4, \ldots., ab^{n-2}\}, \{ab, ab^3, ab^5, \ldots., ab^{n-1}\}, \{b, b^{n-1}\}, \{b^2, b^{n-2}\}, \ldots.., \left\{b^{\frac{n}{2}}\right\}$$

**Proof. Case 1** When $n$ is odd.

We have, $b^n = e$, so $b^n \in Z(D_n)$.

Therefore, conjugacy class of $b^n$, ie, $\{b^n\}$ consists of a single element.

Consider two elements of $D_n$ of the form $ab^i$ and $ab^j$ where $1 \leq i, j \leq n$ and $i < j$.

**Case (i)** $i$ and $j$ are both odd and $j - i = 2$. Then

$$\begin{aligned} b^{-1}ab^ib &= b^{-1}ab^{i+1} \\ &= abb^{i+1} (\because aba = b^{-1} \Rightarrow ab = b^{-1}a) \\ &= ab^{i+2} \\ &= ab^j \end{aligned}$$

Thus $ab^i \sim ab^j$.

**Case (ii)** $i$ and $j$ are both even and $j - i = 2$.

Then there exists $\alpha = ab^{i+1} \in D_n$ such that

$$\alpha^{-1}ab^i\alpha = (ab^{i+1})^{-1}ab^i(ab^{i+1})$$

$$= b^{-(i+1)}a^{-1}ab^iab^{i+1}$$

$$= b^{-1}ab^{i+1}$$

$$= abb^{i+1}(\because aba = b^{-1} \Rightarrow ab = b^{-1}a)$$

$$= ab^{i+2}$$

$$= ab^j$$

Therefore $ab^i \sim ab^j$.

**Case (iii)** Let $i$ is odd and $j$ even and $j = i + 1$.

Then for $\alpha = b^{\frac{n+1}{2}} \in D_n$, we can similarly (like the previous two cases) show that $\alpha^{-1}ab^i\alpha = ab^j$. Thus $ab^i \sim ab^j$.

Combining these three cases, we have,

$$ab \sim ab^2 \sim ab^3 \sim ... \sim ab^{n-1} \sim ab^n$$

and hence they belong to the same conjugacy class.

Let $b^i$ and $b^j$ be two elements of $D_n$ where $1 \le i, j \le n$, $i < j$ and $i + j = n$.

For $\alpha = ab^2 \in D_n$, we have $\alpha^{-1}b^i\alpha = b^j$ and so $b^i \sim b^j$.

Thus we have, $b \sim b^{n-1}$, $b^2 \sim b^{n-2}$, $b^3 \sim b^{n-3}$ and so on.

Consider the normalizer of an element of the form $b^i$ where $1 \le i < n$.

Then $N(b^i) = \{b, b^2, b^3, ..., b^n\}$.

Suppose $ab^j \in N(b^i)$ where $i \le j \le n$, then

$$b^i(ab^j) = (ab^j)b^i$$

which gives $2i = n$, which is a contradiction to the fact that $n$ is odd.

Therefore, $ab^j \notin N(b^i)$.

So the number of elements of the conjugacy class $\{b^i\}$ is given by :

$$|\{b^i\}| = \frac{|D_n|}{|N(b^i)|} = \frac{2n}{n} = 2$$

Thus conjugacy classes containing $b^i$, $1 \le i < n$, contains only two elements and the number of conjugacy classes which contain only 2 elements must be $\frac{n-1}{2}$.

Also conjugacy class containing $ab$ is $\{ab, ab^2, ab^3, ..., ab^{n-1}, ab^n\}$.

Thus total number of conjugacy classes will be $\frac{n-1}{2} + 1 + 1 = \frac{n+3}{2}$ (for odd $n$).

**Case 2** When $n$ is even.

We have, $b^n = e$, so $b^n \in Z(D_n)$.

Therefore, conjugacy class of $b^n$, ie, $\{e\}$ consists of a single element.

Let us consider the element $b^{n/2}$ of $D_n$.

Let $b^j$, $1 \le j \le n$, be an element of $D_n$.

Then $b^{n/2}b^j = b^{(n/2)+j} = b^j b^{n/2}$, which implies that $b^{n/2}$ commutes with all the elements of the form $b^j$, $1 \le j \le n$.

Consider an element of the form $ab^i$, $1 \le j \le n$.

Then $b^{n/2}(ab^i) = (ab^i)b^{n/2}$.

Thus, $\alpha b^{n/2} = b^{n/2}\alpha \Rightarrow b^{n/2} = \alpha^{-1}b^{n/2}\alpha \Rightarrow b^{n/2} \in Z(D_n)$, for all $\alpha \in D_n$.

Thus the conjugacy class $\{b^{n/2}\}$ consists of the single element $b^{n/2}$.

Consider two elements of $D_n$ of the form $ab^i$ and $ab^j$ where $1 \le i < j \le n$.

**Case (i)** Both $i$ and $j$ are odd and $j - i = 2$.

Then $b^{-1}ab^ib = ab^j$ so that $ab^i \sim ab^j$.

Thus, $ab \sim ab^3 \sim ab^5 \sim ... \sim ab^{n-3} \sim ab^{n-1}$.

**Case (ii)** Both $i$ and $j$ are even and $j - i = 2$.

Then there exists an element $\alpha = ab^{i+1} \in D_n$ such that $\alpha^{-1}ab^i\alpha = ab^j$, ie $ab^i \sim ab^j$.

Thus, $ab^2 \sim ab^4 \sim ... \sim ab^{n-2} \sim ab^n$.

Let $b^i$ and $b^j$ be two elements of $D_n$ where $1 \le i < j < n$ and $i + j = n$.

Then there exists $\alpha = ab^2 \in D_n$ such that $\alpha^{-1}b^i\alpha = b^j \Rightarrow b^i \sim b^j$.

Consider the normalizer of an element of the form $b^i$, where $1 \le i < \frac{n}{2}$ and $\frac{n}{2} < i < n$.

Let $b^j \in D_n$ $(1 \le j \le n)$.

Then $N(b^i) = \{b, b^2, b^3, ..., b^n\}$.

Because if $ab^k \in N(b^i)$, $1 \le k \le n$, then we must have $b^i(ab^k) = (ab^k)b^i$ which implies $i = \frac{n}{2}$ which is a contradiction to the fact that $1 \le i < \frac{n}{2}$ and $\frac{n}{2} < i < n$.

Consider an element $b^i$, where $1 \le i < \frac{n}{2}$ and $\frac{n}{2} < i < n$.

Then, $|\{b^i\}| = \frac{|D_n|}{|N(b^i)|} = \frac{2n}{n} = 2$.

Thus conjugacy classes containing $b^i$, where $1 \le i < \frac{n}{2}$ and $\frac{n}{2} < i < n$, contains two elements and the number of conjugacy classes with only two elements is $\frac{n-2}{2}$. We can also prove that the conjugacy classes containing $ab$ or $ab^2$ does not contain any element of the form $b^i$ where $1 \le i < \frac{n}{2}$ and $\frac{n}{2} < i < n$.

Let us now compute the normalizer of $ab^i$, where $1 \le i \le n$.

Suppose $b^j \in N(ab^i)$ where $1 \le i \le n$.

Then $b^j(ab^i) = (ab^i)b^j \Rightarrow j = \frac{n}{2}$ or $j = n$.

Hence $\{b^n, b^{n/2}\} \subseteq N(ab^i)$.

Suppose $ab^j \in N(ab^i)$, $1 \le j \le n$, $j \ne i$.

Then $(ab^i)(ab^j) = (ab^j)(ab^i)$ gives $j = i + \frac{n}{2}$.

Thus $\{ab^i, ab^j\} \subseteq N(ab^i)$ where $j = i + \frac{n}{2}$.

Therefore, $N(ab^i) = \{b^n, b^{n/2}, ab^i, ab^{i+\frac{n}{2}}\}$.

So $|\{ab\}| = \frac{|D_n|}{|N(ab)|} = \frac{2n}{4} = \frac{n}{2}$ and $|\{ab^2\}| = \frac{|D_n|}{|N(ab^2)|} = \frac{2n}{4} = \frac{n}{2}$.

Thus conjugacy class of $ab$ and $ab^2$, contains $\frac{n}{2}$ elements and they are $ab, ab^3, ab^5, ..., ab^{n-3}, ab^{n-1}$ and $ab^2, ab^4, ab^6, ..., ab^{n-2}, ab^n$ respectively. Hence the total number of conjugacy classes will be $\frac{n-2}{2} + 2 + 2 = \frac{n+6}{2}$ (for even $n$).

**Theorem 3.1.1** The conjugacy class graph of $D_n$, $n \ge 3$ are given as follows:

$$\Gamma_g^{cl}(D_n) = \begin{cases} K_{\frac{n-1}{2}} \cup K_1 & \text{when } n \equiv 1\ (mod\ 2) \\ K_{\frac{n}{2}+1} & \text{when } n \equiv 0\ (mod\ 4) \\ K_{\frac{n}{2}-1} \cup K_2 & \text{when } n \equiv 2\ (mod\ 4) \end{cases}$$

**Proof.** Clearly the number of vertices in $\Gamma_g^{cl}(D_n)$ are $\frac{n+1}{2}$ and $\frac{n}{2}+1$ respectively.

**Case 1** $n$ odd.

The $\frac{n-1}{2}$ vertices of the form $\{b^k, b^{n-k}\}$ have cardinalities 2 and hence all vertices are adjacent. The only vertex of the form $\{a, ab, ab^2, ab^3, ..., ab^{n-2}, ab^{n-1}\}$ have cardinality $n$ and $n$ being odd, it is non-adjacent to the other vertices. Hence, we have a disconnected graph with 1 isolated vertex and a complete graph $K_{\frac{n-1}{2}}$.

**Case 2** $n$ even.

The $\frac{n}{2}-1$ vertices of the form $\{b^k, b^{n-k}\}$ have cardinalities 2 and hence all vertices are adjacent. The two vertices of the form $\{a, ab^2, ab^4, ..., ab^{n-2}\}$, $\{ab, ab^3, ab^5, ..., ab^{n-1}\}$ have cardinality $\frac{n}{2}$ each, so they are adjacent to each other and $n$ being even, they are adjacent to the other set of vertices if $\frac{n}{2}$ is even and non-adjacent if $\frac{n}{2}$ is odd. Hence we have

$$\Gamma_g^{cl}(D_n) = \begin{cases} K_{\frac{n}{2}+1} & \text{when } n \equiv 0\ (mod\ 4) \\ K_{\frac{n}{2}-1} \cup K_2 & \text{when } n \equiv 2\ (mod\ 4) \end{cases}$$

**Corollary 3.1.1** $\Gamma_g^{cl}(D_n)$ is non-planar for $n = 8$ and $n \geq 11$.

**Definition 3.1.2** The generalized quaternion group $Q_{4n}$ of order $4n$ is defined by the representation:

$$< x, y : x^{2n} = y^4 = e,\ x^n = y^2,\ y^{-1}xy = x^{-1} >$$

for $n \geq 2$.

**Lemma 3.1.2** The number of conjugacy classes of $Q_{4n}$ is $n + 3$ and the conjugacy classes are given by $\{a^{2n} = e\}$, $\{a, a^{2n-1}\}$, $\{a^2, a^{2n-2}\}$, ..., $\{a^{n-1}, a^{n+1}\}$, $\{a^n\}$, $\{ab, a^3b, ..., a^{2n-1}b\}$, $\{b, a^2b, a^4b, ..., a^{2n-2}b\}$.

The proof of this lemma can be derived similarly just as in the case of dihedral group.

Following results are obtained for $\Gamma_g^{cl}(Q_{4n})$:

**Theorem 3.1.2** The conjugacy class graph of $Q_{4n}$, $n \geq 2$ are given as follows:

$$\Gamma_g^{cl}(Q_{4n}) = \begin{cases} K_{n-1} \cup K_2 & \text{when } n \equiv 1\ (mod\ 2) \\ K_{n+1} & \text{when } n \equiv 0\ (mod\ 2) \end{cases}$$

**Proof.** Clearly, the number of vertices in $\Gamma_g^{cl}(Q_{4n})$ is $n + 1$.

**Case 1** $n$ is odd.

The $n - 1$ vertices of the form $\{a^k, a^{2n-k}\}$ have cardinality 2 and are adjacent to each other. The two vertices $\{ab, a^3b, ..., a^{2n-1}b\}$ and $\{b, a^2b, a^4b, ..., a^{2n-2}b\}$ have cardinality $n$ each so they are adjacent to each other, and $n$ being odd these two vertices are not adjacent to the $n - 1$ vertices having cardinality 2. Thus we get a disconnected graph with two components which are complete graphs $K_2$ and $K_{n-1}$.

**Case 2** $n$ is even.

The $n - 1$ vertices of the form $\{a^k, a^{2n-k}\}$ have cardinality 2 and are adjacent to each other. The two vertices $\{ab,$

$a^3b$, ..., $a^{2n-1}b\}$ and $\{b, a^2b, a^4b, ..., a^{2n-2}b\}$ have cardinality $n$ each so they are adjacent to each other, and $n$ being even these two vertices are adjacent to the $n-1$ vertices having cardinality 2. Thus we get a complete graph $K_{n+1}$.

**Corollary 3.1.2** $\Gamma_g^{cl}(Q_{4n})$ is non-planar for $n = 4$ and $n \geq 6$.

**Definition 3.1.3** The symmetric group $S_n$ defined over a set $X$ is the group whose elements are all bijective functions from $X$ to $X$, the group operation being composition of functions. The symmetric group on a set of $n$ elements has order $n!$.

**Definition 3.1.4** The *partition* of an integer $n$ is the number of ways of writing $n$ as a sum of positive integers. We denote the number of partition of $n$ by $p(n)$.

**Proposition 3.1.1** The number of conjugacy classes of $S_n$ in equal to the number of integer partitions of $n$. The size of the conjugacy class $|Cl|$ of $S_n$ is given as:

Let $\lambda$ be an unordered integer partition of $n$ such that $\lambda$ has $a_i$ parts of size $i$ for each $i$.

Then,

$$|Cl| = \frac{n!}{\prod (i)^{a_i}(a_i!)}$$

So, each partition of $n$ determines a conjugacy class of $S_n$.

Following are some results obtained for conjugacy class graph of $S_n$.

**Theorem 3.1.3** Conjugacy class graph of $S_n$ are complete graphs for $n = 5, 6, 7, 8$. The graph structures are given by:

$$\Gamma_g^{cl}(S_n) = \begin{cases} K_6 & \text{for } n = 5 \\ K_{10} & \text{for } n = 6 \\ K_{14} & \text{for } n = 7 \\ K_{21} & \text{for } n = 8 \end{cases}$$

**Proof.** Consider $n = 5$.

Now,

$$\begin{aligned} p(5) &= 5 \\ &= 4+1 \\ &= 3+2 \\ &= 3+1+1 \\ &= 2+2+1 \\ &= 2+1+1+1 \\ &= 1+1+1+1+1 \end{aligned}$$

Corresponding to each partition the number of elements in each conjugacy class are respectively 24, 30, 20, 20, 15, 10, 1. By definition, the conjugacy class graph does not contain the centre of the group, so the 6 vertices are 10, 15, 20, 20, 24, 30.

Since $p(6) = 11$, $p(7) = 15$, $p(8) = 22$ so there are 10, 14 and 21 vertices respectively in $\Gamma_g^{cl}(S_6)$, $\Gamma_g^{cl}(S_7)$, $\Gamma_g^{cl}(S_8)$. By direct calculation, size of the conjugacy classes are found out to be as follows (Table 1):

**Table 1.** Size of conjugacy classes of $S_5$, $S_6$, $S_7$, $S_8$

| Group | Size of conjugacy classes |
|---|---|
| $S_5$ | 1, 10, 15, 20, 20, 24, 30 |
| $S_6$ | 1, 15, 15, 40, 40, 45, 90, 90, 120, 120, 144 |
| $S_7$ | 1, 21, 70, 105, 105, 210, 210, 280, 420, 420, 504, 504, 630, 720, 840 |
| $S_8$ | 1, 28, 105, 112, 210, 420, 420, 1,120, 1,120, 1,120, 1,260, 1,260, 1,344, 1,680, 2,520, 2,688, 3,360, 3,360, 3,360, 4,032, 5,040, 5,760 |

Clearly, every pair of vertices have a common factor, so each pair of vertices are adjacent and hence we get a complete graph.

**Theorem 3.1.4** The conjugacy class graph of $S_p$, where $p$ is prime, is the complete graph $K_{n(p)-1}$.

**Proof.** There are $n(p)$ number of conjugacy classes in $S_p$. The size of the conjugacy classes are given by

$$|Cl| = \frac{p!}{\prod (i)^{a_i}(a_i!)}$$

where $a_i$ is the number of times $i$ occurs in the partition of $p$. So the conjugacy classes will always have $p$ as a common factor. Therefore, every pair of vertex is adjacent and since number of vertices in $\Gamma_g^{cl}(S_p)$ is $n(p) - 1$,

$$\Gamma_g^{cl}(S_p) = K_{n(p)-1}$$

**Conjecture** $\Gamma_g^{cl}(S_n)$ is non-planar for $n \geq 5$.

## 3.2 *Conjugacy class graph of $D_n \times D_m$*

**Definition 3.2.1** For two groups $G_1$ and $G_2$, the direct product $G_1 \times G_2$ is defined as:

$$G_1 \times G_2 = \{(g_1, g_2) : g_1 \in G_1, g_2 \in G_2\}$$

Two elements $(g_1, h_1)$ and $(g_2, h_2)$ are conjugate in $G \times H$ if and only if $g_1$ and $g_2$ are conjugate in $G$ and $h_1$ and $h_2$ are conjugate in $H$.

**Theorem 3.2.1** For odd $m$ and $n$, the conjugacy class graph of $D_n \times D_m$ comprises of a maximal complete subgraph $K_{n+m-2+\frac{(n-1)(m-1)}{4}}$ if $gcd(m, n) = 1$ and a maximal complete $K_{\frac{n-1}{2}+\frac{m-1}{2}+3}$ if $gcd(m, n) > 1$.

**Proof.** Conjugacy classes of $D_n$ are

$$\{e\}, \{b, b^{n-1}\}, \{b^2, b^{n-2}\}, \ldots, \left\{b^{\frac{n-1}{2}}, b^{\frac{n+1}{2}}\right\}, \{a, ab, ab^2, \ldots, ab^{n-1}\}$$

There are $\frac{n-1}{2}$ conjugacy classes of order 2, one conjugacy class of order one and one conjugacy class with $n$ number of elements so there will be $\left(\frac{n+3}{2}\right)\left(\frac{m+3}{2}\right)$ conjugacy classes in $D_n \times D_m$. Since conjugacy class graph do not contain the centre, there are $\left(\frac{n+3}{2}\right)\left(\frac{m+3}{2}\right) - 1$ vertices in $\Gamma_g^{cl}(D_n \times D_m)$. Orders of the conjugacy classes will be as follows:

$$\underbrace{2,2,...,2}_{\left(\frac{n-1}{2}+\frac{m-1}{2}\right)times}, \quad \underbrace{4,4,...,4}_{\left(\frac{n-1}{2}\right)\left(\frac{m-1}{2}\right)times}, \ m, \ n, \ \underbrace{2m,2m,...,2m}_{\left(\frac{n-1}{2}\right)times}, \ \underbrace{2n,2n,...,2n}_{\left(\frac{m-1}{2}\right)times}, \ mn$$

Since both $m$, $n$ are odd, the graph structure will be either of the two types depending on whether $gcd(m, n) = 1$ (Figure 1(a)) or $gcd(m, n) > 1$ (Figure 1(b)).

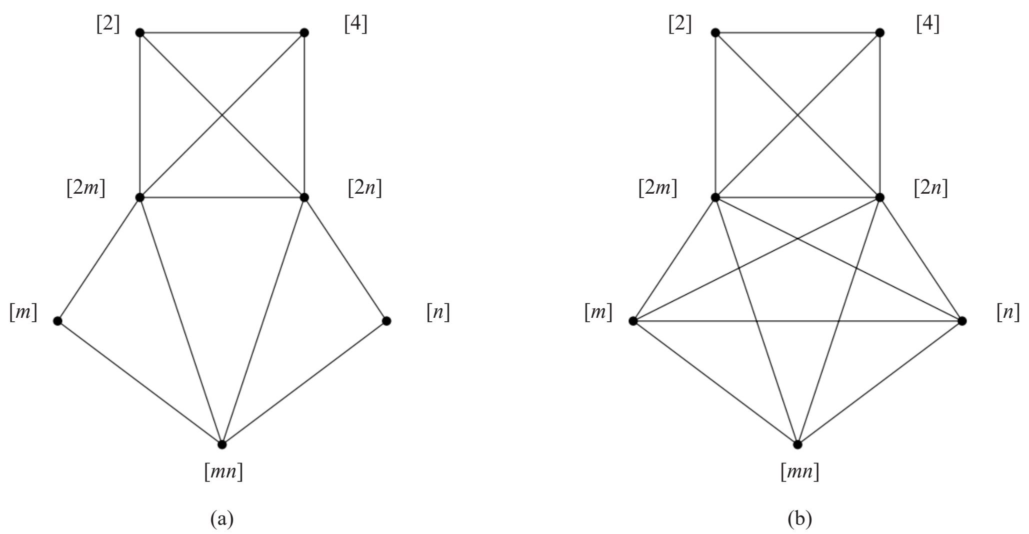


**Figure 1.** Conjugacy class graph of $D_n \times D_m$ (for odd values of $m$ and $n$)

Each vertex represents a set of vertices with same order and adjacent to each other. The adjacency between any two pair of vertices indicates that every point in a vertex is adjacent to each point in the other vertex and so on.

**Theorem 3.2.2** For even values of $m$ and $n$, the conjugacy class graph of $D_m \times D_n$ is either a complete graph $K_{(\frac{m+6}{2})(\frac{n+6}{2})-4}$ or comprises of a maximal complete subgraph $K_{2m+2n+(\frac{m}{2}-1)(\frac{n}{2}-1)}$ or $K_{2m+2n-8+(\frac{m}{2}-1)(\frac{n}{2}-1)}$.

**Proof.** Conjugacy classes of $D_m$ are

$$\{e\}, \left\{b, b^{m-1}\right\}, \left\{b^2, b^{m-2}\right\}, ..., \left\{b^{\frac{m}{2}}\right\}, \left\{a, ab^2, ab^4, ..., ab^{m-2}\right\}, \left\{ab, ab^3, ab^5, ..., ab^{m-1}\right\}$$

There are $\frac{m}{2}-1$ conjugacy classes of order 2, two conjugacy classes of order 1 and two conjugacy classes of order $\frac{m}{2}$. The number of conjugacy classes of $D_m$ being $\frac{m+6}{2}$, there will be $\left(\frac{m+6}{2}\right)\left(\frac{n+6}{2}\right)$ conjugacy classes in $D_m \times D_n$. So the number of vertices in $\Gamma_g^{cl}(D_m \times D_n)$ is $\left(\frac{m+6}{2}\right)\left(\frac{n+6}{2}\right) - 4$. There will be $m + n - 4$ vertices of order 2, $\left(\frac{m}{2}-1\right)\left(\frac{n}{2}-1\right)$ vertices of order 4, $n - 2$ vertices of order $m$, $m - 2$ vertices of order $n$, 4 vertices of order $\frac{m}{2}$, 4 vertices of order $\frac{n}{2}$ and 4 vertices of order $\frac{mn}{4}$.

We have the following cases:

**Case 1** Both $\frac{m}{2}$ and $\frac{n}{2}$ are even.

In this case, all the vertices will have even order and so *gcd* of the orders of any two vertices will be atleast 2. Hence we get a complete graph $K_{(\frac{m+6}{2})(\frac{n+6}{2})-4}$ (Figure 2).

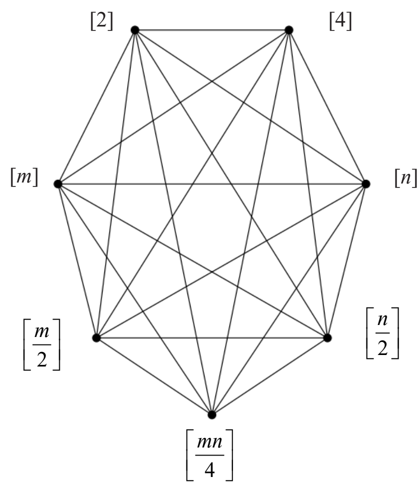


**Figure 2.** Conjugacy class graph of $D_n \times D_m$ (when $m$, $n = 0$ ($mod$ 4))

**Case 2** One of $\frac{m}{2}$ and $\frac{n}{2}$ is odd and the other even.

In either of the cases, we get a complete subgraph $K_{2m+2n+(\frac{m}{2}-1)(\frac{n}{2}-1)}$ and the graph structures are any one of the following depending on whether $n = 2$ ($mod$ 4), $m = 0$ ($mod$ 4) (Figure 3(a)) or $m = 2$ ($mod$ 4) and $n = 0$ ($mod$ 4) (Figure 3(b)):

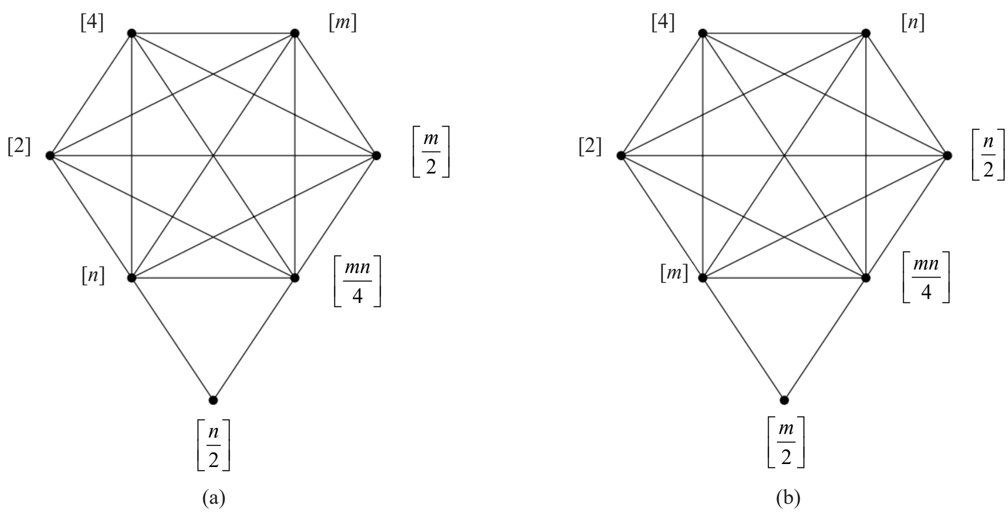


**Figure 3.** Conjugacy class graph of $D_n \times D_m$ (when one of $\frac{m}{2}$ or $\frac{n}{2}$ is odd and the other even)

**Case 3** Both $\frac{m}{2}$ and $\frac{n}{2}$ are odd.

In this case, the maximal complete subgraph of $\Gamma_g^{cl}(D_m \times D_n)$ will be $K_{2m+2n-8+(\frac{m}{2}-1)(\frac{n}{2}-1)}$.

In this case we will have three different graph structures depending on whether $gcd(m/2, n/2) = gcd(m, n/2) = gcd(n, m/2) = 1$ (Figure 4(a)) or $gcd(m, n/2) = gcd(n, m/2) = 1$ (Figure 4(b)) or $gcd(m/2, n/2)$, $gcd(m, n/2)$, $gcd(n, m/2) > 1$ (Figure 4(c)).

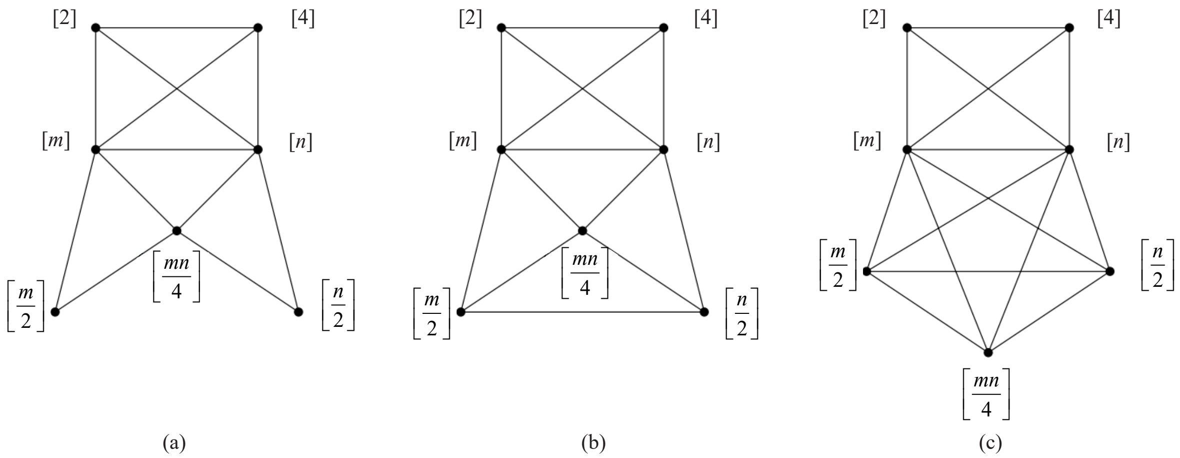


**Figure 4.** Conjugacy class graph of $D_n \times D_m$ (when $m, n = 2\ (mod\ 4)$)

**Theorem 3.2.3** For odd $n$ and even $m$, the conjugacy class graph of $D_n \times D_m$ comprises of a maximal complete subgraph $K_{2n+m+(\frac{m}{2}-1)(\frac{n-1}{2})}$ or $K_{2n+m-4+(\frac{m}{2}-1)(\frac{n-1}{2})}$.

**Proof.** Conjugacy classes of $D_n$ are

$$\{e\}, \{b, b^{n-1}\}, \{b^2, b^{n-2}\}, ..., \left\{b^{\frac{n-1}{2}}, b^{\frac{n+1}{2}}\right\}, \{a, ab, ab^2, ..., ab^{n-1}\}$$

Conjugacy classes of $D_m$ are

$$\{e\}, \{b, b^{m-1}\}, \{b^2, b^{m-2}\}, ..., \left\{b^{\frac{m}{2}}\right\}, \{a, ab^2, ab^4, ..., ab^{m-2}\}, \{ab, ab^3, ab^5, ..., ab^{m-1}\}$$

The number of conjugacy classes of $D_n$ is $\frac{n+3}{2}$ and the number of conjugacy classes of $D_m$ is $\frac{m+6}{2}$, so there will be $\left(\frac{m+6}{2}\right)\left(\frac{n+3}{2}\right)$ conjugacy classes in $D_n \times D_m$ and the number of vertices in $\Gamma_g^{cl}(D_m \times D_n)$ is $\left(\frac{m+6}{2}\right)\left(\frac{n+3}{2}\right) - 2$. Orders of the conjugacy classes will be as follows:

$$\underbrace{2, 2, ..., 2}_{\left(\frac{m}{2}+n-2\right) times}, \underbrace{4, 4, ..., 4}_{\left(\frac{m}{2}-1\right)\left(\frac{n-1}{2}\right) times}, \underbrace{m, m, ..., m}_{(n-1) times}, n, n\frac{m}{2}, \frac{m}{2}, \underbrace{2n, 2n, ..., 2n}_{\left(\frac{m}{2}-1\right) times}, \frac{mn}{2}, \frac{mn}{2}$$

We have the following cases:

**Case 1** $\frac{m}{2}$ is even.

In this case, all the vertices have even order except the vertex with order $n$. So $\Gamma_g^{cl}(D_m \times D_n)$ will have a complete subgraph $K_{2n+m+(\frac{m}{2}-1)(\frac{n-1}{2})}$.

**Case 2** $\frac{m}{2}$ is odd.

We have two subcases here: $gcd\left(n, \frac{m}{2}\right) = 1$ and $gcd\left(n, \frac{m}{2}\right) > 1$. Here, the vertices with orders 2, 4, $m$, $2n$ are always adjacent to each other. In any case, we get the maximal complete subgraph $K_{2n+m-4+(\frac{m}{2}-1)(\frac{n-1}{2})}$ of $\Gamma_g^{cl}(D_m \times D_n)$.

## 3.3 *Regularity of line graphs*

The Line graph $L(G)$ of a graph $G$ is another graph that represents the adjacencies between the edges of $G$. It follows without saying that the structure of a connected graph can be completely retrieved from its line graph.

**Proposition 3.3.1** [8] The line graph $L(K_n) = G$ of the complete graph $K_n$ has the following properties:

- $G$ has $\binom{n}{2}$ vertices.
- $G$ is regular of degree $2(n-2)$
- Every two non-adjacent points are mutually adjacent to exactly four points.
- Every two adjacent points are mutually adjacent to exactly $n-2$ points.

**Theorem 3.3.1** For odd $n$, line graph of $\Gamma_g^{cl}(D_n)$ is regular of degree $n-5$.

**Proof.** For odd $n$,

$$\Gamma_g^{cl}(D_n) = K_{\frac{n-1}{2}} \cup K_1$$

The number of edges of $\Gamma_g^{cl}(D_n) = {}^{\frac{n-1}{2}}C_2$.

Therefore the line graph of $\Gamma_g^{cl}(D_n)$ will have ${}^{\frac{n-1}{2}}C_2$ number of vertices. These vertices in $L(\Gamma_g^{cl}(D_n))$ correspond to the number of edges of the complete graph $K_{\frac{n-1}{2}}$ in $\Gamma_g^{cl}(D_n)$.

Line graph of a complete graph $K_n$ being regular of degree $2(n-2)$, so $L(\Gamma_g^{cl}(D_n))$ is regular of degree $2\left(\frac{n-1}{2}-2\right)$ $= n-5$.

**Corollary 3.3.1** For odd $n$, line graph of $\Gamma_g^{cl}(D_n)$ is eulerian.

**Proof.** From the previous theorem, $L(\Gamma_g^{cl}(D_n))$ is regular of degree $2\left(\frac{n-1}{2}-2\right) = n-5$.

Since $n$ is odd, vertex degree of each vertex is even. Hence $L(\Gamma_g^{cl}(D_n))$ is eulerian.

**Theorem 3.3.2** For even values of $n$ and $\frac{n}{2}$, line graph of $\Gamma_g^{cl}(D_n)$ is regular of degree $n-2$ and has an euler cycle.

**Proof.** For $n \equiv 0\ (mod\ 4)$,

$$\Gamma_g^{cl}(D_n) = K_{\frac{n}{2}+1}$$

So $L(\Gamma_g^{cl}(D_n))$ will have ${}^{\frac{n}{2}+1}C_2$ vertices and is regular of degree $n-2$.
$n$ being even, $n-2$ is even and $L(\Gamma_g^{cl}(D_n))$ being regular, degree of each vertex is even.
Hence, $L(\Gamma_g^{cl}(D_n))$ is eulerian where $n \equiv 0\ (mod\ 4)$.

**Theorem 3.3.3** For even $n$ and odd $\frac{n}{2}$, line graph of $\Gamma_g^{cl}(D_n)$ is disconnected with 1 isolated vertex and a regular graph component of degree $n-6$.

**Proof.** For $n \equiv 2$ ($mod$ 4),

$$\Gamma_g^{cl}(D_n) = K_2 \cup K_{\frac{n}{2}-1}$$

The line graph of $\Gamma_g^{cl}(D_n)$ will have one isolated vertex corresponding to the graph $K_2$ in $\Gamma_g^{cl}(D_n)$ and a regular graph with ${}^{\frac{n-1}{2}}C_2$ vertices, the degree of each vertex being $2\left(\frac{n}{2}-1-2\right) = n-6$.

**Theorem 3.3.4** For odd $n$, the line graph of $\Gamma_g^{cl}(Q_{4n})$ is disconnected with 1 isolated vertex and a component which is regular of degree $2n-6$.

**Proof.** For odd $n$,

$$\Gamma_g^{cl}(Q_{4n}) = K_2 \cup K_{n-1}$$

The line graph corresponding to $\Gamma_g^{cl}(Q_{4n})$ will have one isolated vertex and regular graph with ${}^{n-1}C_2$ vertices, each vertex degree being $2(n-1-2) = 2n-6$.

**Theorem 3.3.5** For even $n$, the line graph of $\Gamma_g^{cl}(Q_{4n})$ is regular of degree $2n-2$ and hence eulerian.

**Proof.** For even $n$,

$$\Gamma_g^{cl}(Q_{4n}) = K_{n+1}$$

Line graph of a complete graph being regular, $L(\Gamma_g^{cl}(Q_{4n}))$ will be reular with ${}^{n+1}C_2$ number of vertices, degree of each vertex being $2(n+1-2) = 2n-2$.

Also, $n$ being even, degree of each vertex is even and hence the graph is eulerian.

## 3.4 *Energy: Hyperenergetic and hypoenergetic graphs*

The concept of graph energy has its origin in chemistry involving study of conjugated hydrocarbons using a tight-binding method known as the *Hückel molecular orbital* (HMO) method. Motivated by its chemical implications, this concept was then extended to graphs and thereafter many research articles (see [9, 10]) have been published on *energy* of different types of graph structures. Although the concept loses its chemical meaning when extended to non-bipartite graphs, as a graph invariant the energy of a graph provides structural information about the graph. Other applications of energy include data mining, quality assessment, network analysis, satellite communications, crystallography, theory of macromolecules and biology.

**Theorem 3.4.1** $\Gamma_g^{cl}(D_n)$ is non-hyperenergetic for all $n$.

**Proof.** From the structure of $\Gamma_g^{cl}(D_n)$ the spectrum can be calculated as:

$$\left\{0, -1^{\left(\frac{n-3}{2}\right)}, \frac{n-3}{2}\right\}$$

$$\left\{-1^{\left(\frac{n}{2}\right)}, \frac{n}{2}\right\}$$

$$\left\{1, -1^{\left(\frac{n}{2}-1\right)}, \frac{n}{2}-2\right\}$$

for $n \equiv 1\ (mod\ 2)$, $n \equiv 0\ (mod\ 4)$ and $n \equiv 2\ (mod\ 4)$ respectively.

Thus, energy of $\Gamma_g^{cl}(D_n)$.

$$E(\Gamma_g^{cl}(D_n)) = \begin{cases} n-3 \\ n \\ n-2 \end{cases}$$

correspondingly.

Clearly, $\Gamma_g^{cl}(D_n)$ is non-hyperenergetic for $n \equiv 0\ (mod\ 4)$ the graph being a complete graph.

The number of vertices in $\Gamma_g^{cl}(D_n)$, for odd n, being $\frac{n+1}{2}$ and

$$E(\Gamma_g^{cl}(D_n)) = n-3$$

$$\ngtr 2\left(\frac{n+1}{2}\right) - 2 = n-1$$

so $\Gamma_g^{cl}(D_n)$ is non-hyperenergetic for odd values of $n$.

The number of vertices in $\Gamma_g^{cl}(D_n)$, for $n \equiv 2\ (mod\ 4)$, being $\frac{n}{2}+1$ and

$$E(\Gamma_g^{cl}(D_n)) = n-2$$

$$\ngtr 2(\frac{n}{2}+1) - 2 = n$$

so $\Gamma_g^{cl}(D_n)$ is non-hyperenergetic for $n \equiv 2\ (mod\ 4)$.

**Theorem 3.4.2** $\Gamma_g^{cl}(D_n)$ is hypoerenergetic for $n = 3, 5$.

**Proof.** $E(\Gamma_g^{cl}(D_n)) = n - 3$, for odd values of $n$.

The number of vertices of $\Gamma_g^{cl}(D_n)$ being $\frac{n+1}{2}$ and

$$E(\Gamma_g^{cl}(D_n)) = n-3 < \frac{n+1}{2},$$

for $n = 3, 5$.

Therefore, $\Gamma_g^{cl}(D_3)$ and $\Gamma_g^{cl}(D_5)$ are hypoenergetic.

**Theorem 3.4.3** $\Gamma_g^{cl}(Q_{4n})$ is non-hyperenergetic as well as non-hypoenergetic for all $n$.

**Proof.** The spectrum of $\Gamma_g^{cl}(Q_{4n})$ are

$$\left\{1, -1^{(n-1)}, n-2\right\}$$

$$\left\{-1^{(n)}, n\right\}$$

for odd and even values of $n$ respectively.

Clearly, $\Gamma_g^{cl}(Q_{4n})$ is non-hyperenergetic as well as non-hypoenergetic for even $n$, the graph being a complete graph.

For odd $n$,

$$E(\Gamma_g^{cl}(Q_{4n})) = 2n - 2$$

The number of vertices in $\Gamma_g^{cl}(Q_{4n})$ being $n + 1$, and $E(\Gamma_g^{cl}(Q_{4n})) = 2n - 2 \not> 2(n+1) - 2 = 2n$, so $\Gamma_g^{cl}(Q_{4n})$ is non-hyperenergetic for all values of $n$.

Also, $E(\Gamma_g^{cl}(Q_{4n})) = 2n - 2 \not< n + 1$, so $\Gamma_g^{cl}(Q_{4n})$ is non-hypoenergetic for all values of $n$.

**Theorem 3.4.4** $\Gamma_g^{cl}(D_n)$ and $\Gamma_g^{cl}(Q_{4n})$ are equienergetic where m is odd and $n = 2m$.

**Proof.** Since $m$ is odd,

$$\Gamma_g^{cl}(Q_{4m}) = K_{m-1} \cup K_2$$

and $n = 2m$ being even,

$$\Gamma_g^{cl}(D_n) = \begin{cases} K_{\frac{n}{2}+1}, & \text{if } n \equiv 0 \ (mod \ 4) \\ K_{\frac{n}{2}-1} \cup K_2, & \text{if } n \equiv 2 \ (mod \ 4) \end{cases}$$

Number of vertices in $\Gamma_g^{cl}(Q_{4n}) = m + 1$ and number of vertices in $\Gamma_g^{cl}(D_n) = \frac{n}{2} + 1 = m + 1$, so they have the same number of vertices.

From Theorem 3.4.1 and Theorem 3.4.3,

$$E(\Gamma_g^{cl}(Q_{4m})) = 2m - 2 = n - 2 = E(\Gamma_g^{cl}(D_n))$$

Hence they are equienergetic.

## Future study

The partition dimension of conjugacy class graphs related to different groups will be an interesting topic of further research and bounds can be established if it exists.

## Conflict of interest

On behalf of all the authors, the corresponding author states that there is no conflict of interest. No funding was received for conducting this study. This manuscript has not been submitted for publication elsewhere and has not been published in any other journal.

## References

[1] Bertram EA, Herzog M, Mann A. On a graph related to conjugacy classes of groups. *Bulletin of the London Mathematical Society*. 1990; 22(6): 569-575.

[2] Zulkarnain A, Sarmin NH, Noor AHBM. On the conjugate graphs of finite $p$-groups. *Malaysian Journal of Fundamental and Applied Sciences*. 2017; 13(2): 100-102.

[3] Sarmin NH, Bilhikmah NH, Omer SMS, Noor AHBM. The conjugacy classes, conjugate graph and conjugacy class graph of some finite metacyclic 2-groups. *Menemui Matematik (Discovering Mathematics)*. 2016; 38(1): 1-12.

[4] Zamri SNA, Sarmin NH, El-Sanfaz MA. The conjugate and generalized conjugacy class graphs for metacyclic 3-groups and metacyclic 5-groups. *AIP Conference Proceedings*. 2017; 1830(1): 070015. Available from: doi:

10.1063/1.4980964.
[5] Alolaiyan H, Yousaf A, Ameer M, Razaq A. Non-conjugate graphs associated with finite groups. *IEEE Access*. 2019; 7: 122849-122853. Available from: doi: 10.1109/ACCESS.2019.2938083.
[6] Gutman I. The energy of a graph. *Berichte der Mathematisch-Statistischen Sektion im Forschungszentrum Graz*. 1978; 103: 1-22.
[7] Walikar HB, Ramane HS, Hampiholi P. On the energy of a graph. *Graph Connections*. New Delhi: Allied Publishers; 1999. pp. 120-123.
[8] Frank H. *Graph Theory*. Narosa Publishing House; 2001.
[9] Kumar S, Sarkar P, Pal A. A study on the energy of graphs and its applications. *Polycyclic Aromatic Compounds*. 2023. Available from: doi: 10.1080/10406638.2023.2245104.
[10] Filipovski S, Jajcay R. Bounds for the energy of graphs. *Mathematics*. 2021; 9(14): 1687. Available from: doi: 10.3390/math9141687.
[11] Erfanian A, Tolue B. Conjugate graphs of finite groups. *Discrete Mathematics, Algorithms and Applications*. 2012; 4(2): 1250035. Available from: doi: 10.1142/S1793830912500358.
[12] Vaidya SK, Popat KM. On equienergetic, hyperenergetic and hypoenergetic graphs. *Kragujevac Journal of Mathematics*. 2020; 44(4): 523-532.
[13] Hou Y, Gutman I. Hyperenergetic line graphs. *Match*. 2001; 43: 29-39.