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# Analyze the Eigenvalues of a Neutral Delay Differential Equation by Employing the Generalized Lambert W Function

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**Abstract:** In this paper the authors discuss a class of differential equations known as neutral delay differential equations (NDDEs) in which the delay occurs in the derivative with the highest order. They are encountered in various fields of applied sciences and are essential for mathematically representing real-world phenomena. It was challenging to discover approximate analytical solutions for certain methods. In this study, the researchers utilized the generalized Lambert W function to derive the characteristic roots for a first-order neutral delay differential equation featuring random delays. Numerical examples were then utilized to validate the obtained results.

*Keywords*: neutral delay differential equation, Lambert W function, characteristic root

**MSC:** 34A05, 34A25, 34A30, 34E05, 34K06, 42A10, 44A10

## 1. Introduction

In Delay Differential Equations (DDEs), the values of the solution and its derivatives in a particular instant depend on the time derivatives at a distinct earlier time instant. The general form of these equation described as

$$x^k(t) = f(t, x(t), \ldots, x^{k-1}(t), x(d_0), \ldots, x^k(d_k)), \text{ where } d_j \equiv d_j(t, x(t)) \tag{1}$$

is known as the delay, sustaining $d_j \leq t$ everywhere in the time interval $[t_0, t_1]$ given, for $j = 0, \ldots, k$. Neutral delay differential equations (NDDEs) are frequently encountered in many practical areas. DDEs, in particular NDDEs, offer an effective mathematical tool to represent different phenomena from issues in real life. New analytical techniques are being developed quickly in recent times with the goal of solving various classes of DDEs. It is clear that many of these newly

proposed methodologies have had some trouble locating approximate analytical solutions that lead to an accurate solution of DDEs, more specifically NDDEs. Due to their distinct transcendental nature, finding analytical solutions for Neutral Delay Differential Equations (NDDEs) with constant and proportional delays is extremely difficult. Therefore, numerical techniques are commonly employed for their solution [1–4]. Hence, it becomes increasingly vital to develop an innovative analytical approach for addressing these equations.

Neutral Delay Differential Equations find applications across a diverse range of fields such as neural networks, engineering, bioscience, economics, fluid dynamics, physics, and chemistry. Challenges within these domains have frequently prompted researchers and physicists to invest significant efforts in exploring intriguing phenomena, such as the impact of vibrating systems attached to an elastic bar. Within the realm of biology, a neutral logistic equation is employed to represent the growth of a population of a single species. Numerical evaluation of the Lambert W function and its application in generating generalized Gaussian noise with an exponent [5]. In addition, [6] utilizes this model to depict the torsional characteristics of a flexible rod that incorporates mass.

In 2020, Normah M and Bardez J [7, 8] proposed a resolution for NDDE by employing HAM and the Natural transform method. However, it proves challenging to identify an initial approximate solution due to its tedious nature. Piriadarshani D and Sasikala K [9] achieved stabilization of the equation by employing numerical methods with the Lambert $W$ function, successfully obtaining approximate roots for the equation.

## 1.1 *The Lambert $W$ function*

In this the authors presented some basic concept of Lambert $W$ function.

Euler and Lambert examined the transcendental equation $xe^x = a$ before [10, 11]. The inverse of the above equation is determined by $W$ indicates the Lambert function and. It gives the solution denoted by $W(a)$. The well-known A transcendental equation that appears in a variety of applications can be solved using the Lambert-W function, including population dynamics, physics, and combinatorics (see [12–14]).

## 1.2 *The Generalized Lambert $W$ function*

In this the authors discussed Generalized Lambert $W$ function.

The Generalized Lambert $W$ function, as introduced by T.C. Scott et al., serves as the solution to the equation described as follows:

$$e^x \frac{(x-t_1)(x-t_2)\dots(x-l_n)}{(x-s_1)(x-s_2)\dots(x-s_m)} = z. \tag{2}$$

The solution of equation (2) can be represented [10],

$$W \begin{pmatrix} t_1 \ t_2 \ \dots \ t_n \\ s_1 \ s_2 \ \dots \ s_n \end{pmatrix} ; Z \end{pmatrix}. \tag{3}$$

Specifically,

$$W(t;z) = \log(z), \ W(t;z) = W(z), \ W(t;z) = W\left(-\frac{1}{Z}\right)$$

$$W(t;z) = t + W(ze^{-t}), \ W(t;z) = s - W\left(-\frac{e^s}{Z}\right).$$

The solution function of the Lambert function is actually the equation $xe^x + rx = z$. Where $r$ is real or arbitrary parameter.

$$W_r(z)e^{W_r(z)} + rW_r(z) = z.$$

## 1.3 *Existence and computation of characteristic roots of NDDE*

In this the authors discussed some the Existence and computation of Characteristic roots of NDDE.

**Theorem 1.** *For the first order NDDE*

$$x'(t) + ax(t) + a_d x(t-h) + b_d x'(t-h) = 0,\ t \in [0,\ T_1];\ \ x(t) = \varphi(t),\ T \in [-h,\ 0] \tag{4}$$

*The characteristic root is given by*

$$\lambda = \frac{1}{h} W_{b_d e^{(ah)}}(-b_d\ h\ e^{(ah)}(-a + \frac{a_d}{b_d})) - a.$$

*Proof.* The characteristic equation of NDDE (4) is

$$\lambda e^{\lambda t} + a e^{\lambda t} + a_d e^{\lambda(t-h)} + b_d \lambda e^{\lambda(t-h)} = 0. \tag{5}$$

It can be written as

$$e^{-\lambda h} = \frac{-(\lambda - (-a))}{b_d\left(\lambda - \frac{b_d}{a_d}\right)}$$

$$e^{\lambda h} \frac{(\lambda + a)}{(\lambda + \frac{b_d}{a_d})} = -b_d.$$

Which can be denoted as

$$hW\begin{pmatrix} -ha \\ -h\frac{a_d}{b_d} & -b_d \end{pmatrix}. \tag{6}$$

The characteristic equation of Neutral Delay Differential Equation is

$$\lambda e^{\lambda t} + a e^{\lambda t} + a_d e^{\lambda(t-h)} + b_d \lambda e^{\lambda(t-h)} = 0$$

$$(\lambda + a)e^{\lambda t} + (a_d + b_d \lambda)e^{\lambda(t-h)} = 0.$$

Multiplying by $h e^{h(\lambda+a)-\lambda t}$ yields

$$(\lambda + a)e^{\lambda t} h e^{h(\lambda+a)-\lambda t} + (a_d + b_d \lambda)e^{\lambda(t-h)} h e^{h(\lambda+a)-\lambda t} = 0$$

$$(\lambda + a)e^{\lambda t} h e^{h(\lambda+a)-\lambda t} + a_d e^{\lambda(t-h)} h e^{h(\lambda+a)-\lambda t} + b_d \lambda e^{\lambda(t-h)} h e^{h(\lambda+a)-\lambda t} = 0$$

$$h e^{h(\lambda+a)}(\lambda + a) + h e^{(ah)} a_d + h e^{h(a)} b_d \lambda = 0$$

$$h(\lambda + a)e^{h(\lambda+a)} + a_d h e^{(ah)} + b_d h \lambda e^{(ah)} = 0$$

$$h(\lambda + a)e^{h(\lambda+a)} + b_d h \lambda e^{(ah)} = -a_d\ h\ e^{(ah)}.$$

Adding and subtracting $b_d ahe^{(ah)}$ and after rearranging the equation's terms, we obtain

$$h(\lambda + a) \, e^{h(\lambda+a)} + b_d h\lambda e^{(ah)} + b_d ahe^{(ah)} = -a_d \, he \, (ah) + b_d ahe^{(ah)}$$

$$h(\lambda + a)e^{h(\lambda+a)} + b_d \, he \, (ah)(\lambda + a) = -a_d \, he \, (ah) + b_d ahe^{(ah)}$$

It follows that

$$\lambda = \frac{1}{h}W_{b_d e^{(ah)}}(b_d ahe^{(ah)} - a_d he^{(ah)}) - a. \tag{7}$$

Comparing (6) and (7), we get

$$hW\left(\begin{array}{cc} -ha \\ -h\frac{a_d}{b_d} & -b_d \end{array} ; -b_d\right) = \frac{1}{h}W_{b_d e^{(ah)}}\left(-b_d he^{(ah)}\left(-a + \frac{a_d}{b_d}\right)\right) - a.$$

The $r$ - Lambert $W$ function have five branch structures

**(i)** $r = 0$;

**(ii)** $r > \dfrac{1}{e^2}$;

**(iii)** $0 < r < \dfrac{1}{e^2}$;

**(iv)** $r = \dfrac{1}{e^2}$; and

**(v)** $r < 0$

**Case 1:**

$$r = b_d e^{(ah)} = 0 \rightarrow b_d = 0 \text{ or } e^{ah} = 0.$$

Since $e^{ah} \neq 0$ hence $b_d = 0$.

If $r = 0 \rightarrow b_d = 0$, then the Lambert $W$ function is obtained.

Hence $\lambda = \dfrac{1}{h}W_{b_d e^{(ah)}}(b_d ahe^{(ah)} - a_d he^{(ah)}) - a$ reduces to

$$\lambda = \frac{1}{h}W_0(-a_d he) - a.$$

**Case 2:** If $r > \dfrac{1}{e^2}$, then $sgnW(x) = sgn(x)$, where $W_r(x) : R \to R$ is differentiable and strictly increasing function universally.

$$r > \frac{1}{e^2} \Rightarrow b_d e^{(ah)} > \frac{1}{e^2}$$

$$e^2 e^{(ah)} > \frac{1}{b_d} \Rightarrow e^{2+ah} > \frac{1}{b_d}$$

$$2 + ah > \log \overline{b_d} \Rightarrow 2 + ah > \log b_d^{-1}$$

$$2 + ah > -\log b_d \Rightarrow ah > -2 - \log b_d$$

$$sgn \ W_r(x) = sgn(x).$$

*sgn* is a signum function defined as

$$sgn(x) = \begin{cases} -1 \ if \ x < 0 \\ 0 \ if \ x = 0 \\ 1 \ if \ x > 0 \end{cases}$$

Here $x = W_{b_d e^{(ah)}}(b_d ahe^{(ah)} - a_d he(ah))$.

$$sgn(W_{b_d e^{(ah)}}(b_d ahe^{(ah)} - a_d he^{(ah)})) = sgn(b_d ahe^{(ah)} - a_d he^{(ah)})$$

$$sgn(b_d ahe^{(ah)} - a_d he^{(ah)}) = \begin{cases} -1 \ if \ b_d ahe^{(ah)} - a_d \ h \ e^{\ (ah)} < 0 \\ 0 \ if \ b_d ahe^{(ah)} - a_d \ h \ e^{\ (ah)} = 0 \\ 1 \ if \ b_d ahe^{(ah)} - a_d \ h \ e^{\ (ah)} > 0 \end{cases}$$

$$sgn(b_d ahe^{(ah)} - a_d he^{(ah)}) = \begin{cases} -1 \ if \ b_d ahe^{(ah)} < a_d he^{(ah)} \\ 0 \ if \ b_d ahe^{(ah)} = a_d \ h \ e^{\ (ah)} \\ 1 \ if \ b_d ahe^{(ah)} > a_d he^{(ah)} \end{cases}$$

$$sgn(b_d ahe^{(ah)} - a_d he^{(ah)}) = \begin{cases} -1 \ if \ b_d a < a_d \\ 0 \ if \ b_d a = a_d \\ 1 \ if \ b_d a > a_d \end{cases}$$

**Case 3:** If $r = \dfrac{1}{e^2}$, then

$$b_d e^{(ah)} = \frac{1}{e^2} \Rightarrow e^2 e^{(ah)} = \frac{1}{b_d} \Rightarrow e^{2+ah} = \frac{1}{b_d} \Rightarrow 2 + ah = \log \frac{1}{b_d}.$$

Hence, $b_d e^{(ah)} = \dfrac{1}{e^2}$, then $W_r(x) : R \to R$ differentiable function everywhere and is a strictly increasing, on $R \backslash \{-\dfrac{4}{e^2}\}$, where $sgnW_r(x) = sgn(x)$.

**Case 4:** If $0 < r < \dfrac{1}{e^2}$, then three branches of $W_r$ are represented by $w_{r,-2}, w_{r,-1}, w_{r,0}$. Assume $\alpha_r = W_{-1}(-re) - 1$ and $\beta_r = W_0(-re) - 1$.

In which $W_0$ and $W_{-1}$ denote two branches of the Lambert function and can be represented as follows:

$w_{r,-2} :\, ]-\infty, f_r(\alpha_r)] \rightarrow ]-\infty, \alpha_r]$ is a function increase strictly.

$w_{r,-1} : [f_r(\alpha_r), f_r(\beta_r)] \rightarrow [-\alpha_r/\beta_r]$ is a function decrease strictly.

$w_{r,0} : [f_r\infty [\, \rightarrow [\beta_r/\infty]$ is a function decrease strictly.

The above functions are differentiable on their respective domains interiorly.

**Case 5:** $w_{r,-1}$ and $w_{r,0}$ are the two branches of $W_r(x)$ for $r < 0$. Let

$$\gamma_r = W(-re) - 1,$$

where classical Lambert function is denoted by $W$.

Henceforth, we have that for these branches.

$w_{r,-1} : [f_r(\gamma_r), +\infty[ \rightarrow ]-\infty, \gamma_r]$ is a function increases strictly.

$w_{r,0} : [f_r + \infty[ \rightarrow [\gamma_r + \infty[$ is a function increases strictly.

The above functions are differentiable on their respective domains interiorly.

The branches $w_{r,-2}$ and $w_{r,-1}$ takes non positive values, where $w_{r,0}$ is positive for all values of $r$, only if it is less than $\dfrac{1}{e^2}$.

$\square$

**Corollary 1.** *As*

$$-b_d h e^{(ah)}\left(-a + \frac{a_d}{b_d}\right) \rightarrow \infty,$$

$$W_{b_d e^{(ah)}}\left(b_d a h e^{(ah)} - a_d h e^{(ah)}\right) \sim \frac{-b_d h e^{(ah)}\left(-a + \frac{a_d}{b_d}\right)}{b_d e^{(ah)}},$$

*gives* $\lambda = \dfrac{(a_d)}{b_d}$.

*Proof.* We know that for a generalized lambert $W$ function we have as $x \rightarrow \infty W_r(x) \sim \dfrac{x}{r}$.

The characteristic root of the NDDE is

$$\lambda = \frac{1}{h} W_{b_d e^{(ah)}}\left(b_d a h e^{(ah)} - a_d h e^{\,(ah)}\right) - a$$

$$W_{b_d e^{(ah)}}\left(b_d a h e^{(ah)} - a_d h e^{\,(ah)}\right) \sim \frac{-b_d h e^{(ah)}\left(-a + \frac{a_d}{b_d}\right)}{b_d e^{(ah)}}$$

$$\lambda = \frac{1}{h} \frac{-b_d h e^{(ah)}\left(-a + \frac{a_d}{b_d}\right)}{b_d e^{(ah)}} - a$$

$$\lambda = \frac{(a_d)}{b_d}.$$

Hence $\lambda$ is stable, $\lambda < 0 \rightarrow \dfrac{a_d}{b_d} < 0, \rightarrow a_d < b_d$. □

**Example 1.** *Neutral Delay Differential Equation where*

$$x'(t) - +ax(t) + bx(t-h) + cx'(t-h) = 0, x(t) = \emptyset(t), t \in [-h,\ 0],$$

*where $a = 1, b = 2.3, c = 0, h = 1.7$.*
   *Apply case 1, we get*

$$\lambda = \frac{1}{1.7} W_0(-2.3 * 1.7e^{1.7}) - 1$$

$$\lambda = 0.1541 + 1.3418i.$$

**Example 2.** *We show that Neutral Delay Differential Equation $x'(t) + 0.3x(t) + 0.5x(t-2) + 0.7x'(t-2) = 0$ is stable when $a_d < b_d$ and $a = 0.3, a_d = 0.5, b_d = 0.7, h = 2$.*
   *Black dots are denoted by the spectrum values, the imaginary parts and real parts of the characteristic equation are denoted by red and blue lines respectively. We have all the roots present left half z-plane, which can be verified using the Figure 1, QPmR spectrum [13] of this characteristic roots*
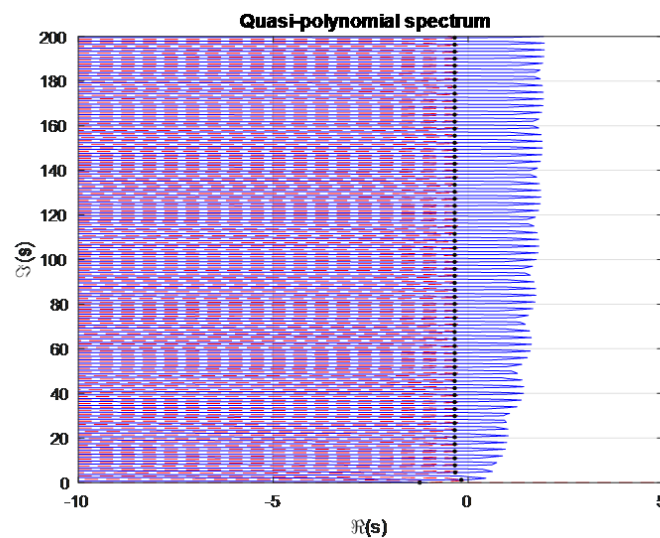


**Figure 1.** Quasi polynomial spectrum when *ad < bd*.

**Corollary 2.** *When $b_d e^{(ah)} = 1/e^2$ is continuous and $a\dfrac{-a_d}{b_d} - \dfrac{4}{h}$, we get $W_{b_d e^{(ah)}}(b_d a h e^{(ah)} - a_d h e^{(ah)}) = -2$, which shows stability of equation.*

*Proof.* We know that when $r = \dfrac{1}{e^2} = 0.135335$ is continuous implies $b_d e^{(ah)} = \dfrac{1}{e^2} = 0.135335$. $e^{ah} = \dfrac{1}{e^2 b_d} = \dfrac{0.135335}{b_{c1}}$.

Taking log on both sides $ah = \log\left[\dfrac{0.135335}{b_d}\right]$

$$a = \frac{1}{h}[-0.8686 - \log b_d]$$

$$W_{b_d e^{(ah)}}\left(b_d a h e^{(ah)} - a_d h e^{(ah)}\right) = W_{\frac{1}{e^2}}\left(-\frac{4}{e^2}\right)$$

Implies that

$$b_d a h \frac{1}{e^2 b_d} - a_d h \frac{1}{e^2 b_d} = -\frac{4}{e^2}$$

$$ah - a_d h \frac{1}{b_d} = -4$$

$$a = \frac{a_d}{b_d} - \frac{4}{h}.$$

When $b_d = \dfrac{e^{-ah}}{e^2}$ and $a\dfrac{-a_d}{b_d} - \dfrac{4}{h}$, we get $W_{b_d e^{(ah)}}\left(b_d a h e^{(ah)} - a_d h e^{(ah)}\right) = -2$ then

$$\lambda = \frac{1}{h} W_{b_d e^{(ah)}}\left(b_d a h e^{(ah)} - a_d h e^{(ah)}\right) - a$$

$$\lambda = \frac{1}{h}(-2) - a$$

$$\lambda = \frac{-2}{h} - a < 0.$$

Which shows the stability. □

**Example 3.** *The NDDE is stable when $a = \dfrac{a_d}{b_d} - \dfrac{4}{h}$ for the equation $x'(t) - 0.0902x(t) + 1.4324x(t-2) + 0.75x'(t-2) = 0$ which is shown in the spectrum of roots (see Figure 2).*
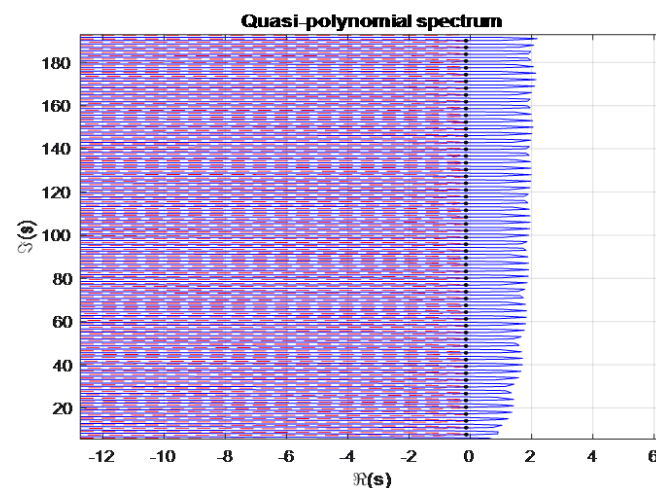
**Figure 2.** Quasi-polynomial spectrum when $a = \frac{a_d}{b_d} - \frac{h}{4}$.

## 2. Conclusion

The generalized Lambert $W$ function of different branches was employed to compute and analyze the characteristic roots of neutral delay differential equation of first order with delays. This offers an alternative perspective on the roots by incorporating our results into the equation. Numerical examples were also generated to validate the obtained results. Moreover, it can be extended to encompass second-order dynamics and variable delays.

## Author contributions

All authors are equally contributed in the research work.

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## Conflict of interest

The authors declare no conflict of interest.

## References

[1]  Bhrawy AH, Alghamdi MA, Baleanu D. Numerical solution of a class of functional-differential equations using Jacobi pseudospectral method. *Abstr. Appl. Anal.* 2013; 2013: 9.

[2]  Sakar MG. Numerical solution of neutral functional-differential equations with proportional delays. *Int. J. Optim. Control. Theor. Appl. IJOCTA*. 2017; 7: 186194.

[3]  Vanani SK, Aminataei A. On the numerical solution of neutral delay differential equations using multiquadric approximation scheme. *Bull. Korean Math. Soc.* 2008; 45: 663-670.

[4]    Zhao JJ, Cao Y, Xu Y. Legendre-Gauss collocation methods for nonlinear neutral delay diff erential equations. *Adv. Difference Equ.* 2015; 2015: 24.

[5]    Corless RM, Gonnet GH, Hare DEG, et al. On the LambertW function. *Advance in Computational Mathematics*. 1996; 5: 329-359. https://doi.org/10.1007/BF02124750.

[6]    Scott TC, Mann R, Martinez RE. General relativity and quantum mechanics: towards a generalization of the Lambert W function. *Appl. Algebra Engg. Comm. Comput.* 2006; 17(1): 41-47.

[7]    Mező I, Baricz Á. On the generalization of the Lambert W function. *Trans. Amer. Math. Soc.* 2017; in press. available online: http://arxiv.org/abs/1408.3999.

[8]    Mező I, Keady G. Some physical applications of generalized Lambert functions. *Eur. J. Phys.* 2016; 37: 065802. http://dx.doi.org/10.1088/0143-0807/37/6/065802.

[9]    Piriadarshani D, Sasikala K. Lambert W Function Approach on Stabilization of Neutral Delay Differential Equation. *Journal of Adv Research in Dynamical & Control Systems*. 2020; 12: 480-488.

[10]   Waleed M, Abd-Elhameed M, Tenreiro Machado JA, Youssri H. Hypergeometric fractional derivatives formula of shifted Chebyshev polynomials: tau algorithm for a type of fractional delay differential equations. *International Journal of Nonlinear Sciences and Numerical Simulation*. 2022; 23(7-8): 1253-1268.

[11]   Hafez RM, Youssri YH. Shifted Gegenbauer-Gauss Collocation Method for Solving Fractional Neutral Functional-Differential Equations With Proportional Delays. *Kragujevac Journal of Mathematics*. 2022; 46(6): 981-996.

[12]   Maan N, Barde J. Analytical technique for neutral delay differential equations with proportional and constants delays. *Journal of Mathematics and Computer Science*. 2020; 20: 334-348.

[13]   Chapeau Blondeau F, Monir A. Numerical evaluation of the Lambert W function and application to generation of generalized Gaussian noise with exponent. *IEEE Transactions on Signal Processing*. 2002; 50(9): 2160-2165. Available from: 10.1109/TSP.2002.801912.

[14]   Corless RM, Gonnet GH, Hare DEG, Jeffrey DJ, Knuth DE. On the Lambert *W* Function. *Adv. Comput. Math.* 1996; 5: 329-359.