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Research Article

Analyze the Eigenvalues of a Neutral Delay Differential Equation by Employing the Generalized Lambert W Function

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Abstract: In this paper the authors discuss a class of differential equations known as neutral delay differential equations (NDDEs) in which the delay occurs in the derivative with the highest order. They are encountered in various fields of applied sciences and are essential for mathematically representing real-world phenomena. It was challenging to discover approximate analytical solutions for certain methods. In this study, the researchers utilized the generalized Lambert W function to derive the characteristic roots for a first-order neutral delay differential equation featuring random delays. Numerical examples were then utilized to validate the obtained results.

***Keywords*:** neutral delay differential equation, Lambert W function, characteristic root

MSC: 34A05, 34A25, 34A30, 34E05, 34K06, 42A10, 44A10

1. Introduction

In Delay Differential Equations (DDEs), the values of the solution and its derivatives in a particular instant depend on the time derivatives at a distinct earlier time instant. The general form of these equation described as

$$x^k(t) = f(t, x(t), \ \ldots, \ x^{k-1}(t), x(d_0), \ \ldots, x^k(d_k)), \ \text{where } d_j \equiv d_j(t, x(t)) \tag{1}$$

is known as the delay, sustaining $d_j \leq t$ everywhere in the time interval $[t_0, \ t_1]$ given, for $j = 0, \ldots, k$. Neutral delay differential equations (NDDEs) are frequently encountered in many practical areas. DDEs, in particular NDDEs, offer an effective mathematical tool to represent different phenomena from issues in real life. New analytical techniques are being

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developed quickly in recent times with the goal of solving various classes of DDEs. It is clear that many of these newly proposed methodologies have had some trouble locating approximate analytical solutions that lead to an accurate solution of DDEs, more specifically NDDEs. Due to their distinct transcendental nature, finding analytical solutions for Neutral Delay Differential Equations (NDDEs) with constant and proportional delays is extremely difficult. Therefore, numerical techniques are commonly employed for their solution [1–4]. Hence, it becomes increasingly vital to develop an innovative analytical approach for addressing these equations.

Neutral Delay Differential Equations find applications across a diverse range of fields such as neural networks, engineering, bioscience, economics, fluid dynamics, physics, and chemistry. Challenges within these domains have frequently prompted researchers and physicists to invest significant efforts in exploring intriguing phenomena, such as the impact of vibrating systems attached to an elastic bar. Within the realm of biology, a neutral logistic equation is employed to represent the growth of a population of a single species. Numerical evaluation of the Lambert W function and its application in generating generalized Gaussian noise with an exponent [5]. In addition, [6] utilizes this model to depict the torsional characteristics of a flexible rod that incorporates mass.

In 2020, Normah et al. [7, 8] proposed a resolution for NDDE by employing HAM and the Natural transform method. However, it proves challenging to identify an initial approximate solution due to its tedious nature. Piriadarshani et al. [9] achieved stabilization of the equation by employing numerical methods with the Lambert W function, successfully obtaining approximate roots for the equation.

1.1 *The Lambert W function*

In this the authors presented some basic concept of Lambert W function.

Euler and Lambert examined the transcendental equation $xe^x = a$ before [10, 11]. The inverse of the above equation is determined by W indicates the Lambert function and. It gives the solution denoted by $W(a)$. The well-known A transcendental equation that appears in a variety of applications can be solved using the Lambert-W function, including population dynamics, physics, and combinatorics (see [12–14]).

1.2 *The generalized Lambert W function*

In this the authors discussed Generalized Lambert W function.

The Generalized Lambert W function, as introduced by [6] serves as the solution to the equation described as follows:

$$e^x \frac{(x-t_1)(x-t_2)\ldots(x-l_n)}{(x-s_1)(x-s_2)\ldots(x-s_m)} = z. \tag{2}$$

The solution of equation (2) can be represented [10],

$$W\left(\begin{matrix} t_1\ t_2\ \ldots\ t_n \\ s_1\ s_2\ \ldots\ s_n \end{matrix}; Z\right). \tag{3}$$

Specifically,

$$W(t;z) = \log(z),\ W(t;z) = W(z),\ W(t;z) = W\left(-\frac{1}{Z}\right)$$

$$W(t;z) = t + W(ze^{-t}),\ W(t;z) = s - W(-\frac{e^s}{Z}).$$

The solution function of the Lambert function is actually the equation $xe^x + rx = z$. Where r is real or arbitrary parameter.

$$W_r(z)e^{W_r(z)} + rW_r(z) = z.$$

1.3 *Existence and computation of characteristic roots of NDDE*

In this the authors discussed some the Existence and computation of Characteristic roots of NDDE.

Theorem 1 For the first order NDDE

$$x'(t) + ax(t) + a_d x(t-h) + b_d x'(t-h) = 0,\ t \in [0,\ T_1];\quad x(t) = \varphi(t),\ T \in [-h,\ 0] \tag{4}$$

The characteristic root is given by

$$\lambda = \frac{1}{h} W_{b_d e^{(ah)}}(-b_d\ h\ e^{(ah)}(-a + \frac{a_d}{b_d})) - a.$$

Proof. The characteristic equation of NDDE (4) is

$$\lambda e^{\lambda t} + ae^{\lambda t} + a_d e^{\lambda(t-h)} + b_d \lambda e^{\lambda(t-h)} = 0. \tag{5}$$

It can be written as

$$e^{-\lambda h} = \frac{-(\lambda - (-a))}{b_d\left(\lambda - \frac{b_d}{a_d}\right)}$$

$$e^{\lambda h}\frac{(\lambda + a)}{(\lambda + \frac{b_d}{a_d})} = -b_d.$$

Which can be denoted as

$$hW\begin{pmatrix} -ha & \\ -h\frac{a_d}{b_d} & -b_d \end{pmatrix}. \tag{6}$$

The characteristic equation of Neutral Delay Differential Equation is

$$\begin{aligned} \lambda e^{\lambda t} + ae^{\lambda t} + a_d e^{\lambda(t-h)} + b_d \lambda e^{\lambda(t-h)} &= 0 \\ (\lambda + a)e^{\lambda t} + (a_d + b_d \lambda)e^{\lambda(t-h)} &= 0. \end{aligned}$$

Multiplying by $he^{h(\lambda+a)-\lambda t}$ yields

$$(\lambda+a)e^{\lambda t}he^{h(\lambda+a)-\lambda t}+(a_d+b_d\lambda)e^{\lambda(t-h)}he^{h(\lambda+a)-\lambda t}=0$$

$$(\lambda+a)e^{\lambda t}he^{h(\lambda+a)-\lambda t}+a_de^{\lambda(t-h)}he^{h(\lambda+a)-\lambda t}+b_d\lambda e^{\lambda(t-h)}he^{h(\lambda+a)-\lambda t}=0$$

$$he^{h(\lambda+a)}(\lambda+a)+he^{(ah)}a_d+he^{h(a)}b_d\lambda=0$$

$$h(\lambda+a)e^{h(\lambda+a)}+a_dhe^{(ah)}+b_dh\lambda e^{(ah)}=0$$

$$h(\lambda+a)e^{h(\lambda+a)}+b_dh\lambda e^{(ah)}=-a_d\ h\ e^{(ah)}.$$

Adding and subtracting $b_dahe^{(ah)}$ and after rearranging the equation's terms, we obtain

$$\begin{aligned} h(\lambda+a)\ e^{h(\lambda+a)}+b_dh\lambda e^{(ah)}+b_dahe^{(ah)} &= -a_d\ he\ (ah)+b_dahe^{(ah)} \\ h(\lambda+a)e^{h(\lambda+a)}+b_d\ he\ (ah)(\lambda+a) &= -a_d\ he\ (ah)+b_dahe^{(ah)} \end{aligned}$$

It follows that

$$\lambda=\frac{1}{h}W_{b_de^{(ah)}}(b_dahe^{(ah)}-a_dhe^{(ah)})-a. \tag{7}$$

Comparing (6) and (7), we get

$$hW\left(\begin{matrix}-ha & \\ -h\frac{a_d}{b_d} & -b_d\end{matrix}\ ;-b_d\right)=\frac{1}{h}W_{b_de^{(ah)}}\left(-b_dhe^{(ah)}\left(-a+\frac{a_d}{b_d}\right)\right)-a.$$

The r-Lambert W function have five branch structures
(i) $r=0$;
(ii) $r>\frac{1}{e^2}$;
(iii) $0<r<\frac{1}{e^2}$;
(iv) $r=\frac{1}{e^2}$; and
(v) $r<0$
Case 1:

$$r=b_de^{(ah)}=0\rightarrow b_d=0\ \text{ or } e^{ah}=0.$$

Since $e^{ah}\neq 0$ hence $b_d=0$.

If $r = 0 \to b_d = 0$, then the Lambert W function is obtained.

Hence $\lambda = \frac{1}{h} W_{b_d e^{(ah)}} (b_d a h e^{(ah)} - a_d h e^{(ah)}) - a$ reduces to

$$\lambda = \frac{1}{h} W_0(-a_d h e) - a.$$

Case 2: If $r > \frac{1}{e^2}$, then $sgnW(x) = sgn(x)$, where $W_r(x)$: $R \to R$ is differentiable and strictly increasing function universally.

$$r > \frac{1}{e^2} \Rightarrow b_d e^{(ah)} > \frac{1}{e^2}$$

$$e^2 e^{(ah)} > \frac{1}{b_d} \Rightarrow e^{2+ah} > \frac{1}{b_d}$$

$$2 + ah > \log \overline{b_d} \Rightarrow 2 + ah > \log b_d^{-1}$$

$$2 + ah > -\log b_d \Rightarrow ah > -2 - \log b_d$$

$$sgn\ W_r(x) = sgn(x).$$

sgn is a signum function defined as

$$sgn(x) = \begin{cases} -1\ if\ x < 0 \\ 0\ if\ x = 0 \\ 1\ if\ x > 0 \end{cases}$$

Here $x = W_{b_d e^{(ah)}} (b_d a h e^{(ah)} - a_d h e(ah))$.

$$sgn(W_{b_d e^{(ah)}}(b_d ahe^{(ah)} - a_d he^{(ah)})) = sgn(b_d ahe^{(ah)} - a_d he^{(ah)})$$

$$sgn(b_d ahe^{(ah)} - a_d he^{(ah)}) = \begin{cases} -1 \; if \; b_d ahe^{(ah)} - a_d \; h \; e^{\;(ah)} < 0 \\ 0 \; if \; b_d ahe^{(ah)} - a_d \; h \; e^{\;(ah)} = 0 \\ 1 \; if \; b_d ahe^{(ah)} - a_d \; h \; e^{\;(ah)} > 0 \end{cases}$$

$$sgn(b_d ahe^{(ah)} - a_d he^{(ah)}) = \begin{cases} -1 \; if \; b_d ahe^{(ah)} < a_d he^{(ah)} \\ 0 \; if \; b_d ahe^{(ah)} = a_d \; h \; e^{\;(ah)} \\ 1 \; if \; b_d ahe^{(ah)} > a_d he^{(ah)} \end{cases}$$

$$sgn(b_d ahe^{(ah)} - a_d he^{(ah)}) = \begin{cases} -1 \; if \; b_d a < a_d \\ 0 \; if \; b_d a = a_d \\ 1 \; if \; b_d a > a_d \end{cases}$$

Case 3: If $r = \frac{1}{e^2}$, then

$$b_d e^{(ah)} = \frac{1}{e^2} \Rightarrow e^2 e^{(ah)} = \frac{1}{b_d} \Rightarrow e^{2+ah} = \frac{1}{b_d} \Rightarrow 2 + ah = \log \frac{1}{b_d}.$$

Hence, $b_d e^{(ah)} = \frac{1}{e^2}$, then $W_r(x)$: $R \to R$ differentiable function everywhere and is a strictly increasing, on $R \backslash \{-\frac{4}{e^2}\}$, where $sgnW_r(x) = sgn(x)$.

Case 4: If $0 < r < \frac{1}{e^2}$, then three branches of W_r are represented by $w_{r,\,-2}$, $w_{r,\,-1}$, $w_{r,\,0}$. Assume $\alpha_r = W_{-1}(-re) - 1$ and $\beta_r = W_0(-re) - 1$.

In which W_0 and W_{-1} denote two branches of the Lambert function and can be represented as follows:

$w_{r,\,-2}$: $]-\infty,\, f_r(\alpha_r)] \to]-\infty,\, \alpha_r]$ is a function increase strictly.

$w_{r,\,-1}$: $[f_r(\alpha_r),\, f_r(\beta_r)] \to [-\alpha_{r/}\beta_r]$ is a function decrease strictly.

$w_{r,\,0}$: $[f_r \infty\, [\, \to [\beta_r/\infty]$ is a function decrease strictly.

The above functions are differentiable on their respective domains interiorly.

Case 5: $w_{r,\,-1}$ and $w_{r,\,0}$ are the two branches of $W_r(x)$ for $r < 0$. Let

$$\gamma_r = W(-re) - 1,$$

where classical Lambert function is denoted by W.

Henceforth, we have that for these branches.

$w_{r,\,-1}$: $[f_r(\gamma_r),\; +\infty[\to] - \infty,\, \gamma_r]$ is a function increases strictly.

$w_{r,\,0}$: $[f_r + \infty[\, \to\; [\gamma_r + \infty[$ is a function increases strictly.

The above functions are differentiable on their respective domains interiorly.

The branches $w_{r,\,-2}$ and $w_{r,\,-1}$ takes non positive values, where $w_{r,\,0}$ is positive for all values of r, only if it is less than $\frac{1}{e^2}$. □

Corollary 1 As

$$-b_d he^{(ah)}(-a+\frac{a_d}{b_d}) \to \infty,$$

$$W_{b_d e^{(ah)}}(b_d ahe^{(ah)} - a_d he^{(ah)}) \sim \frac{-b_d he^{(ah)}(-a+\frac{a_d}{b_d})}{b_d e^{(ah)}},$$

gives $\lambda = \frac{(a_d)}{b_d}$.

Proof. We know that for a generalized lambert W function we have as $x \to \infty W_r(x) \sim \frac{x}{r}$.

The characteristic root of the NDDE is

$$\lambda = \frac{1}{h} W_{b_d e^{(ah)}}(b_d ahe^{(ah)} - a_d he^{\ (ah)}) - a$$

$$W_{b_d e^{(ah)}}(b_d ahe^{(ah)} - a_d he^{\ (ah)}) \sim \frac{-b_d he^{(ah)}(-a+\frac{a_d}{b_d})}{b_d e^{(ah)}}$$

$$\lambda = \frac{1}{h}\frac{-b_d he^{(ah)}(-a+\frac{a_d}{b_d})}{b_d e^{(ah)}} - a$$

$$\lambda = \frac{(a_d)}{b_d}.$$

Hence λ is stable, $\lambda < 0 \to \frac{a_d}{b_d} < 0, \to a_d < b_d$. □

Example 1 Neutral Delay Differential Equation where

$$x'(t) - +ax(t) + bx(t-h) + cx'(t-h) = 0,\ x(t) = \emptyset(t),\ t \in [-h,\ 0],$$

where $a = 1$, $b = 2.3$, $c = 0$, $h = 1.7$.

Apply case 1, we get

$$\lambda = \frac{1}{1.7} W_0(-2.3 * 1.7e^{1.7}) - 1$$

$$\lambda = 0.1541 + 1.3418i.$$

Example 2 We show that Neutral Delay Differential Equation $x'(t) + 0.3x(t) + 0.5x(t-2) + 0.7x'(t-2) = 0$ is stable when $a_d < b_d$ and $a = 0.3$, $a_d = 0.5$, $b_d = 0.7$, $h = 2$.

Black dots are denoted by the spectrum values, the imaginary parts and real parts of the characteristic equation are denoted by red and blue lines respectively. We have all the roots present left half z-plane, which can be verified using the Figure 1, QPmR spectrum [13] of this characteristic roots

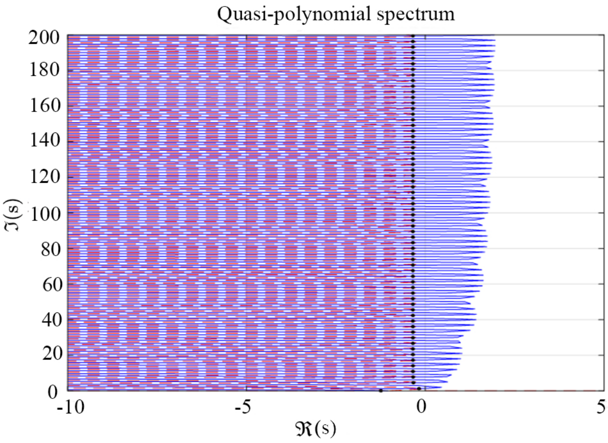

Figure 1. Quasi polynomial spectrum when *ad* < *bd*

Corollary 2 When $b_d e^{(ah)} = 1/e^2$ is continuous and $a\frac{-a_d}{b_d} - \frac{4}{h}$, we get $W_{b_d e^{(ah)}}(b_d a h e^{(ah)} - a_d h e^{(ah)}) = -2$, which shows stability of equation.

Proof. We know that when $r = \frac{1}{e^2} = 0.135335$ is continuous implies $b_d e^{(ah)} = \frac{1}{e^2} = 0.135335$. $e^{ah} = \frac{1}{e^2 b_d} = \frac{0.135335}{b_{c1}}$.

Taking log on both sides $ah = \log\left[\frac{0.135335}{b_d}\right]$

$$a = \frac{1}{h}[-0.8686 - \log b_d]$$

$$W_{b_d e^{(ah)}}(b_d a h e^{(ah)} - a_d h e^{(ah)}) = W_{\frac{1}{e^2}}(-\frac{4}{e^2})$$

Implies that

$$b_d a h \frac{1}{e^2 b_d} - a_d h \frac{1}{e^2 b_d} = -\frac{4}{e^2}$$

$$ah - a_d h \frac{1}{b_d} = -4$$

$$a = \frac{a_d}{b_d} - \frac{4}{h}.$$

When $b_d = \frac{e^{-ah}}{e^2}$ and $a\frac{-a_d}{b_d} - \frac{4}{h}$, we get $W_{b_d e^{(ah)}}(b_d a h e^{(ah)} - a_d h e^{(ah)}) = -2$ then

$$\lambda = \frac{1}{h} W_{b_d e^{(ah)}} (b_d a h e^{(ah)} - a_d h e^{\,(ah)}) - a$$

$$\lambda = \frac{1}{h}(-2) - a$$

$$\lambda = \frac{-2}{h} - a < 0.$$

Which shows the stability. □

Example 3 The NDDE is stable when $a = \dfrac{a_d}{b_d} - \dfrac{4}{h}$ for the equation $x'(t) - 0.0902x(t) + 1.4324x(t-2) + 0.75x'(t-2) = 0$ which is shown in the spectrum of roots (see Figure 2).

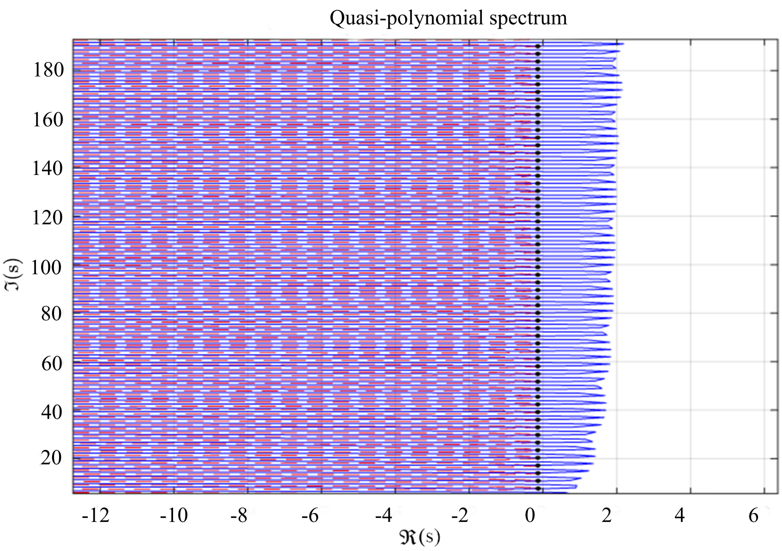

Figure 2. Quasi-polynomial spectrum when $a = \frac{a_d}{b_d} - \frac{h}{4}$

2. Conclusion

The generalized Lambert W function of different branches was employed to compute and analyze the characteristic roots of neutral delay differential equation of first order with delays. This offers an alternative perspective on the roots by incorporating our results into the equation. Numerical examples were also generated to validate the obtained results. Moreover, it can be extended to encompass second-order dynamics and variable delays.

Author contributions

All authors are equally contributed in the research work.

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Conflict of interest

The authors declare no competing financial interest.

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