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Research Article

Application of Clifford Algebra on Group Theory

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Abstract: The orthogonal operators defined as similarity transformations on Euclidean space *E* can also be considered as group actions on the Clifford Algebra. In this paper, we investigate the finite subgroup of Euclidian space *E* of Geometric Algebra over a finite dimension vector space *E*. The hierarchy of the finite subgroups of Clifford Algebra *C(E)* is depicted through the lattice structure and we discussed the group action of these subgroups on the vector space *E*. Further, we shall address the number of non-trivial finite subgroups, Normal subgroups, and subnormal series of the subgroup $B_3$of Clifford Algebra *C(E)* constructed over the vector space *E* by performing group action $\Psi: B_3 \times B_3 \rightarrow B_3$ over the subgroup B_3 of Clifford Algebra *C(E)*.

Keywords: geometric algebra, Clifford Algebra, group action, equivalence class, principle homogeneous space, subnormal series, solvable group

MSC: 03C05, 03E20

1. Introduction

William K. Clifford introduced the Geometric Algebra which is now usually known as Clifford Algebra [1] by creating a new multiplication rule in Grassmann's Exterior Algebra [2, 3]. Clifford was able to unite two seemingly unrelated mathematical frameworks: the algebra of extensions, which Hermann Grassmann invented, and the quaternion, which Sir William Rowan Hamilton constructed [4]. Clifford observed that Hamilton and Grassmann were tackling the same problem from different perspectives, and consequently, he combined both ideas to define the Geometric Product in 1876. In 1878, Clifford made a significant contribution by integrating the framework of quaternion into Grassmann's algebra of extensions [5]. This combination resulted in a system specifically designed to align with the orthogonal geometry of any given space. Within this system, quaternion emerged as a specific instance of a broader category known as Clifford algebra. However, his ideas were not further developed by his contemporary mathematicians for many years [6].

The Clifford algebras have been developed with the involvement of several Mathematicians and physicists such as Rudolf Lipschits, Theodor Valen, Elie Cartan, Claude

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Chevalley reinvented the ”Clifford Algebra” [7] and established its power as a formal mathematics and physics language. Specifically, David Hesten and Elie Cartan are notable contributors to the progress and development of Clifford algebra. Elie Cartan presented the idea of the spinor in 1913 and in 1938 the idea of the pure spinor and he defined Clifford algebra's as algebras of matrices and found that 8 has a periodicity inside these algebraic structures, for more info, refer [8]. David Hesten extended the concept of "Clifford Algebra" to devise a formalism and calls it Geometric Algebra [6]. He defines orthogonal operators as similarity transformations on Euclidian space E, which can also be considered as group actions in Clifford Algebra on the underlying Vector Space. In this paper, we investigate the finite subgroup of Euclidian space of Clifford Algebra over a finite dimension vector space E and we will the show that the subgroup B_3 of Clifford algebra is Solvable.

2. Preliminaries:

2.1 Clifford Algebra:

The algebra denoted as $C(E)$, which is defined over an n-dimensional vector space E, includes a Clifford map denoted as $\rho: E \rightarrow C(E)$. This Clifford map must fulfil the condition that for any algebra L and any other Clifford map $\psi: E \rightarrow L$, there exists a unique algebra homomorphism $\psi_*: C(E) \rightarrow L$ see the Figure 1, such that $\psi_* o \rho = \psi$ [5,7].

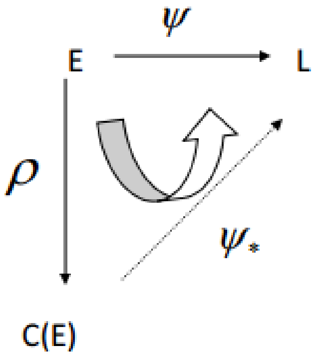

Figure 1: unique algebra homomorphism

2.2 Clifford's original definition:

Grassmann's exterior algebra $\wedge \mathrm{R}^n$ of the linear space R^n is an associative algebra of dimension 2^n. In terms of a basis $\{ \sigma_1, \sigma_2, \sigma_3, \dots \dots, \sigma_n \}$ for R^n the exterior algebra $\wedge \mathrm{R}^n$ has a basis,

1

$\sigma_1, \sigma_2, \sigma_3, \dots \dots, \sigma_n$

$\sigma_1 \wedge \sigma_2,\ \sigma_1 \wedge \sigma_3, \dots . \sigma_1 \wedge \sigma_n,\ \sigma_2 \wedge \sigma_3, \dots \dots \sigma_(n-1) \wedge \sigma_n$

.

.

.

$\sigma_1 \wedge \sigma_2 \wedge \sigma_3 \dots \dots \wedge \sigma_n$

The exterior algebra has unit 1 and satisfies the multiplication rules

$$\sigma_i \wedge \sigma_j = -\sigma_j \wedge \sigma_i \text{ for } i \neq j$$

$$\sigma_i \wedge \sigma_i = 0 \quad (1)$$

Clifford in 1882 kept the first rule but altered the second rule, and arrived at the multiplication rule

$$\sigma_i \sigma_j = -\sigma_j \sigma_i \text{ for } i \neq j$$

$$\sigma_i \sigma_i = 1 \quad (2)$$

This time $\{ \sigma_1, \sigma_2, \sigma_3, \dots \dots, \sigma_n \}$ is an orthonormal basis for the positive definite Euclidean space R^n. An associative algebra of dimension 2^n so defined is the Clifford algebra Cl_n.

Clifford in 1878, considered the multiplication rules

$$\sigma_i \sigma_j = -\sigma_j \sigma_i \text{ for } i \neq j$$

$$\sigma_i \sigma_i = -1 \quad (3)$$

of the Clifford algebra $Cl_{(0,n)}$ of the negative definite space $R^{(0,n)}$ [7, 9].

2.3 Clifford Mapping:

A linear mapping $\psi: E \to L$ is referred to as a Clifford map from E to L provided that it meets the following criteria:

$$[\psi(x)]^2 = g(x,x)1_L = \| x \|^2 \, 1_L$$

(4)

Which is equivalent to,

$$\psi(x)\psi(y) + \psi(y)\psi(x) = 2g(x,y)1_L \quad \forall x, y \varepsilon E \quad (5)$$

2.4 Even Clifford Algebra:

Assume that E is a vector space having n dimensions and $\{\sigma_i\}_{i=1}^{n}$ is a basis of E then the 2^n elements $\sigma_{i_1}, \sigma_{i_2}, \dots \dots \sigma_{i_s}$ for $i_1 < i_2 < \cdots < i_s$, $0 \leq s \leq n$ form a basis for the Clifford Algebra *C(E)*. Let *L*=*C(E)* and $\psi: E \to C(E)$ be given by $\psi(u) = -u$ then we find a isomorphism ψ_* of *C(E)* into itself so that $\psi_*(u) = -u, u \in E$, and may be written as $\psi_*(a) = a'$.

Now we put,

$$C^+(E) = \{a \in C(E): a' = a \quad \& \quad C^-(E) = \{a \in C(E): a' = -a\}, \text{ also } \sigma_s' = (-1)^{|s|}\sigma_s$$

Therefore the product $\sigma_1\sigma_2\sigma_3 \dots \sigma_s$ belongs to $C^+(E)$ or $C^-(E)$ according as 's' is even or odd, then the elements $\sigma_{i_1}\sigma_{i_2} \dots \dots \sigma_{i_s}$ for $i_1 < i_2 < \cdots . < i_s$, $0 \leq s \leq n$ with even or odd s forms a basis of $C^+(E)$ or $C^-(E)$ respectively. Thus, Clifford Algebra is the direct sum of even part $C^+(E)$ and odd $C^-(E)$, i.e,

$$C(E) = C^+(E) \oplus C^-(E)$$

(6)

The even part is not only a subspace but also a sub-algebra of *C(E)* which we call Even Clifford Algebra [5,7,10].

2.5 Arbitrary Element of Clifford Algebra:

The following formal polynomial represents an arbitrary element $\mathcal{A}$ in the Clifford Algebra $Cl_{(p,q)}$:

$$\mathcal{A} = a^0\sigma_0 + \sum_{i=1}^{n} a^i\sigma_i + \sum_{i=1}^{n}\sum_{j=1}^{n} a^{ij}\sigma_{ij} + \cdots + \sum_{i_1=1}^{n} \dots \sum_{i_k=1}^{n} a^{i_1\dots i_k}\sigma_{i_1\dots i_k} + + \cdots + a^{12\dots n}\sigma_{12\dots n} = \sum_{k=0}^{n} a^{i_1 i_2\dots i_k}\sigma_{i_1 i_2\dots i_k} \quad (7)$$

2.6 Fundamental Automorphism of Clifford Algebra:

Clifford Algebra $Cl_{(p,q)}$ has four fundamental automorphism, which are as follows [7, 11, 12]:

2.6.1 *Identity:*

Let $\mathcal{A}$ be any random element of Clifford Algebra $Cl_{(p,q)}$, the Identity automorphism from $\mathcal{A} \to \mathcal{A}$ is one which carries $\sigma_i \to \sigma_i$.

2.6.2 *Involution:*

Let $\mathcal{A} = \mathcal{A}' + \mathcal{A}''$ be the decomposition of an element of Clifford Algebra $Cl_{(p,q)}$, where $\mathcal{A}'$ and $\mathcal{A}''$ contains homogeneous odd and even components individually, then automorphism $\mathcal{A} \to \mathcal{A}^*$ is the Involution so that the sign of the elements of $\mathcal{A}''$ doesn't change and the sign of elements of $\mathcal{A}'$ changes, i.e,

$$\mathcal{A}^* = -\mathcal{A}' + \mathcal{A}''$$

In general Involution automorphism caries $\sigma_i \to -\sigma_i$, for any element of $\mathcal{A}$.

The Involution automorphism can also be expressed with the help of the volume element ω, i.e $\omega = \sigma_{i_1 i_2} \cdots \cdot_{i_{p+q}}$, such that, $\mathcal{A} = \omega\mathcal{A}\omega^{-1}$, where $\omega^{-1} = (-1)^{\frac{(p+q)(p+q-1)}{2}}\omega$. [1, 11]

2.6.3 *Reversion:*

The Reversion of any element of Clifford Algebra $Cl_{(p,q)}$ is the Antiautomorphism from $\mathcal{A} \to \tilde{\mathcal{A}}$, that is an alternative to any basis element $\sigma_{i_1 i_2} \cdots \cdot_{i_k} \in \mathcal{A}$ by an element of $\sigma_{i_k i_{k-1}} \cdots \cdot_{i_1}$, such that:

$$\sigma_{i_1 i_2} \cdots \cdot_{i_k} = (-1)^{\frac{k(k-1)}{2}} \sigma_{i_k i_{k-1}} \cdots \cdot_{i_1},$$

Hence for any element $\mathcal{A}$ of Clifford Algebra $Cl_{(p,q)}$,

$$\tilde{\mathcal{A}} = (-1)^{\frac{k(k-1)}{2}}\mathcal{A}$$

2.6.4 *Conjugation:*

The Conjugation of any element $\mathcal{A}$ of Clifford Algebra $Cl_{(p,q)}$ is the Antiautomorphism from $\mathcal{A} \to \tilde{\mathcal{A}}^*$ which is the composition of Involution and Reversion Automorphism, [11] such that

$$\tilde{\mathcal{A}}^* = (-1)^{\frac{k(k-1)}{2}}\mathcal{A}$$

3. The Lattice Structure of finite sub-group of Clifford Algebra:

In this section, we give the classification of finite sub-groups of Clifford Algebra. The hierarchy of sub groups is represented in the lattice structure format which is given in the Figure 2.

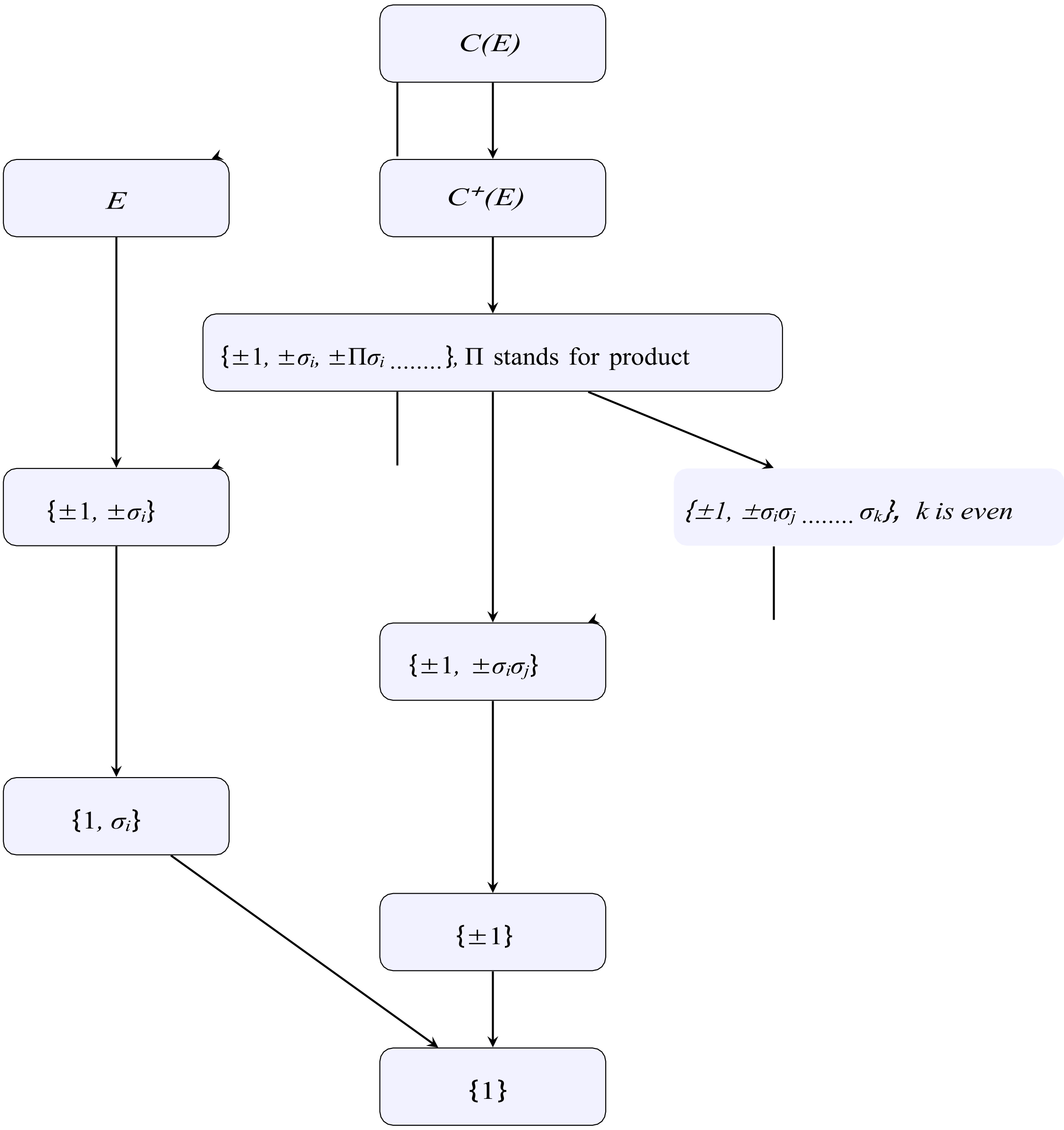

Figure 2: Lattice Structure of Clifford Algebra

4. Group action:

Let G be a set of any elements and a binary operation (*) together satisfied the Closure law, Associative law, and if there exist Unit element and Inverse of that element then we say that the set G is a group under the operation (*). If the above group G also satisfied the commutative law then we called it as Abelian group [13].

Similarly, we can easily define the Subgroup, Normal group, Permutation group etc, for clarity refer [13,14]. Here our main aim is to explain the Group Action which we will explain below.

Cayley's theorem states that it is possible to find a subgroup of the Permutations group *S(X)*, which is isomorphic to any given group . Let G be a subgroup of *S(X)*. Then, for any element $a \in G$ and any element $x \in X$, there exists an element $a(x) \in X$, and this relationship satisfies certain conditions.

- $e(x) = x$, where e embodies the identity in G,
- $(a, b)(x) = a(b(x)), \ \forall x \in X \ \& \ a, b \ \in G,$

The definition that follows abstracts this.
If X is any non-empty set, then let G be a group. Any mapping $\theta: G \times X \to X$ that satisfies the following criteria is an action of G on X [15].

1. $\theta(e, x) = x \ \ \forall x \in X$, where e embodies the identity in G
2. $\theta(ab, x) = \theta(a, \theta(b, x)) \ \ \forall \ a, b \in G.$

If there exists a situation where G performs an action on X , it can be said that G acts on X . We study the finite groups and their action on various subgroup E of *C (E)* and their subgroups which in turn are subgroups of C (E).

Let us consider the following finite subgroup with respect to the operation geometric product.

$$B_3 = \{\pm 1, \pm\sigma_1, \pm\sigma_2, \pm\sigma_3, \pm\sigma_1\sigma_2, \pm\sigma_1\sigma_3, \pm\sigma_2\sigma_3, \pm\sigma_1\sigma_2\sigma_3\} \quad (8)$$

The composition table given in the Table 1 shows the action of B_3 on itself defined by: $\varphi: B_3 \times B_3 \to B_3$ so that $\varphi(x, y) = xy \ \ \forall \ x, y \in B_3$ and the subgroup is,

$$B_3 = \pm 1, \pm\sigma_1, \pm\sigma_2, \pm\sigma_1\sigma_2 \quad (9)$$

It can be verified that φ represents a group action on B_3.

	1	-1	σ_1	$-\sigma_1$	σ_2	$-\sigma_2$	σ_3	$-\sigma_3$	$\sigma_1\sigma_2$	$\sigma_2\sigma_1$	$\sigma_1\sigma_3$	$\sigma_3\sigma_1$	$\sigma_2\sigma_3$	$\sigma_3\sigma_2$	$\sigma_1\sigma_2\sigma_3$	$\sigma_3\sigma_2\sigma_1$
1	1	-1	σ_1	$-\sigma_1$	σ_2	$-\sigma_2$	σ_3	$-\sigma_3$	$\sigma_1\sigma_2$	$\sigma_2\sigma_1$	$\sigma_1\sigma_3$	$\sigma_3\sigma_1$	$\sigma_2\sigma_3$	$\sigma_3\sigma_2$	$\sigma_1\sigma_2\sigma_3$	$\sigma_3\sigma_2\sigma_1$
-1	-1	1	$-\sigma_1$	σ_1	$-\sigma_2$	σ_2	σ_3	$-\sigma_3$	$\sigma_1\sigma_2$	$\sigma_2\sigma_1$	$\sigma_1\sigma_3$	$\sigma_3\sigma_1$	$\sigma_2\sigma_3$	$\sigma_3\sigma_2$	$\sigma_3\sigma_2\sigma_1$	$\sigma_1\sigma_2\sigma_3$
σ_1	σ_1	$-\sigma_1$	1	-1	$\sigma_1\sigma_2$	$\sigma_2\sigma_1$	$\sigma_1\sigma_3$	$\sigma_3\sigma_1$	σ_2	$-\sigma_2$	$\sigma_1\sigma_2\sigma_3$	$\sigma_3\sigma_2\sigma_1$	$-\sigma_3$	σ_3	$\sigma_2\sigma_3$	$\sigma_3\sigma_2$
$-\sigma_1$	$-\sigma_1$	σ_1	-1	1	$\sigma_2\sigma_1$	$\sigma_1\sigma_2$	$\sigma_3\sigma_1$	$\sigma_1\sigma_3$	$-\sigma_2$	σ_2	$\sigma_3\sigma_2\sigma_1$	$\sigma_1\sigma_2\sigma_3$	σ_3	$-\sigma_3$	$\sigma_3\sigma_2$	$\sigma_2\sigma_3$
σ_2	σ_2	$-\sigma_2$	$\sigma_2\sigma_1$	$\sigma_1\sigma_2$	1	-1	$\sigma_2\sigma_3$	$\sigma_3\sigma_2$	$-\sigma_1$	σ_1	σ_3	$-\sigma_3$	$\sigma_1\sigma_2\sigma_3$	$\sigma_3\sigma_2\sigma_1$	$\sigma_3\sigma_1$	$\sigma_1\sigma_3$
$-\sigma_2$	$-\sigma_2$	σ_2	$\sigma_1\sigma_2$	$\sigma_2\sigma_1$	-1	1	$\sigma_3\sigma_2$	$\sigma_2\sigma_3$	σ_1	$-\sigma_1$	$-\sigma_3$	σ_3	$\sigma_3\sigma_2\sigma_1$	$\sigma_1\sigma_2\sigma_3$	$\sigma_1\sigma_3$	$\sigma_3\sigma_1$
σ_3	σ_3	$-\sigma_3$	$\sigma_3\sigma_1$	$\sigma_1\sigma_3$	$\sigma_3\sigma_2$	$\sigma_2\sigma_3$	1	-1	$\sigma_1\sigma_2\sigma_3$	$\sigma_3\sigma_2\sigma_1$	$-\sigma_2$	σ_2	σ_1	$-\sigma_1$	$\sigma_1\sigma_2$	$\sigma_2\sigma_1$
$-\sigma_3$	$-\sigma_3$	σ_3	$\sigma_1\sigma_3$	$\sigma_3\sigma_1$	$\sigma_2\sigma_3$	$\sigma_3\sigma_2$	-1	1	$\sigma_3\sigma_2\sigma_1$	$\sigma_1\sigma_2\sigma_3$	σ_2	$-\sigma_2$	$-\sigma_1$	σ_1	$\sigma_2\sigma_1$	$\sigma_1\sigma_2$
$\sigma_1\sigma_2$	$\sigma_1\sigma_2$	$\sigma_2\sigma_1$	$-\sigma_2$	σ_2	σ_1	$-\sigma_1$	$\sigma_1\sigma_2\sigma_3$	$\sigma_3\sigma_2\sigma_1$	-1	1	$\sigma_1\sigma_3$	$\sigma_3\sigma_1$	$\sigma_2\sigma_3$	$\sigma_3\sigma_2$	$-\sigma_3$	σ_3
$\sigma_2\sigma_1$	$\sigma_2\sigma_1$	$\sigma_1\sigma_2$	σ_2	$-\sigma_2$	$-\sigma_1$	σ_1	$\sigma_3\sigma_2\sigma_1$	$\sigma_1\sigma_2\sigma_3$	1	-1	$\sigma_3\sigma_1$	$\sigma_1\sigma_3$	$\sigma_3\sigma_2$	$\sigma_2\sigma_3$	σ_3	$-\sigma_3$
$\sigma_1\sigma_3$	$\sigma_1\sigma_3$	$\sigma_3\sigma_1$	$-\sigma_3$	σ_3	$\sigma_3\sigma_2\sigma_1$	$\sigma_1\sigma_2\sigma_3$	σ_1	$-\sigma_1$	$\sigma_2\sigma_3$	$\sigma_3\sigma_2$	$\sigma_2\sigma_1$	$\sigma_1\sigma_2$	1	-1	σ_2	$-\sigma_2$
$\sigma_3\sigma_1$	$\sigma_3\sigma_1$	$\sigma_1\sigma_3$	σ_3	$-\sigma_3$	$\sigma_1\sigma_2\sigma_3$	$\sigma_3\sigma_2\sigma_1$	$-\sigma_1$	σ_1	$\sigma_3\sigma_2$	$\sigma_2\sigma_3$	$\sigma_1\sigma_2$	$\sigma_2\sigma_1$	-1	1	$-\sigma_2$	σ_2
$\sigma_2\sigma_3$	$\sigma_2\sigma_3$	$\sigma_3\sigma_2$	$\sigma_1\sigma_2\sigma_3$	$\sigma_3\sigma_2\sigma_1$	$-\sigma_3$	σ_3	σ_2	$-\sigma_2$	$\sigma_3\sigma_1$	$\sigma_1\sigma_3$	-1	1	$\sigma_2\sigma_1$	$\sigma_1\sigma_2$	$-\sigma_1$	σ_1
$\sigma_3\sigma_2$	$\sigma_3\sigma_2$	$\sigma_2\sigma_3$	$\sigma_3\sigma_2\sigma_1$	$\sigma_1\sigma_2\sigma_3$	σ_3	$-\sigma_3$	$-\sigma_2$	σ_2	$\sigma_1\sigma_3$	$\sigma_3\sigma_1$	1	-1	$\sigma_1\sigma_2$	$\sigma_2\sigma_1$	σ_1	$-\sigma_1$
$\sigma_1\sigma_2\sigma_3$	$\sigma_1\sigma_2\sigma_3$	$\sigma_3\sigma_2\sigma_1$	$\sigma_2\sigma_3$	$\sigma_3\sigma_2$	$\sigma_3\sigma_1$	$\sigma_1\sigma_3$	$\sigma_1\sigma_2$	$\sigma_2\sigma_1$	$-\sigma_3$	σ_3	σ_1	$-\sigma_1$	$-\sigma_2$	σ_2	-1	1
$\sigma_3\sigma_2\sigma_1$	$\sigma_3\sigma_2\sigma_1$	$\sigma_1\sigma_2\sigma_3$	$\sigma_3\sigma_2$	$\sigma_2\sigma_3$	$\sigma_1\sigma_3$	$\sigma_3\sigma_1$	$\sigma_2\sigma_1$	$\sigma_1\sigma_2$	σ_3	$-\sigma_3$	$-\sigma_1$	σ_1	σ_2	$-\sigma_2$	1	-1

Table 1: Composition table

. The key findings from Table 1 are summarized as follows:

1. B_3 is non Abelian Group.
2. All the finite sub-groups of B_3 are:

$\{\pm 1\}, \{1, \sigma_1\}, \{1, -\sigma_1\}, \{1, \sigma_2\}, \{1, -\sigma_2\}, \{1, \sigma_3\}, \{1, -\sigma_3\}, \{\pm 1, \pm\sigma_1\}, \{\pm 1, \pm\sigma_2\}, \{\pm 1, \pm\sigma_3\}$, $\{\pm 1, \pm\sigma_1\sigma_2\}, \{\pm 1, \pm\sigma_1\sigma_3\}, \{\pm 1, \pm\sigma_2\sigma_3\}, \{\pm 1, \pm\sigma_1\sigma_2\sigma_3\}, \{\pm 1, \pm\sigma_1\sigma_2, \pm\sigma_1\sigma_3, \pm\sigma_2\sigma_3\}$, $\{\pm 1, \pm\sigma_1, \pm\sigma_2, \pm\sigma_1\sigma_2\}, \{\pm 1, \pm\sigma_2, \pm\sigma_3, \pm\sigma_2\sigma_3\}, \{\pm 1, \pm\sigma_1, \pm\sigma_3, \pm\sigma_1\sigma_3\}, G$

3. We define a relation $\sim$ in B_3 such that $x \sim y$ if and only if $\exists\ g \in B_3$ such that $gx = y$. Clearly $\sim$ is an equivalence relation. Hence it forms a partition on B_3.
4. Orbit of $x = \{y \in B_3 : y \sim x\}$.
   Thus the Orbits are equivalence classes.
5. B_3contains only one equivalence class.
6. The action of B_3 on itself or any other subgroup of it are:
   a) Sharply transitive as for every pair of element $x, y \in B_3$ $\exists$ a unique $g \in B_3$ such that $gx = y$.
   b) Faithful or Effective as different elements of B_3 induce different permutations of B_3.
   c) Free or semi-regular as $gx = x$ for some $x \in B_3$, then $g = e = 1$.

i.e, the stabilizer of $x = Bx = \{g \in B_3\ / gx = x\} = \{e = 1\}$, The kernel 'k'of the homomorphism $\phi: B_3 \to B_3$ is given by the intersection of the stabilizers Bx for all $x \in B$ which is the trivial subgroup $\{1\}$.

Note: Every free action on a non-empty set is faithful. As subgroup (subset) of B_3 is both transitive and regular, it is called a principle homogeneous space

.

5. Conjugate table of the subgroup B_3:

Further, we can draw conclusions on the action of B_3on all its subgroups. We define the operation by conjugation, i.e $\Psi: B_3 \times B_3 \to B_3$ such that $\Psi(x, y) = y^x = x^{-1}yx$.

We can see that, the following are satisfied [16]:

1. $y^e = y$
2. $(y^x)^z = z^{-1}(x^{-1}yx)z = (xz)^{-1}y(xz) = y^{xz}$

which shows that Ψ also is a group action.

The group action of B_3 on its subgroups has some important properties which we shall discuss with the help of the conjugacy table from Table 2.

	1	-1	σ_1	$-\sigma_1$	σ_2	$-\sigma_2$	σ_3	$-\sigma_3$	$\sigma_1\sigma_2$	$\sigma_2\sigma_1$	$\sigma_1\sigma_3$	$\sigma_3\sigma_1$	$\sigma_2\sigma_3$	$\sigma_3\sigma_2$	$\sigma_1\sigma_2\sigma_3$	$\sigma_3\sigma_2\sigma_1$
1	1	-1	σ_1	$-\sigma_1$	σ_2	$-\sigma_2$	σ_3	$-\sigma_3$	$\sigma_1\sigma_2$	$\sigma_2\sigma_1$	$\sigma_1\sigma_3$	$\sigma_3\sigma_1$	$\sigma_2\sigma_3$	$\sigma_3\sigma_2$	$\sigma_1\sigma_2\sigma_3$	$\sigma_3\sigma_2\sigma_1$
-1	1	-1	σ_1	$-\sigma_1$	σ_2	$-\sigma_2$	σ_3	$-\sigma_3$	$\sigma_1\sigma_2$	$\sigma_2\sigma_1$	$\sigma_1\sigma_3$	$\sigma_3\sigma_1$	$\sigma_2\sigma_3$	$\sigma_3\sigma_2$	$\sigma_1\sigma_2\sigma_3$	$\sigma_3\sigma_2\sigma_1$
σ_1	1	-1	σ_1	$-\sigma_1$	$-\sigma_2$	σ_2	$-\sigma_3$	σ_3	$\sigma_2\sigma_1$	$\sigma_1\sigma_2$	$\sigma_3\sigma_1$	$\sigma_1\sigma_3$	$\sigma_2\sigma_3$	$\sigma_3\sigma_2$	$\sigma_1\sigma_2\sigma_3$	$\sigma_3\sigma_2\sigma_1$
$-\sigma_1$	1	-1	σ_1	$-\sigma_1$	$-\sigma_2$	σ_2	$-\sigma_3$	σ_3	$\sigma_2\sigma_1$	$\sigma_1\sigma_2$	$\sigma_3\sigma_1$	$\sigma_1\sigma_3$	$\sigma_2\sigma_3$	$\sigma_3\sigma_2$	$\sigma_1\sigma_2\sigma_3$	$\sigma_3\sigma_2\sigma_1$
σ_2	1	-1	$-\sigma_1$	σ_1	σ_2	$-\sigma_2$	$-\sigma_3$	σ_3	$\sigma_2\sigma_1$	$\sigma_1\sigma_2$	$\sigma_1\sigma_3$	$\sigma_3\sigma_1$	$\sigma_3\sigma_2$	$\sigma_2\sigma_3$	$\sigma_1\sigma_2\sigma_3$	$\sigma_3\sigma_2\sigma_1$
$-\sigma_2$	1	-1	$-\sigma_1$	σ_1	σ_2	$-\sigma_2$	$-\sigma_3$	σ_3	$\sigma_2\sigma_1$	$\sigma_1\sigma_2$	$\sigma_1\sigma_3$	$\sigma_3\sigma_1$	$\sigma_3\sigma_2$	$\sigma_2\sigma_3$	$\sigma_1\sigma_2\sigma_3$	$\sigma_3\sigma_2\sigma_1$
σ_3	1	-1	$-\sigma_1$	σ_1	$-\sigma_2$	σ_2	σ_3	$-\sigma_3$	$\sigma_1\sigma_2$	$\sigma_2\sigma_1$	$\sigma_3\sigma_1$	$\sigma_1\sigma_3$	$\sigma_3\sigma_2$	$\sigma_2\sigma_3$	$\sigma_1\sigma_2\sigma_3$	$\sigma_3\sigma_2\sigma_1$
$-\sigma_3$	1	-1	$-\sigma_1$	σ_1	$-\sigma_2$	σ_2	σ_3	$-\sigma_3$	$\sigma_1\sigma_2$	$\sigma_2\sigma_1$	$\sigma_3\sigma_1$	$\sigma_1\sigma_3$	$\sigma_3\sigma_2$	$\sigma_2\sigma_3$	$\sigma_1\sigma_2\sigma_3$	$\sigma_3\sigma_2\sigma_1$
$\sigma_1\sigma_2$	1	-1	$-\sigma_1$	σ_1	$-\sigma_2$	σ_2	σ_3	$-\sigma_3$	$\sigma_1\sigma_2$	$\sigma_2\sigma_1$	$\sigma_3\sigma_1$	$\sigma_1\sigma_3$	$\sigma_3\sigma_2$	$\sigma_2\sigma_3$	$\sigma_1\sigma_2\sigma_3$	$\sigma_3\sigma_2\sigma_1$
$\sigma_2\sigma_1$	1	-1	$-\sigma_1$	σ_1	$-\sigma_2$	σ_2	σ_3	$-\sigma_3$	$\sigma_1\sigma_2$	$\sigma_2\sigma_1$	$\sigma_3\sigma_1$	$\sigma_1\sigma_3$	$\sigma_3\sigma_2$	$\sigma_2\sigma_3$	$\sigma_1\sigma_2\sigma_3$	$\sigma_3\sigma_2\sigma_1$
$\sigma_1\sigma_3$	1	-1	$-\sigma_1$	σ_1	σ_2	$-\sigma_2$	$-\sigma_3$	σ_3	$\sigma_2\sigma_1$	$\sigma_1\sigma_2$	$\sigma_1\sigma_3$	$\sigma_3\sigma_1$	$\sigma_3\sigma_2$	$\sigma_2\sigma_3$	$\sigma_1\sigma_2\sigma_3$	$\sigma_3\sigma_2\sigma_1$
$\sigma_3\sigma_1$	1	-1	$-\sigma_1$	σ_1	σ_2	$-\sigma_2$	$-\sigma_3$	σ_3	$\sigma_2\sigma_1$	$\sigma_1\sigma_2$	$\sigma_1\sigma_3$	$\sigma_3\sigma_1$	$\sigma_3\sigma_2$	$\sigma_2\sigma_3$	$\sigma_1\sigma_2\sigma_3$	$\sigma_3\sigma_2\sigma_1$
$\sigma_2\sigma_3$	1	-1	σ_1	$-\sigma_1$	$-\sigma_2$	σ_2	$-\sigma_3$	σ_3	$\sigma_2\sigma_1$	$\sigma_1\sigma_2$	$\sigma_3\sigma_1$	$\sigma_1\sigma_3$	$\sigma_2\sigma_3$	$\sigma_3\sigma_2$	$\sigma_1\sigma_2\sigma_3$	$\sigma_3\sigma_2\sigma_1$
$\sigma_3\sigma_2$	1	-1	σ_1	$-\sigma_1$	$-\sigma_2$	σ_2	$-\sigma_3$	σ_3	$\sigma_2\sigma_1$	$\sigma_1\sigma_2$	$\sigma_3\sigma_1$	$\sigma_1\sigma_3$	$\sigma_2\sigma_3$	$\sigma_3\sigma_2$	$\sigma_1\sigma_2\sigma_3$	$\sigma_3\sigma_2\sigma_1$
$\sigma_1\sigma_2\sigma_3$	1	-1	σ_1	$-\sigma_1$	σ_2	$-\sigma_2$	σ_3	$-\sigma_3$	$\sigma_1\sigma_2$	$\sigma_2\sigma_1$	$\sigma_1\sigma_3$	$\sigma_3\sigma_1$	$\sigma_2\sigma_3$	$\sigma_3\sigma_2$	$\sigma_1\sigma_2\sigma_3$	$\sigma_3\sigma_2\sigma_1$
$\sigma_3\sigma_2\sigma_1$	1	-1	σ_1	$-\sigma_1$	σ_2	$-\sigma_2$	σ_3	$-\sigma_3$	$\sigma_1\sigma_2$	$\sigma_2\sigma_1$	$\sigma_1\sigma_3$	$\sigma_3\sigma_1$	$\sigma_2\sigma_3$	$\sigma_3\sigma_2$	$\sigma_1\sigma_2\sigma_3$	$\sigma_3\sigma_2\sigma_1$

Table 2: Conjugate table

The key findings from Table 2 are outlined as follows:

Let $B_3 = \mathrm{G}$,

1. The Normal subgroup are,

   $\{\pm 1\}, \{\pm 1, \pm\sigma_1\}, \{\pm 1, \pm\sigma_2\}, \{\pm 1, \pm\sigma_3\}, \{\pm 1, \pm\sigma_1\sigma_2\}, \{\pm 1, \pm\sigma_1\sigma_3\}, \{\pm 1, \pm\sigma_2\sigma_3\},$
   $\{\pm 1, \pm\sigma_1\sigma_2\sigma_3\}, \{\pm 1, \pm\sigma_1\sigma_2, \pm\sigma_1\sigma_3, \pm\sigma_2\sigma_3\}, \{\pm 1, \pm\sigma_1, \pm\sigma_2, \pm\sigma_1\sigma_2\},$
   $\{\pm 1, \pm\sigma_2, \pm\sigma_3, \pm\sigma_2\sigma_3\}, \{\pm 1, \pm\sigma_1, \pm\sigma_3, \pm\sigma_1\sigma_3\}$

2. The Subnormal series of the above group have isomorphic refinement.
   - $\{1\} \subset \{\pm 1\} \subset \{\pm 1, \pm\sigma_1\} \subset \{\pm 1, \pm\sigma_1, \pm\sigma_2, \pm\sigma_1\sigma_2\} \subset \mathrm{G}$
   - $\{1\} \subset \{\pm 1\} \subset \{\pm 1, \pm\sigma_2\} \subset \{\pm 1, \pm\sigma_1, \pm\sigma_2, \pm\sigma_1\sigma_2\} \subset \mathrm{G}$
   - $\{1\} \subset \{\pm 1\} \subset \{\pm 1, \pm\sigma_1\} \subset \{\pm 1, \pm\sigma_1, \pm\sigma_3, \pm\sigma_1\sigma_3\} \subset \mathrm{G}$
   - $\{1\} \subset \{\pm 1\} \subset \{\pm 1, \pm\sigma_3\} \subset \{\pm 1, \pm\sigma_1, \pm\sigma_3, \pm\sigma_1\sigma_3\} \subset \mathrm{G}$
   - $\{1\} \subset \{\pm 1\} \subset \{\pm 1, \pm\sigma_2\} \subset \{\pm 1, \pm\sigma_2, \pm\sigma_3, \pm\sigma_2\sigma_3\} \subset \mathrm{G}$
   - $\{1\} \subset \{\pm 1\} \subset \{\pm 1, \pm\sigma_3\} \subset \{\pm 1, \pm\sigma_2, \pm\sigma_3, \pm\sigma_2\sigma_3\} \subset \mathrm{G}$

   These are a few examples of many.

3. This shows that *G* is not a Simple Group as it has Normal subgroups.
4. *G* is Solvable.

6. Conclusion:

In this paper,we have shown the total number of Normal subgroup and subnormal series of the subgroup B_3 of Clifford Algebra and by using group action and conjugation operation we given that the given subgroup B_3 is Solvable. we have taken *C(E)*, an algebra known as Clifford Algebra is constructed over a vector space *E* that has *n*-dimensions and have investigated by Group Actions the nature of its subgroups. The subgroup B_3 is a non-abelian group which has seventeen non-trivial finite subgroups with only one equivalence class and the Kernel of the homomorphism $\phi: B_3 \to B_3$ is the trivial subgroup $\{1\}$.

The Action of the subgroup $B_3$with the conjugation, i.e, $\Psi: B_3 \times B_3 \to B_3$ such that $\Psi(x, y) = y^x = x^{-1}yx$ has twelve Normal subgroup which implies that the subgorup B_3 is not a Simple group, further we establish that it has a subnormal series, hence we show that the subgorup B_3 is a Solvable group.

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