Contemporary Mathematics
http://ojs.wiserpub.com/index.php/CM/

Research Article

Non-Markovian Feedback Retrial Queue with Two Types of Customers and Delayed Repair Under Bernoulli Working Vacation

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Received: November 16 2023; **Revised:** March 12 2024; **Accepted:** March 21 2024

Abstract: This paper deals with an M/G/1 feedback retrial G-queue and delayed repair incorporating Bernoulli working vacation. Arrivals of both favorable and unfavorable consumers are two independent Poisson processes. The server is subjected to breakdown during the regular busy period due to the arrival of unfavorable customers and then the server will be down for a short period of time. Further, the concept of delay time is also discussed. After the fulfillment of regular service, the dissatisfied consumer may re-join the orbit to receive another service as a feedback consumer. During the working vacation period, the server provides the service to consumers at a reduced rate. By applying the supplementary variable technique (SVT), we have analyzed the steady-state probability generating function (PGF) for the system size and orbit size. In addition, we have presented the system performance, reliability indicators, and stochastic decomposition property. Finally, we provided numerical examples to illustrate our model's mathematical results.

***Keywords*:** retrial queue, G-queue, working vacation, delayed repair, feedback, supplementary variable technique

MSC: 60K25, 68M20, 90B22

1. Introduction

Queueing theory can be utilized as a mathematical framework to successfully analyze a wide range of practical applications in realistic circumstances. The queueing theory has been used to address issues in a variety of industries, including production methods, wireless communications, IT, and control of traffic, as well as domains that require the provision of services in unpredictable demand situations. Furthermore, the queuing theory (QT) might be beneficial in two areas: planning and management.

The study of retrial queues has been conducted due to their application in a multitude of disciplines, namely service centers, switching systems for telephones, and retransmission connectivity in internet-based telephone systems. The retrial queueing model is required for consumers to receive their service. For example, if arriving consumers find all servers occupied or breakdown, they will join the retrial group (referred to as retrial orbit) and try to get service after a certain amount of time (referred to as retrial time). Falin's [1] survey article, as well as the manuscript by Artalejo [2] and Falin [3] provide a comprehensive summary of the main results and methodologies for the retrial queuing model. Manoharan and Subathra [4] have explored the balking behavior of consumers with working vacation (WV) incorporating the non-Markovian retrial queue model. Micheal and Indhira [5] have examined a retrial queue with Bernoulli working vacation

DOI: https://doi.org/10.37256/cm.5220243940

(BWV) and they validated their model by introducing adaptive neuro-fuzzy inference system (ANFIS) computing and cost optimization of nonlinear metaheuristics. They also compared the outcomes by various approaches like particle swarm optimization, genetic algorithm, and artificial bee colony.

Many academicians have analyzed the arrival of favorable and unfavorable consumers into the queueing model during the past two decades owing to their extensive utilization across various domains such as computers, telephones, and production lines. Van Do [6] has obtained a survey of G-network queueing systems, unfavorable consumers, and applications. Krishnakumar et al. [7], Gao and Wang [8] have recently investigated several techniques of queueing models that operate with the appearance of unfavorable entries. Murugan et al. [9] have explored single-server queues incorporating the concept of G-queue. Agarwal et al. [10], have analyzed the bulk arrival retrial queue, including the concept of priority G-queue with WV. Gupta et al. [11] investigated a queueing structure by introducing BWV with interruption and dissatisfied consumers in the process of classical retrial policy.

The entry of an unfavorable consumer who doesn't receive service removes the favorable consumer and causes the server to crash. Once the server faced a problem, it was taken for repair. The process of making repairs does not start right away and requires some time. Malik et al. [12] investigated M[X]/G/1 retrial queue delayed repair with negative consumers, incorporating maximum entropy results and particle swarm optimization. Singh et al. [13] have examined Bulk arrival retrial G-queue, optional additional services include the concept of delayed repair. Recently, many authors have analyzed the delayed repair from diverse perspectives [14–16] along with references to them. There are some queuing scenarios in which consumers are served multiple times for a particular cause until they receive a satisfied service. This type is known as a feedback queueing model. Ke and Chang [17] have examined M/G/1 retrial queue underdoing feedback along consumers in a different situation. Ayyappan [18] has proposed the model of a single server queue with immediate feedback. Varalakshmi [19] explored M/G/1 model accompanied by immediate feedback from Bernoulli. Madhu Jain [20] investigated the bulk arrival retrial queue with Bernoulli feedback. Recently, Niranjan and Latha [21] have explored two-phase essential services and breakdowns in two phases, and feedback incorporating server vacation.

The server will take a WV once the orbit is empty. During the WV period, the server will supply the consumer with service at a slower rate than usual. This approach to the queueing model is applicable in a broad variety of contexts, such as communication networks, software platforms, data transfer services, and email service providers. A WV is the first use of the Markovian queueing model, initially presented by Servi and Finn [22]. Jain et al. [23] have investigated the Markovian queue with disaster failure and MWVs. In further development of the M/M/1/WV queueing model, Wu and Takagi [24] have constructed an M/G/1 model that established a WV extension. Bharathy and Saravanarajan [25] investigate unreliable retrial queues by including WV. Bouchentouf et al. [26] have investigated finite capacity Markovian queues with different WV. Liu et al. [27] have presented the Markovian queue with preemptive priority and WV interruption. Yang [28] has established the retrial queue with WVs together with the starting fail server. The M/G/1 retrial queueing system (RQS) with preemptive priority consumers under working breakdown has been researched by Ammar and Rajadurai [29].

1.1 *Motivation*

This article investigates a non-Markovian retrial G-queue along feedback consumers, delayed repair under Bernoulli working vacation (BWV). The motivation of this study is that such retrial queues occur in a multitude of practical sectors, including telecommunication networks and industrial management. It characterizes not only the retrial phenomenon among consumers but also considers delayed repairs towards unforeseen failures. Another motive to take into account such a retrial model is to obtain an analytical solution in terms of closed-form expression using the procedure of supplementary variables and to assess the performance metrics and reliability of the system under consideration, which may be appropriate in several communication networks.

The significance of this proposed queuing model when compared to existing litertures is that, in our model both feedback and delayed repair with BWV in the presence of G-queue are considered. Also, the proposed method discusses non-Markovian queues with general retrial times with BWV, which is also a unique characteristic that has not been discussed in any of the past literature. This methodology has notable applications in computer processing systems, production systems and medical care telephone consultation systems.

The objective of this research is to investigate the performance of a retrial queue system with a single server assuming both positive and negative consumers, along with delayed repair and breakdown cases. In addition, the study also aims to analyze the impact of feedback and BWVs on the system's behavior.

Sections of the study are as follows: Section 2 provides a thorough illustration of the proposed model, Section 3 presents the Steady State probabilities, Section 4 discusses the performance characteristics of the model, Section 5 examines special cases, Section 6 provides numerical illustrations on the effect of various system performance. Finally, Section 7 presents the article's conclusion.

2. Formulation of the model

In this section, we present the description of the M/G/1 retrial G-queue with WV under delayed repair along with its real-time implementation in the computer processing system. Also, the model's transition diagram is given in Figure 1.

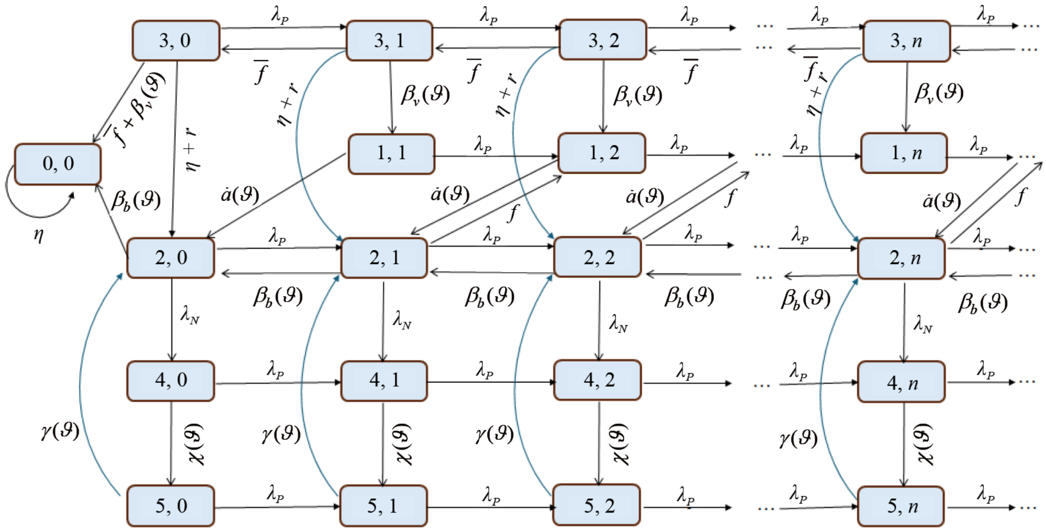

Figure 1. The state diagram in our proposed model

The detailed model description is as follows:

• **Characteristics of consumers:** Two types of consumers called favorable and unfavorable are considered for our model. The favorable consumer joins the queue and gets service when the server is unoccupied. Unlike favorable consumer, the unfavorable consumer will not get service rather they will breakdown the server by interrupting the process.

• **Arrival rule:** When the server is unoccupied, the favorable consumers enter the system based on the Poisson arrival rate λ_P.

• **The retrial rule:** If a favorable consumer arrives and discovers that the server is available, the server will start serving right away. Conversely, if the server is occupied, breakdown, or on vacation, the consumer may enter the orbit's retrial group and retry for service after a random interval of time. Inter-retrial period has the distribution function $A(t)$ and Laplace Stieltjes Transform (LST) as $A^*(\vartheta)$.

• **Normal service rule:** When a new favorable consumer or a retry consumer comes to the service station while the server is unoccupied, the server instantly begins normal busy service to the consumers. The service time follows a general distribution and it is considered by the random variable (RV) B_b with the distribution function $B_b(t)$ and the LST as $B_b^*(\vartheta)$.

• **Feedback rule:** After each consumer's service is completed, clients who are dissatisfied may re-enter the orbit as feedback consumers to obtain repeated service along with the prob. f $(0 \leq f \leq 1)$ else they will probably leave the system, $\overline{f} = 1 - f$.

• **Bernoulli working vacation rule:** If the orbit seems to be unoccupied, the server begins his WV, and the period of vacation assumes to be exponentially distributed along with the parameter η. If a consumer appears during vacation time, the server offers the service at slow rates. During WV, if the orbit contains any consumer, the server will end its vacation and resume normal operations, causing the vacation to be interrupted. When there are no consumers on the system at the end of a vacation, the server either remains to service prospective consumers with the prob. r (Single WV) or going on one more WV with prob. $l = 1 - r$ (Multi-WVs). The lower service period considers a distribution $B_v(\mathfrak{t})$ and LST as $B_v^*(\vartheta)$.

• **The removal and repair rule:** The unfavourable consumers enter from ouside of the system based an Poisson arrival rate λ_N. These unfavorable consumers only enter during the usual service period of favorable consumers. The arrival of an unfavorable consumer removes the favorable consumer in the system and causes the server to crash. As soon as the server fails, it is sent for repair and stops serving consumers during the waiting period, this amount of time is referred to as the server's waiting time. We define the amount of time spent waiting is referred to be delayed period. The delay times with function of distribution $Dy(\mathfrak{t})$ and $Dy^*(\varsigma)$ be its LST, and the first and second moments be $\mathfrak{w}^{(1)}$ and $\mathfrak{w}^{(2)}$ respectively. The repair times are considered with distribution function $G(\mathfrak{t})$ and LST as $G^*(\varsigma)$, and the first two moments be $\mathfrak{g}^{(1)} and\ \mathfrak{g}^{(2)}$.

2.1 *Real-life application of the method*

Our model serves as an efficient practical application in the domain of a computer processing system. The buffer size (orbit), which assists in accumulating messages, is finite and the messages (favorable consume) enter into the system one by one in a computer processing system. The processor (server) takes on the role of performing the messages. The viruses (unfavorable consumer) can affect the working mail server and the system may be subjected to an electronic fault (breakdown) during the normal service period. After the breakdown, the server will stop its service and sent to repair then the maintanance task take some time (called a delay). If the processor is accessible by not engaging with some other tasks, a message will be processed. By the rule of first-come, first-served (FCFS), the messages will be stored for a short time in the buffer to be processed sometime later (retrial time) if the processor is inaccessible. If any failures occur in the previous process, the internet service may require the same service from the processor (the feedback) a number of times after the message processing is completed. When there is no arrival of new consumers after processing all the received messages, the processor is committed to operating specific maintenance jobs in sequence, like scanning the virus (WVs) to enhance the performance of the computer. The system can reduce the cost by having the processor deal with the messages at a slower rate during the maintenance period (WV period). The processor verifies the messages on every completion of maintenance to determine whether or not to restart the ordinary service rate (first WV). The processor will perform another maintenance activity (multiple WVs) when the messages are not received in the system. Such WV discipline is a proficient approximation in computer processing systems.

3. Steady-state probabilities

In this section, we formulate the steady-state governing equations (Eqs) by treating the elapsed times of retrial, service, WV, delayed and repair as supplementary variables (SVs). The PGF for the server-states and for the no. of consumers in the orbit and system are obtained.

We consider that $A(0) = 0$, $A(\infty) = 1$, $B_b(0) = 0$, $B_b(\infty) = 1$, $B_v(0) = 0$, $B_v(\infty) = 1$, are continuous at and $\vartheta = 0$, $W_y(0) = 0$, $W_y(\infty) = 1$ and $G(0) = 0$, $G(\infty) = 1$ are continuous at $\varsigma = 0$. We assume that the hazard rate functions as $\dot{a}(\vartheta)$, $\beta_b(\vartheta)$, $\beta_v(\vartheta)$, $\chi(\varsigma)$, and $\gamma(\varsigma)$ for retrial, regular service, slow rate service, delay repair and for the maintainance in that order respectively.

$$\dot{a}(\vartheta)\ d(\vartheta) = \frac{d(A(\vartheta))}{(1-A(\vartheta))};\ \ \beta_b(\vartheta)d\vartheta = \frac{d(B_b(\vartheta))}{(1-B_b(\vartheta))};\ \ \beta_v(\vartheta)d\vartheta = \frac{d\ (B_v(\vartheta))}{(1-B_v(\vartheta))};$$

$$\chi(\varsigma)d\varsigma = \frac{d(W(\varsigma))}{(1-W(\varsigma))};\ \ \gamma(\varsigma)d\varsigma = \frac{d(G(\varsigma))}{(1-G(\varsigma))}.$$

In addition, let A^0, B_b^0, B_v^0, G^0 and W^0 be the expired retrial, busy, working vacation (WV), delay to repair and repair times are shown at period $\mathfrak{t}$. We also assume the random variable (RV),

$$H(\mathfrak{t}) = \begin{cases} 0, & \text{server is unoccupied and in the lower service mode,} \\ 1, & \text{server is unoccupied and in normal service mode,} \\ 2, & \text{server is occupied and in normol service mode} \\ 3, & \text{server is occupied and in lower service mode} \\ 4, & \text{server is under delayed repair} \\ 5, & \text{the server is undergoing maintenance.} \end{cases}$$

Further, bivariate Markov process $\{H(\mathfrak{t}), Y(\mathfrak{t});\ \mathfrak{t} \geq 0\}$, such as $Y(\mathfrak{t})$ is the no. of consumers in the orbit at time $\mathfrak{t}$, $H(\mathfrak{t})$ represents the server's states as (0, 1, 2, 3, 4, 5) depending on whether the server is unoccupied, regular busy, WV period, delayed repair, and repair period. Let us consider the limiting probabilities $V_0(\mathfrak{t}) = P\{H(\mathfrak{t}) = 0, Y(\mathfrak{t}) = 0\}$; $I_0(\mathfrak{t}) = P\{H(\mathfrak{t}) = 0, Y(\mathfrak{t}) = 0\}$ and the probability (prob.) densities are

$$I_{\mathfrak{n}}(\vartheta,\ \mathfrak{t})d\vartheta = \lim_{\mathfrak{t}\to\infty} Prob.\{H(\mathfrak{t}) = 1, Y(\mathfrak{t}) = \mathfrak{n},\ \vartheta \leq A^0(\mathfrak{t}) < \vartheta + d\vartheta\}$$

$$\Xi_{b,\mathfrak{n}}(\vartheta,\ \mathfrak{t})d\vartheta = \lim_{\mathfrak{t}\to\infty} Prob.\{H(t) = 2, Y(\mathfrak{t}) = \mathfrak{n},\ \vartheta \leq B_b^0(\mathfrak{t}) < \vartheta + d\vartheta\}$$

$$\Phi_{v,\mathfrak{n}}(\vartheta,\ \mathfrak{t})d\vartheta = \lim_{\mathfrak{t}\to\infty} Prob.\{H(\mathfrak{t}) = 3, Y(\mathfrak{t}) = \mathfrak{n},\ \vartheta \leq B_v^0(\mathfrak{t}) < \vartheta + d\vartheta\}$$

$$\omega_{\mathfrak{n}}(\vartheta,\ \varsigma,\ \mathfrak{t})d\vartheta = \lim_{\mathfrak{t}\to\infty} Prob.\{H(\mathfrak{t}) = 4, Y(\mathfrak{t}) = \mathfrak{n},\ \varsigma \leq W^0(\mathfrak{t}) < \varsigma + d\varsigma / B_b^0(\mathfrak{t}) = \vartheta\}$$

$$\Theta(\vartheta,\ \varsigma,\ \mathfrak{t})d\vartheta = \lim_{\mathfrak{t}\to\infty} Prob.\{H(\mathfrak{t}) = 5, Y(\mathfrak{t}) = \mathfrak{n},\ \varsigma \leq G^0(\mathfrak{t}) < \varsigma + d\varsigma / B_b^0(\mathfrak{t}) = \vartheta\}$$

$$\forall\ \mathfrak{t} \geq 1,\ \vartheta \geq 1,\ \mathfrak{n} \geq 1.$$

The time ($\mathfrak{t}_{\mathfrak{n}}$; $\mathfrak{n}$ = 1, 2, ...) indicate the series of epoch correlating to WV completion times, or the conclusion of the delay repaired and repair period. A Markov-chain is formed by the collection of random vectors $\psi_{\mathfrak{n}} = \{H(\mathfrak{t}n+), Y(\mathfrak{t}n+)\}$ and is embedded in the retrial queueing (RQ) system.

3.1 *Notations and probabilities*

We defined following notation and probabilities in our model:

$\dot{a}(\vartheta) \longrightarrow$ the hazard rate for retrial of $A(\vartheta)$.

$\beta_b(\vartheta) \longrightarrow$ the hazard rate for service of $B_b(\vartheta)$.

$\beta_v(\vartheta) \longrightarrow$ the hazard rate for slow rate service of $B_v(\vartheta)$.

$\chi(\varsigma) \longrightarrow$ the hazard rate delayed repair of $W(\varsigma)$.

$\gamma(\varsigma) \longrightarrow$ the hazard rate for repair of $G(\varsigma)$.

$A^0 \longrightarrow$ the elapsed retrial time.

$B_b^0 \longrightarrow$ the elapsed service time.

$B_v^0 \longrightarrow$ the elapsed slow rate service time.

$W^0 \longrightarrow$ the elapsed delay time.

$G^0 \longrightarrow$ the elapsed repair time.

$Y(\mathfrak{t}) \longrightarrow$ the no. of consumers in the orbit at time $\mathfrak{t}$.

$H(\mathfrak{t}) \longrightarrow$ the server states at time $\mathfrak{t}$.

$V_0(\mathfrak{t}) \longrightarrow$ the prob. that the system's unoccupied at period $\mathfrak{t}$ and the server is on WV.

$I_0(\mathfrak{t}) \longrightarrow$ the prob. that the system's unoccupied at period $\mathfrak{t}$ and the server is in a normal service period.

$I_{\mathfrak{n}}(\vartheta, \mathfrak{t}) \longrightarrow$ the prob. that at the period $\mathfrak{t}$ there are $\mathfrak{n}$ consumers in the waiting space, with each consumer's elapsed retrial time occurring between (b/w) ϑ and $\vartheta + d\vartheta$.

$\Xi_{b,\mathfrak{n}}(\vartheta, \mathfrak{t}) \longrightarrow$ the prob. that at the period $\mathfrak{t}$ there are $\mathfrak{n}$ consumers in the waiting space, with each consumer's elapsed normal service period occurring b/w ϑ and $\vartheta + d\vartheta$.

$\Phi_{v,\mathfrak{n}}(\vartheta, \mathfrak{t}) \longrightarrow$ the prob. that at the period $\mathfrak{t}$ there are $\mathfrak{n}$ consumers in the waiting space, with each consumer's elapsed slower service period occuring b/w ϑ and $\vartheta + d\vartheta$.

$\omega_{\mathfrak{n}}(\vartheta, \varsigma, \mathfrak{t}) \longrightarrow$ the prob. that at the period $\mathfrak{t}$ there are $\mathfrak{n}$ consumers in the waiting space with the elapsed service time of the test consumer undergoing service is ϑ and the server's elapsed delay period is ς.

$\Theta(\vartheta, \varsigma, \mathfrak{t}) \longrightarrow$ the prob. that at the period $\mathfrak{t}$ there are $\mathfrak{n}$ consumers in the waiting space with the elapsed service time of the test consumer undergoing service is ϑ and the server's elapsed repair period is ς.

Theorem 1 The Embedded Markov-Chain $\{\psi_{\mathfrak{n}}; \mathfrak{n} \in N\}$ is Ergodic if and only if $\Gamma < A^*(\lambda_P)$, where $\Gamma = (\frac{\lambda_P}{\lambda_N})(1 - B_b^*(\lambda_N))(1 + \lambda_N(\mathfrak{g}^{(1)} + \mathfrak{w}^{(1)})) + fB_b^*(\lambda_N)$.

proof. It is really convenient to use Foster's criterion to demonstrate the sufficient condition of ergodicity (Pakes [30]), this specifies the chain $\{\psi_{\mathfrak{n}}; \mathfrak{n} \in N\}$ irreducibility and aperiodicity Markov-Chain is ergodic if $\exists$ a function $f(j) \geq 0$, $j \in N$ and $\forall \varepsilon > 0$, such that mean drift $\phi_j = E[f(\psi_{\mathfrak{n}} + 1) - f(\psi_{\mathfrak{n}})/\psi_{\mathfrak{n}} = j] < \infty, \forall j \in N$ and $\phi_j \leq -\varepsilon \forall j \in N$. We assume the function as $f(j) = j$. then we get

$$\phi_j = \begin{cases} \Gamma - 1, & j = 0 \\ \Gamma - A^{(*)}(\lambda_P), & j = 1, 2, 3, \ldots \end{cases}$$

It is obvious that the inequality $\Gamma < A^*(\lambda_P)$ is a sufficient condition for Ergodicity.

The Markov-chain $\{\psi_{\mathfrak{n}}; \mathfrak{n} \geq 1\}$ fulfills Kaplan's condition. It is used to establish the necessary condition, as mentioned in [31], if $\phi_j < \infty$, $\forall j \geq 0$ and $\exists\, j_0 \in N$ in such a way that $\phi_j \geq 0$ for $j \geq j_0$, then $\Gamma \geq A^*(\lambda_P)$ implies that Non-Ergodicity of the Markov-Chain.

3.2 *Steady-state conditions*

In this section, we generate the system of governing equations for this approach by using the SVT as follows:

$$\lambda_P I_0 = \eta r V_0 \tag{1}$$

$$(\lambda_P + \eta)V_0 = \eta l V_0 + \overline{f}\left(\int_0^\infty \Xi_{b,0}(\vartheta)\beta_b(\vartheta)d\vartheta + \int_0^\infty \Phi_{v,0}(\vartheta)\beta_v(\vartheta)d\vartheta\right) + \int_0^\infty \Theta_0(\vartheta)\gamma(\vartheta)d\vartheta \tag{2}$$

$$\frac{d}{d\vartheta}I_\mathfrak{n}(\vartheta) = -[\lambda_P + \dot{a}(\vartheta)]I_\mathfrak{n}(\vartheta), \quad \mathfrak{n} \geq 1 \tag{3}$$

$$\frac{d}{d\vartheta}\Xi_{b,0}(\vartheta) = -[\lambda_P + \lambda_N + \beta_b(\vartheta)]\Xi_{b,0}(\vartheta), \quad \mathfrak{n} = 0 \tag{4}$$

$$\frac{d}{d\vartheta}\Xi_{b,\mathfrak{n}}(\vartheta) = -[\lambda_P + \lambda_N + \beta_b(\vartheta)]\Xi_{b,\mathfrak{n}}(\vartheta) + \lambda_P \Xi_{b,\mathfrak{n}-1}(\vartheta), \quad \mathfrak{n} \geq 1 \tag{5}$$

$$\frac{d}{d\vartheta}\Phi_{v,0}(\vartheta) = -[\lambda_P + \eta + \beta_v(\vartheta)]\Phi_{v,0}(\vartheta), \quad \mathfrak{n} = 0 \tag{6}$$

$$\frac{d}{d\vartheta}\Phi_{v,\mathfrak{n}}(\vartheta) = -[\lambda_P + \eta + \beta_v(\vartheta)]\Phi_{v,n}(\vartheta) + \lambda_P \Phi_{v,\mathfrak{n}-1}(\vartheta), \quad \mathfrak{n} \geq 1 \tag{7}$$

$$\frac{d}{d\varsigma}\omega_0(\vartheta, \varsigma) = -[\lambda_P + \chi(\varsigma)]\omega_0(\vartheta, \varsigma), \quad \mathfrak{n} = 0 \tag{8}$$

$$\frac{d}{d\varsigma}\omega_\mathfrak{n}(\vartheta, \varsigma) = -[\lambda_P + \chi(\varsigma)]\omega_\mathfrak{n}(\vartheta, \varsigma) + \lambda_P \omega_{\mathfrak{n}-1}(\vartheta, \varsigma), \quad \mathfrak{n} \geq 1 \tag{9}$$

$$\frac{d}{d\varsigma}\Theta_0(\vartheta, \varsigma) = -(\lambda_P + \gamma(\varsigma))\Theta_0(\vartheta, \varsigma), \quad \mathfrak{n} = 0 \tag{10}$$

$$\frac{d}{d\varsigma}\Theta_{\mathfrak{n}}(\vartheta,\varsigma) = -[\lambda_P + \gamma(\varsigma)]\Theta_{\mathfrak{n}}(\vartheta,\varsigma) + \lambda_P\Theta_{\mathfrak{n}-1}(\vartheta,\varsigma), \quad \mathfrak{n} \geq 1 \tag{11}$$

To solve Eqs (3) to (11), steady state boundary conditions at $\vartheta = 0$ and $\varsigma = 0$ are given below:

$$\begin{aligned} I_{\mathfrak{n}}(0) = {} & \overline{f}\left(\int_0^\infty \Xi_{b,\mathfrak{n}}(\vartheta)\beta_b(\vartheta)d\vartheta + \int_0^\infty \Phi_{v,\mathfrak{n}}(\vartheta)\beta_v(\vartheta)d\vartheta\right) \\ & + f\left(\int_0^\infty \Xi_{b,\mathfrak{n}-1}(\vartheta)\beta_b(\vartheta)d\vartheta + \int_0^\infty \Phi_{v,\mathfrak{n}-1}(\vartheta)\beta_v(\vartheta)d\vartheta\right) + \int_0^\infty \Theta_{\mathfrak{n}}(\vartheta)\gamma(\vartheta)d\vartheta, \quad \mathfrak{n} \geq 1 \end{aligned} \tag{12}$$

$$\Xi_{b,0}(0) = \left(\int_0^\infty I_1(\vartheta)\dot{a}(\vartheta)d\vartheta + \eta\int_0^\infty \Phi_{v,0}(\vartheta)d\vartheta + \lambda_P I_0\right), \quad \mathfrak{n} = 0 \tag{13}$$

$$\Xi_{b,\mathfrak{n}}(0) = \left(\int_0^\infty I_{\mathfrak{n}+1}(\vartheta)\dot{a}(\vartheta)d\vartheta + \lambda_P\int_0^\infty I_{\mathfrak{n}}(\vartheta)d\vartheta + \eta\int_0^\infty \Phi_{v,\mathfrak{n}}(\vartheta)d\vartheta\right), \quad \mathfrak{n} \geq 1 \tag{14}$$

$$\Phi_{v,\mathfrak{n}}(0) = \begin{cases} \lambda_P V_0, & \mathfrak{n} = 0 \\ 0, & \mathfrak{n} \geq 1 \end{cases} \tag{15}$$

$$\omega_{y,\mathfrak{n}}(\vartheta,0) = \lambda_N\int_0^\infty \Xi_{b,\mathfrak{n}}(\vartheta)d\vartheta, \quad \mathfrak{n} \geq 0 \tag{16}$$

$$\Theta_{\mathfrak{n}}(\vartheta,0) = \int_0^\infty \omega_{y,\mathfrak{n}}(\varsigma)\gamma(\vartheta)d\vartheta, \quad \mathfrak{n} \geq 0 \tag{17}$$

The system's normalizing condition is given by

$$\begin{aligned} & I_0 + V_0 + \sum_{\mathfrak{n}=1}^{\infty}\int_0^\infty I_{\mathfrak{n}}(\vartheta)d\vartheta \\ & + \sum_{\mathfrak{n}=0}^{\infty}\left(\int_0^\infty \Xi_{b,\mathfrak{n}}(\vartheta)d\vartheta + \int_0^\infty \Phi_{v,\mathfrak{n}}(\vartheta)d\vartheta + \int_0^\infty\int_0^\infty \Theta_{\mathfrak{n}}(\vartheta,\varsigma)d\vartheta d\varsigma + \int_0^\infty\int_0^\infty \omega_{y,\mathfrak{n}}(\vartheta,\varsigma)d\vartheta d\varsigma\right) = 1 \end{aligned} \tag{18}$$

3.3 *Steady-state solution*

Using the PGFs technique, we create the equation of steady-state to the RQ model. As a result, the PGFs utilized to solve the equations above are defined for $|\tau| \leq 1$ as follows:

$$I(\vartheta,\,\tau)=\sum_{\mathfrak{n}=1}^{\infty} I_{\mathfrak{n}}(\vartheta)\;\tau^{\mathfrak{n}};\quad I(0,\,\tau)=\sum_{\mathfrak{n}=1}^{\infty} I_{\mathfrak{n}}(0)\;\tau^{\mathfrak{n}}$$

$$\Xi_b(\vartheta,\,\tau)=\sum_{\mathfrak{n}=0}^{\infty} \Xi_{b,\mathfrak{n}}(\vartheta)\;\tau^{\mathfrak{n}};\quad \Xi_b(0,\,\tau)=\sum_{\mathfrak{n}=0}^{\infty} \Xi_{b,\mathfrak{n}}(0)\;\tau^{\mathfrak{n}}$$

$$\Phi_v(\vartheta,\,\tau)=\sum_{\mathfrak{n}=0}^{\infty} \Phi_{v,\mathfrak{n}}(\vartheta)\;\tau^{\mathfrak{n}};\quad \Phi_v(0,\,\tau)=\sum_{\mathfrak{n}=0}^{\infty} \Phi_{v,\mathfrak{n}}(0)\;\tau^{\mathfrak{n}}$$

$$\omega_y(\vartheta,\,\varsigma,\,\tau)=\sum_{\mathfrak{n}=0}^{\infty} \omega_{y,\mathfrak{n}}(\vartheta,\,\varsigma)\;\tau^{\mathfrak{n}};\quad \omega_y(\vartheta,\,0,\,\tau)=\sum_{\mathfrak{n}=0}^{\infty} \omega_{y,\mathfrak{n}}(\vartheta,\,0)\;\tau^{\mathfrak{n}}$$

$$\Theta(\vartheta,\,\varsigma,\,\tau)=\sum_{\mathfrak{n}=0}^{\infty} \Theta_{\mathfrak{n}}(\vartheta,\,\varsigma)\;\tau^{\mathfrak{n}};\quad \Theta(\vartheta,\,0,\,\tau)=\sum_{\mathfrak{n}=0}^{\infty} \Theta_{\mathfrak{n}}(\vartheta,\,0)\;\tau^{\mathfrak{n}}$$

On multiplying the Eqs (2) to (17) by $\tau^{\mathfrak{n}}$ and then summing over $\mathfrak{n}$, ($\mathfrak{n}$ = 0, 1, 2, ...), we get

$$\frac{\partial}{\partial\vartheta}\,I(\vartheta,\,\tau)=-[\lambda_P+\dot{a}(\vartheta)]\;I(\vartheta,\,\tau) \tag{19}$$

$$\frac{\partial}{\partial\vartheta}\,\Xi_b(\vartheta,\,\tau)=-[\lambda_P(1-\tau)+\lambda_N+\beta_b(\vartheta)]\;\Xi_b(\vartheta,\,\tau) \tag{20}$$

$$\frac{\partial}{\partial\vartheta}\,\Phi_v(\vartheta,\,\tau)=-[\lambda_P(1-\tau)+\eta+\beta_v(\vartheta)]\;\Phi_v(\vartheta,\,\tau) \tag{21}$$

$$\frac{\partial}{\partial\varsigma}\,\omega_y(\vartheta,\,\varsigma,\,\tau)=-[\lambda_P(1-\tau)+\chi(\varsigma)]\;\omega_y(\vartheta,\,\varsigma,\,\tau) \tag{22}$$

$$\frac{\partial}{\partial\varsigma}\,\Theta(\vartheta,\,\varsigma,\,\tau)=-[\lambda_P(1-\tau)+\gamma(\varsigma)]\;\Theta(\vartheta,\,\varsigma,\,\tau) \tag{23}$$

$$I(0,\tau)=(f\tau+\overline{f})\left(\int_0^{\infty}\Xi_b(\vartheta,\tau)\beta_b(\vartheta)d\vartheta+\int_0^{\infty}\Phi_v(\vartheta,\tau)\beta_v(\vartheta)d\vartheta\right)$$
$$+\int_0^{\infty}\Theta(\vartheta,\tau)\gamma(\vartheta)d\vartheta-((\lambda_P+\eta)V_0-\eta lV_0) \tag{24}$$

$$\Xi_b(0,\tau)=\frac{1}{\tau}\int_0^{\infty}I(\vartheta,\tau)\dot{a}(\vartheta)d\vartheta+\lambda_P\int_0^{\infty}I(\vartheta,\tau)d\vartheta+\eta\int_0^{\infty}\Phi_v(\vartheta,\tau)d\vartheta+\lambda_PI_0 \tag{25}$$

$$\Phi_v(0,\tau)=\lambda_PV_0 \tag{26}$$

$$\omega_y(\vartheta,0,\tau)=\lambda_N\int_0^{\infty}\Xi_b(\vartheta,\tau)d\vartheta \tag{27}$$

$$\Theta_{\mathrm{n}}(\vartheta,0,\tau)=\int_0^{\infty}\omega_{y,\mathrm{n}}(\vartheta,\varsigma,\tau)(\vartheta)\gamma(\varsigma)d\varsigma \tag{28}$$

Solving the partial differential Eqs (19) to (23), we get

$$I(\vartheta,\tau)=I(0,\tau)\,[1-A(\vartheta)]\;exp\{-\lambda_P\vartheta\} \tag{29}$$

$$\Xi_b(\vartheta,\tau)=\Xi_b(0,\tau)[1-B_b(\vartheta)]\;exp\{-A_b(\tau)\vartheta\} \tag{30}$$

$$\Phi_v(\vartheta,\tau)=\Phi_v(0,\tau)[1-B_v(\vartheta)]\;exp\{-A_v(\tau)\vartheta\} \tag{31}$$

$$\Theta(\vartheta,\varsigma,\tau)=\Theta(\vartheta,0,\tau)[1-G(\vartheta)]\;exp\{-b(\tau)\varsigma\} \tag{32}$$

$$\omega_y(\vartheta,\varsigma,\tau)=\omega_y(\vartheta,0,\tau)[1-W(\vartheta)]\;exp\{-h(\tau)\varsigma\} \tag{33}$$

where $A_b(\tau)=(\lambda_N+\lambda_P\;(1-\tau))$, $A_v(\tau)=(\eta+\lambda_P\;(1-\tau))$, $b(\tau)=\lambda_P\;(1-\tau)$ and $h(\tau)=\lambda_P\;(1-\tau)$.

Inserting Eqs (29) to (33) in (25) and then making few modulation, we obtain as,

$$\Xi_b(0,\tau)=(I(0,\tau)/\tau)(A^*(\lambda_P)+\tau(1-A^*(\lambda_P)))+\lambda_PI_0+\lambda_PV_0V(\tau) \tag{34}$$

where, $V(\tau)=\dfrac{\eta\;[1-B_v^*(A_v(\tau))]}{\eta+(1-\tau)}$.

Using Eqs (30) to (33) in (24), which gives

$$I(0,\,\tau)=(f\tau+\overline{f})\left(\Xi_b(0,\,\tau)\,B_b^*(A_b(\tau))+\Phi_v(0,\,\tau)\,B_v^*(A_v(\tau))\right)+\Theta(0,\,\tau)\,G^*(b(\tau))-(\lambda_P+r\eta)V_0 \tag{35}$$

Using Eq (30) in Eq (27), we get

$$\omega_y(\vartheta,\,0,\,\tau)=\lambda_N\Xi_b(0,\,\tau)\left(\frac{1-B_b(A_b(\tau))}{A_b(\tau)}\right) \tag{36}$$

Using the Eq (33) in Eq (28), we have

$$\Theta(\vartheta,\,0,\,\tau)=\omega_y(\vartheta,\,0,\,\tau)(\chi^*(h(\tau))) \tag{37}$$

Using Eqs (26), (33) and (34) in Eq (35), we get

$$I(0,\,\tau)=\frac{\mathrm{Nu}(\tau)}{\mathrm{De}(\tau)} \tag{38}$$

$$\mathrm{Nu}(\tau)=\tau V_0\times\left\{\begin{array}{l}\left(\lambda_P\left((f\tau+\overline{f})B_v^*(A_v(\tau))-1\right)-\eta r\right)+(\lambda_P V(\tau)+\eta r)\\ \left((f\tau+\overline{f})B_b^*(A_b(\tau))+\dfrac{\lambda_N G^*(b(\tau))W^*(h(\tau))(1-B_b^*(A_b(\tau)))}{A_b(\tau)}\right)\end{array}\right\}$$

$$\mathrm{De}(\tau)=\left\{\begin{array}{l}\tau-(A^*(\lambda_P)+\tau(1-A^*(\lambda_P)))\\ \left((f\tau+\overline{f})B_b^*(A_b(\tau))+\dfrac{\lambda_N G^*(b(\tau))W^*(h(\tau))\left(1-B_b^*(A_b(\tau))\right)}{A_b(\tau)}\right)\end{array}\right\}$$

Using Eq (38) in Eq (34), we get

$$\Xi_b(0,\,\tau)=\frac{V_0}{De(\tau)}\left\{\;\lambda_P\left((f\tau+\overline{f})B_v^*(A_v(\tau))-1\right)-\eta r(A^*(\lambda)+\tau(1-A^*(\lambda_P)))+\tau(\lambda_P V(\tau)+\eta r)\;\right\} \tag{39}$$

Using the Eq (39) in Eq (36), we get

$$\omega_y(\vartheta,\,0,\,\tau)=\frac{\lambda_N V_0\left(1-B_b^*(A_b(\tau))\right)}{A_b(\tau)\times\mathrm{De}(\tau)}\times\left\{\begin{array}{l}\left(\lambda_P\left((f\tau+\overline{f})B_v^*(A_v(\tau))-1\right)-\eta r\right)\\ (A^*(\lambda_P)+\tau(1-A^*(\lambda_P)))+\tau(\lambda_P V(\tau)+\eta r)\end{array}\right\} \tag{40}$$

Using the Eq (40) in Eq (37), we get

$$\Theta(\vartheta, 0, \tau) = \frac{\lambda_N V_0 \left(1 - B_b^*(A_b(\tau))\right) W^*(h(\tau))}{A_b(\tau) \times \mathrm{De}(\tau)} \times \left\{ \begin{array}{l} \left(\lambda_P \left((f\tau + \overline{f}) B_v^*(A_v(\tau)) - 1\right) - \eta r\right) \\ \\ (A^*(\lambda_P) + \tau(1 - A^*(\lambda_P))) + \tau(\lambda_P V(\tau) + \eta r) \end{array} \right\} \tag{41}$$

Similarly, the Eq (26) and Eq (38) to (41) substitute in Eqs (29) to (33) then, we obtain the results for the subsequent PGFs as $I(\vartheta, \tau)$, $\Xi_b(\vartheta, \tau)$, $\Phi_v(\vartheta, \tau)$, $\omega_y(\vartheta, \tau)$ and $\Theta(\vartheta, \tau)$.

Theorem 2 The stationary joint distributions of the no. of consumers consumers in the orbit when the server is unoccupied, occupied, on WV, delay, and under repair are provided by stability condition $\Gamma < 1$ is given by

$$I(\tau) = \frac{Nu(\tau)}{De(\tau)} \tag{42}$$

$$Nr(\tau) = \tau V_0 \left(\frac{1 - A^*(\lambda_P)}{\lambda_P}\right) \times \left\{ \begin{array}{l} \left(\lambda_P \left((f\tau + \overline{f}) B_v^*(A_v(\tau)) - 1\right) - \eta r\right) + (\lambda_P V(\tau) + \eta r) \\ \\ \left((f\tau + \overline{f}) B_b^*(A_b(\tau)) + \dfrac{\lambda_N G^*(b(\tau)) W^*(h(\tau)) \left(1 - B_b^*(A_b(\tau))\right)}{A_b(\tau)}\right) \end{array} \right\}$$

$$De(\tau) = \left\{ \begin{array}{l} \tau - (A^*(\lambda_P) + \tau(1 - A^*(\lambda_P))) \\ \\ \left((f\tau + \overline{f}) B_b^*(A_b(\tau)) + \dfrac{\lambda_N G^* b(\tau) W^* h(\tau) \left(1 - B_b^*(A_b(\tau))\right)}{A_b(\tau)}\right) \end{array} \right\}$$

$$\Xi_b(\tau) = \frac{V_0(1 - B_b^*(A_b(\tau)))}{A_b(\tau) De(\tau)} \times \left\{ \begin{array}{l} (\lambda_P((f\tau + \overline{f}) B_v^*(A_v(\tau)) - 1) - \eta r)(A^*(\lambda_P) \\ \\ +\tau(1 - A^*(\lambda_P))) + \tau(\lambda_P V(\tau) + \eta r) \end{array} \right\} \tag{43}$$

$$\Phi_v(\tau) = \{\lambda_P V_0 V(\tau)/\eta\} \tag{44}$$

$$\omega_y(\tau) = \frac{\left(\lambda_N V_0 (1 - W^*(h(\tau)))\left(1 - B_b^*(A_b(\tau))\right)\right)}{h(\tau) A_b(\tau) De(\tau)} \times \left\{\left(\lambda_P \left((f\tau + \overline{f}) B_v^*(A_v(\tau)) - 1\right) - \eta r\right) (A^*(\lambda_P) + \tau(1 - A^*(\lambda_P))) + \tau(\lambda_P V(\tau) + \eta r)\right\} \tag{45}$$

$$\Theta(\tau)=\frac{\left(\lambda_N V_0\left(1-B_b^*(A_b(\tau))\right)(1-G^*(b(\tau))W^*(h(\tau))\right)}{b(\tau)A_b(\tau)De(\tau)}\times$$

$$\left\{\left(\lambda_P\left((f\tau+\overline{f})B_v^*(A_v(\tau))-1\right)-\eta r\right)(A^*(\lambda_P)+\tau(1-A^*(\lambda_P)))+\tau(\lambda_P V(\tau)+\eta r)\right\} \tag{46}$$

where

$$V_0=\frac{A^*(\lambda_P)-\Gamma}{\{(\lambda_P/\eta)(1-B_v^*(\eta))+A^*(\lambda_P)(1+(\eta r/\lambda_P))-\Gamma B_v^*(\eta)\}} \tag{47}$$

$$I_0=\frac{\eta r(A^*(\lambda_P)-\Gamma)}{\lambda_P\{(\lambda_P/\eta)(1-B_v^*(\eta))+A^*(\lambda_P)(1+(\eta r/\lambda_P))-\Gamma B_v^*(\eta)\}} \tag{48}$$

$$\Gamma=(\lambda_P/\lambda_N)(1-B_b^*(\lambda_N))\left(1+\lambda_N(\mathfrak{g}^{(1)}+\mathfrak{w}^{(1)})\right)+fB_b^*(\lambda_N)$$

proof. Integrating the Eqs (29) to (33) with respect to ϑ, we define the PGFs as,

$$I(\tau)=\int_0^\infty I(\vartheta,\tau)d\vartheta,\ \Xi_b(\tau)=\int_0^\infty \Xi_b(\vartheta,\tau)d\vartheta,\ \Phi_v(\tau)=\int_0^\infty \Phi_v(\vartheta,\tau)d\vartheta,$$

$$\Theta(\tau)=\int_0^\infty\int_0^\infty \Theta(\vartheta,\varsigma,\tau)d\vartheta d\varsigma,\ \omega_y(\tau)=\int_0^\infty\int_0^\infty \omega_y(\vartheta,\varsigma,\tau)d\vartheta d\varsigma$$

We can compute the prob. that the server is unoccopaid I_0 by utilizing the normalized condition. Thus, by applying $\tau=1$ in the Eqs (42) to (46) and using L-Hospital's wherever applicable, we obtain

$$I_0+V_0+I(1)+\Xi_b(1)+\Phi_v(1)+\Theta(1)+\omega(1)=1.$$

3.4 *Corollary*

Under the condition of stability $\Gamma<1$, the PGF of no. of consumers in the orbit size $K_o(\tau)$ and system size $K_s(\tau)$ distribution at stationary point of time is

$$K_o(\tau)=\frac{Nu_0(\tau)}{De_s(\tau)}=I_0+V_0+I(\tau)+\Xi_b(\tau)+\Phi_v(\tau)+\Theta(\tau)+\omega(\tau) \tag{49}$$

Where,

$$
Nu_0(\tau)=\left\{\begin{array}{l}
V_0(1-\tau)\left[\begin{array}{l}
\left\{\begin{array}{l}
(\lambda_P/\eta)((\eta+\lambda_P V(\tau))+\eta r) \\
\left(\begin{array}{l}
\tau A_b(\tau)-[A^*(\lambda_P)+\tau(1-A^*(\lambda_P))] \\
\left(A_b(\tau)B_b^*(A_b(\tau))+\lambda_N G^*(b(\tau))W^*(h(\tau))(1-B_b^*(A_b(\tau)))\right)
\end{array}\right)
\end{array}\right\} \\
+\tau(1-A^*(\lambda_P))\left\{\begin{array}{l}
(\lambda_P V(\tau)+\eta r)\left(\begin{array}{l}
A_b(\tau)B_b^*(A_b(\tau)) \\
+\lambda_P G^*(b(\tau))W^*(h(\tau))(1-B_b^*(A_b(\tau)))
\end{array}\right) \\
+A_b(\tau)(\lambda_P(B_v^*(A_v(\tau))-1)-\eta r)
\end{array}\right\}
\end{array}\right] \\
+V_0\left\{\begin{array}{l}
(1-B_b^*)(A_b(\tau))(b(\tau)+\lambda_N(1-G^*(b(\tau))))W^*(h(\tau)) \\
\{(\lambda_P(B_v^*(A_v(\tau))-1)-\eta r)(A^*(\lambda_P))+\tau(1-A^*(\lambda_P))+\tau(\lambda_P V(\tau)+\eta r)\}
\end{array}\right\}
\end{array}\right\}
$$

$$
De_s(\tau)=b(\tau)\times\left\{\begin{array}{l}
\tau A_b(\tau)-(A^*(\lambda_P)+\tau(1-A^*(\lambda_P))) \\
(A_b(\tau)(B_v^*(A_v(\tau)))+\lambda_N G^*(b(\tau))W^*(h(\tau))(1-B_b^*(A_b(\tau))))
\end{array}\right\}
$$

$$
K_s(\tau)=\frac{Nu_s(\tau)}{De_s(\tau)}=I_0+V_0+I(\tau)+\tau(\Xi_b(\tau)+\Phi_v(\tau))+\Theta(\tau)+\omega(\tau) \tag{50}
$$

Where,

$$Nu_s(\tau)=\left\{\begin{array}{l} V_0(1-\tau)\left[\begin{array}{l}\left\{\begin{array}{l}\tau(\lambda_P/\eta)((\eta+\lambda_P V(\tau))+\eta r) \\ \left(\begin{array}{l}\tau A_b(\tau)-[A^*(\lambda_P)+\tau(1-A^*(\lambda_P))] \\ \left(A_b(\tau)B_b^*(A_b(\tau))+\lambda_N G^*(b(\tau))W^*(h(\tau))(1-B_b^*(A_b(\tau))\right)\end{array}\right)\end{array}\right\} \\ +\tau(1-A^*(\lambda_P))\left\{\begin{array}{l}(\lambda_P V(\tau)+\eta r)\left(\begin{array}{l}A_b(\tau)B_b^*(A_b(\tau)) \\ +\lambda_P G^*(b(\tau))W^*(h(\tau))(1-B_b^*(A_b(\tau)))\end{array}\right) \\ +A_b(\tau)(\lambda_P(B_v^*(A_v(\tau))-1)-\eta r)\end{array}\right\}\end{array}\right] \\ +V_0\left\{\begin{array}{l}(1-B_b^*)(A_b(\tau))(\tau b(\tau)+\lambda_N(1-G^*(b(\tau))))W^*(h(\tau)) \\ \{(\lambda_P(B_v^*(A_v(\tau))-1)-\eta r)(A^*(\lambda_P))+\tau(1-A^*(\lambda_P))+\tau(\lambda_P V(\tau)+\eta r)\}\end{array}\right\}\end{array}\right\}$$

4. Performance characteristic

This section evaluates the various system probabilities, the avarage no. of consumers in the orbit and system, reliability indicators, mean occupied time, and average busy cycle of our system.

4.1 *Probabilities of the system state*

We determine some probabilities for the system in various states and assess metrics for the model's system performance. The following outcomes are obtained from Eqs (42) to (46) by applying $\tau\to 1$ then using L-Hospital's wherever appropriate.

1. The prob. that the server will be available for the retrial period:

$$I(1)=V_0\left(1-A^*(\lambda_P)\right)\times$$

$$\left\{\frac{(\lambda_P/\eta)\left(1-B_v(\eta)\right)+\left\{\begin{array}{l}\left(\lambda_P\left(1-B_v(\eta)\right)+\eta r\right) \\ \left(\left(1-B_b^*(\lambda_N)\right)\left(1+\lambda_N(\mathfrak{g}^{(1)}+\mathfrak{w}^{(1)})\right)/\lambda_N+fB_b^*(\lambda_N)\right)\end{array}\right\}}{A^*(\lambda_P)-\left(\lambda_P\left(1-B_b^*(\lambda_N)\right)\left(1+\lambda_N(\mathfrak{g}^{(1)}+\mathfrak{w}^{(1)})\right)/\lambda_N\right)+fB_b^*(\lambda_N)}\right\} \tag{51}$$

2. The prob. when the service is regular busy:

$$\Xi_b(1) = \frac{V_0((1-B_b^*(\lambda_N))}{\lambda_N}\left\{\frac{((\lambda_P)^2/\eta)(1-B_b^*(\lambda_N))+A^*(\lambda_P)(\lambda_P(1-B_v^*(\eta))+\eta r)+\lambda_P fB_v^*(\eta)}{A^*(\lambda_P)-(\lambda_P(1-B_b^*(\lambda_N))(1+\lambda_N(\mathfrak{g}^{(1)}+\mathfrak{w}^{(1)}))/\lambda_N)+fB_b^*(\lambda_N)}\right\} \tag{52}$$

3. The prob. when the server is on a slow service rate:

$$\Phi_v(1) = \{\lambda_P V_0(1-B_v^*(\eta))/\eta\} \tag{53}$$

4. The prob. when the server is undergoing delayed repair:

$$\omega_y(1) = V_0((1-B_b^*(\lambda_N)))\mathfrak{w}^{(1)} \times \left\{\frac{((\lambda_P)^2/\eta)(1-B_v^*(\eta))+A^*(\lambda_P)(\lambda_P(1-B_v^*(\eta))+\eta r)+f\lambda_P B_v^*(\eta)}{A^*(\lambda_P)-\left(\lambda_P\left(1-B_b^*(\lambda_N)\right)\left(1+\lambda_P(\mathfrak{g}^{(1)}+\mathfrak{w}^{(1)})\right)/\lambda_N\right)+fB_b^*(\lambda_N)}\right\} \tag{54}$$

5. The server's prob. under repair is given by:

$$\Theta(1) = V_0((1-B_b^*(\lambda_N)))\mathfrak{g}^{(1)} \times \left\{\frac{((\lambda_P)^2/\eta)(1-B_v^*(\eta))+A^*(\lambda_P)(\lambda_P(1-B_v^*(\eta))+\eta r)+f\lambda_P B_v^*(\eta)}{A^*(\lambda_P)-\left(\lambda_P\left(1-B_b^*(\lambda_N)\right)\left(1+\lambda_N(\mathfrak{g}^{(1)}+\mathfrak{w}^{(1)})\right)/\lambda_N\right)+fB_b^*(\lambda_N)}\right\} \tag{55}$$

4.2 *Average orbit size and system size*

i) Differentiate Eq (49) with respect to τ and calculate at $\tau = 1$, we determine the number of consumers in the orbit L_q as below.

$$L_q = K_0'(1) = \lim_{\varsigma\to 1} K_0'(\tau) = V_0\left[\frac{Nu_q'''(1)De_q''(1)-De_q'''(1)Nu_q''(1)}{3(De_q''(1))^2}\right] \tag{56}$$

$$Nu_q''(1) = -2\left\{\begin{array}{l}(\lambda_P)^2(1-B_b^*(\lambda_N))\left\{(1-B_v^*(\eta))-(1+\lambda_N(\mathfrak{g}^{(1)}+\mathfrak{w}^{(1)})B_v^*(\eta))\right\}+\lambda_N A^*(\lambda_P)(\lambda_P+\eta r)\\ +\dfrac{\lambda_N(\lambda_P)^2}{\eta}(1-B_v^*(\eta))+(1-B_b^*(\lambda_N))\lambda_P fB_v^*(\eta)(\lambda_P(1+\lambda_N(\mathfrak{g}^{(1)}+\mathfrak{w}^{(1)})))\\ -\lambda_N fB_b^*(\lambda_N)((1-A^*)(\lambda_P+\lambda_P^2/\eta(1-B_v^*(\eta))+\eta p))\end{array}\right\}$$

$$Nu_q^{'''}(1) = 6\left((\frac{\lambda_P}{\eta})(\eta + \lambda_P(1 - B_v(\eta)) + \eta r) \right)$$

$$- 6(1 - A^*(\lambda_P))\big(- \lambda_P(1 - B_b(\lambda_N)) + \lambda_N\lambda_P(\mathfrak{g}^{(1)} + \mathfrak{w}^{(1)})(1 - B_b(\lambda_N))\big)$$

$$\left[\lambda_P V(\tau) - ((\frac{\lambda_P}{\eta})(\eta + \lambda_P(1 - B_v(\eta)) + \eta r))\right]$$

$$- 6(\lambda_P)^2\left((\frac{\lambda_N}{\eta})\big(V^{'}(1) + (1 - A^*(\lambda_P))(1 - B_v(\eta))\big) \right)$$

$$- 6(\lambda_P)^3 B_v^{*'}(\eta)(1 - A^*(\lambda_P)) - 6\lambda_P(1 - B_b^*(\lambda_N))(1 - A^*(\lambda_P))(1 + \lambda_N(\mathfrak{g}^{(1)} + \mathfrak{w}^{(1)}))$$

$$\times \big(\eta r + \lambda_P(1 - B_v^*(\eta))\big) + 6(\lambda_P)^3 B_b^{*'}(\lambda_N)B_v^*(\eta)(1 + \lambda_N(\mathfrak{g}^{(1)} + \mathfrak{w}^{(1)}))$$

$$- 6\left(\frac{(\lambda_P)^3}{\eta}(1 - B_b(\lambda_N))((1 + \lambda_N(\mathfrak{g}^{(1)} + \mathfrak{w}^{(1)})))\right)\left((1 - B_v(\eta)) + \eta A^*(\lambda_P)B_v^{*'}(\eta)\right)$$

$$+ 3\lambda_N(\lambda_P)^2(\mathfrak{g}^{(2)} + \mathfrak{w}^{(2)})(1 - B_b(\lambda_N))B_v^*(\eta) + 6(\lambda_P)^3\mathfrak{g}^{(1)}\mathfrak{w}^{(1)}(1 - B_b(\lambda_N))B_v(\eta)$$

where, $V^{'}(1) = \lambda^2\left(2\left(\frac{B_v^{*'}(\eta)}{\eta} + \frac{(1 - B_v^*(\eta))}{\eta^2}\right) - B_v^{*''}(\eta)\right)$, $B_v^{*'} \int_0^\infty \vartheta e^{-\eta} dB_v(\vartheta)$, $B_b^{*'} \int_0^\infty \vartheta e^{-\lambda_N} dB_b(\vartheta)$

$$De_q^{''}(1) = -2\lambda_P\lambda_N(A^*(\lambda_P) - \Gamma) \tag{57}$$

$$De_q^{'''}(1) = 3\lambda_P\left((\lambda_P)^2(2B_b^{*'}(\lambda_N)(1 + \lambda_N(\mathfrak{g}^{(1)} + \mathfrak{w}^{(1)}))) + \lambda_N(1 + \mathfrak{g}^{(2)}\mathfrak{w}^{(2)})(1 - B_b^{*'}(\lambda_N)) + 2\lambda_P\lambda_N\Gamma \right)$$

$$- 6\lambda_P A^*(\lambda_P) \times \left((\lambda_N\lambda_P(\mathfrak{g}^{(1)} + \mathfrak{w}^{(1)}))(1 - B_b^{*'}(\lambda_N) - \lambda_P B_b^*(\lambda_N)) \right)$$

$$+ 6(\lambda_P)^3\lambda_N\mathfrak{g}^{(1)}\mathfrak{w}^{(1)} \times (1 - B_b^*(\lambda_N))$$

$$- 6\lambda_P A^*(\lambda_P)\lambda_N f B_b^*(\lambda_N) - 6\lambda_P^2 f B_b^*(\lambda_N) + 6\lambda_P^2\lambda_N f B_b^{*'}(\lambda_N)$$

where, $\Gamma = (\lambda_P/\lambda_N)\left(1 - B_b^*(\lambda_N)\right)\left(1 + \lambda_N(\mathfrak{g}^{(1)} + \mathfrak{w}^{(1)})\right) + f B_b^*(\lambda_N)$.

ii) Differentiate Eq (49) with respect to τ and calculate at $\tau = 1$, we determine the no. of consumers in the system L_s as below.

$$L_s = K_s'(1) = \lim_{\tau \to 1} K_s'(\tau) = V_0 \left[\frac{Nu_s'''(1) De_q''(1) - De_q'''(1) Nu_q''(1)}{3(De_q''(1))^2} \right] \tag{58}$$

$$Nu_s'''(1) = Nu_q'''(1) + 6(\frac{\lambda_P^3}{\eta})\lambda_N(\mathfrak{g}^{(1)} + \omega^{(1)})(1 - B_b^*(\lambda_N))(1 - B_v^*(\eta)) - 6\lambda_P(1 - A^*(\lambda_P))$$

$$\times (1 - B_b^*(\lambda_N))(\lambda_P((1 - B_v^*(\eta))(\eta)) + \eta r) - 6(1 - A^*(\lambda_P)\frac{\lambda_P^2 \lambda_N}{\eta}(1 - B_v^*)(\eta))$$

4.3 *Reliability measures*

In a queueing system with an unreliable server, reliability metrics give the data needed for the improvement of the system. To create an understanding of and evaluate the model of analytical results, the availability measures (S_{Av}) and failure incidence $(Fail_f)$ are derived as follows:

1. The server's steady-state stability is described as follows:

$$S_{Av} = 1 - \lim_{\tau \to 1}(\omega_y(\tau) + \Theta(\tau)) = 1 - (\omega_y(1) + \Theta(1))$$

$$= 1 - \left\{ \begin{array}{l} (\mathfrak{g}^{(1)} + \mathfrak{w}^{(1)}) \times \left(\left(1 - B_b^*(\lambda_N)\right) / \lambda_N \right) \\ \times \left\{ \dfrac{\left((\lambda_P)^2/\eta\right)(1 - B_v^*(\eta)) + A^*(\lambda_P)\left(\lambda_P\left(1 - B_v^*(\eta)\right) + \eta r\right) + f\lambda_P B_v^*(\eta)}{\{(\lambda_P/\eta)(1 - B_v^*(\eta)) + A^*(\lambda_P)(1 + (\eta r/\lambda_P)) - \Gamma B_v^*(\eta)\}} \right. \end{array} \right\}$$

2. The steady-state system failure occurrence as follows:

$$Fail_f = \lambda_N * \Xi_b(1)$$

$$= \left\{ V_0(1 - B_b^*(\eta)) \left\{ \frac{(\lambda_P)^2/\eta(1 - B_v^*(\eta)) + A^*(\lambda_P)\left(\lambda_P\left(1 - B_v^*(\eta)\right) + \eta r\right) + f\lambda_P B_v^*(\eta)}{A^*(\lambda_P) - (\lambda_P(1 - B_b^*(\lambda_N))(1 + \lambda_N(\mathfrak{g}^{(1)} + \mathfrak{w}^{(1)}))/\lambda_N)} \right\} \right\}$$

4.4 *Average busy time and the busy cycle*

Let the average length of the busy cycle and busy period $M(T_{bp})$ and $M(T_{bc})$ respectively. Using the explanation for an alternate renewal approach, as in [32], yields the findings in a direct manner,

$$I_0 = \frac{M(T_0)}{M(T_0) + M(T_{bp})};\ M(T_{bp}) = \frac{1}{\lambda_P}\left(\frac{1}{I_0} - 1\right) \text{ and } M(T_{bc}) = \frac{1}{(\lambda_P) I_0} = M(T_0) + M(T_{bp}) \tag{59}$$

where the length of the system is unoccupied state denoted by T_0, and $M(T_0) = (1/\lambda_P)$. By substituting the Eq (48) in (49), we get the expected result to be

$$M(T_{bp}) = \left\{ \frac{((\lambda_P/\eta)(1 - B_v^*(\eta)) + A^*(\lambda_P) - \Upsilon)}{\eta r (A^*(\lambda_P) - \Gamma)} \right\} \tag{60}$$

Where, $\Upsilon = ((1 - B_b^*(\lambda_N))/\lambda_N)(1 + \lambda_N(\mathfrak{g}^{(1)} + \mathfrak{w}^{(1)}))(\lambda_P B_v^*(\eta) + \eta r)$

$$M(T_{bc}) = \left\{ \frac{((\lambda_P/\eta)(1 - B_v^*(\eta)) + A^*(\lambda_P)(1 + (\eta r/\lambda_P)) - B_v^*(\eta)\,\Gamma)}{\eta r (A^*(\lambda_P) - \Gamma)} \right\} \tag{61}$$

4.5 *Stochastic decomposition (SD) of system size distribution*

This section examines, the system size distribution's SD property. Fuhrman and Cooper [33] have mentioned the process of SD among types of M/G/1 queues along server vacations. A significant result from this investigation is the distribution of no. of consumers in the system during steady-state at a random point of time can be represented as the sum of two independent RVs. The first variable represents no. of consumers in a normal queueing model in absents of vacations. The second variable, however, varies in its probabilistic interpretations depending on the specific scheduling of vacations. Let $\varphi(\tau)$ represent the PGF of the no. of consumers in the normal busy or on WV in the model steady-state at random intervals of time, $\psi(\tau)$ represents the PGF of the no. of consumers in the retrial area while it is unoccupied, or under maintenance, and $\hat{S}_D(\tau)$ the PGF of the no. of consumers of the given system to be decomposed. $\check{S}_D(\tau) = \psi(\tau) \times \varphi(\tau)$ has been used to represent the SD procedure.

$$\check{S}_D(\tau) = \psi(\tau) \times \varphi(\tau)$$

$$\psi(\tau) = \frac{V_0(1 + \frac{\eta p}{\lambda_P}) + I(\tau) + \Theta(\tau) + \omega(\tau)}{V_0(1 + \frac{\eta p}{\lambda_P}) + I(1) + \Theta(1) + \omega(1)} \tag{62}$$

Using the Eqs (42), (45), (46), (47) and also the Eqs (51), (54), (55) substitute in Eq (62), we get

$$
\psi(\tau)=\frac{
\left\{\begin{array}{l}
(1+\dfrac{\eta p}{\lambda_P})(\lambda(1-\tau))\left\{\begin{array}{l}\tau A_b(\tau)-(A^*(\lambda_P)+\tau(1-A^*(\lambda_P)))\\[2ex](A_b(\tau)(f\tau+\overline{f})B_b^*(A_b(\tau))+\lambda_N G^* b(\tau)W^* h(\tau)(1-B_b^*(A_b(\tau)))\end{array}\right\}\\[4ex]
+\tau(\lambda_P(1-\tau))\dfrac{(1-A^*(\lambda_P))}{\lambda_P}\left\{\begin{array}{l}\big(A_b(\tau)\lambda_P\big((f\tau+\overline{f})B_v^*(A_v(\tau))-1\big)-\eta r\big)\\[2ex]+(\lambda_P V(\tau)+\eta r)\left(\begin{array}{l}(f\tau+\overline{f})B_b^*(A_b(\tau))+\lambda_N G^*(b(\tau))\\[2ex]W^*(h(\tau))(1-B_b^*(A_b(\tau)))\end{array}\right)\end{array}\right\}\\[4ex]
+(\lambda_N(1-B_b^*(A_b(\tau)))((1-G^*(b(\tau)))W^*(h(\tau))+(1-W^*(h(\tau)))))\\[2ex]
\big\{\big(\lambda_P\big((f\tau+\overline{f})B_v^*(A_v(\tau))-1\big)-\eta r\big)(A^*(\lambda_P)+\tau(1-A^*(\lambda_P)))+\tau(\lambda_P V(\tau)+\eta r)\big\}
\end{array}\right\}
\\[2ex]
\times A^*(\lambda_P)-\Big(\lambda_P(1-B_b^*(\lambda_N))\Big(1+\lambda_N(\mathfrak{g}^{(1)}+\mathfrak{w}^{(1)})\Big)/\lambda_N\Big)+fB_b^*(\lambda_P)
}{
\left\{\begin{array}{l}
(1+\dfrac{\eta p}{\lambda})\Big(A^*(\lambda_P)-\Big(\lambda_P(1-B_b^*(\lambda_N))\Big(1+\lambda_N(\mathfrak{g}^{(1)}+\mathfrak{w}^{(1)})\Big)/\lambda_N\Big)+fB_b^*(\lambda_P)\Big)\\[4ex]
+\left\{(\lambda_P/\eta)(1-B_v(\eta))+\left\{\begin{array}{l}(\lambda_P(1-B_v(\eta))+\eta r)\\[2ex]\Big((1-B_b^*(\lambda_N))\Big(1+\lambda_N(\mathfrak{g}^{(1)}+\mathfrak{w}^{(1)})\Big)/\lambda_N+fB_b^*(\lambda_P)\Big)\end{array}\right\}\right\}\\[4ex]
+((1-B_b^*(\lambda_N)))\,\mathfrak{g}^{(1)}+\mathfrak{w}^{(1)}\times\left\{\begin{array}{l}\big((\lambda_P)^2/\eta\big)(1-B_v^*(\eta))+A^*(\lambda_P)\\[2ex](\lambda_P(1-B_v^*(\eta))+\eta r)+f\lambda_P B_v^*(\eta)\end{array}\right\}
\end{array}\right\}
\\[2ex]
\times\left\{\begin{array}{l}\tau A_b(\tau)-(A^*(\lambda_P)+\tau(1-A^*(\lambda_P)))\\[2ex]\big(A_b(\tau)(f\tau+\overline{f})B_b^*(A_b(\tau))+\lambda_N G^* b(\tau)W^* h(\tau)(1-B_b^*(A_b(\tau)))\big)\end{array}\right\}
}
$$

$$\varphi(\tau)=\frac{\Xi_b(\tau)+\Phi_v(\tau)}{\Xi_b(1)+\Phi_v(1)} \tag{63}$$

Using the Eqs (43), (44), (52) and (53) in Eq (63), we get

$$\varphi(\tau)=$$

$$\frac{\left\{\begin{array}{l}(1-B_b^*(A_b(\tau)))\left\{\begin{array}{l}(\lambda_P((f\tau+\overline{f})B_v^*(A_v(\tau))-1)-\eta r)\\ \\(A^*(\lambda_P)+\tau(1-A^*(\lambda_P)))+\tau(\lambda_P V(\tau)+\eta r)\end{array}\right\}\\ \\+(\lambda_P V(\tau)/\eta)\left\{\begin{array}{l}\tau A_b(\tau)-(A^*(\lambda_P)+\tau(1-A^*(\lambda_P)))\\ \\(A_b(\tau)(f\tau+\overline{f})B_b^*(A_b(\tau))+\lambda_N G^*b(\tau)W^*h(\tau)(1-B_b^*(A_b(\tau))))\end{array}\right\}\end{array}\right\} \times\left\{A^*(\lambda_P)-\left(\lambda_P(1-B_b^*(\lambda_N))\left(1+\lambda_N(\mathfrak{g}^{(1)}+\mathfrak{w}^{(1)})\right)/\lambda_N\right)+fB_b^*(\lambda_P)\right\}}{\left\{\begin{array}{l}((1-B_b^*(\lambda_N))/\delta)\left\{((\lambda_P)^2/\eta)(1-B_b^*(\eta))+A^*(\lambda_P)(\lambda_P(1-B_v^*(\eta))+\eta r)+\lambda_P fB_v^*(\eta)\right\}\\ \\+(\lambda_P(1-B_v^*(\eta))/\eta)\left\{A^*(\lambda_P)-\left(\lambda_P(1-B_b^*(\lambda_N))\left(1+\lambda_N(\mathfrak{g}^{(1)}+\mathfrak{w}^{(1)})\right)/\lambda_N\right)+fB_b^*(\lambda_P)\right\}\end{array}\right\} \times\left\{\tau A_b(\tau)-(A^*(\lambda_P)+\tau(1-A^*(\lambda_P)))\begin{pmatrix}A_b(\tau)(f\tau+\overline{f})B_b^*(A_b(\tau))+\lambda_N G^*b(\tau)\\ \\W^*h(\tau)(1-B_b^*(A_b(\tau)))\end{pmatrix}\right\}}$$

5. Special cases

In this section, we examine a few specific instances of our model that coincide with the existing collection of literature.

Case (i): No unfavorable arrival, no delay repair, and with multiple WV. We consider that $(\lambda_N,\ \omega)_{\rightarrow}(0,\ 0)$. Our model reduced to an *SRQ* with Bernoulli WV and vacations interruption (VI) concepts. Here, $K_s(\tau)$ indicates the PGF of the no. of consumers in the system, whereas (K_0) indicates the PGF of no. of the consumers in the orbit. The average length of the system and orbit was assumed respectively L_s and L_q.

$$K_s(\tau)=\frac{I_0\left\{\left[\begin{array}{l}\left\{(1-\tau)\ (\lambda_P/\eta)((\eta+\lambda_P\tau V(\tau))+\eta r)\begin{pmatrix}\tau-[A^*(\lambda_P)+\tau(1-A^*(\lambda_P))]\\ \\ (B_b^*(A_b(\tau)))\end{pmatrix}\right\}\\ \\ +(1-\tau)\ \tau(1-A^*(\lambda_P))\left\{\begin{array}{l}(\lambda_P V(\tau)+\eta r)(B_b^*(A_b(\tau)))\\ \\ +(\lambda_P(B_v^*(A_v(\tau))-1)-\eta r)\end{array}\right\}\end{array}\right]+\varpi\right\}}{A_b(\tau)\left\{\tau-(A^*(\lambda_P)+\tau(1-A^*(\lambda_P)))+\tau(\lambda_P V(\tau)+\eta r)\right\}} \tag{64}$$

where,

$$\varpi=\tau(1-B_b^*(A_b(\tau)))\left\{(\lambda_P(B_v^*(A_v(\tau))-1)-\eta r)(A^*(\lambda_P)+\tau(1-A^*(\lambda_P)))+\tau(\lambda_P V(\tau)+\eta r)\right\}$$

This matches the outcome obtained by Rajadurai et al. [34].

Case (ii): No arrival of unfavorable consumer and MWV. Let $\lambda_N=r=0$, our approach can be simplified to a M/G/1 RQ with general retrial times, WVs and VI. Here $K_s(\tau)$ was resulted as

$$K_s(\tau)=\frac{I_0\left\{(1-\tau)\left[\begin{array}{l}\left\{\left(\tau-[A^*(\lambda_P)+\tau(1-A^*(\lambda_P))](B_b^*(A_b(\tau)))\right)\right\}\\ \\ +\tau(1-A^*(\lambda_P))\left\{(B_v^*(A_v(\tau)))+(V(\tau)(B_b^*(A_b(\tau))-1))\right\}\end{array}\right]+\varpi_1\right\}}{A_b(\tau)\left\{\tau-(A^*(\lambda_P)+\tau\ (1-A^*(\lambda_P)))+\tau(\lambda_P V(\tau))\right\}} \tag{65}$$

$$\varpi_1=(1-B_b^*(A_b(\tau)))\ \left\{(\lambda_P(B_v^*(A_v(\tau))-1))\ (A^*(\lambda_P)+\tau(1-A^*(\lambda_P)))+\tau\ (V(\tau))\right\}$$

This matches the outcome obtained by Gao et al. [35].

Case (iii): When there is no arrival of unfavorable consumer no vacation interruption and WV. Let $\lambda_N=\eta=0$. Our approach can be simplified to a M/G/1 retrial queue with WV. Here $K_s(\tau)$ was resulted as

$$K_s(\tau)=\left[\frac{I_0\left\{[\tau+(1-\tau)A^*(\lambda_N)]\ [1-B_v^*(\lambda_P(1-\tau))]+A^*(\lambda_P)B_v^*(\lambda_P)\ (1-\tau)\right\}}{B_v^*(\lambda_P)\left\{[\tau+(1-\tau)A^*(\lambda_P)]\ B_b^*(\lambda_P(1-\tau))-\tau\right\}}\right]B_b^*(\lambda_P(1-\tau)) \tag{66}$$

This matches the outcome obtained by Arivudainambi et al. [36].

6. Numerical example

In this section, we examined the effect that a number of various variables have on the system efficiency indicators of our system using a number of different numerical demonstrations. The examples are based on the assumption that all retrial instances, periods of service, slow-rate service periods, delay, and maintenance times follow an exponentially, and Erlangianly distributed. Consequently, the parameters are chosen with arbitrary values which satisfying the stability condition. The findings are numerically illustrated using MATLAB software. It is worth noting that the equation $f(\vartheta) = \nu e^{-\nu\vartheta}$, $\vartheta > 0$ is exponential distribution and Erlang-2 stage distribution is $f(\vartheta) = \nu^2\vartheta e^{-\nu\vartheta}$, $\vartheta > 0$.

Table 1, displays that when the repeated attempts rate $\dot{a}$ grows, the probability of the orbit size L_q, the prob. of the server is being unoccupied while retrial time I, and the average waiting time in the orbit W_q is fall consistently. while the prob. of the server unoccopaid I_0 also grows. For the values of $\lambda_P = 2$; $\eta = 3$; $B_b = 8$; $r = 0.5$; $\chi = 4$; $\gamma = 4$; $\lambda_N = 0.3$; $f = 0.2$; $B_v = 4$. Table 2, the impacts of the prob. on the performance metrics of the system for the value of $\lambda_P = 2$; $\eta = 3$; $B_b = 8$; $r = 0.5$; $\chi = 4$; $\gamma = 4$; $\dot{a} = 5$; $B_v = 4$; $f = 0.2$ are outlined and reported. It has been brought to our attention that the prob. of unfavorable rate λ_N steadily rises as the value of the prob. of orbit size L_q, the prob. of server unoccopaid rate I, the average waiting time W_q, and the server failure frequency $fail_f$ rises. The tendencies that are shown by the tables are consistent with what was anticipated. Table 3 indicates when the lower service rate B_v rises, then server unoccopaid I_0 and length of the orbit Lq rise, prob. that the server is on slow service or WV (Φ_v) and expected waiting time W_q is declines. For the values of $\lambda_P = 2$; $\lambda_N = 0.3$; $B_b = 8$; $r = 0.5$; $\chi = 4$; $\gamma = 4$; $\dot{a} = 5$; $\eta = 3$; $f = 0.2$.

Table 4: When the feedback rate f rises, then average length of the orbit L_q, server unoccopaid during retrial period I, and expected waiting time W_q rise, and server unoccopaid I_0 declines. For the values of $\lambda_P = 2$; $\lambda_N = 0.3$; $B_b = 8$; $r = 0.5$; $\chi = 4$; $\gamma = 4$; $\dot{a} = 5$; $\eta = 3$.

Table 1. The impact of retrial rate $\dot{a}$ on I_O, L_q, I, and W_q

Retrial rate	Exponential				Erlang 2 stage			
$\dot{a}$	I_0	L_q	I	W_q	I_0	L_q	I	W_q
6	0.2566	2.7193	0.2857	1.3596	0.0645	3.0620	0.1874	1.5310
7	0.2674	2.4992	0.2444	1.2496	0.0823	2.9011	0.1597	1.4505
8	0.2726	2.3292	0.2136	1.1646	0.0958	2.7652	0.1392	1.3826
9	0.2784	2.1937	0.1897	1.0968	0.1064	2.6490	0.1233	1.3245
10	0.2831	2.0832	0.1707	1.0416	0.1149	2.5483	0.1106	1.2741

Table 2. The impact of unfavorable consumer arrival rate λ_N on L_q, I, $Fail_f$, and W_q

unfavorable rate	Exponential				Erlang 2 stage			
λ_N	L_q	I	$Fail_f$	W_q	L_q	I	$Fail_f$	W_q
0.20	1.0329	0.3809	0.0805	0.4198	0.8322	0.6696	0.0833	0.4161
0.30	1.1491	0.3958	0.1180	0.9562	0.8719	0.6849	0.1284	0.4359
0.40	1.2635	0.4116	0.1588	1.3701	0.9331	0.7033	0.1760	0.4665
0.50	1.3821	0.4284	0.2506	1.7843	1.0365	0.7264	0.2262	0.5325
0.60	1.5090	0.4463	0.3018	2.2658	1.2396	0.7575	0.2791	0.6198

Table 3. The impact of WV period B_v on I_0, L_q, Φ_v, and W_q

Slow service rate	Exponential				Erlang 2 stage			
B_v	I_0	L_q	Φ_v	W_q	I_0	L_q	Φ_v	W_q
4	0.2414	1.9125	0.0313	0.9562	0.0402	3.9413	0.0278	1.9706
5	0.2522	1.8184	0.0292	0.9092	0.0441	3.8617	0.0267	1.9308
6	0.2613	1.7342	0.0273	0.8671	0.0477	3.7939	0.0257	1.8969
7	0.2691	1.6588	0.0257	0.8294	0.0512	3.7360	0.0248	1.8680
8	0.2759	1.5914	0.0243	0.7957	0.0544	3.6864	0.0239	1.8432

Table 4. The impact of feedback rate f on I_0, L_q, I, and W_q

Feedback rate	Exponential				Erlang 2 stage			
f	I_0	L_q	I	W_q	I_0	L_q	I	W_q
0.01	0.2684	1.7605	0.1224	0.8802	1.1074	2.6436	0.1219	1.3218
0.02	0.2673	1.8056	0.1306	0.9028	0.1046	2.6683	0.1265	1.3341
0.03	0.2663	1.8525	0.1390	0.9262	0.1018	2.6936	0.1313	1.3468
0.04	0.2652	1.9015	0.1477	0.9507	0.0989	2.7195	0.1361	1.3597
0.05	0.2641	1.9525	0.1568	0.9762	0.0959	2.7459	0.1410	1.3729

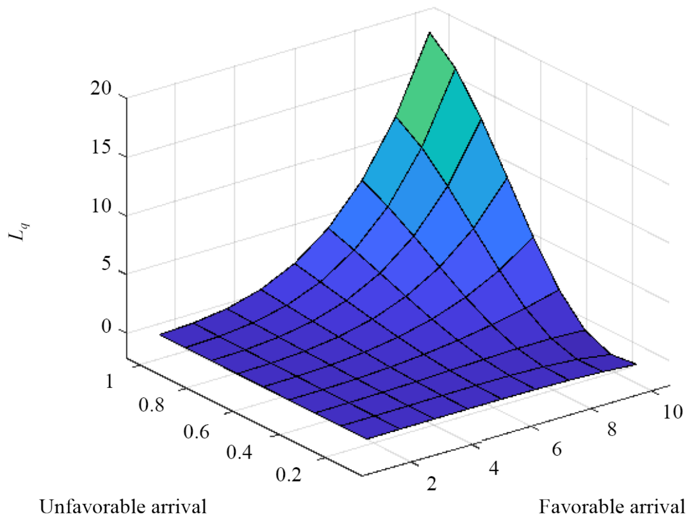

Figure 2. L_q versus λ_P and λ_N

Figure 2-6 depict the influence of the variables λ_P, λ_N, r, $\dot{a}$, η, B_b, and B_v on the 3-D graph based on system performance metrics.

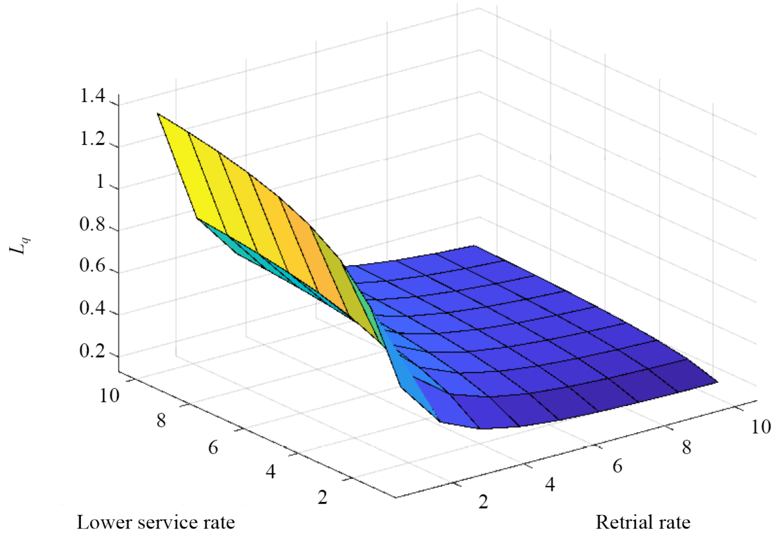

Figure 3. L_q versus B_v and $\dot{a}$

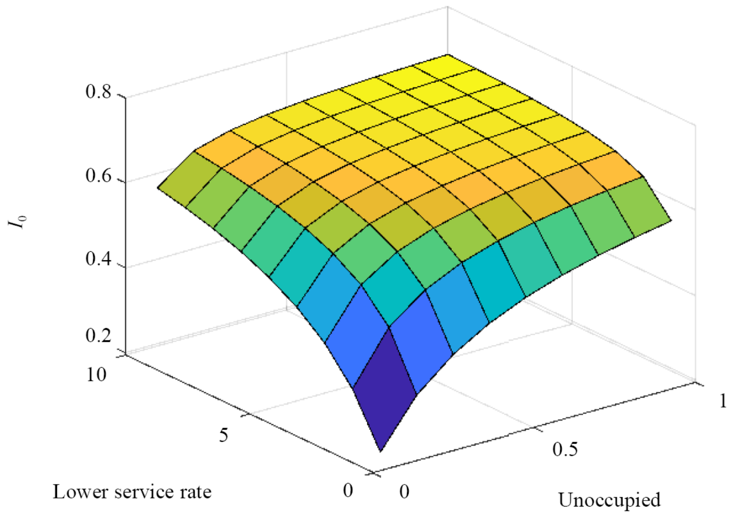

Figure 4. I_0 versus r and η

In Figure 2, the surface exhibits a steadily increasing pattern as it increases the rate of the favorable consumer λ_P and the rate of the arrival of unfavorable consumer λ_N against the average orbit size (L_q). In Figure 3, the average orbit size (L_q) declines when the lower service rate B_v and the rate of retrial $\dot{a}$ grows. Figure 4 shows the unoccupied prob. (I_0) increases as the single WV probability (r) and rate of the vacation (η) is increase. In Figure 5, the unoccupied prob. (I_0) increases when the server gives the rate of the slower service B_v and rate of the normal service B_b also increases. In Figure 6, the unoccupied prob. (I_0) increases when the server gives the rate of the WV rate η and rate of the delay on χ also decreases. Figure 7-10 depict the influence of the variables λ_P, λ_N, r, $\dot{a}$, η, B_b, and B_v on the 2-D graph based on system performance metrics. Note that the exponential distribution is $f(\vartheta) = ve^{-v\vartheta}$, $\vartheta > 0$, Erlang-2 stage distribution is $f(\vartheta) = v^2\vartheta e^{-v\vartheta}$, $\vartheta > 0$ and hyper-exponential distribution is $f(\vartheta) = cve^{-v\vartheta} + (1-c)v^2e^{-v^2\vartheta}$, $\vartheta > 0$. Figure 7 shows that when the probability p rises, the average size of the orbit L_q grows. Figure 8 displays that the average size of orbit L_q declines for the rising value of the retrial rate $\dot{a}$. Figure 9 displays that the average size of orbit L_q declines for the growing value of the WV rate η. Figure 10 displays that the unoccupied prob. I_0 rises for escalating value of the delay rate χ.

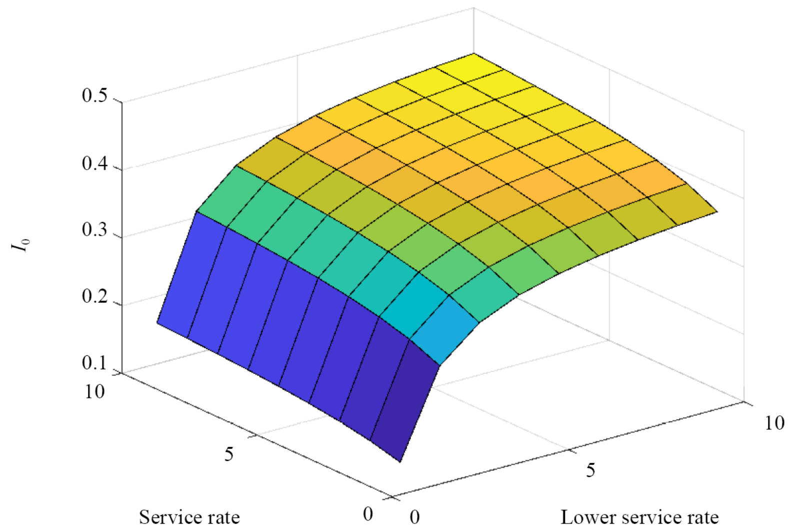

Figure 5. I_0 versus B_v and B_b

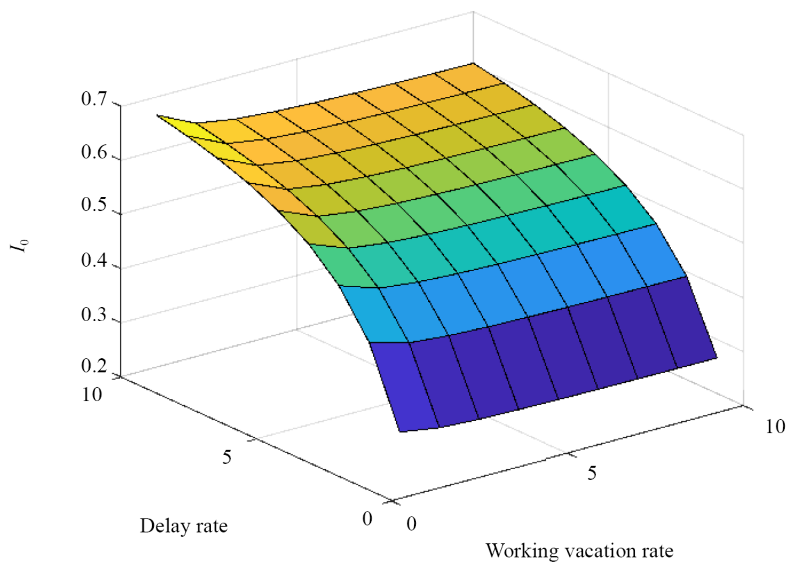

Figure 6. I_0 versus η and χ

The data visualizations below demonstrate the impact of the characteristics on system performance metrics, and it is clear that the results apply to real-world situations.

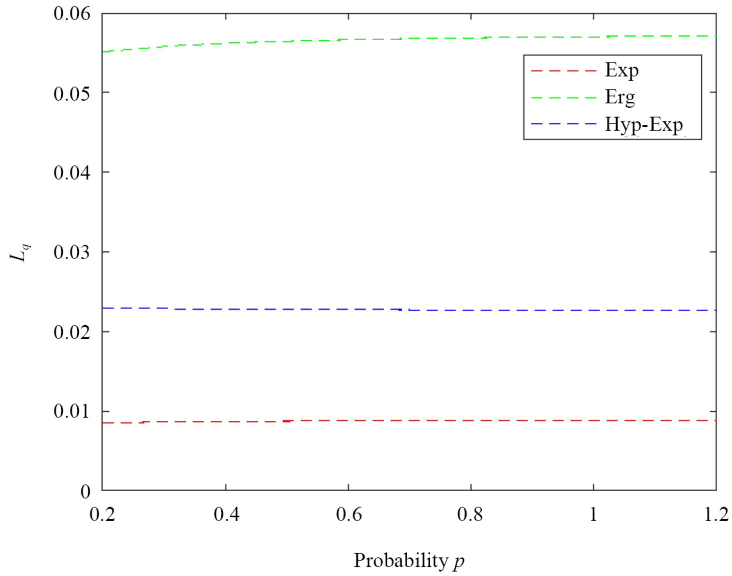

Figure 7. L_q versus p

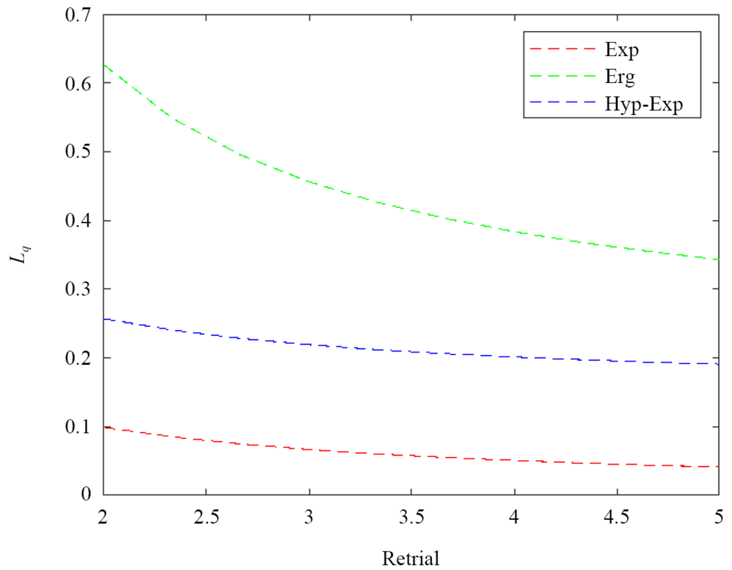

Figure 8. L_q versus ȧ

7. Conclusion

In this research, we have investigated a single server feedback retrial queue and delayed repair with Bernoulli working vacation. We have derived the probability-generating function of the average number of consumers in the system and orbit by using the supplementary variable method. The implementation of our proposed model can enhance the performance of various systems by providing explicit data for the measurement of various system parameters such as queue length, waiting time of the consumer and the system capacity. The application of this model can be extended to various other fields such as computer processing, manufacturing process and communication channels. Our future work is focused on extension of the proposed model by augmenting bulk arrival, optional phase services, re-service, and starting failure to the existing model.

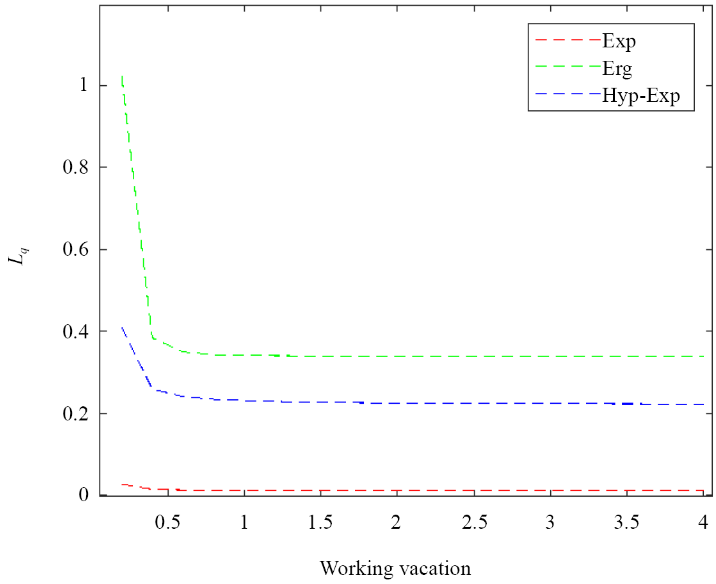

Figure 9. L_q versus η

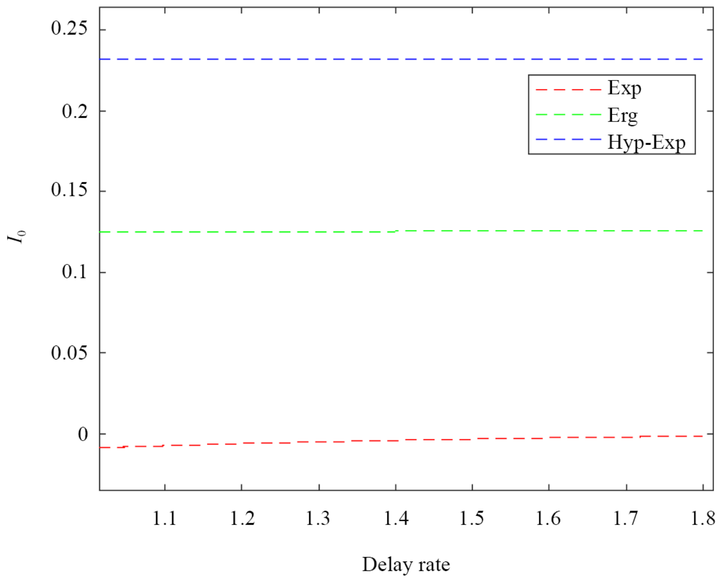

Figure 10. I_0 versus χ

Authors contribution

Dr. Nandhini S and Sundarapandiyan S contributed equally to the design, implementation and analysis of the research and to the writing of the manuscript.

Conflict of interest

The authors declare that there is no conflict of interest regarding the publication of this manuscript.

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