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Finite Groups Containing No Blocks of Defect Zero

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Abstract: The irreducible characters of finite groups are always contained in blocks of defects which are nonnegative integers. Even though blocks always exist in finite groups, it is not the case that blocks of defect zero would always exist as well. Blocks of defect zero contain only one irreducible ordinary character each of defect zero and the defect group of blocks of defect zero is always the trivial subgroup of a finite group. Some finite groups do not have characters of defect zero and hence no blocks of defect zero either. The object in this paper is to study finite groups containing no blocks of defect zero. Finite abelian groups and p-groups will serve as special cases in this regard, with all blocks of finite abelian groups being of full/highest defect. We shall also determine an upper bound for the number of blocks in finite groups which contain no blocks of defect zero.

Keywords: blocks of characters, defect groups of blocks, defect zero characters, defect zero blocks, the highest defect of blocks, the highest defect of characters, Sylow subgroups, linear characters, deficiency classes, full defective groups

MSC: 20C15, 20C20, 20D20, 20E28

1. Introduction

Let G be a finite group, $Irr(G)$ the set of all the irreducible ordinary characters of G, $IBr(G)$ the set of all the irreducible Brauer characters of G and $S = Irr(G) \bigcup IBr(G)$. Then S gets partitioned into disjoint subsets called blocks of G, where the set of all blocks of G is denoted by $Bl(G)$. Thus for $\chi \in Irr(G)$ and $\phi \in IBr(G)$, we have that $\chi \in Irr(B(G))$ and $\phi \in IBr(B(G))$ for some block $B(G) \in Bl(G)$ of G.

For $\chi \in Irr(B(G))$ for some block $B(G) \in Bl(G)$ of G, we define $d(\chi)$, $d(B(G))$, $h(\chi)$ which are the defect of $\chi \in Irr(B(G))$, the defect of $B(G) \in Bl(G)$ and the height of $\chi \in Irr(B(G))$ respectively as follows (cf. [1] and [2, Proposition 13.5.15]):

$$p^{d(\chi)} = (|G|/\chi(1))_p,$$

$$d(B(G)) = \max\{d(\chi) \mid \chi \in Irr(B(G))\} \quad \text{and}$$

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$$h(\chi) = d(B(G)) - d(\chi).$$

Fukushima in [3, 4] studied the existence of p-blocks of defect 0 in p-nilpotent and solvable groups respectively. Various authors have proved that any finite group G possesses a character of defect zero if and only if it possesses a p-block of defect zero. Tsushima in [5, 6] studied blocks of defect zero and the existence of characters of defect zero respectively. Furthermore, it is mentioned in [6] that by Clifford's Theorem, if a group G possesses a character of p-defect zero, then every proper normal subgroup of G also possesses a character of p-defect zero.

The object in this paper is to study finite groups containing no blocks of defect zero. In §2 we give preliminaries, in §3 we give the general case, in §4 we give the special case of p-groups and in §5 we give the special case of abelian groups. We also determine an upper bound for the number of blocks in finite groups containing no blocks of defect zero.

Throughout, all our groups G are finite, p is a prime that divides the order $|G|$ of G, $B_0(G) \in Bl(G)$ is the principal block of G which is the block of G that contains the identity character of G and $B(G) \in Bl(G)$ is any block of G unless otherwise specified to the contrary. By $1_G \in G$ we shall mean the identity element of the finite group G and by $|G|_p$ we shall mean the p-part of the order $|G|$ of G.

2. Preliminaries

Definition 2.1 (cf. [7–9]) If $n \in \mathbb{Z}^+ \bigcup \{0\}$, then a group G is said to be of p-deficiency class n if all non-principal blocks $B(G) \in Bl(G)$ of G are such that $d(B(G)) < n$. Also G is said to be of exact p-deficiency class n if $n = 0$ or $n > 0$ and G is not of p-deficiency class $n-1$.

Remark 2.2 We shall simply say that G is of deficiency class n instead of p-deficiency class n.

Thus G is of deficiency class 0 if and only if $B_0(G) \in Bl(G)$ is the only block of G and G is of deficiency class 1 if and only if all the non-principal blocks $B(G) \in Bl(G)$ if any exist are such that $d(B(G)) = 0$. By [9, Theorem 4], M_{22} and M_{24} are the only finite simple groups of 2-deficiency class zero. According to [10], a finite nontrivial p'-group is of exact deficiency class 1. If G is of deficiency class n, then it certainly is of deficiency class $n+1$. Moreover if $|G|_p = p^k$, then G is always of deficiency class $k+1$ (cf. [7, 9]).

By [7, Remark 4], if Q is a p-subgroup of $Z(G)$ of order p^n, then G is of deficiency class r if and only if G/Q is of deficiency class $r-n$. By [11, Theorem 3.1], a finite group G with a nontrivial cyclic Sylow p-subgroup $P \in Syl_p(G)$ is of deficiency class 0 if and only if $P \in Syl_p(G)$ is normal in G and $O_{p'}(G) = \{1_G\}$.

Lemma 2.3 Let G be a group and $B(G) \in Bl(G)$ be a block of G containing a linear character. Then $k(B(G)) > 1$.

Proof. By the definition in Section 1, a linear character cannot be of defect zero and in fact by [12, Exercise 5.2.9], a linear character will always be of the highest/full defect and thus cannot sit alone in a block $B(G) \in Bl(G)$ of G. So the result follows immediately by [9, Lemma 3.3] and [13, Theorem 3.18] and this completes the proof. □

According to [14] a finite group G such that every block of G is of the highest/full defect is called full defective.

Corollary 2.4 Let $B(G) \in Bl(G)$ be a block of G containing a linear character, then $B(G) \in Bl(G)$ is of the highest defect.

Proof. For any linear character $\chi \in Irr(G)$ and any prime p, we have that $(\chi(1_G),\ p) = 1$. Thus by [12, Exercise 5.2.9] and from the definition above of a defect of a character, we get that every linear character will always be of the highest defect. Thus the result follows immediately and the proof is complete. □

By Corollary 2.4 and [15, Lemma 7.8.3], it immediately follows that $B_0(G) \in Bl(G)$ is always of the highest defect and $\delta(B_0(G))$ is always a Sylow p-subgroup of G. By [9, 14] we have that every group of exact deficiency class 0 is always full defective.

3. The general case

Various authors e.g. Navarro in [13, Theorem 3.18] proved that any finite group G possesses a character of defect zero if and only if it possesses a block of defect zero. In the introduction of [16], following R. Brauer, Ito mentions that a conjugacy class of elements of a group G is of defect d if the order of the centralizer of an element of the class is divisible by exactly p^d and a block of characters $B(G) \in Bl(G)$ of G is of defect d if the degrees of all the characters in that block are divisible by p^{a-d} and at least one of them is not divisible by p^{a-d+1}, where p^a is the highest power of p dividing the order $|G|$ of G.

We also obtain from [17, Lemma 61.3 (i)] following [18] that if $B(G) \in Bl(G)$ is a block of G with defect $d(B(G))$, $\chi \in Irr(B(G))$ and $P \in Syl_p(G)$ such that $|P| = p^a$, then $p^{a-d(B(G))} \mid \chi(1_G)$. By [19, Theorem 15.41, Corollary 15.42], we have that if $|G| = p^a m$, where $p \nmid m$ and $B(G) \in Bl(G)$ is a block of G of defect $d(B(G))$, then $p^{a-d(B(G))}$ is the largest power of p which divides all $\chi(1_G)$, where $\chi \in Irr(B(G))$ and also $p^{a-d(B(G))}$ is the largest power of p which divides all $\phi(1_G)$, where $\phi \in IBr(B(G))$.

Proposition 3.1 Let G be a finite nontrivial group and p be a prime such that G contains a nontrivial normal p-subgroup. Then G contains no blocks of defect zero.

Proof. If the nontrivial normal p-subgroup of G has order p^k, $k > 0$, then by [20, Lemma 87.26] for all blocks $B(G) \in Bl(G)$ of G, we get that $d(B(G)) \geq k > 0$. Moreover by [8, Corollary III.6.9] and [12, Theorem 5.2.8 (i)], the normal p-subgroup of order p^k would be contained in a defect group of every block of G. Hence the result follows and the proof is complete. □

If G contains a nontrivial normal p-subgroup of order p^k, $k > 0$ such that G is of deficiency class $r \leq k$, then by [8, Lemma V.11.3] G is of deficiency class 0. By [21, Problem 5.1] we have that every normal p-subgroup of G is contained in $O_p(G)$. Thus if G contains a nontrivial normal p-subgroup, then $O_p(G) \neq \{1_G\}$ so that the triviality of $O_p(G)$ implies the triviality of every normal p-subgroup of G.

Corollary 3.2 Let G be a finite nontrivial group and p be a prime such that $O_p(G) \neq \{1_G\}$. Then G contains no blocks of defect zero.

Proof. We have by [8, Corollary III.6.9] and [12, Theorem 5.2.8 (i)] that any normal p-subgroup of G would be contained in a defect group of every block of G. Thus $O_p(G)$ would be contained in a defect group of every block of G and this fact is also echoed by [19, Corollary 15.39]. Thus the result follows by Proposition 3.1 and the proof is complete. □

Corollary 3.3 If a group G possesses no nontrivial normal p'-subgroup, then G contains no blocks of defect zero.

Proof. We have that $O_{p'}(G) = \{1_G\}$ and thus by [19, Problem 8.7], G contains a normal Sylow p-subgroup. Hence by Proposition 3.1, G contains no blocks of defect zero. □

Corollary 3.4 If a group G possesses no nontrivial normal p'-subgroup, then G is full defective.

Proof. We have by Corolary 3.3 that G contains no blocks of defect zero. By [19, Problem 8.7], G contains a normal Sylow p-subgroup which would be the defect group of the principal block and all other blocks of full/highest defect. Thus by [8, Corollary III.6.9, 12, Theorem 5.2.8 (i), 20, Lemma 87.26] and [22, Lemma 9] we obtain that every block of G will be of the highest defect and hence the result follows immediately completing the proof. □

Corollary 3.5 If a group G possesses a normal Sylow p-subgroup, then G is full defective.

Proof. We have by [8, Corollary III.6.9] and [12, Theorem 5.2.8 (i)] that the normal Sylow p-subgroup of G would be contained in a defect group of every block of G thus rendering G not to have any blocks of defect zero which also follows by Proposition 3.1. Thus every block of G would have this normal Sylow p-subgroup as its defect group. Hence the result follows immediately and the proof is complete. □

Corollary 3.6 If G contains a normal Sylow p-subgroup, then any normal subgroup of G containing the normal Sylow p-subgroup of G also becomes full defective.

Proof. We have by Corollary 3.5 that G is full defective. So by [23, Proposition 4.2], the result follows immediately completing the proof. □

Proposition 3.7 Let G be a finite nontrivial group containing no blocks of defect zero. Then $1 \leq |Bl(G)| \leq \dfrac{|Irr(G)|}{2}$.

Proof. We obtain generally that $|Bl(G)|$ does not exceed the number of conjugacy classes of p'-elements of G. Since G contains no blocks of defect zero, we get for each block $B(G) \in Bl(G)$ of G that $k(B(G)) \geq 2$. Hence the result follows immediately completing the proof. □

By Proposition 3.7, we obtain that for G a finite nontrivial group containing no blocks of defect zero, the number of conjugacy classes of p'-elements of G cannot exceed $\frac{|Irr(G)|}{2}$.

Corollary 3.8 Let G be a finite nontrivial group containing no blocks of defect zero such that $|Bl(G)| = \frac{|Irr(G)|}{2}$. Then for all blocks $B(G) \in Bl(G)$ of G, we get that $k(B(G)) = 2$.

Proof. Since G contains no blocks of defect zero, all its blocks will be of positive defect. Thus for all blocks $B(G) \in Bl(G)$ of G, we obtain that $k(B(G)) \geq 2$. However since $|Bl(G)| = \frac{|Irr(G)|}{2}$, the result thus follows immediately. □

Corollary 3.9 Let G be a finite nontrivial group containing no blocks of defect zero such that $|Bl(G)| = \frac{|Irr(G)|}{2}$. Then for all blocks $B(G) \in Bl(G)$ of G and $\chi \in Irr(B(G))$, we get that $h(\chi) = 0$.

Proof. Since all the blocks of G are of positive defect, we obtain from [2, Lemma 87.26] that for all blocks $B(G) \in Bl(G)$ of G, $k_0(B(G)) \geq 2$. However from Corollary 3.8, we obtain that $k(B(G)) = 2$ which thus gives the desired result and the proof is complete. □

From Corollary 3.8, we observe that all such blocks $B(G) \in Bl(G)$ of G with $k(B(G)) = 2$, satisfy the Brauer's $k(B(G))$-Conjecture and from Corollary 3.9, we have for all $\chi \in Irr(G)$ that $h(\chi) = 0$.

4. The special case of *p*-groups

If G is a finite nontrivial p-group, where p is a prime that divides the order $|G|$ of G, then we get that $O_{p'}(G) = \{1_G\}$. We have by [24, Theorem 3.8.3] that for a finite p-group G, every maximal subgroup of G is normal and by [24, Corollary 3.8.4], every maximal subgroup of G has index p. By [24, Corollary 4.11.5] we have that finite p-groups are solvable and by [24, Theorem 3.8.7, Lemma 4.11.14] we have that finite p-groups are nilpotent. We furthermore have by [25, Exercise 19.1] that a finite p-group P has a faithful irreducible complex character if and only if $Z(P)$ is cyclic.

The following definitions can be found on pages 30, 92 and 121 of [21, 25, 26] respectively.

Definition 4.1

1. A group G is said to have a normal p-complement if it has a normal p'-subgroup N such that $G = PN$ for some Sylow p-subgroup P of G (cf. [26]).

2. A normal p-complement in a group G is a normal subgroup N such that if S is a Sylow p-subgroup of G, then $G = SN$, $S \cap N = \{1_G\}$, $|S| = [G:N]$, $|N| = [G:S]$ (cf. [25]).

3. Let G be a finite group, p a prime, N a normal subgroup of G having index a power of p and order not divisible by p. Then N is called a normal p-complement in G. In other words, a normal p-complement N in G is a normal subgroup whose index in G is equal to the order of a Sylow p-subgroup of G (cf. [21]).

Thus a normal p-complement N in G is actually a normal p'-subgroup of G whose index in G is equal to the order of a Sylow p-subgroup of G. We furthermore obtain that

(i) According to [17, 25], if $P \in Syl_p(G)$ such that $P \subseteq Z(N_G(P))$, then G has a normal p-complement. This result is in [17, Theorem 18.7, Appendix] and [25, Theorem 18.7] and is known as Burnside's Theorem.

(ii) According to [19, Theorem 6.9], if $\chi(1_G)$ is a power of a prime p for every $\chi \in Irr(G)$, then G has an abelian normal p-complement. Hence an abelian group has an abelian normal p-complement.

(iii) It is mentioned in [21] that [21, Corollary 9.11, Lemma 9.12] can be combined to give a very useful sufficient condition for a group to have a normal p-complement.

Also [19, Theorem 8.22] (The Brauer-Suzuki Theorem) gives conditions under which a normal p-complement would exist.

Proposition 4.2 Let G be a finite nontrivial p-group. Then G has no normal p-complement.

Proof. We have by Definition 4.1 that a normal p-complement N in G is a normal p'-subgroup of G. Moreover we have that $O_{p'}(G) = \{1_G\}$ and by [26, Theorem 6.12], we obtain that $N \subseteq O_{p'}(G) = \{1_G\}$. Hence the result follows immediately completing the proof. $\square$

Proposition 4.3 Let G be a finite nontrivial p-group. Then G has no blocks of defect zero.

Proof. We have that

$$\sum_{\chi \in Irr(G)} (\chi(1_G))^2 = |G|$$

and by [19, Lemma 3.11], for any $\chi \in Irr(G)$, we have that $\chi(1_G) \mid |G|$. Also for any $\chi \in Irr(G)$ of defect zero, we have that $|G|_p \mid \chi(1_G)$. Thus a finite nontrivial p-group G cannot have any characters of defect zero and hence no blocks of defect zero. Moreover if $Z(G)$ is cyclic, then by [25, Exercise 19.1] G would have a faithful irreducible complex character, by [24, Corollary 4.11.5] G is solvable and by Proposition 4.2 G possesses no nontrivial normal p-complement. We also have that $O_p(G) \neq \{1_G\} = O_{p'}(G)$. Thus by Proposition 3.1 and Corollary 3.2, G would have no blocks of defect zero and the proof is complete. $\square$

Corollary 4.4 Let G be a finite nontrivial p-group. Then G is of deficiency class 0.

Proof. We have from Proposition 4.3 that G possesses no blocks of defect zero. We furthermore have by [24, Theorem 4.11.15] that G has a normal Sylow p-subgroup and that every maximal subgroup is normal in G. By Corollary 3.5, we have that G is full defective. Thus we obtain that $O_p(G) \neq \{1_G\} = O_{p'}(G)$ and has its centralizer in G being a p-group. Hence by [22, Theorem 7], we obtain that G is of deficiency class 0. $\square$

Corollary 4.4 also follows by [7, Remark 2, 8, Lemma V.11.2, 9, Lemma 3.1, 10, Lemma 2.2, Corollary 2.6 (i), 12, Exercise 5.2.10, 13, Problems 4.8, 4.9, 20, Exercise 87.3].

Corollary 4.5 Let G be a finite nontrivial p-group. Then the unique Sylow p-subgroup of G is the defect group of its unique block.

Proof. We have that $O_{p'}(G) = \{1_G\}$ and by [24, Theorems 3.8.7, 4.11.14] and [20, Result 5.6, p 15] we obtain that G is nilpotent. Thus by [24, Proposition 3.8.3, Theorem 4.11.15] and [19, Problem 8.7] G has a unique Sylow p-subgroup. By Proposition 4.4, we have that G has only the principal block. Thus the unique Sylow p-subgroup of G becomes the defect group of the unique block of G which is the principal block. $\square$

We obtain by [20, Definition 6.2] that if G is a p-group, then G is its own Sylow p-subgroup. Thus the unique Sylow p-subgroup of G in Corollary 4.5 is actually G itself thus making G the defect group of its unique block.

5. The special case of abelian groups

It thus becomes abundantly clear that finite abelian groups would not possess elements, blocks and characters, all of defect zero. By [9, Definition 1], we obtain that an abelian group cannot be of deficiency class 1 and according to [21, Theorem 5.28] (Brodkey's Theorem), for a finite group G with an abelian Sylow p-subgroup $S \in Syl_p(G)$, there exists $T \in Syl_p(G)$ such that $S \bigcap T = O_p(G)$.

We have that blocks of finite abelian groups are amongst other characteristics, separable over their centers and satisfy $k(B(G)) = |D|$ and $l(B(G)) = 1$ for every block $B(G) \in Bl(G)$ of G and $D \in \delta(B(G))$ being a defect group of $B(G) \in Bl(G)$ (cf. [2, Proposition 13.5.15] and [27, Proposition 1]). By [24, Theorem 3.8.7] we obtain that finite abelian groups are nilpotent and by [24, Proposition 3.11.13] we have that finite abelian groups are solvable. By [28, Remark 6.1.1] we obtain that the hierachy of structure among groups is given by:

$$\text{cyclic groups} \subset \text{abelian groups} \subset \text{nilpotent groups} \subset \text{solvable groups} \subset \text{all groups}$$

where all the above containments are proper.

Proposition 5.1 If G is a finite abelian group, then G contains no blocks of defect zero.

Proof. We have by [24, Theorem 3.8.7] that G is nilpotent and by [24, Theorem 4.11.15] G contains a normal Sylow p-subgroup. Thus by [22, Lemma 9] the desired result is obtained. Moreover all the irreducible characters of G are linear and by Lemma 2.3, the result follows and the proof is complete. □

Corollary 5.2 Let G be a finite abelian group with $P \in Syl_p(G)$ such that $C_G(P)$ is a p-group. Then G is of deficiency class 0.

Proof. We have by Proposition 5.1 that G contains no blocks of defect zero. We also have that $P \in Syl_p(G)$ is normal in G. Thus the result follows immediately by [12, Exercise 5.2.10] and [22, Theorem 7] and the proof is complete. □

Proposition 5.3 Let G be a finite abelian group. Then all the blocks of G have as their defect group the unique Sylow p-subgroup of G.

Proof. If $|Bl(G)| = 1$, then that will be the principal block and the result follows immediately by Proposition 2.4 and [15, Lemma 7.8.3]. If $|Bl(G)| > 1$, then all the blocks of G will contain linear characters and by Proposition 2.4 and [12, Exercise 5.2.10], all these blocks will be of the highest defect. Since G is abelian, it will contain a unique Sylow p-subgroup and so render the result to follow immediately by [8, Corollary III.6.9], [12, Theorem 5.2.8 (i)] and [14, Corollary 5.2] and this completes the proof. □

Corollary 5.4 If G is a finite abelian group, then G is full defective.

Proof. We have by Proposition 5.3 that all the blocks of G have as their defect group the unique Sylow p-subgroup of G. Moreover all $\chi \in Irr(G)$ are linear and satisfy that $(\chi(1_G),\ p) = 1$. Thus by Proposition 2.4, Corollary 3.5, [12, Exercise 5.2.9] and [14, Corollary 5.2] the result follows immediately. □

Corollary 5.5 If G is a finite abelian group, then all the blocks of G are conjugate.

Proof. We have by Proposition 5.3 together with [8, Corollary III.6.9] and [12, Theorem 5.2.8 (i)] that all the blocks of G have the unique Sylow p-subgroup of G as their defect group. The result follows immediately by [2, Lemma 10.2.1 (iii)] and the proof is complete. □

If G is a finite nontrivial abelian group, then G will have no blocks of defect zero. Since $|Irr(G)| = |G|$, we get by Proposition 3.7 that $1 \leq |Bl(G)| \leq \dfrac{|G|}{2}$. In the case G has prime order $p > 2$, we thus obtain from Proposition 3.7 that $1 \leq |Bl(G)| < \dfrac{p}{2}$. Thus for $|G| \in \{2,\ 3\}$, we get by Lemma 2.3 and Proposition 3.7 that G must have exactly the principal block only. However if $|Bl(G)| = \dfrac{|G|}{2}$, then for all blocks $B(G) \in Bl(G)$ of G, we get by Lemma 2.3 and Corollary 3.8 that $k(B(G)) = 2$.

Corollary 5.6 Let G be a finite nontrivial abelian p-group. Then G is the defect group of its unique block.

Proof. We have from Proposition 4.4, [7, Remark 2, 8, Lemma V.11.2, 9, Lemma 3.1, 10, Lemma 2.2, Corollary 2.6 (i), 12, Exercise 5.2.10, 13, Problems 4.8, 4.9, 20, Exercise 87.3] that G has only the unique block which is the principal block. However we obtain by [20, Definition 6.2] that G is its own unique Sylow p-subgroup. Hence the result follows immediately completing the proof. □

We have by a remark from [7] that all linear characters in the principal block form a group under multiplication and the number of such characters in $B_0(G) \in Bl(G)$ divides $[G : G']$, where according to [19, Corollary 2.23 (b)], $[G : G']$ is the number of linear characters of G. Thus for G abelian, the number of irreducible characters in $B_0(G) \in Bl(G)$ always divides $|G|$.

A necessary condition for a finite group G to be of exact deficiency class 0 is that $d(\chi) \neq 0$ for all $\chi \in Irr(G)$. This is of course satisfied in the case of abelian groups since all the irreducible characters are linear and therefore of the highest defect.

According to [10], a finite group of characteristic p is always of deficiency class 0 and by [9, Corollary 3.12] and [10, Lemma 2.4], if a group G is of deficiency class 0, then $O_{p'}(G) = \{1_G\}$.

Furthermore by [9, Lemma 4.8], in characteristic 2, a finite group G is of deficiency class 0 if and only if G is of $U2B$-type. It is mentioned in [29] that according to [20, Exercise 1], if a finite group G has a nontrivial normal p-subgroup N such that the centralizer of N is contained in N, then G is of exact deficiency class 0.

6. Concluding remarks

It is an irrefutable fact that not all finite groups possess blocks of defect zero with p-groups and abelian groups serving as special examples in this regard, where p-groups are of deficiency class 0 and abelian groups are full defective. In the case of finite groups possessing no blocks of defect zero, the numbers of blocks in such groups is bounded above by $\frac{|Irr(G)|}{2}$. Thus finite groups possessing no blocks of defect zero also possess no characters of defect zero and also no elements of defect zero.

Conflict of interest

The authors declare that there is no conflict of interest.

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