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Research Article

Comparison of Various Ranking Methods with Pareto Distribution Methods in Imprecise Data

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Abstract: The Pareto distribution has seen wide application in many different fields recently. Nowadays, the data we collect may be imprecise and contain uncertainties. This means that the values we gather cannot necessarily be considered a single, definitive value. In this paper, it is explored how various estimation methods like the method of moments, maximum likelihood estimation, and least squares estimation are used to determine the parameters of a Pareto distribution when the data is fuzzy or uncertain in nature. When making decisions, ranking methods are often heavily relied upon. For this reason, the application of several different ranking techniques to triangular fuzzy numbers (TFNs) is considered. The ranking methods for TFNs are found to tend to produce estimated parameters that are quite similar or even identical. The estimating parameters of a Pareto distribution can be solved for when dealing with fuzziness by the use of ranking methods such as alpha-cut, Yeager's method, sub-interval method, Pascal method, magnitude method, and centroid method on the TFNs. By comparing the results of these various ranking techniques, an analysis is conducted to whether the estimated parameters of the Pareto distribution are consistent across the different ranking methods for TFNs get the uniqueness of the solution or differ. The decision making done by among the various ranking techniques, which method preferable for future comparison with extended of new ranking techniques. The conclusion is that the simulation results match up well with actual application data, serving as a good example for explaining this overall strategy.

Keywords: triangular fuzzy number, ranking method, pareto, originality, maximum likelihood estimator

MSC: 62E86, 60E99

Abbreviation

TFN	Triangular Fuzzy Number
FPD	Fuzzy Pareto Distribution
MOM	Method of Moments
MLE	Maximum Likelihood Estimator
LSE	Least Square Estimation

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1. Introduction

The Pareto's original based on the income descriptive of economic statistics. Vilfred Pareto was introduced the pareto distribution at 1912. The definition of the Pareto Law was "unique pioneering work of econometric investigation". According to Cirillo [1] Pareto's 'Law' of Income Distribution can be expressed by the equation as

$$N_0 = AX^{-a}$$

Here X shows as Income and A and a are the parameters in empirical statistics and N_0 as the number of persons. The pareto distribution has been applied in social, hydrology, quality control, insurance, scientific, finance, geophysical. Based on the works of Amin [2] , Jiayu Fu et al. [3], Riham Elhabyan et al. [4], Amer Ibrahim Al-Omari et al. [5] and Reinartz [6] applied in modelling in various phenomena. Recently now a days much attention paid to the statistical distribution such as socio-economic fields such as medicine, business, queuing, real life situation analysis etc. The real life decision making depends on the statistical distribution. In this paper explained the real life simulation decision in uncertainty cases for this we proposed the fuzzy sets. The fuzzy sets defined by their membership values presence of the uncertainty. This paper framework of the fuzzy logic and estimating the parameters of pareto distribution in uncertainty [7].

Most of the authors are discussed about the pareto distribution on imprecise data. Such as reliability Shu [8], the triangular intuitionistic fuzzy number (TIFN) Mahapatra [9], data envelopment analysis Puri [10], type-1 fuzzy sets and IFSs Atanassov [11], transportation problem Mahmoodirad [12], Minkowski score functions of intuitionistic fuzzy values (IFV) Feng [13], decision making Pekala [14], the Pendant, Hexant and Octant fuzzy numbers Varghese [15].

In Zadeh [16], was introduced fuzzy sets theory. It presents the information about the data in nonstatistical i.e., imprecise or uncertain. The fuzzy numbers exists the possible value between 0 and 1. When it is convex, also called the normalized fuzzy set. By using these concepts, we solving the problems arising uncertainty. For the decision-making procedure ordering fuzzy numbers and their comparison play a key role. The ranking fuzzy number method was first introduced by Jain [17]. According to practical purpose and research, several kinds of ranking methods are introduced and some are introduced by based on the membership function such as triangular, trapezoidal, pentagon, hexagon and so on.

Most of the authors are discussed about the several procedures of the ranking methods on fuzzy numbers. Some of these are, Area compensation Fortemps et al. [18], the method of induced function using weighted average of fuzzy numbers Facchinetti [19], multi criteria decision making model with a polygon fuzzy number Bekheet [20], generalized hexagonal using centroid method Thiruppathi et al. [21] and quadrilateral shape fuzzy number using centroid technique Thiruppathi et al. [22].

The structure of the present paper is organized as follows. In Section 2, we discussed about preliminaries of the imprecise data. Next Section 3, we recall about the various ranking methods in TFN and in Section 4, we introduced FPD and explained properties of FPD. In Section 5, estimating parameter methods in pareto distribution in fuzzy. In Section 6, we used simulation study by using R software update version. In Section 7, we collect the real-life data from online i.e., chess game from Litchess platform and estimate the parameters of the pareto distribution in various ranking methods in TFN. In Section 8, we conclude that the parameters in various raking methods in TFN is existed as unique.

2. Preliminaries

In this section we recall the basic definition of Fuzzy sets and Fuzzy numbers. We used the ranking method in the special case of the membership function of the triangular fuzzy number.

2.1 *Definition-1*

Fuzzy set: According to Zadeh [16], a fuzzy set $\widetilde{A}$ in a nonempty set X is categorized by its membership function $\mu_{\widetilde{A}}: X \to [0, 1]$ and $\mu_{\widetilde{A}}(x)$ is meant as the degree of membership of element x in fuzzy set $\widetilde{A}$ for each x belongs to X.

$$\widetilde{A} = \{(x, \mu_{\widetilde{A}}(x)) / x \in X\}$$

2.2 *Definition-2*

According to Ganesh [23], a fuzzy set $\widetilde{A}$ is defined on the universe set X is said to be normal if and only if $\mu_{\widetilde{A}}(x) = 1$, where $\mu_{\widetilde{A}}(x)$ is the degree of membership of x in X in fuzzy set $\widetilde{A}$.

2.3 *Definition-3*

According to Ebrahimnejad [24], a fuzzy set $\widetilde{A}$ is defined on the universe set X is said to be convex if and only if $\mu_{\widetilde{A}}(x) = 1$.

$$\mu_{\widetilde{A}}(\delta x + (1-\delta) y) \geq min\left(\mu_{\widetilde{A}}(x), \mu_{\widetilde{A}}(y)\right), \forall x, y \in X \; and \; \delta \in [0, 1]$$

2.4 *Definition-4*

Fuzzy number: According to Zimmermann [25], A fuzzy number $\widetilde{A}$ is a fuzzy set of the real line with a normal, convex and continuous membership function of bounded support. The family of fuzzy numbers will be denoted by F. i.e., fuzzy number iff normal and convex.

Properties of a Fuzzy number:

1. Fuzzy set $\widetilde{A}$ is normal,
2. Fuzzy set $\widetilde{A}$ is convex,
3. Sup $\widetilde{A}$ is Bounded.

2.5 *Definition-5*

Triangular fuzzy number Figure 1: According to Klir [26], A triangular fuzzy number A is defined by a triplet $A = (a_1, a_2, a_3)$ and has the membership function as

$$\mu_A(x) = \begin{cases} 0, & x < a_1 \\ \dfrac{x - a_1}{a_2 - a_1}, & a_1 \leq x \leq a_2 \\ \dfrac{a_3 - x}{a_3 - a_2} & a_2 \leq x \leq a_3 \\ 0, & x > a_3. \end{cases}$$

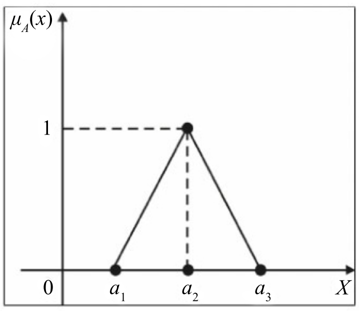

Figure 1. Triangular Fuzzy Number

3. Various ranking methods in TFN

The ranking methods on fuzzy numbers were discussed in this section. In the special case of the triangular fuzzy number, a huge amount of methods were considered, but only a few were discussed here for further procedure.

Ranking methods are ordering the fuzzy number. It describes the several characteristics of the fuzzy number. According to Rao [27], a ranking function is defined which is the Euclidean distance between the circumcenters point and the original point to rank fuzzy numbers. For example, centroid, an area enclosed by the membership function of any two fuzzy numbers.

There are many different "triangular fuzzy numbers" used in ranking methods. All of the rankings found here for "Triangular fuzzy numbers" throughout the previous few years have been gathered from researching numerous research articles. Here, just the most crucial of them are mentioned. It's them (Lavanya). Here the TFN is in the form of (a, b, c).

- "α-Cut" for Triangular Fuzzy Numbers: Srinivasan [28]

$$\left(a_\alpha^L,\, a_\alpha^U\right) = (b-a)\,\alpha + a,\, c-(c-b)\,\alpha \text{ where } \alpha \in [0,\, 1].$$

- "Yeager's ranking" method for Triangular Fuzzy Numbers: Wutsq et al. [29]

$$Y\,(a,\, b,\, c) = \frac{1}{2}\int \left(a_\alpha^L,\, a_\alpha^U\right)\, d\alpha \text{ , where } \left(a_\alpha^L,\, a_\alpha^U\right) = (b-a)\,\alpha + a,\, c-(c-b)\,\alpha \text{ and } \alpha \in [0,\, 1].$$

- "Sub interval Average" method for Triangular Fuzzy Numbers: Dinagar [30]

$$R(a,\, b,\, c) = \left(\frac{4(a+b+c)}{12}\right)$$

- "Sub interval Addition" method for Triangular Fuzzy Numbers: Dinagar et al. [31]

$$R(a,\, b,\, c) = \left(\frac{4(a+b+c)}{6}\right)$$

- "Pascal Triangular Graded Mean" for Triangular Fuzzy Numbers: Srinivasan [28]

$$P(a,\, b,\, c) = \left(\frac{a+2b+c}{4}\right)$$

- "Magnitude Ranking" for Triangular Fuzzy Numbers: Srinivasan [28]

The function $f(r)$ is a weighting function and a non-negative and increasing function on $[0, 1]$ with $f(0) = 0, f(1) = 1$ and $\int_0^1 f\,(r) = \frac{1}{2}$.

$$Mag\,(a,\, b,\, c) = \frac{1}{2}\int_0^1 (c+3a-b)\, r dr$$

- "Centroid approach" for Triangular Fuzzy Numbers: Kamble [32]

A fuzzy number $A = (a, b, c; w)$ is called a triangular fuzzy number where $0 < w < 1$ is a constant and if its membership function $A(x)$ has the following form:

$$A(x) = \begin{cases} w(x-a)/b-a, & a \leq x \leq b \\ w(c-x)/c-b, & b \leq x \leq c \\ 0, & \text{Otherwise} \end{cases}$$

Then the centroid approach for TFN as follows,

$$C(a, b, c; w) = \left(\frac{2a+14b+2c}{6}\right)\left(\frac{7w}{6}\right)$$

4. Pareto distribution in fuzzy

If X is a random variable in imprecise with a Pareto (Type I) distribution, Arnold [33] then the probability that X is greater than some number $\widetilde{x}$, i.e., the survival function is given by

$$\bar{F}\left(\widetilde{x}\right) = Pr\left(X > \widetilde{x}\right) = \begin{cases} \frac{\zeta^{\psi}}{\widetilde{x}} & \tilde{x} \geq \zeta \\ 0 & \tilde{x} < \zeta \end{cases}$$

where ζ is the minimum positive value of X, and ψ is a positive parameter. The Pareto Type I distribution is specialized by a scale parameter or location parameter and a shape parameter ψ. The shape parameter shown as the tail index. The shape parameter is known as Pareto index because given distribution is used to model the distribution of wealth. The parameters are $(\zeta, \psi) \to [0, 1]$. Now we define the cdf, pdf and properties of fuzzy pareto distribution as follows as,

4.1 *Cumulative distribution function*

From the definition, the cumulative distribution function of a Fuzzy Pareto random variable with parameters ψ and ζ is

$$F_x\left(\widetilde{x}\right) = \begin{cases} 1 - \frac{\zeta^{\psi}}{\widetilde{x}} & \tilde{x} \geq \zeta \\ 0 & \tilde{x} < \zeta \end{cases}$$

4.2 *Probability density function*

The probability density function of a Fuzzy Pareto random variables differentiated as follows that, with parameters ψ and ζ is

$$f_x(\tilde{x}) = \begin{cases} \frac{\psi\zeta^{\psi}}{\tilde{x}^{\psi+1}} & \tilde{x} \geq \zeta \\ 0 & \tilde{x} > \zeta \end{cases}$$

When displayed on linear axes, the distribution takes the well-known J-shaped curve that approaches each orthogonal axis asymptotically. All curve segments are self-similar (subject to proper scaling factors). A straight line represents the distribution when plotted in a log-log plot Pareto [34]. Similarly, this property applied on the imprecise data so, which exists in the fuzzy pareto distribution. The FPD is also one of the power distributions. It is defined as a power law distribution as the property that large numbers are rare, but smaller numbers are more common.

4.3 *Properties*

- The expected value of a random variable following a fuzzy pareto distribution is

$$E\left(\widetilde{X}\right) = \frac{\psi\varsigma}{\psi - 1}$$

- The variance of a random variable following a fuzzy pareto distribution is

$$Var\left(\widetilde{x}\right) = \left(\frac{\varsigma}{\psi - 1}\right)^2 \frac{\psi}{\psi - 2}$$

- The raw moments of fuzzy pareto distribution are

$$\mu_n' = \frac{\psi\,\varsigma^n}{\psi - n}$$

- The quantile function of the fuzzy pareto distribution as follows

$$Q\left(u\right) = \varsigma\left(1 - u\right)^{\frac{1}{\psi}}$$

- Hazard Function

$$h\left(x\right) = \frac{f\left(x\right)}{S\left(x\right)}$$

where $f(x)$ is the pdf and $S(x)$ is the survival function.

The survival function is just defined as $S\left(x\right) = 1 - F\left(x\right)$ where $F(x)$ is the cdf. then we get a hazard rate as follows for the fuzzy pareto distribution

$$h\left(\widetilde{x}\right) = \frac{f\left(\widetilde{x}\right)}{1 - F\left(\widetilde{x}\right)} = \frac{\frac{\psi\zeta}{\widetilde{x}^{\psi+1}}^{\psi}}{1\left(1 - \frac{\zeta}{\widetilde{x}}^{\psi}\right)} = \frac{\psi}{\widetilde{x}}$$

After introducing the pareto distribution's features, Zaher [35] came to the conclusion that the shape parameter needed to be positive and at least 1. This work used the TMF to introduce and conclude the characteristics of the fuzzy pareto distribution. The values of the parameters must fall between 0 and 1. The hazard rate function of the Pareto distribution was presented by Ijet [36], who came to the conclusion that it could discriminate between two kinds of stocks. Here, we deduce that the right tail index skew in the imprecise data also contrasted with the hazard function.

5. Estimating methods of parameters

The purpose of this study is to analyse various ways of estimating the Pareto distribution, which has been regarded as one of the most outstanding alternatives for explaining the distribution of incomes, assets, and so on. Some of the strategies mentioned are classic; one is likely original and appears to be sufficiently promising to be generalize in problems of estimating distribution parameters.

The distinguish between $F(\widetilde{x})$ and $f(\widetilde{x})$ of a random variable $\widetilde{x}$ in the fuzzy pareto distribution is given by

$$F\left(\widetilde{x}\right) = 1 - \frac{\varsigma}{\widetilde{x}}^{\psi}$$

where ζ is the location parameter and ψ is the shape parameter. We estimated these parameters by taking the three methods moments, MLE and LSE.

5.1 *Method of moments*

Pareto [37] introduced the concept of method of moments as the k^{th} moment of the Pareto distribution is given as (Pareto)

$$E\left(\widetilde{x}\right) = \int x\, dF\left(\widetilde{x}\right) = \frac{\psi\varsigma}{\psi - 1}$$

calculate now the shape parameter of ψ by equating to the sample mean $\bar{\widetilde{x}}$, yielding

$$\hat{\psi} = \frac{\bar{\widetilde{x}}}{\bar{\widetilde{x}} - \hat{\varsigma}}$$

where $\hat{\zeta}$ is some estimate of ζ. Thus, the probability distribution of the lowest sample value is

$$G\left(\widetilde{x}\right) = 1 - \frac{\varsigma}{\widetilde{x}}^{\psi n}$$

and the expected value of the lowest sample observation is

$$\hat{\varsigma} = \frac{\psi n \varsigma}{\psi n - 1}$$

Equating the lowest sample value, x_0, to the expected value, we obtain

$$\hat{\varsigma} = \frac{\left(\psi n - 1\right)\widetilde{x}_0}{\psi n}$$

And therefore

$$\hat{\psi} = \frac{n\widetilde{\bar{x}} - \widetilde{x}_0}{n\left(\widetilde{\bar{x}} - \widetilde{x}_0\right)}$$

The method of moments estimators is consistent because the sample parameters are consistent Victoria, 2018 [38].

5.2 *Maximum likelihood estimator*

For MLE we take likelihood function for samples $(x_1, x_2, x_3, \ldots, x_n)$ as

$$L = L(x, \varsigma, \psi) = \frac{\psi^n \varsigma^{n\psi}}{\prod \widetilde{x}_i^{\psi + 1}}$$

apply natural logarithm then we obtain

$$L = n\, log\psi + \psi n\, log\varsigma - (\psi + 1) \sum log\, \widetilde{x}_i$$

Hence

$$\frac{\partial L}{\partial \psi} = \frac{n}{\psi} + n\, log\varsigma - \sum log\widetilde{x}_i = 0$$

We get as

$$\hat{\psi} = \frac{n}{\sum ln\, \frac{\widetilde{x}_i}{\varsigma}}$$

where $\hat{\zeta}$ is some estimate of ζ. Although the value of $\hat{\zeta} = min(\widetilde{x}_i)$ is the minimum positive value in $\widetilde{x}_i$. Therefore,

$$\hat{\varsigma} = min\left(\widetilde{x}_i\right) \text{ and } \widehat{\psi} = \frac{n}{\sum ln\, \frac{x_i}{\widetilde{x}_m}}$$

The MLE estimators are having the consistence with unbounded random variable was consistent Quandt [39].

5.3 *Least square estimator*

The Cumulative distribution can be taken for this,

$$F\,(\,\widetilde{x}\,)\;=\;1\;-\;\frac{\varsigma}{\widetilde{x}}^{\psi}$$

$$1\;-\;F\,(\,\widetilde{x}\,)=\frac{\varsigma}{\widetilde{x}}^{\psi}$$

$$log\,(\,1\;-\;F\,(\,\widetilde{x}\,))\;=\psi\;log\varsigma\;-\;\psi\;log\widetilde{x}$$

The equation can be written in the form of

$$Y_i\;=\;A\;+\;B\,X_i$$

where

$$Y_i\;=\;log\,(1\;-\;F(\widetilde{x})),\;A\;=\;\psi\;log\varsigma,\;B\;=\;-\;\psi\;and\;X_i\;=\;log\widetilde{x}_i$$

we know that estimate the LSE by using

$$A\;=\;\frac{1}{n}\sum log\,(1\;-\;F(\widetilde{x}_i))\;-\;\frac{1}{n}\,B\sum log\widetilde{x}_i$$

$$B\;=\;\frac{\sum log\widetilde{x}_i\;\;\sum log\,(1\;-\;F(\widetilde{x}_i))\;-\;n\sum log\widetilde{x}_i\;log\,(1\;-\;F(\widetilde{x}_i))}{(\sum log\widetilde{x}_i)^2\;-\;n\sum log\widetilde{x}_i{}^2}$$

where $F(x)=\frac{i\,-\,0.3}{N\,+\,0.4}$ by using the median rank method, small to large assign the ranks. Now calculated the values of ζ and ψ by using the above values Aruna [40].

The estimator is consistent by least squares where the dependent variable is the logarithm of 1 minus the sample cumulative distribution.

6. Simulation study

In this section choosen the simulation data from R software related to the satisfied of the pareto distribution. In the simulation, 1000 random integers were generated using a pareto distribution with shape parameter = 5 and scale parameter = 10. Different samples of sizes 10, 20, 40, 60, 80 and 100 were taken from the simulated population and their corresponding fits were estimated 100 times.

Now converted the simulation data into TFN by taking $(x-\delta,\,x,\,x+\delta)$ here δ takes approximately as 0.5. For the TFN data had calculate membership values on different various methods according into sample sizes. When the data in membership values by the ranking methods in TFN, using the moments, MLE and LSE methods to estimated parameters as shape and scale. By observing the estimated parameters and actual parameters of the shape and scale in various ranking methods of the TFN. The various ranking methods for the triangular fuzzy number of the estimated parameters are unique and same. The resultant observed in the given Table 1.

Table 1. Simulation table for estimated parameters of pareto by various ranking methods by shape = 5 and scale = 10 in TFN

Size	Methods	Estimated parameters by various ranking methods															
		Left alphacut		Right alphacut		Yogers		SI AVg		SI Add		Pascal		Magnitude		Centroid	
		Sh	Sc	Sh	Sc	Sh	Sc	Sh	Sc	Sh	Sc	Sh	Sc	Sh	Sc	Sh	Sc
10	MOM	3.4	8.9	3.9	10	3.7	9.7	3.7	9.7	3.7	19	3.7	9.7	3.5	13	3.7	34
	MLE	3.6	9.2	4.1	10	3.8	10	3.8	10	3.8	20	3.8	10	3.6	14	3.8	35
	LSE	3.1	9.0	3.5	10	3.3	9.8	3.3	9.8	3.3	19	3.3	9.8	3.2	13	3.3	34
20	MOM	4.2	9.1	4.8	10	4.5	9.9	4.5	9.9	4.5	19	4.5	9.9	4.2	13	4.5	34
	MLE	4.1	9.2	4.7	10	4.4	10	4.4	10	4.4	20	4.4	10	4.2	14	4.4	35
	LSE	4.8	9.6	5.5	11	5.2	10	5.2	10	5.2	20	5.2	10	4.9	14	5.2	36
40	MOM	3.7	9.1	4.2	10	3.9	9.9	3.9	9.9	3.9	19	3.9	9.9	3.7	13	3.9	34
	MLE	3.8	9.2	4.3	10	4.0	10	4.0	10	4.0	20	4.0	10	3.8	14	4.0	35
	LSE	3.4	9.0	3.7	10	3.6	9.7	3.6	9.7	3.6	19	3.6	9.7	3.4	13	3.6	34
60	MOM	5.4	9.2	6.2	10	5.8	10	5.8	20	5.8	20	5.8	10	5.5	14	5.8	35
	MLE	5.6	9.3	6.4	10	6.0	10	6.0	10	6.0	20	6.0	10	5.7	14	6.0	35
	LSE	4.5	8.9	5.0	10	4.8	9.7	4.8	9.7	4.8	19	4.8	9.7	4.5	13	4.8	34
80	MOM	5.1	9.1	5.8	10	5.4	9.9	5.4	9.9	5.4	19	5.4	9.9	5.1	13	5.4	34
	MLE	5.2	9.2	5.9	10	5.5	10	5.5	10	5.5	20	5.5	10	5.2	14	5.5	35
	LSE	4.8	9.9	5.4	10	5.1	9.8	5.1	9.8	5.1	19	5.1	9.8	4.8	13	5.1	34
100	MOM	4.0	9.1	4.6	10	4.3	9.9	4.3	9.9	4.3	19	4.3	9.9	4.1	13	4.3	34
	MLE	3.9	9.2	4.5	10	4.2	10	4.2	10	4.2	20	4.2	10	3.9	14	4.2	35
	LSE	4.3	9.4	4.9	11	4.6	10	4.6	10	4.6	20	4.6	10	4.4	14	4.6	35

By observing the above table we know the MLE method best estimator, the shape and scale value lies between in the left and right alpha-cut, in ranking methods of yogers, sub-interval average, sub-interval addition, and pascal has the same values in MLE method, in magnitude and centroid has the huge different values as compared with true values.

7. Real life application

In this section, consider the estimation of the parameters of the pareto distribution of dataset from the insurance industry in imprecise. By using to estimate the parameter of the pareto in different methods as moments, MLE and Least square. The data considered as TFN mode to calculate the fuzzy parameters for imprecise data. By using R (4.2.3) version software repeated above to analyse the consistent.

A online chess game data from Lichess which consists of 10 claims on a day of appropriately. We will refer to this dataset as the Lichess claims in this study in Table 2. To the best of our knowledge, this dataset has never been used in the extreme value theory literature.

Table 2. The real data set of the Online chess game from Lichess

Game ID	Rated	Turns	Victory
1	FALSE	13	Out of
2	TRUE	16	Resign
3	TRUE	61	Mate
4	TRUE	61	Mate
5	TRUE	95	Mate
6	FALSE	5	Draw
7	TRUE	33	Resign
8	FALSE	9	Resign
9	TRUE	66	Resign
10	TRUE	119	Mate

This data is used for proving this paper as estimated parameters in pareto distribution in imprecise. For that purpose, convert this data into triangular fuzzy number (a, x, b) randomly chooses a and b values and calculate the membership values for each ranking method i.e., the imprecise we convert into crisp data by defuzzification.

From the above Table 3, it can be observed that the membership values from Yogers, SI average, Pascal has the same values. From the alpha-cut it will lies between left and right but in the remaining methods of ranking SI addition, Magnitude and Centroid it has the different membership value.

Table 3. The membership values for each ranking method in TFN of data set

TFN (a, x, b)	LEFT alpha-cut	RIGHT alpha-cut	YOGERS	SI AVG	SI ADD	PASCAL	MAGNITUDE	CENTROID
(10, 13, 16)	10.6	15.4	13	13	26	13	16.5	45.5
(10, 16, 25)	11.2	23.2	17.2	17	34	16.75	19.5	57.1
(45, 61, 70)	48.2	68.2	58.2	58.6	117.3	59.25	72	210.7
(45, 61, 70)	48.2	68.2	58.2	58.6	117.3	59.25	72	210.7
(75, 95, 105)	79	103	91	91.6	183.3	92.5	117.5	328.6
(0, 5, 10)	1	9	5	5	10	5	2.5	17.5
(25, 33, 38)	26.6	37	31.8	32	64	32.25	40	114.3
(6, 9, 12)	6.6	11.4	9	9	18	9	10.5	31.5
(58, 66, 74)	59.6	72.4	66	66	132	66	91	231
(105, 119, 135)	107.8	131.8	119.8	119.6	239.3	119.5	165.5	417.2

The values of the estimated parameters of pareto distribution by taking the membership in the various ranking methods in TFN as shown in the below Table 4. By knowing of the among three estimated methods MOM, MLE and LSE, conclude by based on MLE the shape is 0.5 and scale is 5 for real data set online chess game from Lichess. By observing the Table 4 conclude that each ranking methods have the estimators are unique and consistent in yogers, sub interval average and pascal. In the ranking method of sub interval addition, the estimated parameter has doubled the scale but same as the shape. The values in the alpha-cut lies between left and right, remaining methods of magnitude and centroid has the different estimated value, compared with others.

Table 4. Shape and Scale values of the chess game different methods by various ranking methods

Methods	MOM		MLE		LSE	
	Shape	Scale	Shape	Scale	Shape	Scale
Left alphacut	1.023148	0.9022624	0.3243294	1	0.4818028	3.0999339
Right alphacut	1.191399	8.244585	0.705835	9	0.7865109	11.2265782
Yogers	1.110837	4.549889	0.5531286	5	0.6947245	7.8747238
SI Avg	1.109845	4.549487	0.5519975	5	0.6924208	7.8684355
SI Add	1.109845	9.098974	0.5519975	10	0.6924208	15.7368709
Pascal	1.109845	4.549487	0.5519975	5	0.6895049	7.8588906
Magnitude	1.039788	2.259566	0.3787593	2.5	0.5468855	6.2768858
Centroid	1.106663	15.91867	0.5484141	17.5	0.684566	27.436535

8. Conclusion

In this section, present our findings on the estimation of pareto distribution parameters in TFN. By utilized the method of moments, maximum likelihood estimation, and least square estimation method to estimate these parameters. Our decision-making process heavily relied on ranking methods. Therefore, employed various ranking methods for the triangular fuzzy number. It is worth noting that the estimated parameters in the ranking methods of TFN were almost identical. The simulation results indicated that the sample size did not affect the uniqueness of the solution, making it applicable in all situations. Additionally, observed that the scale parameter values varied in the centroid, sub interval addition, and magnitude methods, while the shape parameter remained constant.

To validate our findings, utilized real-life data from online chess in Litchess. By observing concluded that the estimated parameters of the pareto distribution using the MLE method in various ranking methods of TFN exhibited uniqueness in the shape parameter for Yogers, Sub interval average, and Pascal ranking methods. In the alpha-cut ranking method, the parameter values fell between the left and right values. In the sub interval addition ranking method, the shape parameter remained the same, but the scale was halved compared to the real value. The remaining ranking methods, magnitude and centroid, yielded different results.

Currently, there are numerous ranking methods established in TFN for decision making. The comparison and acute value play crucial roles in this process. In the future, when proposing new ranking methods, it is essential to compare them with Yogers, Sub interval average, and Pascal ranking methods to ensure better results.

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Disclosure

The author declares that has no relevant or material financial interests that relate to the research described in this paper.

Conflict of interest

The authors declare no competing financial interest.

References

[1] Cirillo R. *The Economics of Vilfredo Pareto*. London, UK: Frank Cass and Company; 1979.

[2] Zeinab HA. Bayesian inference for the Pareto lifetime model under progressive censoring with binomial removals. *Journal of Applied Statistics*. 2008; 35(11): 1203-1217.

[3] Fu J. Objective bayesian analysis of Pareto distribution under progressive type-II censoring. *Statistics Probability Letters*. 2012; 82(10): 1829-1836.

[4] Riham E. A Pareto optimization-based approach to clustering and routing in wireless sensor networks. *Journal of Network and Computer Applications*. 2018; 114: 57-69.

[5] Amer Ibrahim Al-Omari E. Estimation of the stress-strength reliability for exponentiated Pareto distribution using median and ranked set sampling methods. *Computers, Materials Continua*. 2020; 64(2): 835-857.

[6] Reinartz W. Modeling customer lifetime value, retention, and churn. In *Hand-Book of Market Research*. Berlin/Heidelberg, Germany: Springer; 2022. p.1001-1033.

[7] Zahra R. The reliability analysis based on the generalized intuitionistic fuzzy two parameter pareto distribution. *Soft Computing*. 2023; 27(6): 3095-3113.

[8] Shu MH. Using intuitionistic fuzzy sets for fault-tree analysis on printed circuit board assembly. *Microelectron Reliab*. 2006; 46(12): 2139-2148.

[9] Mahapatra GS. Reliability evaluation using triangular intuitionistic fuzzy numbers arithmetic operations. *World Acadamic Science Engineering Technology*. 2009; 50: 574-580.

[10] Puri J. Intuitionistic fuzzy data envelopment analysis: An application to the banking sector in India. *Expert System Applications*. 2015; 42(11): 4982-4998.

[11] Atanassov KT. Type-1 fuzzy sets and intuitionistic fuzzy sets. *Algorithms*. 2017; 10(3): 106.

[12] Mahmoodirad A. A new effective solution method for fully intuitionistic fuzzy transportation problem. *Soft Computing*. 2019; 23(12): 4521-4530.

[13] Feng F. Minkowski weighted score functions of intuitionistic fuzzy values. *Mathematics*. 2020; 8(7): 1143.

[14] Pekala B. New transitivity of Atanassov's intuitionistic fuzzy sets in a decision making model. *International Journal Applied Mathematical Computing Science*. 2021; 31(4): 563-576.

[15] Varghese PJ. A study on reliability using pendant, hexant, octant fuzzy numbers. *Journal Reliability Statistics Studied*. 2021; 14: 491-526.

[16] Zadeh LA. Fuzzy sets. *Information and Control*. 1965; 8(2): 338-353.

[17] Jain R. Decision-making in the presence of fuzzy variables. *IEEE Transactions on Systems, Man, and Cybernetics*. 1976; 6: 698-703.

[18] Fortemps P, Roubens M. Ranking and defuzzification methods based on area compensation. *Fuzzy Sets and Systems*. 1996; 82: 319-330.

[19] Facchinetti G. Ranking function induced by weighted average of fuzzy numbers. *Fuzzy Optimization and Decision Making*. 2002; 1, 313-327.

[20] Samah Bekheet. An enhanced fuzzy multi criteria decision making model with a proposed polygon fuzzy number. *International Journal of Advanced Computer Science and Applications* 2014; 5: 118.

[21] Thiruppathi A, Kirubhashankar CK. New ranking of generalized hexagonal fuzzy number using centroids of centroided method. *Advances in Mathematics: Scientific Journal*. 2020; 9(8): 6229-6240.

[22] Thiruppathi A, Kirubhashankar CK. New ranking of generalized quadrilateral shape fuzzy number using centroid technique. *Intelligent Automation & Soft Computing*. 2023; 36(2): 2253-2266.

[23] Ganesh AHM. Ordering of generalised trapezoidal fuzzy numbers based on area method using euler line of centroids. *Advances in Fuzzy Mathematics*. 2017; 12: 783-791.

[24] Ebrahimnejad AJ. *Fuzzy sets-based methods and techniques for modern analytics*. Bern, Switzerland, Springer; 2018. p.364.

[25] Zimmermann KH. *Fuzzy sets and theory*. 4th ed. Berlin, Germany: Springer Science, Business Media, LLC; 2021.

[26] Klir GJ, Folger TA. *Fuzzy sets, uncertainity and information*. Hoboken, NJ, USA; Prentice-Hall, Inc.; 2003.

[27] Phani Bushan Rao P, Ravi Shankar N. Ranking fuzzy numbers with a distance method using circumcenter of centroids and an index of modality. *Advances in Fuzzy Systems*. 2011; 2011: 1-7.

[28] Srinivasan R. Evaluation of fuzzy non-preemptive priority queues in intuitionistic pentagonal fuzzy numbers using centroidal approach. *Malaya Journal of Matematik*. 2020; 1: 427-430.

[29] Wutsq DU, Insani N. Yager's ranking method for solving the trapezoidal fuzzy number linear programming. *Journal of Physics: Conference Series*. 2018; 983(1): 012135.

[30] Dinagar S. Sub interval average method for ranking of linear fuzzy numbers. *International Journal of Pure and Applied Mathematics*. 2017; 114(6): 119-130.

[31] Dinagar DS, Kamalanathan S. Sub interval addition method for ranking of fuzzy numbers. *International Journal of Pure and Applied Mathematics*. 2017; 115(9): 159-169.

[32] Kamble AJ. Some notes on pentagonal fuzzy numbers. *International Journal of Fuzzy Mathematical Archive*. 2017; 13: 113-121.

[33] Arnold C. *Pareto Distributions*. New York, NY, USA: International Co-Operative Publishing House; 1983.

[34] Pareto Distribution. (n.d.). *Wikipedia*. Available from: https://en.wikipedia.org/wiki/Pareto_distribution.

[35] Zaher HM. Estimation of pareto parameters using fuzzy least squares method and other known techniques with a comparison. *British Journal of Mathematics Computer Science*. 2014; 4(14): 2067-2088.

[36] Jjet. Hazard rate of the pareto distribution. *Mathematics Stack Exchange*. 2016. Available from: https://math.stackexchange.com/questions/2060333/hazard-rate-of-the-pareto-distribution.

[37] Pareto V. La Courbe de la Repartition de la Richesse. *Busino G Editor*. 1965; 1-5. Available from: https://doi.org/10.3917/DROZ.PARET.1967.01.

[38] Akpan VM. Efficiency of some estimation methods of the parameters of a two-parameter pareto distribution. *American Journal of Mathematics and Statistics*. 2018; 8(5): 105-110.

[39] Quandt RE. Old and New methods of estimation in the pareto distribution. *Metrika*. 1966; 10(1): 55-82.

[40] Aruna CV, Sekharam OVR, Anjaneyulu G. Estimation of shape and scale of two-parameter lomax distribution (LD) using median ranks method. *International Journal for Research in Applied Science Engineering Technology*. 2022; 10: 1316-1322. Available from: https://doi.org/10.22214/ijraset.2022.39900.