

## Research Article

# Theorem of Alternative and Scalarization of Optimization Problems in Topological Vector Spaces

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**Abstract:** In this paper, we introduce the definitions of affinelike, preaffinelike, generalized affinelike, and generalized preaffinelike functions by use of “pointed convex cone”, and prove that those definitions of generalized affine functions are all different. We discuss optimization problems in topological vector spaces and obtain a theorem of alternative and a scalarization theorem. Our inequalities are given by partial order relations. Our generalized affineness may be used for many discussions in mathematics or applied mathematics wherever the affineness is a condition.

**Keywords:** affine functions, affinelike functions, preaffinelike functions, generalized affinelike functions, generalized preaffinelike functions

**MSC:** 90C26, 90C30, 90C29, 90C48

## 1. Introduction and preliminary

There were many generalizations of convexity, most of them are meaningful and useful. However, this paper has a different approach-generalizes the definition of affine functions and finds out that the affineness and generalized affinenesses are also interesting, meaningful, and useful. The author is the first one to generalize the affineness.

Let  $Y$  be a topological vector space. A subset  $Y_+$  of  $Y$  is called a cone if  $\lambda y \in Y_+$  for all  $y \in Y_+$  and  $\lambda \geq 0$ . We denote by  $0_Y$  the zero element in the topological vector space  $Y$  and simply by  $0$  if there is no confusion. A convex cone is one for which  $\lambda_1 y_1 + \lambda_2 y_2 \in Y_+$  for  $\forall y_1, y_2 \in Y_+$  and  $\forall \lambda_1, \lambda_2 \geq 0$ . A pointed cone is one for which  $Y_+ \cap (-Y_+) = \{0\}$ .

A functional on the vector space  $Y$  is a real-valued function on  $Y$ . The set  $Y^*$  of all continuous linear functionals on  $Y$  is called the topological dual of  $Y$ . The dual cone  $Y_+^*$  of  $Y_+$  is defined as

$$Y_+^* = \{\xi \in Y^*: \xi(y) \geq 0, \forall y \in Y_+\}.$$

Let  $Y$  be a topological vector space with pointed convex cone  $Y_+$ . We denote the partial order induced by  $Y_+$  as follows:

$$y_1 \succ y_2 \text{ iff } y_1 - y_2 \in Y_+, \text{ or, } y_1 \prec y_2 \text{ iff } y_1 - y_2 \in -Y_+;$$

$$y_1 \succ y_2 \text{ iff } y_1 - y_2 \in \text{int}Y_+, \text{ or } y_1 \prec\prec y_2 \text{ iff } y_1 - y_2 \in -\text{int}Y_+,$$

where  $\text{int} Y_+$  denotes the topological interior of the set  $Y_+$ .

It is known that, a function  $f: X \rightarrow Y$  is called linear on  $D \subseteq X$  if

$$f(\alpha x_1 + \beta x_2) = \alpha f(x_1) + \beta f(x_2),$$

whenever  $x_1, x_2 \in D \subseteq X, \alpha, \beta \in R$ ;

A function  $f: X \rightarrow Y$  is said to be affine on  $D \subseteq X$  if

$$f(\alpha x_1 + (1 - \alpha)x_2) = \alpha f(x_1) + (1 - \alpha)f(x_2)$$

whenever  $x_1, x_2 \in D \subseteq X, \alpha \in R$ ; and A function  $f$  is said to be  $Y_+$ -convex on  $D \subseteq X$  if

$$\alpha f(x_1) + (1 - \alpha)f(x_2) \prec f(\alpha x_1 + (1 - \alpha)x_2)$$

whenever  $x_1, x_2 \in D \subseteq X, \alpha \in [0, 1], Y_+$  is a pointed convex cone of  $Y$ .

It is true that linearity  $\Rightarrow$  affineness  $\Rightarrow$  convexity. A function is linear if and only if it is in the form of  $f(x) = ax$ . A function is an affine function if and only if it is in the form of  $f(x) = ax + b$  (a translation of a linear function).

In the next section, we will introduce some definitions of generalized affine functions.

## 2. Definitions of generalized affine functions

We introduce the definitions of generalized affine functions as follows.

Our generalized affineness can be used for many discussions in mathematics or applied mathematics wherever the affineness is a condition.

**Definition 1** A function  $f: D \subseteq X \rightarrow Y$  is said to be affinelike on  $D$  if  $\forall x_1, x_2 \in D, \forall \alpha \in R, \exists x_3 \in D$  such that

$$\alpha f(x_1) + (1 - \alpha)f(x_2) = f(x_3).$$

**Definition 2** A function  $f: D \subseteq X \rightarrow Y$  is said to be preaffinelike on  $D$  if  $\forall x_1, x_2 \in D, \forall \alpha \in R, \exists x_3 \in D, \exists \tau \in R \setminus \{0\}$  such that

$$\alpha f(x_1) + (1 - \alpha)f(x_2) = \tau f(x_3).$$

In the following Definitions 3 and 4,  $Y_+$  is a pointed convex cone of  $Y$  and  $\tau \neq 0$  is a scalar.

**Definition 3** A function  $f: D \subseteq X \rightarrow Y$  is said to be generalized  $Y_+$ -affinelike on  $D$  if  $\forall x_1, x_2 \in D, \forall \alpha \in R, \exists x_3 \in D, \exists u \in Y_+$ , such that

$$u + \alpha f(x_1) + (1 - \alpha)f(x_2) = f(x_3).$$

**Definition 4** A function  $f: D \subseteq X \rightarrow Y$  is said to be generalized  $Y_+$ -preaffinelike on  $D$  if  $\forall x_1, x_2 \in D, \forall \alpha \in R, \exists x_3 \in D, \exists u \in Y_+, \exists \tau \in R \setminus \{0\}$  such that

$$u + \alpha f(x_1) + (1 - \alpha)f(x_2) = \tau f(x_3).$$

The following Example 1 shows that our definition of generalized  $Y_+$ -preaffineness is non-trivial. The generalized  $Y_+$ -preaffineness is the weakest definition of generalized affine functions introduced in this article.

**Example 1** Given the function  $f(x, y) = (x^2, y^2)$ ,  $x, y \in R$ , and assume that  $Y_+ = \{(x, -y) : x, y \geq 0\}$ . Take  $\alpha = 5, (x_1, y_1) = (1, 0), (x_2, y_2) = (0, 1)$ , then

$$\alpha f(x_1, y_1) + (1 - \alpha)f(x_2, y_2) = (5, -4).$$

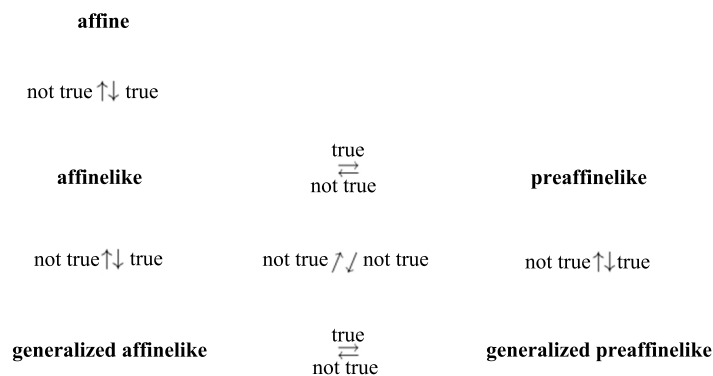
So,  $\forall u = (x, -y) \in Y_+$  one has

$$u + \alpha f(x_1, y_1) + (1 - \alpha)f(x_2, y_2) = (x + 5, -y - 4),$$

where  $x + 5 > 0, -y - 4 < 0$ . Due to the fact that  $f(x_3, y_3) = (x_3^2, y_3^2) \succ 0, \forall (x_3, y_3) \in R^2$  one gets

$$u + \alpha f(x_1, y_1) + (1 - \alpha)f(x_2, y_2) \neq \tau f(x_3, y_3), \forall u \in Y_+, \forall \tau \neq 0.$$

Therefore,  $f(x, y) = (x^2, y^2), x, y \in R$  is not a generalized  $Y_+$ -preaffinelike function. We are going to present some examples to prove the following diagram.



**Example 2** “Affinelike” does not imply “affine”.

Given the function  $f(x) = \tan x, x \in \mathbb{R}$ .

Since an affine function is in the form of  $f(x) = ax + b$ , therefore  $f(x) = \tan x, x \in \mathbb{R}$  is not affine.

However,  $f$  is affinelylike.  $\forall x_1, x_2 \in \mathbb{R}, \forall \alpha \in \mathbb{R}$ , taking

$$x_3 = \tan^{-1}[\alpha \tan x_1 + (1 - \alpha) \tan x_2],$$

then  $\alpha f(x_1) + (1 - \alpha)f(x_2) = f(x_3)$ .

**Example 3** “Preaffinelylike” does not imply “affinelylike”.

Given the function  $f(x) = \sqrt{x}, x \in \mathbb{R}^+ = [0, +\infty)$ .

Take  $x_1 = 0, x_2 = 1, \alpha = 2$ , then  $\alpha f(x_1) + (1 - \alpha)f(x_2) = -1$ ; but

$$\forall x_3 \in [0, +\infty), f(x_3) = \sqrt{x_3} \geq 0.$$

Therefore  $\alpha f(x_1) + (1 - \alpha)f(x_2) \neq f(x_3), \forall x_3 \in \mathbb{R}$ . So  $f$  is not affinelylike.

But  $f$  is an preaffinelylike function. For  $\forall x_1, x_2 \in \mathbb{R}^+, \forall \alpha \in \mathbb{R}$ , taking  $\tau = 1$  if  $\alpha f(x_1) + (1 - \alpha)f(x_2) \geq 0, \tau = -1$  if  $\alpha f(x_1) + (1 - \alpha)f(x_2) < 0$ , then

$$\alpha f(x_1) + (1 - \alpha)f(x_2) = \tau f(x_3),$$

where  $x_3 = [\alpha f(x_1) + (1 - \alpha)f(x_2)]^2$ .

**Example 4** “Generalized affinelylike” does not imply “affinelylike”.

Consider the function  $f(x) = x^3, x \in D = [0, 1]$ , and the pointed convex cone  $Y_+ = \mathbb{R}^+$ .  $\forall x_1, x_2 \in D = [0, 1], \forall \alpha \in \mathbb{R}$ , take  $u \in Y_+$  such that  $0 \leq u + \alpha f(x_1) + (1 - \alpha)f(x_2) \leq 1$ , then

$$u + \alpha f(x_1) + (1 - \alpha)f(x_2) = f(x_3),$$

where  $x_3 = [u + \alpha f(x_1) + (1 - \alpha)f(x_2)]^{1/3}$ . Therefore  $f(x) = x^3, x \in [0, 1]$  is generalized  $Y_+$ -affinelylike on  $D = [0, 1]$ .

$f(x) = x^3, x \in [0, 1]$  is not affinelylike on  $D = [0, 1]$ . Actually, for  $\alpha = -1 \in \mathbb{R}, x_1 = 1 \in D, x_2 = 0 \in D = [0, 1]$ , one has  $\alpha f(x_1) + (1 - \alpha)f(x_2) = -1$ , but

$$f(x_3) = x_3^3 \neq -1, \forall x \in [0, 1],$$

hence

$$\alpha f(x_1) + (1 - \alpha)f(x_2) \neq f(x_3), \forall x_3 \in D = [0, 1].$$

**Example 5** “Generalized preaffinelylike” does not imply “preaffinelylike”.

Given  $f(x) = x^2, x \in D = (0, +\infty)$ , and  $Y_+ = \mathbb{R}^+$ .

$\forall x_1, x_2 \in D = (0, +\infty), \forall \alpha \in \mathbb{R}$ , we may take  $u \in Y_+$  large enough such that  $u + \alpha f(x_1) + (1 - \alpha)f(x_2) > 0$ , and take  $\tau = 1$ , then

$$u + \alpha f(x_1) + (1 - \alpha)f(x_2) = \tau f(x_3),$$

where  $x_3 = [u + \alpha f(x_1) + (1 - \alpha)f(x_2)]^{1/2}$ . Therefore  $f(x) = x^2, x \in D = [0, +\infty)$ , is generalized  $Y_+$ -preaffinelike.

However, for  $\alpha = -1 \in R, x_1 = 1 \in D, x_2 = \sqrt{1/2}$  one has

$$\alpha f(x_1) + (1 - \alpha)f(x_2) = 0,$$

but  $f(x_3) = x_3^2 \neq 0, \forall x \in (0, +\infty)$ . Hence

$$\alpha f(x_1) + (1 - \alpha)f(x_2) \neq \tau f(x_3), \forall x_3 \in (0, +\infty), \forall \tau \neq 0.$$

This shows that the function  $f$  is not preaffinelike.

**Example 6** “Generalized preaffinelike” does not imply “generalized affinelike”.

Given  $f(x, y) = (x^2, y^6), x, y \in R$ , and  $Y_+ = \{(x, y): x \leq 0, y \leq 0, x, y \in R\}$

Let  $\alpha = 2, (x_1, y_1) = (0, 0), (x_2, y_2) = (1, 1)$ , then

$$\alpha f(x_1, y_1) + (1 - \alpha)f(x_2, y_2) = (-1, -1).$$

Therefore,  $\forall u = (x, y) \in Y_+$  one has

$$u + \alpha f(x_1, y_1) + (1 - \alpha)f(x_2, y_2) = (x - 1, y - 1) \neq f(x_3, y_3) = (x_3^2, y_3^6),$$

since  $(x - 1, y - 1) \prec (-1, -1) \prec \prec 0$ . And so,  $f(x, y) = (x^2, y^6), x, y \in R$  is not generalized  $Y_+$ -affinelike.

However,  $f(x, y) = (x^2, y^6), x, y \in R$  is generalized  $Y_+$ -preaffinelike.

$\forall x_1, x_2 \in [0, 1], \forall \alpha \in R$ , we may choose  $u = (x, y) \in Y_+$  such that

$$x + \alpha x_1^2 + (1 - \alpha)x_2^2 < 0, y + \alpha y_1^6 + (1 - \alpha)y_2^6 < 0.$$

Let  $\tau = -1$ , then

$$\begin{aligned} & u + \alpha f(x_1, y_1) + (1 - \alpha)f(x_2, y_2) \\ &= (x + \alpha x_1^2 + (1 - \alpha)x_2^2, y + \alpha y_1^6 + (1 - \alpha)y_2^6) \\ &= \tau (x_3^2, y_3^6) \end{aligned}$$

i.e.,

$$u + \alpha f(x_1, y_1) + (1 - \alpha)f(x_2, y_2) = \tau f(x_3, y_3),$$

where

$$x_3 = |x + \alpha x_1^2 + (1 - \alpha)x_2^2|^{1/2}, y_3 = |y + \alpha y_1^6 + (1 - \alpha)y_2^6|^{1/6}.$$

**Example 7** “Preaffineline” does not imply “generalized affineline”.

Given the function  $f(x, y, z) = (x^2, -x^2, x^2)$ ,  $x, y, z \in \mathbb{R}$ , and

$$Y_+ = \{(x, y, z): x \leq 0, y \geq 0, z \leq 0, x, y, z \in \mathbb{R}\}.$$

Let  $(x_1, y_1, z_1) = (1, 1, 1)$ ,  $(x_2, y_2, z_2) = (0, 1, 1)$ ,  $\alpha = -2$ , then

$$\begin{aligned} & \alpha f(x_1, y_1, z_1) + (1 - \alpha)f(x_2, y_2, z_2) \\ &= (\alpha x_1^2 + (1 - \alpha)x_2^2, -(\alpha x_1^2 + (1 - \alpha)x_2^2), \alpha x_1^2 + (1 - \alpha)x_2^2). \\ &= (-2, 2, -2). \end{aligned}$$

So,  $\forall u = (x, y, z) \in Y_+$ , one has

$$u + \alpha f(x_1, y_1, z_1) + (1 - \alpha)f(x_2, y_2, z_2) = (x - 2, y + 2, z - 2) \neq (x_3^2, -x_3^2, x_3^2),$$

since  $x - 2 < 0$ ,  $y + 2 > 0$ ,  $z - 2 < 0$  but  $x_3^2 \geq 0$ ,  $-x_3^2 \leq 0$ ,  $x_3^2 \geq 0$ .

Consequently,  $f(x, y, z) = (x^2, -x^2, x^2)$ ,  $x, y, z \in \mathbb{R}$  is not generalized  $Y_+$ -affineline.

On the other hand,  $\forall x_1, x_2 \in \mathbb{R}$ ,  $\forall \alpha \in \mathbb{R}$ , let  $\tau = 1$  if  $\alpha x_1^2 + (1 - \alpha)x_2^2 \geq 0$ ; or  $\tau = -1$  if  $\alpha x_1^2 + (1 - \alpha)x_2^2 < 0$ , then

$$\begin{aligned} & \alpha f(x_1, y_1, z_1) + (1 - \alpha)f(x_2, y_2, z_2) \\ &= (\alpha x_1^2 + (1 - \alpha)x_2^2, -(\alpha x_1^2 + (1 - \alpha)x_2^2), \alpha x_1^2 + (1 - \alpha)x_2^2) \\ &= \tau (x_3^2, -x_3^2, x_3^2) \\ &= \tau f(x_3, y_3, z_3) \end{aligned}$$

where  $x_3 = |\alpha x_1^2 + (1 - \alpha)x_2^2|^{1/2}$ .

Therefore,  $f(x, y, z) = (x^2, -x^2, x^2)$ ,  $x, y, z \in R$  is preaffineline.

**Example 8** “Generalized affineline” does not imply “preaffineline”.

Given the function  $f(x, y) = (x^2, y^2)$ ,  $x, y \in R$ , and

$$Y_+ = \{(x, y): x \geq 0, y \geq 0, x, y \in R\} |$$

Let  $(x_1, y_1) = (1, 1)$ ,  $(x_2, y_2) = (0, 1)$ ,  $\alpha = -1$ , then

$$\alpha f(x_1, y_1) + (1 - \alpha)f(x_2, y_2) = (\alpha x_1^2 + (1 - \alpha)x_2^2, \alpha y_1^2 + (1 - \alpha)y_2^2) = (-1, 1)$$

$$\neq \tau f(x_3, y_3) = \tau (x_3^2, y_3^2),$$

since either  $\tau x_3^2$  and  $\tau y_3^2$  are both negative or both non-negative,  $\forall \tau \neq 0$ .

Therefore,  $f(x, y) = (x^2, y^2)$  is not preaffineline.

However,  $f(x, y) = (x^2, y^2)$ ,  $x, y \in R$  is generalized  $Y_+$ -affineline.

In fact,  $\forall x_1, x_2 \in R, \forall \alpha \in R$ , we may choose  $u = (x, y) \in Y_+$  such that

$$u + \alpha f(x_1, y_1) + (1 - \alpha)f(x_2, y_2) = (x + \alpha x_1^2 + (1 - \alpha)x_2^2, y + \alpha y_1^2 + (1 - \alpha)y_2^2) \succ 0.$$

Then,

$$u + \alpha f(x_1, y_1) + (1 - \alpha)f(x_2, y_2) = f(x_3, y_3),$$

where  $x_3 = (x + \alpha x_1^2 + (1 - \alpha)x_2^2)^{1/2}$  and  $y_3 = (y + \alpha y_1^2 + (1 - \alpha)y_2^2)^{1/2}$ .

### 3. Theorem of alternative and scalarization

This section proves a theorem of alternative and a scalarization theorem in vector optimization, which are examples of the applications of our generalized affinenesses.

Our generalized affinenesses can be used for not only scalarizations but also many other aspects of optimization. Our generalized affineness can be used for many other discussions in mathematics or applied mathematics wherever affineness is a condition.

Consider the following vector optimization problem:

$$\begin{aligned} & Y_+ - \min f(x) \\ \text{(VP)} \quad & g_i(x) \prec 0, i = 1, 2, \dots, m; \\ & h_j(x) = 0, j = 1, 2, \dots, n; \\ & x \in D, \end{aligned}$$

where  $f: X \rightarrow Y$ ,  $g_i: X \rightarrow Z_i$ , and  $h_j: X \rightarrow W_j$ ,  $Y_+, Z_{i+}$  are closed convex cones in  $Y$  and  $Z_i$ , respectively, and  $D$  is a nonempty subset of  $X$ .

**Definition 5** A function  $f: D \subseteq X \rightarrow Y$  is said to be generalized  $Y_+$ -preconvexlike on  $D$  if  $\forall u \in \text{int}Y_+, \forall x_1, x_2 \in D, \forall \alpha \in R, \exists x_3 \in D, \exists \tau > 0$  such that

$$u + \alpha f(x_1) + (1 - \alpha)f(x_2) \prec \tau f(x_3).$$

In this section, we assume that  $f, g_i$  are generalized preconvexlike,  $h_j$  are preaffinelike (this article introduces the assumption of preaffinelikeness for the equality constraints  $h_j$  of an optimization problem), i.e., the following condition (A) is satisfied.

(A)  $\forall u_0 \in \text{int}Y_+, \forall u_i \in \text{int}Z_{i+}, \forall x_1, x_2 \in D, \forall \alpha \in [0, 1], \exists x', x'', x''' \in D, \exists \tau_i > 0 (i = 0, 1, 2, \dots, m), \exists t_j \neq 0 (j = 1, 2, \dots, n)$  such that

$$u_0 + \alpha f(x_1) + (1 - \alpha)f(x_2) \prec \tau_0 f(x')$$

$$u_i + \alpha g_i(x_1) + (1 - \alpha)g_i(x_2) \prec \tau_i g_i(x'')$$

$$\alpha h_j(x_1) + (1 - \alpha)h_j(x_2) = t_j h_j(x''),$$

where  $\tau_i, t_j$  are real scalars.

(B)  $\text{int} h_j(D) \neq \emptyset, (j = 1, 2, \dots, n)$ ;

Let  $F$  be the feasible set of (VP), i.e.,

$$F := \{x \in D: g_i(x) \prec 0, i = 1, 2, \dots, m; h_j(x) = 0, j = 1, 2, \dots, n\}.$$

**Theorem 1** (Theorem of Alternative) Let (i) and (ii) denote the systems

(i)  $\exists x \in D, \text{s.t.}, f(x) \prec \prec 0, g_i(x) \prec 0, (i = 1, 2, \dots, m); h_j(x) = 0, (j = 1, 2, \dots, n)$ ;

(ii)  $\exists (\xi, \eta, \zeta) \in (Y^* \times Z^* \times W^*) \setminus \{(0_Y, 0_Z, 0_W)\}$  such that

$$\xi(f(x)) + \eta(g(x)) + \zeta(h(x)) \geq 0, \forall x \in D.$$

If (i) has no solutions then (ii) has a solution.

If (ii) has a solution  $(\xi, \eta, \zeta)$  with  $\xi \neq 0_Y$  then (i) has no solutions.

**Proof.** It is easy to prove that

$$B = \left( \bigcup_{t>0} (tf(D) + \text{int}Y_+) \right) \times \left( \bigcup_{t>0} (tg(D) + \text{int}Z_+) \right) \times \left( \bigcup_{t \neq 0} th(D) \right).$$

is a convex set. From the assumption (B),  $\text{int} B \neq \emptyset$ . We also have  $(0_Y, 0_Z, 0_W) \notin B$  since (i) has no solution. Therefore, according to the separation theorem of convex sets of topological linear space, there exists a nonzero vector  $(\xi, \eta, \zeta) \in Y^* \times Z^* \times W^*$  such that



$$\xi(\tau_0 f(x) + y^0) + \eta_i(\tau_i g_i(x) + z_i^0) + \zeta_j(t_j h_j(x)) \geq 0,$$

for  $\forall x \in D, \forall y^0 \in \text{int}Y_+, \forall z_i^0 \in \text{int}Z_{i+}, \forall \tau_i > 0 (i = 0, 1, \dots, m), \forall t_j \neq 0 (j = 1, 2, \dots, n)$

Since  $\text{int}Y_+, \text{int}Z_{i+}$  are convex cones, one gets

$$\xi(\tau_0 f(x) + \lambda_0 y^0) + \eta_i(\tau_i g_i(x) + \lambda_i z_i^0) + \zeta_j(t_j h_j(x)) \geq 0,$$

for  $\forall x \in D, \forall y^0 \in \text{int}Y_+, \forall z_i^0 \in \text{int}Z_{i+}, \forall \lambda_i > 0 (i = 0, 1, \dots, m), \forall t_j \neq 0 (j = 1, 2, \dots, n)$ .

Let  $\tau_0 = 1, \tau_i \rightarrow 0 (i = 1, 2, \dots, m), \lambda_i \rightarrow 0 (i = 1, 2, \dots, m), t_j \rightarrow 0 (j = 1, 2, \dots, n)$  one has

$$\xi(y^0) \geq 0, \forall y^0 \in \text{int}Y_+.$$

Therefore  $\xi(y) \geq 0, \forall y \in Y_+$ . Hence  $\xi \in Y_+^*$ . Similarly,  $\eta_i \in Z_{i+}^*$ . And one has

$$\xi(f(x)) + \eta(g(x)) + \zeta(h(x)) \geq 0, x \in D,$$

which means that (ii) has solutions.

On the other hand, suppose that (ii) has a solution  $(\xi, \eta, \zeta)$  with  $\xi \neq 0_{Y_+^*}$ . If the system (ii) had a solution  $x \in D$ , there would hold

$$f(x) \prec \prec 0, g_i(x) \prec 0, (i = 1, 2, \dots, m); h_j(x) = 0, (j = 1, 2, \dots, n),$$

since (B) states that  $\text{int} B \neq \emptyset$ . We complete the proof. □

**Definition 6**  $\bar{x} \in D$  is said to be a weakly efficient solution of (VP) if there is no  $x \in D$  such that  $f(\bar{x}) \succ \succ f(x)$ .

**Definition 7** The problem (VP) is said to satisfy the Slater constraint qualification (SC) if  $\forall (\eta, \zeta) \in (Z_+^* \times W^*) \setminus \{O\}$ ,  $\exists x \in D$  such that  $\eta(g(x)) < 0$  and  $\zeta(h(x)) < 0$ .

Consider the scalar optimization problem

$$(VPS) \min_{x \in D} \xi(f(x)).$$

**Definition 8**  $\bar{x} \in D$  is said to be an optimal solution of the scalar optimization problem (VPS) if  $\xi(f(x)) \prec \xi(f(\bar{x})), \forall x \in D$ .

**Theorem 2** (Scalarization Theorem) Suppose  $\bar{x} \in D$ , and

(a)  $f, g_i, h_j$  satisfy Condition (A) in Theorem 1;

(b) (VP) satisfies the Slater constraint qualification (SC), then  $\bar{x}$  is a weakly efficient solution of (VP) if and only if  $\exists \xi \in Y_+^* \setminus \{O\}$  such that  $\bar{x}$  is an optimal solution of the scalar optimization problem (VPS).

**Proof.** Let  $\exists \xi \in Y_+^* \setminus \{O\}$ . If  $\bar{x} \in D$  is an optimal solution of the scalar optimization problem (VPS), then  $\xi(f(x)) \prec \xi(f(\bar{x})), \forall x \in D$ . So, there is no  $x \in D$  such that

$$f(\bar{x}) \prec\prec f(x).$$

Therefore  $\bar{x}$  is a weakly efficient solution of (VP).

On the other hand, suppose that  $\bar{x}$  is a weakly efficient solution of (VP).

From Theorem 1  $\exists \bar{x} \in D$  such that the following system

$$f(x) \prec\prec f(\bar{x}), g(x) \prec 0, h(x) = 0$$

has no solutions for  $x \in D$ . Hence,  $\exists \xi \in Y_+^*, \eta \in Z_+^*, \zeta \in W^*$  with  $(\xi, \eta, \zeta) \neq O$  such that

$$\xi(f(x) - f(\bar{x})) + \eta(g(x)) + \zeta(h(x)) \geq 0, \forall x \in D.$$

i.e.,

$$\xi(f(x)) + \eta(g(x)) + \zeta(h(x)) \geq \xi(\bar{x}), \forall x \in D.$$

If  $\xi = O$ , then  $(\zeta, \eta) \neq O$ , and so

$$\eta(g(x)) + \zeta(h(x)) \geq 0, \forall x \in D.$$

This is contradicting to the Slater constraint qualification (SC). Therefore  $\xi \neq O$ . Therefore, from  $g(x) \prec 0, h(x) = 0$  one has

$$\xi(f(x)) \geq \xi(f(\bar{x})), \forall x \in D.$$

Which means  $\bar{x}$  is an optimal solution of (VPS). □

## 4. Conclusion

A function  $f: X \rightarrow Y$  is called affine on  $D \subseteq X$  if

$$f(\alpha x_1 + (1 - \alpha)x_2) = \alpha f(x_1) + (1 - \alpha)f(x_2),$$

whenever  $x_1, x_2 \in D \subseteq X, \alpha \in R$ . For the convexity, the above equality will be replaced by an inequality.

Although there are many different generalizations of the convexity (some of them are interesting and useful), the author is the first one to generalize the definition of affine functions. Our generalizations are not difficult but interesting, meaningful, and useful.

In this article, we introduce the following definitions of generalized affine functions: affinelikeness, preaffinelikeness, generalized affinelikeness, and generalized preaffinelikeness. We demonstrate that definitions of affine, affinelike, preaffinelike, subaffinelike, and presubaffinelike functions are all different from each other. We also showed that our weakest affineness-the generalized preaffineness-is non-trivial.

The last section of the article, “Theorem of the alternative and scalarization”, is just an example that our generalized affineness could be used for vector optimization problems. For the optimization problems discussed here, we required that the equality constraints are preaffinelike, i.e.,  $\forall x_1, x_2 \in D, \forall \alpha \in [0, 1], \exists x_3 \in D, \exists t_j \neq 0 (j = 1, 2, \dots, n)$  such that

$$\alpha h_j(x_1) + (1 - \alpha)h_j(x_2) = t_j h_j(x_3).$$

We might actually assume that the equality constraints are generalized preaffinelike (the proof would be different). Actually, our generalized affinenesses can be used for not only scalarizations but also many other aspects of optimization, e.g., generalizing the results about Lagrange multiplier in Donato [1], the results about duality in Guu et al. [2], the results about constraints qualification in Zhao [3].

Our Theorem 1 is a generalization or a modification of the theorems of alternatives in [4–10], and our Theorem 2 is a generalization or a modification of the scalarization theorems in [11–14].

**Remark 1** This paper introduced the definitions of various generalized affinelikeness by use of “pointed convex cones”, while [15] used “linear sets” to define generalized affinelikeness, both for vector-valued functions. Moreover, [16] defined generalized affine maps for set-valued situations by using of “affine cones”. We demonstrated that the definitions of corresponding generalized affineness by using of different “auxiliary sets” have similar properties and may have different applications.

**Remark 2** Our generalized affineness may be used for many other discussions in mathematics or applied mathematics wherever affineness is a condition.

Specially, we may discuss generalized affine optimization problems for saddle points, Lagrangian multipliers, proper-effective solutions, etc., our theorem of alternative and scalarization theorem in this article just two working examples.

## Conflict of interest

The author declares no competing financial interest.

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