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Research Article

Solutions for a New Fractional Differential Dynamical System and Yosida Quasi-Inverse Variational Inequality in Hilbert Space

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Abstract: In this article, first we introduce and study a Yosida Quasi-inverse variational inequality problem (in short, YQIVI) in Hilbert space and then developed a new fractional differential dynamical system for the YQIVI. We prove the existence and uniqueness of solution for the suggested dynamical system. Further, using the Lyapunov function we also prove the asymptotic stability of the new dynamical system at the equilibrium point. Furthermore, using Rothe's time discretization method we investigate existence and uniqueness of solution of the proposed dynamical system. Finally, we provide a numerical example to demonstrate the credibility and efficacy of the dynamical system in solving the YQIVI.

Keywords: fractional differential dynamical system, resolvent operator, yosida approximation operator, rothe's time discretization method

MSC: 65P40, 47J20, 49J40

1. Introduction

The variational inequality problem (see, [1]) is a very essential useful tool for addressing a variety of optimisation problems, including ODE, PDEs, complementarity problems, systems of linear or non-linear equations. In 1988, Noor [2] introduced Quasi variational inequality and in 2003, Noor [3] discussed the well-posedness of variational inequality. Facchinei et al. [4] studied variational inequalities with complementarity problems in finite dimensional space. A variety of generalizations have been made about it, for more details see [5–7]. Additionally, numerous analytical and numerical methods have been developed to solve problems in variational inequality. In 2011, Censor et al. [8, 9] obtained the solution of variational inequality by subgradient and extragradient method. The projection and contraction algorithm for solving variational inequality problems is given by Dong et al. [10–12]. In 2019, Ahmad et al. [13] proved strong convergence results of three-step iterative algorithm for generalized mixed ordered quasi-inclusion problem. See [14, 15] for the hybrid steepest descent method, and references therein. The mixed quasi variational inequality problem is one of the most significant generalization of variational inequality problem, and it is defined as:

Let $\mathscr{M}: \mathbb{H} \to \mathbb{H}$ be a set-valued mapping and $\mathscr{F}: \mathbb{H} \to \mathbb{H}$ be a single-valued mapping. The quasi variational inequality problem (QVIP) is to find a point $p^* \in \mathscr{M}(p^*)$ such that

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$$\langle \mathscr{F}(\mathrm{p}^*), \mathrm{w}-\mathrm{p}^* \rangle + \phi(\mathrm{w}) - \phi(\mathrm{p}^*) \geq 0, \quad \forall\, \mathrm{w} \in \mathscr{M}(\mathrm{p}^*), \tag{1}$$

where $\psi\colon \mathbb{H} \to \mathbb{R} \cup \{+\infty\}$ is a proper convex lower semi-continuous functional. Clearly, in the above equation if $\mathscr{M}(\mathrm{p}^*) = \mathscr{K}$ and $\phi(\mathrm{p}^*) = 0,\ \forall \mathrm{p}^* \in \mathbb{H},$ then MQVIP (1) will convert into the classical variational inequality problem studied by Stampacchia [16].

The problem where to find $\mathrm{p}^* \in \mathbb{H}$ such that

$$\mathscr{F}(\mathrm{p}^*) \in \mathscr{K} \quad \text{and} \quad \langle \mathrm{p}^*, \mathrm{w} - \mathscr{F}(\mathrm{p}^*) \rangle \geq 0, \ \ \forall\, \mathrm{w} \in \mathbb{H}, \tag{2}$$

is known as inverse variational inequality (IVI), where $\mathscr{F}\colon \mathbb{H} \to \mathbb{H}$ is a mapping and $\mathscr{K} \subseteq \mathbb{H}$ is a closed convex subset of $\mathbb{H}$. Inverse variational inequalities are used frequently in transportation networks and economic equilibrium problems [17, 18]. It is simple to see that, if the inverse of the mapping $\mathscr{F}$ exists, then IVI (2) can be transformed into classical variational inequality problem.

Numerous applications of IVI problems can be found in science and engineering, see for details [19–21]. Several problems with normative flow control that arise in the transport and telecommunications networks could be reduces to IVI problems in addition to the economics problem of market equilibrium.

In 2016, Zou et al. [22] proved that the following projected equation in finite dimensional space represents IVI (2) as its equivalent:

$$\mathscr{F}(\mathrm{p}^*) = P_{\mathscr{K}}(\mathscr{F}(\mathrm{p}^*) - \alpha\, \mathrm{p}^*), \tag{3}$$

where $\alpha > 0$ is a constant and $P_{\mathscr{K}}\colon \mathbb{R}^n \to \mathscr{K}$ is the projection mapping defined by

$$P_{\mathscr{K}}(\mathrm{p}^*) := \mathop{arg\,min}_{\mathrm{w} \in \mathscr{K}} \|\mathrm{p}^* - \mathrm{w}\|, \ \ \mathrm{p}^* \in \mathbb{R}^n.$$

However, there are only a few numerical techniques available for solving IVI problems, see [23–25] and references therein.

To address the IVI problem, Zou et al. [22] presented a single-layer recurrent neural network (4), which is defined as:

$$\dot{\mathrm{p}} = \lambda \left\{ P_{\mathscr{K}}(\mathscr{F}(\mathrm{p}) - \alpha \mathrm{p}) - \mathscr{F}(\mathrm{p}) \right\}, \tag{4}$$

where $\dot{\mathrm{p}} = \dfrac{d\mathrm{p}}{dt}$ and $\lambda > 0$ is a fixed constant.

Due of their applications in numerous disciplines, some researchers have shown their interest in them. They also characterised them in a variety of ways, see for example [17, 26, 27] and references therein.

It is important to note that the Yosida approximation notion allows monotone mappings on Hilbert spaces to be regularised into single-valued, nonexpansive, Lipschitz continuous monotone mappings. Because of their importance in convex analysis, partial differential equations, variable inclusions, etc., these operators have been extensively researched. The Yosida approximation strategy serves as the foundation for yet another potential method of resolving elliptic boundary value problems, which are multi-valued differential equations. For reference, using the Yosida approximation method, Petterson [28] demonstrated the existence of multi-valued stochastic differential equations with a maximal monotone operator in 1995.

Motivated and influenced by the aforementioned work, in this article, we consider the following Yosida Quasi-inverse Variational Inequality Problem (YQIVIP) in Hilbert space. To find $p^* \in \mathscr{M}(p^*)$ such that,

$$J(p^*) \in \mathscr{M}(p^*), \quad \langle p^*, w - J(p^*)\rangle + \phi(w) - \phi(J(p^*)) \geq 0, \quad \forall\, w \in \mathscr{M}(p^*), \tag{5}$$

where $J\colon \mathbb{H} \to \mathbb{H}$ is Yosida operator which is defined in (9) and $\phi\colon \mathbb{H} \to \mathbb{R} \cup \{+\infty\}$ is proper convex lower semi-continuous functional.

If a convex closed valued set $\mathscr{K}(U)$'s indicator function is $\phi(\cdot)$ in $\mathbb{H}$, that is,

$$\phi(p) = \begin{cases} 0, & p \in \mathscr{K}(p), \\ +\infty, & \text{otherwise} \end{cases}$$

and $J(p) = \mathscr{F}(p)$, then (5) is similar to find $p^* \in \mathscr{M}(p^*)$ such that,

$f(p^*) \in \mathscr{M}(p^*)$, $\langle p^*, w - f(p^*)\rangle \geq 0$, $\forall\, w \in \mathscr{M}(p^*)$ which is studied in [23]. Similarly, by suitable choices of mappings involved in (5) one can obtain many existing problems in the literature.

Proposition 1 The element $p \in \mathbb{H}$ is a solution of (5) if and only if p satisfies the following equation

$$J(p) = R_{\partial\Psi}[J(p) - \alpha p]. \tag{6}$$

Proof. Let $p \in \mathbb{H}$ satisfies (6), then

$$J(p) = R_{\partial\Psi}[J(p) - \rho p]$$

$$J(p) = [I + \rho\partial\psi]^{-1}[J(p) - \rho p]$$

$$J(p) + \rho\partial\psi(J(p)) = J(p) - \rho p$$

which implies that $p \in \partial\psi(J(p))$. That is,

$$\langle p^*, w - J(p^*)\rangle + \phi(w) - \phi(J(p^*)) \geq 0, \quad \forall\, w \in \mathscr{M}(p^*),$$

hence, $p \in \mathbb{H}$ is a solution of (5). □

Based on Proposition 1, we suggest the following fractional dynamical system for solving YQIVI defined in (5),

$${}_{C}D^{\alpha}p(t) = \lambda\left\{R_{\partial\Psi(p)}[J(p) - \alpha p] - J(p)\right\} = Q(p(t)), \tag{7}$$

where ${}_{C}D^{\alpha}$ $(0 < \alpha < 1)$ is the α fractional order's Caputo derivative, $\lambda > 0$ is a scaling factor and $R_{\partial\Psi}$ is the resolvent operator (8).

Our paper is structured as follows. In Section 2, we are presented several fundamental definitions and findings that can be used in following sections. We prove the existence and uniqueness of a solution to YQIVI (5) in Section 3. In Section 4, we look at the neural network's overall stability, including global exponential and global asymptotic stability. We discretize the system (7) in Section 5 and determine the neural network's existence and uniqueness by using the Rothe's method. We give a numerical example to demonstrate the neural network's efficiency in solving YQIVI (5). Finally, we present our analysis's conclusion in Section 6.

Remark 1 The dynamical system can be presented as single-layer recurrent neural network in $\mathbb{R}^n$. Here, $\lambda > 0$ is a scaling factor and $R_{\partial\psi}$ can be apply by using piece-wise activation function. Neural Network's Architecture of Dynamical System (7) in $\mathbb{R}^n$ is given in Figure 1.

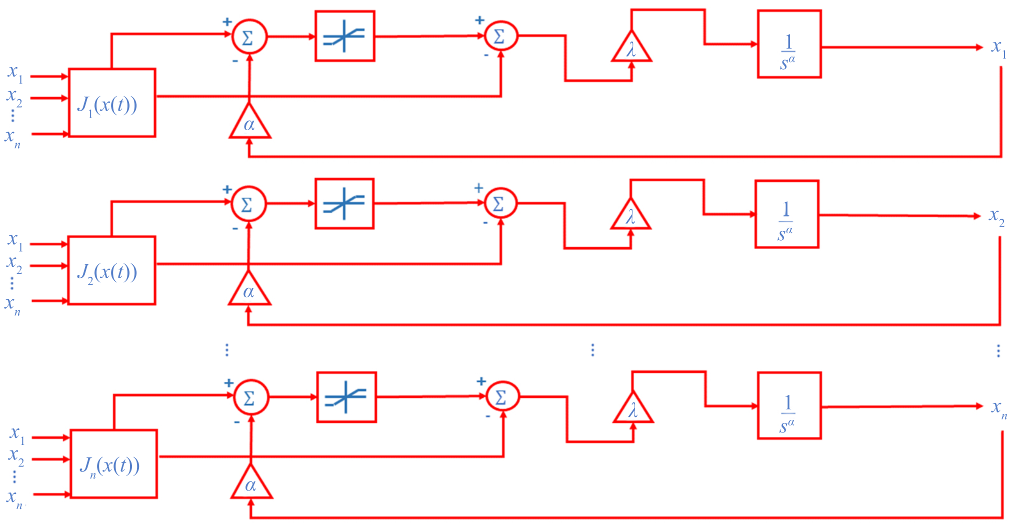

Figure 1. Neural Network's Architecture of Dynamical System (7) in $\mathbb{R}^n$

2. Methods and materials

In this section, we present some fundamental definitions and lemmas that are utilized to prove the main results of our article. For more details, we refer to see [29–31].

Notations: Throughout this article, the Hilbert space over the space of reals with the standard inner product $\langle \cdot, \cdot \rangle$ and norm $\|.\|$ is represented by the symbol $\mathbb{H}$.

Definition 1 ([32]) Let ψ: $\mathbb{H} \to \mathbb{R} \cup \{+\infty\}$ be a proper convex lower semcontinuous functional and I: $\mathbb{H} \to \mathbb{H}$ be an identity operator, then the proximal operator $R_{\partial\psi}$: $\mathbb{H} \to \mathbb{H}$ associated with ψ is defined as

$$R_{\partial\psi}(\mathrm{p}) = [I + \alpha\partial\psi]^{-1}(\mathrm{p}), \quad \forall\, \mathrm{p} \in \mathbb{H}, \tag{8}$$

where $\alpha > 0$ is a constant.

Definition 2 ([32]) The Yosida approximation operator J: $\mathbb{H} \to \mathbb{H}$ of ψ associated with $R_{\partial\psi}$ is defined by

$$J(\mathrm{p}) = \frac{1}{\alpha}[I - R_{\partial\psi}](\mathrm{p}), \quad \forall\, \mathrm{p} \in \mathbb{H}, \tag{9}$$

here $\alpha > 0$ is a constant.

Proposition 2 ([32]) The proximal operator $R_{\partial\psi}$ associated with subdifferential of proper lower semicontinuous functional ψ is θ-Lipschitz continuous, where $\theta > 0$ is a constant.

Proposition 3 ([32]) The Yosida approximation operator J is $\frac{1+\theta}{\alpha}$-Lipschitz continuous.

Definition 3 ([33]) The integral of $\mathrm{p}(t)$ of fractional order $\alpha > 0$ is defined by

$${}_{C}D^{-\alpha}\mathrm{p}(t) := \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\mathrm{p}(s)ds, \quad t > 0, \tag{10}$$

and the Caputo fractional derivative ${}_{C}D^{\alpha}\mathrm{p}(t)$ is defined by

$${}_{C}D^{\alpha}\mathrm{p}(t) := \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}\mathrm{p}^{(n)}(s)ds, \quad t > 0, \tag{11}$$

here $\Gamma(\alpha)$ represents the gamma function.

Proposition 4 (Gronwall's inequality) ([33]) Let $\hat{\mathrm{p}}$ and $\hat{\mathrm{w}}$ be real-valued nonnegative continuous functions, for $t \geq t_0$. Let $a(t) = a_0(|t - t_0|)$, where a_0 is a monotonically increasing function,

$$\hat{\mathrm{p}}(t) \leq a(t) + \int_{t_0}^t \hat{\mathrm{p}}(s)\hat{\mathrm{w}}(s)ds,$$

then

$$\hat{\mathrm{p}}(t) \leq a(t)e^{\int_{t_0}^t \hat{\mathrm{w}}(s)ds}.$$

Lemma 1 ([33]) If $\{\mathrm{p}_n\}$, $\{\mathrm{w}_n\}$ and $\{z_n\}$ are nonnegative sequences and

$$\mathrm{p}_n \leq \mathrm{w}_n + \sum_{0\leq k<n} z_k \mathrm{p}_k \qquad \text{for } n \geq 0,$$

then

$$\mathrm{p}_n \leq \mathrm{w}_n + \sum_{0\leq k<n} \mathrm{w}_k z_k \Big(\sum_{k<j<n} g_j\Big), \qquad \text{for } n \geq 0.$$

3. Existence result

This main aim of this section is to derive some sufficient conditions for the solution of the system (7).

Theorem 1 [23, 22] If there is a constant $\mu > 0$ such that

$$\|R_{\partial\Psi(\mathrm{p})}(p) - R_{\partial\Psi(\mathrm{w})}(p)\| \leq \mu\|\mathrm{p} - \mathrm{w}\|, \tag{12}$$

then there exists a unique dynamical system (7)'s continuous solution , for each $\mathrm{p} \in \mathbb{H},$ in whole interval $[0, \infty)$.

Proof. Let p: $[0, T] \to \mathbb{H},$ such that

$$Q(\mathrm{p}) = \lambda\left\{R_{\partial\Psi(\mathrm{p})}[J(\mathrm{p}) - \alpha\mathrm{p}] - J(\mathrm{p})\right\}. \tag{13}$$

For any $\mathrm{p}, \mathrm{w} \in \mathbb{H},$ we have

$$\begin{aligned}\|Q(\mathrm{p}) - Q(\mathrm{w})\| &\leq \lambda\|R_{\partial\Psi(\mathrm{p})}[J(\mathrm{p}) - \alpha\mathrm{p}] - R_{\partial\Psi(\mathrm{w})}[J(\mathrm{w}) - \alpha\mathrm{w}]\| + \lambda\|J(\mathrm{p}) - J(\mathrm{w})\| \\ &\leq \lambda\|R_{\partial\Psi(\mathrm{p})}[J(\mathrm{p}) - \alpha\mathrm{p}] - R_{\partial\Psi(\mathrm{p})}[J(\mathrm{w}) - \alpha\mathrm{w}]\| \\ &\quad + \lambda\|R_{\partial\Psi(\mathrm{p})}[J(\mathrm{w}) - \alpha\mathrm{w}] - R_{\partial\Psi(\mathrm{w})}[J(\mathrm{w}) - \alpha\mathrm{w}]\| + \lambda\|J(\mathrm{p}) - J(\mathrm{w})\|.\end{aligned}$$

Since $R_{\partial\Psi(\mathrm{p})}$ is $\theta-$Lipschitz continuous and using (12), we have

$$\begin{aligned}\|Q(\mathrm{p}) - Q(\mathrm{w})\| &\leq \lambda\theta\|J(\mathrm{p}) - J(\mathrm{w})\| + \lambda\theta\alpha\|\mathrm{p} - \mathrm{w}\| + \lambda\mu\|\mathrm{p} - \mathrm{w}\| + \lambda\|J(\mathrm{p}) - J(\mathrm{w})\| \\ &= \lambda(1+\theta)\|J(\mathrm{p}) - J(\mathrm{w})\| + \lambda(\theta\alpha + \mu)\|\mathrm{p} - \mathrm{w}\|.\end{aligned}$$

Using Lipschitz continuity of Yosida operator, we have

$$\|Q(\mathrm{p}) - Q(\mathrm{w})\| \leq \lambda\left[(1+\theta)^2/\lambda + (\theta\alpha + \mu)\right]\|\mathrm{p} - \mathrm{w}\|.$$

$$\Longleftrightarrow \quad \|Q(\mathrm{p}) - Q(\mathrm{w})\| \leq K\|\mathrm{p} - \mathrm{w}\|, \text{ where } K = \lambda\left[(1+\theta)^2/\lambda + (\theta\alpha + \mu)\right].$$

This shows that, $Q(\mathrm{p})$ is locally Lipschitz continuous in $\mathbb{H}$. Now, we have

$$\begin{aligned}\|Q(\mathrm{p})\| &= \|\lambda\left\{R_{\partial\Psi(\mathrm{p})}[J(\mathrm{p}) - \alpha\mathrm{p}] - J(\mathrm{p})\right\}\| \\ &\leq \lambda\left[\|R_{\partial\Psi}[J(\mathrm{p}) - \alpha\mathrm{p}] - R_{\partial\Psi(\mathrm{p})}[J(\mathrm{p})]\| + \|R_{\partial\Psi(\mathrm{p})}[J(\mathrm{p})] - R_{\partial\Psi(0)}[J(\mathrm{p})]\|\right. \\ &\quad \left. + \|R_{\partial\Psi(0)}[J(\mathrm{p})] - R_{\partial\Psi(0)}(0)\| + \|R_{\partial\Psi(0)}(0)\| + \|J(\mathrm{p})\|\right].\end{aligned}$$

Using Lipschitz continuity of $R_{\partial\Psi}$ and (12), we have

$$\begin{aligned}\|Q(\mathrm{p})\| &\leq \lambda\left[\theta\|J(\mathrm{p})-\alpha\mathrm{p}-J(\mathrm{p})\|+\mu\|\mathrm{p}\|+\theta\|J(\mathrm{p})\|+\|R_{\partial\Psi(0)}(0)+\|J(\mathrm{p})\|\right] \\ &= \lambda[\theta\alpha\|\mathrm{p}\|+\mu\|\mathrm{p}\|+(1+\theta)\|J(\mathrm{p})\|+\|R_{\partial\Psi(0)}(0)\|],\end{aligned} \tag{14}$$

and

$$\begin{aligned}\|J(\mathrm{p})\| &= \left\|\frac{1}{\lambda}(\mathrm{p}-R_{\partial\Psi(\mathrm{p})}(\mathrm{p}))\right\| \\ &\leq \frac{1}{\lambda}[\|\mathrm{p}\|+\|R_{\partial\Psi(\mathrm{p})}(\mathrm{p})-R_{\partial\Psi(\mathrm{p})}(0)\|+\|R_{\partial\Psi(\mathrm{p})}(0)-R_{\partial\Psi(0)}(0)\|+\|R_{\partial\Psi(0)}(0)\|] \\ &\leq \frac{1}{\lambda}\left[\|\mathrm{p}\|+\theta\|\mathrm{p}\|+\mu\|\mathrm{p}\|+\|R_{\partial\Psi(0)}(0)\|\right] \\ &= \frac{1}{\lambda}\left[(1+\theta+\mu)\|\mathrm{p}\|+\|R_{\partial\Psi(0)}(0)\|\right].\end{aligned} \tag{15}$$

Using (15) in (14), we have

$$\|Q(\mathrm{p})\| \leq \lambda[\theta\alpha\|\mathrm{p}\|+\mu\|\mathrm{p}\|+\frac{(1+\theta)}{\lambda}\left[(1+\theta+\mu)\|\mathrm{p}\|+\|R_{\partial\Psi(0)}(0)\|\right]+\|R_{\partial\Psi(0)}(0)\|],$$

that is,

$$\begin{aligned}&\|Q(\mathrm{p})\| \leq \lambda(\theta\alpha+\mu)\|\mathrm{p}\|+(1+\theta)(1+\theta+\mu)\|\mathrm{p}\|+(\theta+2)\|R_{\partial\Psi(0)}(0)\|, \\ &\|Q(\mathrm{p})\| \leq M\|\mathrm{p}\|+(\theta+2)\|R_{\partial\Psi(0)}(0)\|,\end{aligned} \tag{16}$$

where $M=\lambda(\theta\alpha+\mu)+(1+\theta)(1+\theta+\mu)$.

The dynamical system is

$${}_{C}D^{\alpha}\mathrm{p}(t)=\lambda\left\{R_{\partial\Psi}[J(\mathrm{p})-\alpha\mathrm{p}]-J(\mathrm{p})\right\}=Q(\mathrm{p}),$$

which implies that,

$$\mathrm{p}(t)=\mathrm{p}_0+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}Q(\mathrm{p}(s))\,ds.$$

Taking norm on both sides, we have

$$\|\mathrm{p}(t)\| \leq \|\mathrm{p}_0\| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|Q(\mathrm{p}(s))\|\,ds.$$

Using (16), the above inequality reduce to

$$\|\mathrm{p}(t)\| \leq \|\mathrm{p}_0\| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[M\|\mathrm{p}(s)\| + (\theta+2)\|R_{\partial\Psi(0)}(0)\|]\,ds$$

$$\leq \|\mathrm{p}_0\| + \frac{(\theta+2)\|R_{\partial\Psi(0)}(0)\|}{\Gamma(\alpha)}\frac{t^\alpha}{\alpha} + \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|\mathrm{p}(s)\|\,ds.$$

Let $h(t) = \|\mathrm{p}_0\| + \frac{(\theta+2)\|R_{\partial\Psi(0)}(0)\|}{\alpha\Gamma(\alpha)}t^\alpha$ and $\beta = \frac{M}{\Gamma(\alpha)}$, then above inequality reduce to

$$\|\mathrm{p}(t)\| \leq h(t) + \beta\int_0^t (t-s)^{\alpha-1}\|\mathrm{p}(s)\|\,ds.$$

By Gronwall's inequality, we have

$$\|\mathrm{p}(t)\| \leq h(t)e^{(\beta t)}.$$

Therefore, it is clearly indicate that, $\mathrm{p}(t)$ is bounded on any bounded interval $[0, T]$. By the functional differential equations's continuation theorem a single continuous solution $\mathrm{p}(t)$ exists over the entire range $[0, \infty)$. □

4. Stability result

In this section we are presented the global asymptotic stability of the dynamical system (7).

Theorem 2 [22, 23] Let M: $\mathbb{H} \to 2^{\mathbb{H}}$ be a multi-valued mapping with non-empty, convex and closed values, and let J: $\mathbb{H} \to \mathbb{H}$ be a Yosida approximation operator. Assume that

$$\Theta = [-\beta + (\theta+\mu)(1+\theta)]\lambda \leq 0. \tag{17}$$

Then, the dynamical system (7) is asymptotically stable.

Proof. Let p^* be the unique solution of YQIVI (5). Now at the equilibrium point p^* we will demonstrate that the system's trajectories is globally asymptotically stable. For this, consider the Lyapunov function,

$$L(\mathrm{p}) = \|\mathrm{p} - \mathrm{p}^*\|^2.$$

Then, by using (7), we obtain

$$
\begin{aligned}
{}_CD^{\alpha}L(\mathrm{p}) \leq & 2\langle \mathrm{p}-\mathrm{p}^*,\ {}_CD^{\alpha}\mathrm{p}(t)\rangle \\
= & 2\langle \mathrm{p}-\mathrm{p}^*, \lambda\{R_{\partial\Psi(\mathrm{p})}[J(\mathrm{p})-\alpha\mathrm{p}]-J(\mathrm{p})\}\rangle \\
= & 2\langle \mathrm{p}-\mathrm{p}^*, \lambda\{R_{\partial\Psi(\mathrm{p})}[J(\mathrm{p})-\alpha\mathrm{p}]-J(\mathrm{p})\}-\lambda\{R_{\partial\Psi}[J(\mathrm{p}^*)-\alpha\mathrm{p}^*]-J(\mathrm{p}^*)\}\rangle \\
= & -2\lambda\langle \mathrm{p}-\mathrm{p}^*, J(\mathrm{p})-J(\mathrm{p}^*)\rangle+2\lambda\langle \mathrm{p}-\mathrm{p}^*, R_{\partial\Psi(\mathrm{p})}[J(\mathrm{p})-\alpha\mathrm{p}]-R_{\partial\Psi(\mathrm{p}^*)}[J(\mathrm{p}^*)-\alpha\mathrm{p}^*]\rangle \\
= & -2\lambda\langle \mathrm{p}-\mathrm{p}^*, J(\mathrm{p})-J(\mathrm{p}^*)\rangle+2\lambda\langle \mathrm{p}-\mathrm{p}^*, R_{\partial\Psi(\mathrm{p})}[J(\mathrm{p})-\alpha\mathrm{p}]-R_{\partial\Psi(\mathrm{p})}[J(\mathrm{p}^*)-\alpha\mathrm{p}^*]\rangle \\
& +2\lambda\langle \mathrm{p}-\mathrm{p}^*, R_{\partial\Psi(\mathrm{p})}[J(\mathrm{p}^*)-\alpha\mathrm{p}^*]-R_{\partial\Psi(\mathrm{p}^*)}[J(\mathrm{p}^*)-\alpha\mathrm{p}^*]\rangle.
\end{aligned}
$$

Using Caucy-Schwarz inequality and β-strongly monotonicity of Yosida approximation operator, we have

$$
\begin{aligned}
{}_CD^{\alpha}L(\mathrm{p}) \leq & -2\lambda\beta\|\mathrm{p}-\mathrm{p}^*\|^2+2\lambda\|\mathrm{p}-\mathrm{p}^*\|\,\|R_{\partial\Psi(\mathrm{p})}[J(\mathrm{p})-\alpha\mathrm{p}]-R_{\partial\Psi(\mathrm{p})}[J(\mathrm{p}^*)-\alpha\mathrm{p}^*]\| \\
& +2\lambda\|\mathrm{p}-\mathrm{p}^*\|\,\|R_{\partial\Psi(\mathrm{p})}[J(\mathrm{p}^*)-\alpha\mathrm{p}^*]-R_{\partial\Psi(\mathrm{p}^*)}[J(\mathrm{p}^*)-\alpha\mathrm{p}^*]\|.
\end{aligned}
$$

Since $R_{\partial\Psi}$ is $\theta-$Lipschitz continuous, we have

$$
\begin{aligned}
{}_CD^{\alpha}L(\mathrm{p}) \leq & -2\lambda\beta\|\mathrm{p}-\mathrm{p}^*\|^2+2\lambda\theta\|\mathrm{p}-\mathrm{p}^*\|\,[\|J(\mathrm{p})-J(\mathrm{p}^*)\|+\alpha\|\mathrm{p}-\mathrm{p}^*\|]+2\lambda\mu\|\mathrm{p}-\mathrm{p}^*\|^2 \\
\leq & -2\lambda\beta\|\mathrm{p}-\mathrm{p}^*\|^2+2\lambda\theta(1+\theta)\|\mathrm{p}-\mathrm{p}^*\|^2+2\lambda\theta\alpha\|\mathrm{p}-\mathrm{p}^*\|^2+2\lambda\mu\|\mathrm{p}-\mathrm{p}^*\|^2 \qquad (18) \\
= & -2\lambda\{-\beta+\theta(1+\theta)+\theta\alpha+\mu\}\|\mathrm{p}-\mathrm{p}^*\|^2.
\end{aligned}
$$

Let $\Theta=\lambda\{-\beta+\theta(1+\theta)+\theta\alpha+\mu\}$. Then, we obtain

$$
{}_CD^{\alpha}L(\mathrm{p}) \leq 2\Theta L(\mathrm{p}). \tag{19}
$$

Taking Laplace transform of (19), we have

$$s^{\alpha}L(\mathrm{p}(s)) - s^{\alpha-1}L(\mathrm{p}(0)) \leq 2\Theta L(\mathrm{p}(s))$$

$$(s^{\alpha} - 2\Theta)L(\mathrm{p}(s)) \leq s^{\alpha-1}L(\mathrm{p}_0)$$

$$\Longrightarrow L(\mathrm{p}(s)) \leq \left(\frac{s^{\alpha-1}}{s^{\alpha} - 2\Theta}\right) L(\mathrm{p}_0).$$

Taking inverse Laplace transform, we have

$$L(\mathrm{p}(t)) \leq L(\mathrm{p}_0)E_{\alpha,\,1}(2\Theta t^{\alpha})$$

$$\Longleftrightarrow \|\mathrm{p} - \mathrm{p}^*\| \leq \|\mathrm{p} - \mathrm{p}_0\|E_{\alpha,\,1}(2\Theta t^{\alpha}).$$

By (17), $\Theta < 0$. Clearly, letting $t \to \infty$, we get $\|\mathrm{p} - \mathrm{p}^*\| \to 0$. This implies that, the considered dynamical system (7) at the equilibrium point p^* is asymptotically stable. □

5. The system's discretization

Here, we examine the discretization approach to solve the dynamical system (7) and prove its convergence under certain conditions.

We divide the range $[0,\, T]$ into n sub-intervals of length $h_n = \dfrac{T}{n}$ to use Rothe's method of time discretization. For $j = 1,\, 2,\, \cdots,\, n,$ we denote $t_j^n = jh_n$. At $t = t_j^n$ basically, we approximate the Caputo derivative as,

$$\begin{aligned} {}_C D^{\alpha}\mathrm{p}(t_j^n) &\approx \frac{1}{\Gamma(2-\alpha)} \sum_{i=1}^{j} b_{j-i} \frac{\mathrm{p}_i^n - \mathrm{p}_{i-1}^n}{h_n} h_n^{1-\alpha} \\ &= \sum_{i=1}^{j} (\mathrm{p}_i^n - \mathrm{p}_{i-1}^n) a_i^{j,\,n}, \end{aligned} \tag{20}$$

where $b_k = (k+1)^{1-\alpha} - k^{1-\alpha}$ and $a_i^{j,\,n} = b_{j-i}\dfrac{h_n^{-\alpha}}{\Gamma(2-\alpha)}$. At $t = t_j^n$, we replace equation (7) by the following approximation scheme:

$$\frac{1}{\Gamma(2-\alpha)} \sum_{i=1}^{j} b_{j-i} \left(\frac{\mathrm{p}_i^n - \mathrm{p}_{i-1}^n}{h_n^{\alpha}}\right) = \lambda \left\{ R_{\partial\Psi(\mathrm{p}_j^n)}[J(\mathrm{p}_j^n) - \alpha \mathrm{p}_j^n] - J(\mathrm{p}_j^n) \right\} = Q(\mathrm{p}_j^n). \tag{21}$$

Lemma 2 If all the mappings and conditions are same as in the Theorem 2 and additionally if the following condition holds

$$M = 1 - \{\lambda(\theta\alpha + \mu) + (1+\theta)(1+\theta+\mu)\} > 0,$$

then there exists a constant $\delta_1 > 0$ such that,

$$\|\mathrm{p}_j^n\| < \delta_1, \quad j = 0, 1, 2, \cdots.$$

Proof. For $j = 1$, equation (21) reduces to

$$\frac{1}{\Gamma(2-\alpha)} \frac{\mathrm{p}_1^n - \mathrm{p}_0^n}{h_n^\alpha} = \lambda \left\{ R_{\partial\Psi(\mathrm{p}_1^n)}[J(\mathrm{p}_1^n) - \alpha \mathrm{p}_1^n] - J(\mathrm{p}_1^n) \right\}.$$

Since $\mathrm{p}_0^n = 0,$ we have

$$\frac{1}{\Gamma(2-\alpha)} \|\mathrm{p}_1^n\| \le \lambda h_n^\alpha \|R_{\partial\Psi(\mathrm{p}_1^n)}[J(\mathrm{p}_1^n) - \alpha \mathrm{p}_1^n] - J(\mathrm{p}_1^n)\|. \tag{22}$$

From (16), we have

$$\begin{aligned} \|R_{\partial\Psi(\mathrm{p}_1^n)}[J(\mathrm{p}_1^n) - \alpha \mathrm{p}_1^n] - J(\mathrm{p}_1^n)\| &\le \|R_{\partial\Psi(\mathrm{p}_1^n)}[J(\mathrm{p}_1^n) - \alpha \mathrm{p}_1^n]\| + \|J(\mathrm{p}_1^n)\| \\ &\le \frac{\lambda(\theta\alpha + \mu) + (1+\theta)(1+\theta+\mu)}{\lambda} \|\mathrm{p}_1^n\| \\ &\quad + \left(\frac{1+\lambda+\theta}{\lambda} \right) \|R_{\partial\Psi(0)}(0)\|. \end{aligned} \tag{23}$$

Using (23) in (22), we have

$$\left[\frac{1}{\Gamma(2-\alpha)} - h_n^\alpha \lambda(\theta\alpha + \mu) - h_n^\alpha (1+\theta)(1+\theta+\mu) \right] \|\mathrm{p}_1^n\| \le (1+\lambda+\theta) \|R_{\partial\Psi(0)}(0)\|,$$

which implies that,

$$\|\mathrm{p}_1^n\| \le \frac{\Gamma(2-\alpha)(1+\lambda+\theta)\|R_{\partial\Psi(0)}(0)\|}{1 - \Gamma(2-\alpha) h_n^\alpha [\lambda(\theta\alpha+\mu) + (1+\theta)(1+\theta+\mu)]}.$$

For $j \ge 1,$ applying (20) in (21), we get

$$\sum_{i=1}^{j} (\mathrm{p}_i^n - \mathrm{p}_{i-1}^n) a_i^{j,n} = \lambda \left\{ R_{\partial\Psi(\mathrm{p}_j^n)}[J(\mathrm{p}_j^n) - \alpha \mathrm{p}_j^n] - J(\mathrm{p}_j^n) \right\},$$

this implies that,

$$a_j^{j,n}\mathrm{p}_j^n = \sum_{i=1}^{j-1}(a_{i+1}^{j,n} - a_i^{j,n})\mathrm{p}_i^{j,n} + \lambda\left\{R_{\partial\Psi(\mathrm{p}_j^n)}[J(\mathrm{p}_j^n) - \alpha\mathrm{p}_j^n] - J(\mathrm{p}_j^n)\right\}.$$

Using (16) and the fact that $\|\mathrm{p}_j^n\| \leq a_j^{j,n}\|\mathrm{p}_j^n\|,$ we have

$$\|\mathrm{p}_j^n\| \leq \sum_{i=1}^{j-1}(a_{i+1}^{j,n} - a_i^{j,n})\|\mathrm{p}_i^n\| + (\lambda(\theta\alpha+\mu) + (1+\theta)(1+\theta+\mu))\|\mathrm{p}_j^n\| + (1+\lambda+\theta)\|R_{\partial\Psi(0)}(0)\|$$

$$[1-(\lambda(\theta\alpha+\mu)+(1+\theta)(1+\theta+\mu))]\,\|\mathrm{p}_j^n\| \leq \sum_{i=1}^{j-1}(a_{i+1}^{j,n} - a_i^{j,n})\|\mathrm{p}_i^n\| + (1+\lambda+\theta)\|R_{\partial\Psi(0)}(0)\|.$$

Set $M = 1-(\lambda(\theta\alpha+\mu)+(1+\theta)(1+\theta+\mu))$. By the hypothesis that $M > 0,$ we have

$$\|\mathrm{p}_j^n\| \leq \sum_{i=1}^{j-1}\left(\frac{a_{i+1}^{j,n} - a_i^{j,n}}{M}\right)\|\mathrm{p}_i^n\| + \left(\frac{1+\lambda+\theta}{M}\right)\|R_{\partial\Psi(0)}(0)\|.$$

By Lemma 1, there exists a constant $\delta_1 > 0$ such that,

$$\|\mathrm{p}_j^n\| \leq \delta_1, \qquad \forall \quad j = 0, 1, 2, \ldots.$$

□

Define $\nabla\mathrm{p}_i^n = \dfrac{\mathrm{p}_i^n - \mathrm{p}_{i-1}^n}{h_n},$ for $i = 1, 2, \ldots, n.$

Lemma 3 Suppose that all the mappings and conditions are same as in Lemma 2 then $\|\nabla\mathrm{p}_i^n\| \leq \delta_2$, for $i = 1, 2, \ldots, n$.

Proof. For $j = 1$, (21) can be written as

$$\frac{1}{\Gamma(2-\alpha)}b_0\frac{\mathrm{p}_1^n - \mathrm{p}_0^n}{h_n^\alpha} = \lambda\left\{R_{\partial\Psi(\mathrm{p}_1^n)}[J(\mathrm{p}_1^n) - \alpha\mathrm{p}_1^n] - J(\mathrm{p}_1^n)\right\}.$$

By using (16), we obtain

$$\|\frac{\mathrm{p}_1^n - \mathrm{p}_0^n}{h_n}\| \leq \frac{\Gamma(2-\alpha)}{h_n^{1-\alpha}}[M\|\mathrm{p}_1\| + (\theta+2)\|R_{\partial\psi(0)}(0)\|],$$

that is,

$$\|\frac{\mathrm{p}_1^n - \mathrm{p}_0^n}{h_n}\| \leq \delta_3, \text{ where } \delta_3 = \frac{\Gamma(2-\alpha)}{h_n^{1-\alpha}}[M\|\mathrm{p}_1\| + (\theta+2)\|R_{\partial\psi(0)}(0)\|].$$

For $j \geq 2,$ subtracting (21) for $j-1$ from (21) for j, we have

$$\frac{1}{\Gamma(2-\alpha)}\sum_{i=1}^{j} b_{j-i}\left(\frac{\mathrm{p}_i^n-\mathrm{p}_{i-1}^n}{h_n}\right)h_n^{1-\alpha}-\frac{1}{\Gamma(2-\alpha)}\sum_{i=1}^{j-1} b_{j-i-1}\left(\frac{\mathrm{p}_i^n-\mathrm{p}_{i-1}^n}{h_n}\right)h_n^{1-\alpha}=Q(\mathrm{p}_j^n)-Q(\mathrm{p}_{j-1}^n)$$

$$\frac{1}{\Gamma(2-\alpha)}[b_0(\frac{\mathrm{p}_j-\mathrm{p}_{j-1}}{h_n})+\sum_{i=1}^{j-1}(b_{j-i}-b_{j-i-1}).(\frac{\mathrm{p}_i-\mathrm{p}_{i-1}}{h_n})]=\frac{1}{h_n^{1-\alpha}}[Q(\mathrm{p}_j^n)-Q(\mathrm{p}_{j-1}^n)]$$

$$\frac{\mathrm{p}_j-\mathrm{p}_{j-1}}{h_n}=\sum_{i=1}^{j-1}(b_{j-i-1}-b_{j-i}).(\frac{\mathrm{p}_i-\mathrm{p}_{i-1}}{h_n})]+\frac{\Gamma(2-\alpha)}{h_n^{1-\alpha}}[Q(\mathrm{p}_j^n)-Q(\mathrm{p}_{j-1}^n)],$$

which implies that,

$$\|\frac{\mathrm{p}_j-\mathrm{p}_{j-1}}{h_n}\|\leq\sum_{i=1}^{j-1}(b_{j-i-1}-b_{j-i})\|\frac{\mathrm{p}_i-\mathrm{p}_{i-1}}{h_n}\|+\frac{\Gamma(2-\alpha)}{h_n^{1-\alpha}}[\|Q(\mathrm{p}_j^n)\|+\|Q(\mathrm{p}_{j-1}^n)\|],$$

$$\|\nabla\mathrm{p}_j^n\|\leq\sum_{i=1}^{j-1}(b_{j-i-1}-b_{j-i})\|\nabla\mathrm{p}_i^n\|+\frac{\lambda\Gamma(2-\alpha)}{h_n^{1-\alpha}}\{M||\mathrm{p}_j^n||$$

$$+(\theta+2)\|R_{\partial\Psi(0)}(0)\|+M||\mathrm{p}_{j-1}^n||+(\theta+2)||R_{\partial\Psi(0)}(0)|\}$$

$$\leq\sum_{i=1}^{j-1}(b_{j-i-1}-b_{j-i})\|\nabla\mathrm{p}_i^n\|+\frac{\lambda\Gamma(2-\alpha)}{h_n^{1-\alpha}}\{M(\|\mathrm{p}_j^n\|+\|\mathrm{p}_{j-1}^n\|)$$

$$+2(\theta+2)\|R_{\partial\Psi(0)}(0)\|\},$$

using Lemma 2 and Grownwall's inequality, we have

$$\|\nabla\mathrm{p}_j^n\|\leq\sum_{i=1}^{j-1}(b_{j-i-1}-b_{j-i})\|\nabla\mathrm{p}_i^n\|+\frac{\lambda\Gamma(2-\alpha)}{h_n^{1-\alpha}}\delta_4,$$

that is,

$$\|\nabla\mathrm{p}_j^n\|\leq\delta_5,\quad\text{where }\delta_4=2M[\delta_1+(\theta+2)\|R_{\partial\Psi(0)}(0)\|].$$

□

Lemma 4 ([33]) Assume that all the conditions of Lemma 3 are satisfied. Then

$$\|{}_CD^{\alpha}\mathrm{p}_j^n\|\leq\delta_6,$$

where $\delta_6 > 0$ is a constant.

Proof. For the proof see [33]. $\square$

For a given n, we introduce Rothe's sequence $P^n(t)$ and a piecewise constant interpolation function $\overline{P}$ as follows

$$P^n(t) = \begin{cases} \mathrm{p}_0, & t = 0, \\ \mathrm{p}^n_{j-1} + \frac{t - t^n_{j-1}}{h_n}(\mathrm{p}^n_j - \mathrm{p}_{j-1}), & t \in (t^n_{j-1}, t^n_j], \end{cases}$$

and

$$\overline{P} = \begin{cases} 0, & t = 0, \\ \mathrm{p}^n_j, & t \in (t^n_{j-1}, t^n_j]. \end{cases}$$

Lemma 5 ([33]) There is a subsequence $\{P^{n_k}\}$ of $\{P^n\}$ such that ${}_C D^\alpha P^{n_k} \to {}_C D^\alpha \mathrm{p}$ in $L^2([0, T], \mathbb{H}),$ as $n \to \infty$.

Proof. For the proof see Lemma-10 in [33]. $\square$

Theorem 3 Suppose all the mappings and conditions are same as in Lemmas 2, 3, 4 and 5. Then the dynamical system (7) has the unique strong solution.

Proof. For any $\mathrm{p}^* \in \mathbb{H}$, we have

$$\int_0^t \langle {}_C D^\alpha \mathrm{p}^n(s), \mathrm{p}^* \rangle ds = \int_0^t \langle Q^n(s), \mathrm{p}^* \rangle ds.$$

For the subsequence, we have

$$\int_0^t \langle {}_C D^\alpha \mathrm{p}^{n_k}(s), \mathrm{p}^* \rangle ds = \int_0^t \langle Q(\mathrm{p}_{n_k}(s), \mathrm{p}^* \rangle ds.$$

By applying the bounded convergence theorem, Lemma 2 with $k \to \infty$, we obtain

$$\int_0^t \langle {}_C D^\alpha \mathrm{p}(s), \mathrm{p}^* \rangle ds = \int_0^t \langle Q(\mathrm{p}(s)), \mathrm{p}^* \rangle ds. \tag{24}$$

Since

$$\int_0^t \langle {}_C D^\alpha \mathrm{p}(s), \mathrm{p}^* \rangle ds = \langle I^{1-\alpha} \mathrm{p}(t), \mathrm{p}^* \rangle,$$

then (24) becomes

$$\langle I^{1-\alpha} \mathrm{p}(t), \mathrm{p}^* \rangle = \int_0^t \langle Q(\mathrm{p}(s)), \mathrm{p}^* \rangle ds. \tag{25}$$

Due to the coninuous and uniformly bounded integrand on right hand side for any fixed $\mathrm{p}^* \in \mathbb{H}$, $\langle I^{1-\alpha}\mathrm{p}(t),\, \mathrm{p}^*\rangle$, is continuously differentiable, that is,

$$ {}_C D^{\alpha}\mathrm{p}(t) = Q(\mathrm{p}(t)), $$

therefore, $\mathrm{p}(t)$ is the strong solution of the dynamical system (7).

Now it only remains to prove that $\mathrm{p}(t)$ is unique. For this let $\mathrm{p}_1(t)$ and $\mathrm{p}_2(t)$ be two solutions of the dynamical system (7), then we have

$$ \mathrm{p}_1(t) = \mathrm{p}_0 + \frac{1}{\Gamma(\alpha)}\int_0^t Q(\mathrm{p}_1(s))ds, $$

and

$$ \mathrm{p}_2(t) = \mathrm{p}_0 + \frac{1}{\Gamma(\alpha)}\int_0^t Q(\mathrm{p}_2(s))ds, $$

which implies that

$$ u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (Q(\mathrm{p}_1(s)) - Q(\mathrm{p}_2(s))ds, $$

where $u(t) = \mathrm{p}_1(t) - \mathrm{p}_2(t)$. Using Lipschitz continuity of $Q(\mathrm{p}(t))$, we have

$$ \|u(t)\|^2 \leq \frac{K^2}{\Gamma(\alpha)}\int_0^t \|u(s)\|_s^2 ds, $$

Gronwall's inequality gives us

$$ \|u\|_t = 0, \quad t \in\ [0,\, T]. $$

Hence $u(t) = 0$. This completes the proof. □

Example 1 Let ψ: $\mathbb{R}^3 \to \mathbb{R} \cup \{+\infty\}$ is defined by $\psi(\mathrm{p}) = \|\mathrm{p}\|_1$. Then subdifferential operator $\partial\psi$ is defined by

$$ \begin{cases} \sum_{i=1}^3 \mathrm{sgn}(\mathrm{p}_i)e_i, & \text{if } \mathrm{p} \neq 0, \\ \mathscr{K}, & \text{if } \mathrm{p} = 0, \end{cases} $$

where $\mathrm{sgn}(\mathrm{p}_i)$ is the sign of the i-th slot of p, and e_1, e_2, e_3 are the standard basis in $\mathbb{R}^3$, and $\mathscr{K}$ is the set of all non-negative convex combination vectors forming by the standard basis vectors.

Then, the corresponding resolvent operator and Yosida approximation operators are as follows:

$$R_{\partial\psi}(\mathrm{p}) = \left(\frac{|\mathrm{p}_1|}{1+2\rho}\right)\mathrm{sgn}(\mathrm{p}_1)e_1 + \left(\frac{|\mathrm{p}_2|}{1+2\rho}\right)\mathrm{sgn}(\mathrm{p}_2)e_2 + \left(\frac{|\mathrm{p}_3|}{1+2\rho}\right)\mathrm{sgn}(\mathrm{p}_3)e_3, \tag{26}$$

$$J(\mathrm{p}) = \left(\frac{\mathrm{p}_1 e_1}{\rho+2|\mathrm{p}_1|}\right) + \left(\frac{\mathrm{p}_2 e_2}{\rho+2|\mathrm{p}_2|}\right) + \left(\frac{\mathrm{p}_3 e_3}{\rho+2|\mathrm{p}_3|}\right), \tag{27}$$

for $\mathrm{p} = (\mathrm{p}_1, \mathrm{p}_2, \mathrm{p}_3) \in \mathbb{R}^3$ and a positive constant $\rho > 0$.

Then, the dynamical system (7) has unique equilibrium point $(0, 0, 0)$, that is YQIVI has unique solution $(0, 0, 0)$. By theorem 2, at $(0, 0, 0)$ the dynamical system is asymptotically and exponentially stable. The Figure 2 shows global convergence to the optimal solution $(0, 0, 0)$ of the trajectories of (5) with initial point at $t_0 = 0$.

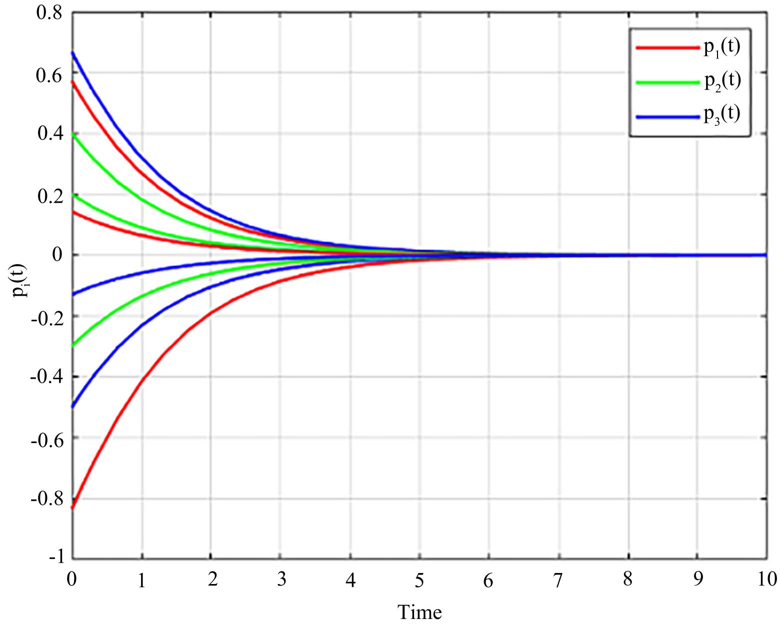

Figure 2. The dynamical system (7)'s transient behavior at three distinct points $\mathrm{p}_1 = [2/3.5,\ 1/5,\ 3/4.5]$, $\mathrm{p}_2 = [-1/1.2,\ 0.4,\ -0.5]$ and $\mathrm{p}_3 = [1/7,\ -0.3,\ -0.13]$ of $\mathbb{R}^3$

6. Conclusion and future directions

This article introduces a Yosida inverse variational inequality problem and for its solution a dynamical system is developed with detorsion simple single-layer structures and cheap implementation complexity. We established its global convergence, asymptotic stability, and exponentiality using the functional differential equation theory and the Lyapunov function. Under some simple assumptions, we determine the existence of a unique solution for our new dynamical system by using the Rothe's approach. Finally, we have given one numerical example to demonstrate the dynamical system's efficiency in solving YQIVI (5). It is noted that some of the stability conditions for the Lipschitz constant K in this paper can be difficult and expensive to achieve in practical implementations. Therefore, additional research is needed to obtain more endurable conditions on the stability result of the proposed system.

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Conflict of interest

The authors declare no competing financial interest.

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