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Research Article

On Linearization Coefficients of Shifted Jacobi Polynomials

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Abstract: This study describes how to represent $x^m p_\ell(x)$, $p_\ell(x)q_m(x)$ and $\prod_{j=1}^{s} p_{\ell_j}(x)$ in terms of shifted Jacobi polynomials (SJPs) using computational methods, where $p_\ell(x)$ and $q_m(x)$ are polynomials of degrees ℓ and m, respectively. The suggested problems are discussed when $p_\ell(x)$ and $q_m(x)$ are generalized Laguerre, Hermite, and SJPs. In particular, these expansions are presented for the cases of shifted ultraspherical, shifted Legendre, and shifted Chebyshev polynomials of the first and second kinds.

***Keywords*:** classical orthogonal polynomials, linearization and connection coefficients, symbolic computation, generalized hypergeometric functions

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1. Introduction

Many fields of applied sciences rely heavily on special functions. Special functions have a crucial role in numerous domains, including quantum mechanics, numerical analysis, and approximation theory; see, for instance, [1–4]. Among these functions are the several kinds of orthogonal polynomials (OPs). These polynomials provide the backbone for solving solutions for distinct issues connected to diverse domains such as science, engineering, and mathematics; see, for example, [5, 6]. We also note the widespread application of orthogonal polynomials in other fields like signal processing, probability theory, and statistics; see [7–10]. Additionally, these polynomials are utilized in approximating integrals using techniques like Gaussian quadrature [11].

Research on the various types of OPs, both theoretical and practical, has attracted the attention of several authors. For example, Ahmed in [12–14] has studied some classical discrete OPs. Another study on Bessel polynomials was given in [15]. There are other studies regarding the classical continuous OPs; see, for example, [16, 17]. Some formulas between orthogonal polynomials and Fibonacci polynomials are developed in [18].

A large number of contributions were devoted to the utilization of the OPs in different applications. For example, the authors of [19, 20] used an operational technique for handling some differential equations (DEs). The authors of [21] applied a Galerkin algorithm to handle some partial DEs. In [22], the author used a shifted Jacobi operational matrix of derivatives to solve some mult-term fractional differential equations. Other fractional DEs were treated in [23] using

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Chebyshev polynomials (CPs). Some generalized CPs were employed in [24] to solve multi-term fractional DES. Hermite polynomials were used in [25] to treat some optimal control problems. Some fractional-integro-DEs were handled via Laguerre polynomials in [26]. A finite OPs class was utilized in [27] to solve some fractional DEs.

One of the most influential families of OPs is the family of Jacobi polynomials (JPs). These polynomials were widely used in different branches of mathematics. They are solutions to particular second-order DEs. They involve two parameters, allowing for the generation of some well-known polynomials. Applications of JPs and their special classes of polynomials have been the subject of several publications. For example, some combinations of Legendre polynomials were utilized in [28] to treat some DEs. CPs were used in [29] to treat some singular DEs. Some generalized CPs were used in [30] to solve other DEs. In [31–33], Jacobi polynomials were used to solve some fractional DEs. In [34], the authors treated some partial DEs.

Addressing the linearization problems for various OPs is crucial. Their importance comes from their appearance in physics and quantum chemistry applications. For example, they are useful in describing quantum-mechanical systems' physical and chemical properties, [35]. They are also required to figure out the logarithmic potentials of OPs when figuring out a quantum system's position and momentum information entropies; see [36]. Furthermore, these formulas help in treating some non-linear DEs; see, for example [37].

Generalized hypergeometric functions (GHFs) have vital roles within special functions. These functions also crop up in other areas, such as combinatorics, probability theory, and mathematical physics. These functions are used to solve several significant issues, including connection, duplication, and linearization. The authors in [17, 38, 39], solved many linearization and connection problems. The connection and linearization coefficients often include GHFs functions that can be reduced in particular cases.

For the polynomials $p_\ell(x)$ and $q_m(x)$ and the set of SJPs: $\mathscr{P}_{M,\,i}^{(\nu,\,\theta)}(x),\ i \geq 0$, we will address the more general linearization problem (LP):

$$p_\ell(x)\, q_m(x) = \sum_{i=0}^{\ell+m} c_i(\ell,\, m)\, \mathscr{P}_{M,\,i}^{(\nu,\,\theta)}(x), \tag{1}$$

where the linearization coefficients $c_i(\ell,\, m)$ to be determined. In addition, $p_\ell(x)$ and $q_m(x) \in T =$ {Jacobi, shifted Jacobi, generalized Laguerre, and Hermite polynomials}. The presented approach is built on employing the closed formulas of $D^q p_\ell(0)$ and $D^q q_m(0)$. Whenever feasible, we create closed representations for these coefficients using some algebraic computations. The LPs have been presented in many investigations using different techniques [40–46]. Furthermore, analyzing the positivity requirements of the coefficients $c_i(\ell,m)$ is essential to solving the linearization issue. [47, Lecture 5 and 6].

In the current paper, we consider the following generalization of the problem 1:

$$\prod_{j=1}^{s} p_{\ell_j}(x) = \sum_{i=0}^{L_s} c_i(\ell_1,\, \ldots,\, \ell_s)\ \mathscr{P}_{M,\,i}^{(\nu,\,\theta)}(x), L_s = \textstyle\sum_{j=0}^{s} \ell_j, \tag{2}$$

will be discussed. We will show that the linearization coefficients $c_i(\ell_1,\, \ldots,\, \ell_s)$ have forms that include GHFs of N variables[48]. Several new closed forms will be obtained when $p_{\ell_j}(x) \in T$. Various applied problems use the products of several classical OPs, see [49–51]. For instance [48, 52, 53], the problem

$$\prod_{j=1}^{s} \mathscr{L}_{\ell_j}^{(\nu_j)}(x) = \sum_{i=0}^{L_s} c_i(\ell_1,\, \ldots,\, \ell_s)\, \mathscr{L}_i^{(\nu)}(x), \tag{3}$$

is gaining a lot of weight in nuclear and atomic shell theories.

This article is structured as follows: Section 2 displays some fundamentals of the SJPs and some properties of the computational tools used. In Sections 3-5, we show and prove three theorems that give new formulas for expanding $x^m p_\ell(x)$, $p_\ell(x)q_m(x)$ and $\prod_{j=1}^{s} p_{\ell_j}(x)$ in terms of SJPs. We calculate the relevant expansions for these theorems when $p_\ell(x)$, $q_m(x)$, and $p_{\ell_j}(x) \in T$. Ultimately, in Section 6, we provide a comprehensive overview and conclusive observations.

2. Some essentials of JPs

It is well-known that JPs $\{\mathscr{P}_\ell^{(\nu,\,\theta)}(t)\}_{\ell=0}^{\infty}$ for $\nu,\ \theta > -1$, satisfy the orthogonality relation [54, pp.300-301]

$$\int_{-1}^{1} (1-t)^{\nu}(1+t)^{\theta}\mathscr{P}_\ell^{(\nu,\,\theta)}(t)\mathscr{P}_m^{(\nu,\,\theta)}(t)\,dt = 2^{\lambda} h_\ell\, \delta_{\ell,\,m}, \tag{4}$$

where

$$h_\ell = \frac{\Gamma(\ell+\nu+1)\Gamma(\ell+\theta+1)}{\ell!(2\ell+\lambda)\Gamma(\ell+\lambda)},\ \lambda = \nu+\theta+1.$$

In addition, they may be constructed using the recurrence relation:

$$\begin{aligned}\mathscr{P}_i^{(\nu,\,\theta)}(t) =& \frac{(\lambda+2i-2)\{\nu^2-\theta^2+t\,(\lambda+2i-1)(\lambda+2i-3)\}}{2i(\lambda+i-1)(\lambda+2i-3)}\mathscr{P}_{i-1}^{(\nu,\,\theta)}(t)\\ &-\frac{(\nu+i-1)(\theta+i-1)(\lambda+2i-1)}{i(\lambda+i-1)(\lambda+2i-3)}\mathscr{P}_{i-2}^{(\nu,\,\theta)}(t), \quad i = 2,\ 3,\ \ldots,\end{aligned} \tag{5}$$

with

$$\mathscr{P}_0^{(\nu,\,\theta)}(t) = 1,\ \mathscr{P}_1^{(\nu,\,\theta)}(t) = \frac{1}{2}(\lambda+1)t + \frac{1}{2}(\nu-\theta).$$

By changing the variable: $t = \dfrac{2x}{M}-1$, these polynomials may be defined on the interval $[0,\,M]$. The new polynomials is the so-called SJPs $\mathscr{P}_i^{(\nu,\,\theta)}(\dfrac{2x}{M}-1)$ and be denoted by $\mathscr{P}_{M,\,i}^{(\nu,\,\theta)}(x)$. In view of the two relations (4) and (5), the polynomials $\mathscr{P}_{M,\,i}^{(\nu,\,\theta)}(x)$ satisfy the following orthogonality relation:

$$\int_{0}^{M} x^{\theta}(M-x)^{\nu}\mathscr{P}_{M,\,i}^{(\nu,\,\theta)}(x)\mathscr{P}_{M,\,j}^{(\nu,\,\theta)}(x)\,dx = M^{\lambda} h_i\, \delta_{ij},\ \nu,\ \theta > -1, \tag{6}$$

and can be generated from

$$\mathscr{P}_{M,\,i}^{(\nu,\,\theta)}(x) = \frac{(\lambda+2i-2)\{\nu^2-\theta^2+(\frac{2x}{M}-1)\,(\lambda+2i-1)(\lambda+2i-3)\}}{2i(\lambda+i-1)(\lambda+2i-3)}\mathscr{P}_{M,\,i-1}^{(\nu,\,\theta)}(x)$$
$$-\frac{(\nu+i-1)(\theta+i-1)(\lambda+2i-1)}{i(\lambda+i-1)(\lambda+2i-3)}\mathscr{P}_{M,\,i-2}^{(\nu,\,\theta)}(x), \quad i \geq 2, \tag{7}$$

with the starting values:

$$\mathscr{P}_{M,\,0}^{(\nu,\,\theta)}(x) = 1,\ \mathscr{P}_{M,\,1}^{(\nu,\,\theta)}(x) = \frac{1}{2}(\lambda+1)\left(\frac{2x}{M}-1\right)+\frac{1}{2}(\nu-\theta).$$

$\mathscr{P}_{M,\,i}^{(\nu,\,\theta)}(x)$ has the analytical form

$$\mathscr{P}_{M,\,i}^{(\nu,\,\theta)}(x) = (-1)^i\frac{(\theta+1)_i}{i!}\sum_{k=0}^{i}\frac{(-i)_k(i+\lambda)_k}{(\theta+1)_k}\frac{M^{-k}}{k!}x^k,$$

which may be represented as:

$$\mathscr{P}_{M,\,i}^{(\nu,\,\theta)}(x) = (-1)^i\frac{(\theta+1)_i}{i!}\,{}_2F_1\left[\begin{array}{c|c} -i,\ i+\lambda & \frac{x}{M} \\ \theta+1 & \end{array}\right], \tag{8}$$

where $(d)_k$ denotes the Pochhammer's symbol and ${}_2F_1$ is the known hypergeometic function. It is not hard to observe that:

$$D^k\mathscr{P}_{M,i}^{(\nu,\theta)}(0) = (-1)^i\frac{(\theta+1)_i(\lambda+i)_k(-i)_k}{i!\,(\theta+1)_k}M^{-k},\ i \geq k,\ \ k = 0, 1, 2, \ldots. \tag{9}$$

Lemma 1 Assume we have a polynomial $Q_\ell(x)$ of degree ℓ with the expansion

$$Q_\ell(x) = \sum_{i=0}^{\ell}a_i(\ell)\ \mathscr{P}_{M,\,i}^{(\nu,\,\theta)}(x), \tag{10}$$

then $a_i(\ell),\ i = 0,\ 1,\ \ldots,\ \ell$, meet the following system:

$$a_i(\ell) = \frac{M^i}{(i+\lambda)_i}Q_\ell^{(i)}(0) - \sum_{k=1}^{\ell-i}(-1)^k\frac{(\theta+i+1)_k(\lambda+2i)_k}{k!\,(\lambda+i)_k}a_{k+i}(\ell),\ i = \ell-1,\ \ldots,\ 1,\ 0,$$
$$a_\ell(\ell) = \frac{M^\ell}{(\ell+\lambda)_\ell}Q_\ell^{(\ell)}(0), \tag{11}$$

and they may be calculated using the following form

$$a_i(\ell) = \sum_{r=0}^{\ell-i} \Omega_r(i,\ \nu,\ \theta,\ M)\, Q_\ell^{(r+i)}(0),\ \ i = \ell,\ \ell-1,\ \ldots,\ 0, \tag{12}$$

where

$$\Omega_r(i,\ \nu,\ \theta,\ M) = \frac{M^{i+r}(2i+\lambda)(\theta+1)_{i+r}(\lambda)_i}{r!\,(\theta+1)_i(\lambda)_{2i+r+1}}.$$

Proof. By using (9) and (10), we have for $i = 0,\ 1,\ \ldots,\ \ell$,

$$Q_\ell^{(i)}(0) = \left[D^i Q_\ell(x)\right]_{x=0} = \sum_{k=i}^{\ell} a_k(\ell)\, D^i \mathscr{P}_{M,\ k}^{(\nu,\ \theta)}(0) \tag{13}$$

$$= \sum_{k=0}^{\ell-i} (-1)^k \frac{(\theta+i+1)_k(\lambda+k+i)_i}{k!} M^{-i} a_{k+i}(\ell), \tag{14}$$

which constitutes a triangular system of dimension $(\ell+1)$ whose unknowns are $a_i(\ell),\ (0 \le i \le \ell)$, and accordingly, the system (14) can be written in the form (11). Now, we have

$$\sum_{k=0}^{\ell-i} (-1)^k \frac{(\theta+i+1)_k(\lambda+k+i)_i}{k!} M^{-i} a_{k+i}(\ell) = \sum_{k=0}^{\ell-i} (-1)^k \frac{(\theta+i+1)_k(\lambda+k+i)_i}{k!} M^{-i}$$

$$\times \sum_{r=0}^{\ell-i-k} \frac{M^{i+k+r}(2i+2k+\lambda)(\theta+1)_{i+k+r}(\lambda)_{i+k}}{r!\,(\theta+1)_{i+k}(\lambda)_{2i+2k+r+1}} Q_\ell^{(r+i+k)}(0),$$

then, the rearrangement of terms turns the last formula into

$$\sum_{k=0}^{\ell-i} (-1)^k \frac{(\theta+i+1)_k(\lambda+k+i)_i}{k!} M^{-i} a_{k+i}(\ell) = \sum_{r=0}^{\ell-i} \frac{M^r}{r!}(\theta+i+1)_r A_r(i) Q_\ell^{(r+i)}(0), \tag{15}$$

where

$$A_r(i) = \sum_{k=0}^{r} \frac{(-r)_k(2i+2k+\lambda)}{k!(2i+k+\lambda)_{r+1}}. \tag{16}$$

Zeilberger's algorithm [55], specifically, by using the 'sumrecursion command' in Maple software, it can be shown that $A_r(i)$ for $r \ge 1$ meets the next recursive formula:

$$(r+1)(2i+2r+\lambda+1)(2i+2r+\lambda+2)A_{r+1}(i)+r(2i+r+\lambda+1)A_r(i)=0, \tag{17}$$

governing by: $A_1(i)=0$. This recurrence relation has the exact solution: $A_r(i)=\delta_{r,\,0}$. Then formula (15) takes the form

$$\sum_{k=0}^{\ell-i}(-1)^k\frac{(\theta+i+1)_k(\lambda+k+i)_i}{k!}M^{-i}a_{k+i}(\ell)=Q_\ell^{(i)}(0),$$

which indicates that the solution of the system (11) takes the form (12). □

The next proposition provides some tools that are employed in the following sections.

Proposition 1 [56, p.467] The following relationships are satisfied:

$$(i)\ (d)_{m+r}=(d)_\ell\,(d+m)_r,\qquad (ii)\ (d)_{m-r}=(-1)^r\frac{(d)_m}{(1-d-m)_r}, \tag{18}$$

$$(iii)\ (d)_{2r}=2^{2r}\left(\frac{d}{2}\right)_r\left(\frac{d+1}{2}\right)_r,\qquad (iv)\ (-m)_r=(-1)^r\frac{m!}{(m-r)!}. \tag{19}$$

3. Connection problem: relationship between $x^m q_\ell(x)$ and a sum of SJPs

In this section, our focus is on determining the explicit formula of the coefficients $a_i(\ell,\,m)$ in the expansion:

$$x^m q_\ell(x)=\sum_{i=0}^{\ell+m}a_i(\ell,\,m)\mathscr{P}_{M,\,i}^{(\nu,\,\theta)}(x). \tag{20}$$

In this regard, the following theorem is given.

Theorem 1 The expansion coefficients $a_i(\ell,\,m),\ i=0,\,1,\,\ldots,\,\ell+m$, can be expressed as

$$a_i(\ell,\,m)=M^m\frac{i!(2i+\lambda)(\theta+1)_m}{(\theta+1)_i(\lambda+i)_{m+1}}\sum_{r=0}^{\ell}\binom{r+m}{i}\frac{M^r(\theta+m+1)_r}{r!(\lambda+i+m+1)_r}q_\ell^{(r)}(0), \tag{21}$$

that can be expressed as

$$\begin{aligned}a_i(\ell,\,m)=m!M^m&\frac{(2i+\lambda)(\theta+1)_m\Gamma(\lambda+i)}{(\theta+1)_i}\\ &\times\sum_{r=0}^{\ell}\frac{M^r(m+1)_r(\theta+m+1)_r}{r!\,(m-i+r)!\,\Gamma(\lambda+i+m+1+r)}q_\ell^{(r)}(0).\end{aligned} \tag{22}$$

Proof. Let $Q_{\ell+m}(x)=x^m q_\ell(x)$, then applying Lemma 1 lead to

$$a_i(\ell,\ m) = \sum_{r=0}^{\ell+m-i} \Omega_r(i,\ \nu,\ \theta,\ M)\, Q_{\ell+m}^{(r+i)}(0). \tag{23}$$

Applying the Leibniz rule, it is easy to obtain

$$Q_{\ell+m}^{(j)}(0) = \begin{cases} m!\binom{j}{m} q_\ell^{(j-m)}(0), & j \geq m, \\ 0, & 0 \leq j \leq m-1. \end{cases} \tag{24}$$

Substitution of (24) into (23)-after some calculations- yields (21). Employing Proposition Theorem 1 leads to (22).□
As an application of Theorem 1, the expression of coefficients $a_i(\ell,\ m)$ when $q_\ell(x) = \mathscr{P}_{M,\ \ell}^{(\gamma,\ \delta)}(x)$ can be obtained.
Corollary 1 In the expansion

$$x^m \mathscr{P}_{M,\ \ell}^{(\gamma,\ \delta)}(x) = \sum_{i=0}^{\ell+m} a_i(\ell,\ m) \mathscr{P}_{M,\ i}^{(\nu,\ \theta)}(x), \tag{25}$$

the coefficients $a_i(\ell,\ m)$ can be expressed as

$$\begin{aligned} a_i(\ell,\ m) = (-1)^\ell m! M^m \frac{(2i+\lambda)(\theta+1)_m \Gamma(\lambda+i)\Gamma(\delta+\ell+1)}{\ell!\,(\theta+1)_i} \\ \times \sum_{r=0}^{\ell} \frac{(m+1)_r(\theta+m+1)_r(\mu+\ell)_r(-\ell)_r}{r!(m-i+r)!\Gamma(\lambda+i+m+r+1)\Gamma(\delta+r+1)}, \end{aligned} \tag{26}$$

which has the following alternative form

$$\begin{aligned} a_i(\ell,\ m) = (-1)^\ell m! M^m \frac{(2i+\lambda)(\theta+1)_m \Gamma(\lambda+i)\Gamma(\delta+\ell+1)}{\ell!\,(\theta+1)_i} \\ \times\ {}_4\bar{F}_3 \left[\begin{array}{c} -\ell,\ m+1,\ m+\theta+1,\ \mu+\ell \\ 1-i+m,\ \lambda+i+m+1,\ \delta+1 \end{array} \middle| 1 \right], \end{aligned} \tag{27}$$

where $\mu = \gamma+\delta+1$.

Proof. Substitution of (9) into (22) gives (26) which can be represented as in (27). □

Note 1 It is noted that the using of series formula of regularized hypergeometric function ${}_\mathrm{p}\bar{F}_\mathrm{q}$ is useful in computations, by using Mathematica, rather than the using of usual series formula ${}_\mathrm{p}F_\mathrm{q}$, where it is defined by

$$
{}_{\mathfrak{p}}\bar{F}_{\mathfrak{q}}\left[\begin{array}{l} a_1,\, a_2,\, \ldots,\, a_{\mathfrak{p}} \\ b_1,\, b_2,\, \ldots,\, b_{\mathfrak{q}} \end{array}\middle| x\right] = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{\mathfrak{p}} (a_i)_k}{\prod_{i=1}^{\mathfrak{q}} \Gamma(b_i+k)} \frac{x^k}{k!}. \tag{28}
$$

In particular, for the special case $\ell = 0$, formula (25) becomes

$$
x^m = m!M^m\,(\theta+1)_m \sum_{i=0}^{m} \frac{(2i+\lambda)}{(m-i)!(\lambda+i)_{m+1}(\theta+1)_i} \mathscr{P}_{M,\,i}^{(\nu,\,\theta)}(x). \tag{29}
$$

Also, for the special case $m = 0$, formula (25) becomes

$$
\mathscr{P}_{M,\,\ell}^{(\gamma,\,\delta)}(x) = \sum_{i=0}^{\ell} a_i(\ell,\,0)\mathscr{P}_{M,\,i}^{(\nu,\,\theta)}(x), \tag{30}
$$

where $a_i(\ell,\,0)$ can be expressed as

$$
a_i(\ell,\,0) = (-1)^{\ell} \frac{(2i+\lambda)(\delta+1)_{\ell}(-\ell)_i(\mu+\ell)_i}{\ell!\,(\lambda+i)_{i+1}(\delta+1)_i} \sum_{r=0}^{\ell-i} \frac{(\theta+i+1)_r(\mu+i+\ell)_r(-\ell+i)_r}{r!(\lambda+2i+1)_r(\delta+i+1)_r}, \tag{31}
$$

by using formula (26), Proposition 1 and some rather manipulation. This formula becomes

$$
a_i(\ell,\,0) = (-1)^{\ell} \frac{(2i+\lambda)(\delta+1)_{\ell}(-\ell)_i(\mu+\ell)_i}{\ell!\,(\lambda+i)_{i+1}(\delta+1)_i}\, {}_3F_2\left[\begin{array}{l} -\ell+i, 1, \theta+i+1, \mu+i+\ell \\ \lambda+2i+1, \delta+i+1 \end{array}\middle| 1\right]. \tag{32}
$$

Using the known relation between the Jacobi and ultraspherical polynomials

$$
\mathscr{C}_{M,\,\ell}^{(\gamma)}(t) = \frac{\ell!}{(\gamma+1/2)_{\ell}} \mathscr{P}_{M,\,\ell}^{(\gamma-1/2,\,\gamma-1/2)}(t),\ \gamma > -1/2, \tag{33}
$$

enable us to prove Corollary 2 as an immediate result of Corollary 1. The main advantage of relation (33) is that the shifted Legendre polynomials $\mathscr{P}_{M,\,\ell}(x)$, and the shifted Chebyshev polynomials of the first and second kind $\mathscr{T}_{M,\,\ell}(x)$ and $\mathscr{U}_{M,\,\ell}(x)$ can be obtained as direct special cases of $\mathscr{C}_{M,\,\ell}^{(\gamma)}(x)$. Explicitly, the following relations are valid:

$$
\begin{aligned}
\mathscr{P}_{M,\,\ell}(t) &= \mathscr{C}_{M,\,\ell}^{(1/2)}(t), \\
\mathscr{T}_{M,\,\ell}(t) &= \mathscr{C}_{M,\,\ell}^{(0)}(t), \\
\mathscr{U}_{M,\,\ell}(t) &= (\ell+1)\mathscr{C}_{M,\,\ell}^{(1)}(t).
\end{aligned} \tag{34}
$$

Corollary 2 In the expansion

$$x^m \mathscr{C}^{(\gamma)}_{M,\,\ell}(x) = \sum_{i=0}^{\ell+m} a_i(\ell,\, m)\mathscr{C}^{(\nu)}_{M,\,i}(x), \tag{35}$$

the coefficients $a_i(\ell,\, m)$ can be expressed as

$$\begin{aligned} a_i(\ell,\, m) = (-1)^{\ell}\frac{m!}{i!} M^m \Gamma(\gamma+1/2)(\nu+1/2)_m(2i+2\nu)\Gamma(2\nu+i) \\ \times\, {}_4\bar{F}_3\left[\begin{array}{l} -\ell,\, m+1,\, m+\nu+1/2,\, 2\gamma+n \\ 1-i+m,\, 2\nu+i+m+1,\, \gamma+1/2 \end{array}\middle|\, 1\right]. \end{aligned} \tag{36}$$

In case of $m=0$, formula (35) gives the connection problem

$$\mathscr{C}^{(\gamma)}_{M,\,\ell}(x) = \sum_{i=0}^{\ell} a_i(\ell,\, 0)\mathscr{C}^{(\nu)}_{M,\,i}(x), \tag{37}$$

where the coefficients $a_i(\ell,\, 0)$ can be written in the form

$$a_i(\ell,\, 0) = (-1)^{\ell}\frac{(-\ell)_i(2\gamma+\ell)_i(\nu+1/2)_i}{i!(2\nu+i)_i(\gamma+1/2)_i}\, {}_3F_2\left[\begin{array}{l} -\ell+i, \nu+i+1/2, 2\gamma+i+\ell \\ 2\nu+2i+1, \gamma+i+1/2 \end{array}\middle|\, 1\right], \tag{38}$$

and by using Watson formula [57]

$${}_3F_2\left[\begin{array}{l} -k,\, k+2\mu+2\nu-1,\, \mu \\ 2\mu,\, \mu+\nu \end{array}\middle|\, 1\right] = \begin{cases} \dfrac{k!\Gamma(\mu+\frac{k}{2})\Gamma(\nu+\frac{k}{2})\Gamma(2\mu)\Gamma(\mu+\nu)}{(\frac{k}{2})!\Gamma(\mu+\nu+\frac{k}{2})\Gamma(2\mu+k)\Gamma(\mu)\Gamma(\nu)}, & k\,\text{even}, \\ 0, & k\,\text{odd}, \end{cases}$$

with $k=\ell-i,\ \nu=\gamma-\nu$ and $\mu=\nu+i+1/2$, it is not difficult to show that formula (37) takes the form

$$\mathscr{C}^{(\gamma)}_{M,\,\ell}(x) = \sum_{i=0}^{[\ell/2]} \frac{\ell!(\nu+\ell-2i)(\gamma-\nu)_i(\gamma)_{\ell-s}(2\nu)_{\ell-2i}}{(2\gamma)_\ell\ i!\,(\nu)_{\ell-s+1}(\ell-2i)!}\mathscr{C}^{(\nu)}_{M,\,\ell-2i}(x). \tag{39}$$

Again, by the application of Theorem 1, the expression of coefficients $a_i(\ell,\, m)$ when $q_\ell(x)=\mathscr{L}^{(\gamma)}_\ell(x)$ can be obtained in terms of hypergeometric function ${}_3\bar{F}_2(M)$ as in the following corollary.

Corollary 3 In the expansion

$$x^m \mathscr{L}^{(\gamma)}_\ell(x) = \sum_{i=0}^{\ell+m} a_i(\ell,\, m)\mathscr{P}^{(\nu,\,\theta)}_{M,\,i}(x), \tag{40}$$

where the coefficients $a_i(\ell,\ m)$ can be expressed as

$$a_i(\ell,\ m) = m!\,M^m \frac{(2i+\lambda)(\theta+1)_m \Gamma(\lambda+i)\Gamma(\gamma+\ell+1)}{\ell!(\theta+1)_i} \times {}_3\bar{F}_2\left[\begin{matrix} -\ell,\ m+1,\ m+\theta+1 \\ 1-i+m,\ \lambda+i+m+1,\ \gamma+1 \end{matrix}\middle| M\right]. \tag{41}$$

Proof. The direct substitution of formula $D^r \mathscr{L}_\ell^{(\nu)}(0)$ into (22) gives (41). $\square$

Remark 1 Using the formulae of $D^i q_\ell(0)$, for $q_\ell(x) = \mathscr{H}_\ell(x),\ \mathscr{P}_\ell^{(\nu,\ \theta)}(x),\ \mathscr{C}_\ell^{(\alpha)}(x),\ \mathscr{P}_\ell(x),\ \mathscr{T}_\ell(x), \mathscr{U}_\ell(x)$ listed in Table A1 and applying Theorem 1, the corresponding formulae of the expansions coefficients $a_i(\ell,\ m)$ can be computed in similar way.

Remark 2 For the special case $m = 0$, the connection problem

$$q_\ell(x) = \sum_{i=0}^{\ell} a_i(\ell,\ 0) \mathscr{P}_{M,\ i}^{(\nu,\ \theta)}(x), \tag{42}$$

is a direct consequence of Theorem 1.

4. Linearization problem: relationship between $q_m(x)\,p_\ell(x)$ and a sum of SJPs

In this part, we look at how to calculate the coefficients $c_i(\ell,\ m)$ in the expansion,

$$p_\ell(x)\,q_m(x) = \sum_{i=0}^{\ell+m} c_i(\ell,\ m)\, \mathscr{P}_{M,\ i}^{(\nu,\ \theta)}(x), \tag{43}$$

where $q_m(x)$ and $p_\ell(x)$ are two OPs of degrees m and ℓ, respectively.

Theorem 2 The coefficients $c_i(\ell,\ m),\ i = 0,\ 1,\ \ldots,\ \ell+m$, in (43) can be represented as

$$c_i(\ell,\ m) = \frac{i!(2i+\lambda)}{(\theta+1)_i(\lambda+i)} \sum_{k=0}^{m} \sum_{r=0}^{\ell} \binom{r+k}{i} \frac{M^{r+k}(\theta+1)_{r+k}}{r!k!(\lambda+i+1)_{r+k}} q_m^{(k)}(0) p_\ell^{(r)}(0), \tag{44}$$

or it can be written in the two forms

$$c_i(\ell,\ m) = \sum_{k=\max(0,\ i-\ell)}^{m} \frac{q_m^{(k)}(0)}{k!} a_i(\ell,\ k) = \sum_{k=0}^{m} \frac{q_m^{(k)}(0)}{k!} a_i(\ell,\ k), \tag{45}$$

and

$$c_i(\ell,\, m) = \sum_{r=\max(0,\, i-m)}^{\ell} \frac{p_\ell^{(r)}(0)}{r!}\, a_i(m,\, r) = \sum_{r=0}^{\ell} \frac{p_\ell^{(r)}(0)}{r!}\, a_i(m,\, r). \tag{46}$$

Proof. We have

$$q_m(x) = \sum_{k=0}^{m} \frac{q_\ell^{(k)}(0)}{k!} x^k,$$

and therefore, we can write

$$q_m(x)\, p_\ell(x) = \sum_{k=0}^{m} \frac{q_\ell^{(k)}(0)}{k!}\, x^k p_\ell(x), \tag{47}$$

then using Theorem 1 leads to (44).

Now, we need to prove (45). Employing formula (20) leads one to express (47) as follows:

$$q_m(x)\, p_\ell(x) = \sum_{k=0}^{m} \sum_{i=0}^{\ell+k} \frac{q_\ell^{(k)}(0)}{k!} a_i(\ell,\, k)\, \mathscr{P}_{M,\, i}^{(\nu,\, \theta)}(x). \tag{48}$$

Then, expanding and collecting similar terms gives

$$q_m(x)\, p_\ell(x) = \sum_{i=0}^{\ell+m} \left[\sum_{k=\max(0,\, i-\ell)}^{m} \frac{q_m^{(k)}(0)}{k!} a_i(\ell,\, k) \right] \mathscr{P}_{M,\, i}^{(\nu,\, \theta)}(x). \tag{49}$$

With the help of (21), we can see that $a_i(\ell,\, k) = 0$ for $i > \ell + k$. Hence, formula (49) takes the form

$$q_m(x)\, p_\ell(x) = \sum_{i=0}^{\ell+m} \left[\sum_{k=0}^{m} \frac{q_m^{(k)}(0)}{k!} a_i(\ell,\, k) \right] \mathscr{P}_{M,\, i}^{(\nu,\, \theta)}(x), \tag{50}$$

and this proves (45). Similarly, formula (46) can be proved. □

Corollary 4 In the LP

$$\mathscr{P}_{M,\, \ell}^{(\nu,\, \theta)}(x)\, \mathscr{P}_{M,\, m}^{(\gamma,\, \delta)}(x) = \sum_{i=0}^{\ell+m} c_i(\ell,\, m) \mathscr{P}_{M,\, i}^{(\alpha,\, \beta)}(x), \tag{51}$$

the coefficients $c_i(\ell,\, m)$ can be expressed as

$$c_i(\ell,\ m) = (-1)^{\ell+m}\frac{(2i+\lambda)\Gamma(\lambda+i)\Gamma(\theta+\ell+1)(\delta+1)_m}{\ell!m!\,(\beta+1)_i}$$

$$\times\sum_{k=0}^{m}\frac{(\beta+1)_k(\gamma+\delta+m+1)_k(-m)_k}{(\delta+1)_k}\,{}_4\bar{F}_3\left[\begin{array}{c}-\ell,\ k+1,\ k+\beta+1,\ \nu+\theta+\ell+1\\ 1-i+k,\ \lambda+i+k+1,\ \theta+1\end{array}\middle|1\right]. \tag{52}$$

Proof. In Theorem 2, consider $q_m(x) = \mathscr{P}_{M,\ m}^{(\gamma,\ \delta)}(x)$ and $p_\ell(x) = \mathscr{P}_{M,\ \ell}^{(\nu,\ \theta)}(x)$. Applying formula (45), gives

$$c_i(\ell,\ m) = \sum_{k=0}^{m}\frac{q_m^{(k)}(0)}{k!}a_i(\ell,\ k),\ i = 0,\ 1,\ \ldots,\ \ell+m. \tag{53}$$

Based on Corollary 1, we get

$$a_i(\ell,\ k) = (-1)^{\ell}k!M^k\frac{(2i+\lambda)(\beta+1)_k\Gamma(\lambda+i)\Gamma(\theta+\ell+1)}{\ell!\,(\beta+1)_i}$$

$$\times\,{}_4\bar{F}_3\left[\begin{array}{c}-\ell,\ k+1,\ k+\beta+1,\ \nu+\theta+\ell+1\\ 1-i+k,\ \lambda+i+k+1,\ \theta+1\end{array}\middle|1\right],\ i = 0,\ \ldots,\ \ell+k. \tag{54}$$

Substitution of (54) and (9) into (53)-after some manipulation- yields (52). □

The following corollary is a direct consequence of Corollary 4 and relation (33).

Corollary 5 In the LP

$$\mathscr{C}_{M,\ \ell}^{(\theta)}(x)\mathscr{C}_{M,\ m}^{(\gamma)}(x) = \sum_{i=0}^{\ell+m}c_i(\ell,\ m)\mathscr{C}_{M,\ i}^{(\nu)}(x), \tag{55}$$

the coefficients $c_i(\ell,\ m)$ can be expressed as

$$c_i(\ell,\ m) = (-1)^{\ell+m}\frac{1}{i!}(2i+2\nu)\Gamma(2\nu+i)\Gamma(\theta+1/2)$$

$$\times\sum_{k=0}^{m}\frac{(\nu+1/2)_k(2\gamma+m)_k(-m)_k}{(\gamma+1/2)_k}\,{}_4\bar{F}_3\left[\begin{array}{c}-\ell,\ k+1,\ k+\nu+1/2,\ 2\theta+n\\ 1-i+k,\ 2\nu+i+k+1,\ \theta+1/2\end{array}\middle|1\right]. \tag{56}$$

Note 2 It is worth to note that in the two LPs

$$\mathscr{P}_{\ell}^{(\nu,\ \theta)}(x)\,\mathscr{P}_{m}^{(\gamma,\ \delta)}(t) = \sum_{i=0}^{\ell+m}c_i(\ell,\ m)\mathscr{P}_{i}^{(\alpha,\ \beta)}(t), \tag{57}$$

and

$$\mathscr{C}_{\ell}^{(\theta)}(t)\mathscr{C}_{m}^{(\gamma)}(t)=\sum_{i=0}^{\ell+m}c_i(\ell,\,m)\mathscr{C}_{i}^{(\nu)}(t), \tag{58}$$

the expansion coefficients $c_i(\ell,\,m)$ are given by (52) and (56), respectively.

Remark 3 Using the formulae of $D^i p_\ell(0)$, for $p_\ell(x)=\mathscr{H}_\ell(x),\ \mathscr{L}_\ell^{(\nu)}(x),\ \mathscr{P}_\ell^{(\gamma,\,\delta)}(x), \mathscr{P}_\ell(x),\ \mathscr{T}_\ell(x),\ \mathscr{U}_\ell(x)$, listed in Table A1 (In the Appendix) and applying Theorem 2, the formulae of expansions coefficients $c_i(\ell,\,m)$ can be computed easily in many different cases.

5. Linearization problem: relationship between $\prod_{j=1}^{s} p_{\ell_j}(x)$ and a sum of SJPs

In this section, the explicit formula of linearization coefficients $c_i(\ell_1,\,\ell_2,\,\ldots,\,\ell_s)$ in the expansion

$$\prod_{j=1}^{s}p_{\ell_j}(x)=\sum_{i=0}^{L_s}c_i(\ell_1,\ell_2,\ldots,\ell_s)\,\mathscr{P}_{M,i}^{(\nu,\theta)}(x), \tag{59}$$

where $L_s=\sum_{j=1}^{s}\ell_j$, are given in following theorem.

Theorem 3 The expansion coefficients $c_i(\ell_1,\,\ell_2,\,\ldots,\,\ell_s)$ in (59) can be expressed as

$$c_i(\ell_1,\,\ell_2,\,\ldots,\,\ell_s)=\frac{i!(2i+\lambda)}{(\theta+1)_i(\lambda+i)}\sum_{r_1=0}^{\ell_1}\cdots\sum_{r_s=0}^{\ell_s}\binom{d_s}{i}\frac{M^{d_s}(\theta+1)_{d_s}}{(\lambda+i+1)_{d_s}}\prod_{j=1}^{s}\frac{p_{\ell_j}^{(r_s)}(0)}{r_j!},\ i=0,\,1,\,\ldots,\,L_s, \tag{60}$$

where $d_s=r_1+\cdots+r_s$.

proof. We proceed by induction on s. When $s=2$, the formula (60) is the same as (44). Let's say that the formula (60) works for s. We want to show that

$$c_i(\ell_1,\,\ell_2,\,\ldots,\,\ell_{s+1})=\frac{i!(2i+\lambda)}{(\theta+1)_i(\lambda+i)}\sum_{r_1=0}^{\ell_1}\cdots\sum_{r_{s+1}=0}^{\ell_{s+1}}\binom{d_{s+1}}{i}\frac{M^{d_{s+1}}(\theta+1)_{d_{s+1}}}{(\lambda+i+1)_{d_{s+1}}}\prod_{j=1}^{s+1}\frac{p_{\ell_j}^{(r_j)}(0)}{r_j!}. \tag{61}$$

Let $q_{L_s}(x)=\prod_{j=1}^{s}p_{\ell_j}(x)$, then $\prod_{j=1}^{s+1}p_{\ell_j}(x)=p_{\ell_{s+1}}(x)\,q_{Ls}(x)$. Applying formula (43) leads to

$$\prod_{j=1}^{s+1}p_{\ell_j}(x)=\sum_{i=0}^{L_{s+1}}c_i(\ell_{s+1},\,L_s)\,\mathscr{P}_{M,\,i}^{(\nu,\,\theta)}(x), \tag{62}$$

then

$$c_i(\ell_1,\,\ell_2,\,\ldots,\,\ell_{s+1}) = c_i(\ell_{s+1},\,L_s)$$

$$= \frac{i!(2i+\lambda)}{(\theta+1)_i(\lambda+i)} \sum_{k=0}^{L_s} \sum_{r_{s+1}=0}^{\ell_{s+1}} \binom{k+r_{s+1}}{i} \frac{M^{k+r_{s+1}}(\theta+1)_{k+r_{s+1}}}{(\lambda+i+1)_{k+r_{s+1}}\, r_{s+1}!k!} q_{L_s}^{(k)}(0)\, p_{\ell_{s+1}}^{(r_{s+1})}(0). \tag{63}$$

Assume that theorem holds for s and employing (9) lead to

$$q_{L_s}^{(k)}(0) = \sum_{j=k}^{L_s} c_j(\ell_1,\,\ell_2,\,\ldots,\,\ell_s)\,(-1)^j \frac{(\theta+1)_j(\lambda+j)_k(-j)_k}{j!\,(\theta+1)_k} M^{-k}, \tag{64}$$

where

$$c_j(\ell_1,\,\ell_2,\,\ldots,\,\ell_s) = \frac{j!(2j+\lambda)}{(\theta+1)_j(\lambda+j)} \sum_{r_1=0}^{\ell_1} \cdots \sum_{r_s=0}^{\ell_s} \binom{d_s}{j} \frac{M^{d_s}(\theta+1)_{d_s}}{(\lambda+j+1)_{d_s}} \prod_{j=1}^{s} \frac{p_{\ell_j}^{(r_s)}(0)}{r_j!},\ j=0,\,1,\,\ldots,\,L_s. \tag{65}$$

Substituting (64) into (63), we get

$$c_i(\ell_{s+1},\,L_s) = \frac{i!(2i+\lambda)}{(\theta+1)_i(\lambda+i)} \sum_{r_1=0}^{\ell_1} \cdots \sum_{r_{s+1}=0}^{\ell_{s+1}} M^{d_s}(\theta+1)_{d_s} \prod_{j=1}^{s+1} \frac{p_{\ell_j}^{(r_s)}(0)}{r_j!} \Theta_{r_{s+1}}^{(i)}(L_s,\,\nu,\,\theta), \tag{66}$$

where

$$\Theta_{r_{s+1}}^{(i)}(L_s,\,\nu,\,\theta) = \sum_{k=0}^{L_s} \sum_{j=k}^{L_s} \frac{1}{k!} \binom{d_s}{j} \binom{k+r_{s+1}}{i} \times \frac{M^{r_{s+1}}(\theta+1)_{k+r_{s+1}}}{(\lambda+i+1)_{k+r_{s+1}}} \frac{(2j+\lambda)}{(\lambda+j)} \frac{(-1)^j(\lambda+j)_k(-j)_k}{(\theta+1)_k}$$

$$= \sum_{k=0}^{L_s} \frac{1}{k!} \binom{k+r_{s+1}}{i} \frac{M^{r_{s+1}}(\theta+1)_{k+r_{s+1}}}{(\lambda+i+1)_{k+r_{s+1}}(\theta+1)_k} \times \left[\sum_{j=k}^{L_s} \binom{d_s}{j} \frac{(-1)^j(2j+\lambda)(\lambda+j)_k(-j)_k}{(\lambda+j)(\lambda+j+1)_{d_s}} \right]. \tag{67}$$

Noting that

$$\binom{d_s}{j} = 0,\ j > d_s,$$

it is easy to note that

$$\sum_{j=k}^{L_s}\binom{d_s}{j}\frac{(-1)^j(2j+\lambda)(\lambda+j)_k(-j)_k}{(\lambda+j)(\lambda+j+1)_{d_s}}=\sum_{j=k}^{d_s}\binom{d_s}{j}\frac{(-1)^j(2j+\lambda)(\lambda+j)_k(-j)_k}{(\lambda+j)(\lambda+j+1)_{d_s}}$$

$$=\frac{d_s!}{(d_s-k)!}A_{d_s-k}(k)$$

$$=\frac{d_s!}{(d_s-k)!}\,\delta_{k,\,d_s},$$

where $A_r(i)$ is defined by formula (16). Hence formula (67) can be expressed as follows

$$\Theta^{(i)}_{r_{s+1}}(L_s,\ \nu,\ \theta)=\sum_{k=0}^{L_s}\frac{1}{k!}\left(\begin{array}{c}k+r_{s+1}\\ i\end{array}\right)\frac{M^{r_{s+1}}(\theta+1)_{k+r_{s+1}}}{(\lambda+i+1)_{k+r_{s+1}}(\theta+1)_k}d_s!\,\delta_{k,\,d_s},$$

$$=\left(\begin{array}{c}d_s+r_{s+1}\\ i\end{array}\right)\frac{M^{r_{s+1}}(\theta+1)_{d_s+r_{s+1}}}{(\lambda+i+1)_{d_s+r_{s+1}}(\theta+1)_{d_s}},$$

that can be represented as

$$\Theta^{(i)}_{r_{s+1}}(L_s,\ \nu,\ \theta)=\left(\begin{array}{c}d_{s+1}\\ i\end{array}\right)\frac{M^{r_{s+1}}(\theta+1)_{d_{s+1}}}{(\lambda+i+1)_{d_{s+1}}(\theta+1)_{d_s}},\ d_{s+1}=d_s+r_{s+1}. \tag{68}$$

Substituting (68) into (66), one can obtain (61), and completes the proof of theorem. □

The generalized hypergeometric series of N variables (GHS-N), ${}^{s}F^{\mathfrak{p}:\mathfrak{q}_1;\ \dots;\ \mathfrak{q}_s}_{l:m_1;\ \dots;\ m_s}$, can be used to describe the coefficients $c_i(\ell_1,\ \ell_2,\ \dots,\ \ell_s)$. These functions are given by Niukkanen [48],

$${}^{s}F^{\mathfrak{p}_0,\ \mathfrak{p}_1,\ \dots,\ \mathfrak{p}_s}_{\mathfrak{q}_0,\ \mathfrak{q}_1,\ \dots,\mathfrak{q}_\ell}\left[\begin{array}{l}\mathbf{a}_0;\ \mathbf{a}_1,\ \dots,\ \mathbf{a}_s;\ x_1,\ \dots,\ x_s\\ \mathbf{b}_0;\ \mathbf{b}_1,\ \dots,\ \mathbf{b}_s;\end{array}\right]=\sum_{r_1,\dots,r_s=0}^{\infty}\frac{(\mathbf{a}_0)_{d_s}}{(\mathbf{b}_0)_{d_s}}\prod_{j=1}^{s}\frac{(\mathbf{a}_j)_{r_j}}{(\mathbf{b}_j)_{r_j}}\frac{x_j^{r_j}}{r_j!}, \tag{69}$$

where

$$d_s=r_1+\ldots+r_s,\ \mathbf{a}_j=\left(a^j_1,\ldots,a^j_{\mathfrak{p}_j}\right),\ \mathbf{b}_j=\left(b^j_1,\ \dots,\ b^j_{\mathfrak{q}_j}\right),$$

$$(\mathbf{a}_j)_{r_j}=\prod_{i=1}^{\mathfrak{p}_j}(a^j_i)_{r_j},\ (\mathbf{b}_j)_{r_j}=\prod_{i=1}^{\mathfrak{q}_j}(b^j_i)_{r_j},\ j=1,\ 2,\ \dots,\ s.$$

Note 3 As a particular case, for $\mathfrak{p}_j=\mathfrak{p}$ and $\mathfrak{q}_j=\mathfrak{q},\ j=1,\dots,s$, the series (69) simply denotes by ${}^{s}F^{\mathfrak{p}_0,\mathfrak{p}}_{\mathfrak{q}_0,\mathfrak{q}}$.

Corollary 6 In the LP

$$\prod_{j=1}^{s} \mathscr{P}_{M,\,\ell_j}^{(\nu_j,\,\theta_j)}(x) = \sum_{i=0}^{L_s} C_i^{\nu,\,\theta}(\ell_1,\,\ell_2,\,\ldots,\,\ell_s)\,\mathscr{P}_{M,\,i}^{(\nu,\,\theta)}(x), \quad s \geq 1, \tag{70}$$

the expansion coefficients $C_i^{\nu,\,\theta}(\ell_1,\,\ell_2,\,\ldots,\,\ell_s)$ can be expressed as

$$C_i^{\nu,\,\theta}(\ell_1,\,\ell_2,\,\ldots,\,\ell_s) = \frac{(-1)^i(2i+\lambda)(\theta+1)_{L_s}(-L_s)_i}{(\theta+1)_i(\lambda+i)(\lambda+i+1)_{L_s}} \prod_{j=1}^{s} \frac{(\lambda_j+\ell_j)_{\ell_j}}{\ell_j!}$$

$$\times\, {}^{s}F_{2,\,1}^{2,\,2}\left[\begin{matrix} i-L_s,\ -i-L_s-\lambda;\ -\ell_1,\ -\theta_1-\ell_1,\ \ldots,\ -\ell_s,\ -\theta_s-\ell_s;\ 1,\ \ldots,\ 1 \\ -L_s-\theta,\ -L_s;\ 1-2\ell_1-\lambda_1,\ \ldots,\ 1-2\ell_s-\lambda_s \end{matrix}\right], \tag{71}$$

$$i = 0,\ 1,\ \ldots,\ L_s,$$

where ν and θ denote to the two arrays $\nu,\ \nu_1,\ \ldots,\ \nu_s$ and $\theta,\ \theta_1,\ \ldots,\ \theta_s$, respectively, and $\lambda_j = \nu_j+\theta_j+1,\ j = 1,\ 2,\ \ldots,\ s$.

Proof. Using Theorem 3 and formula (9), the coefficients $C_i^{\nu,\,\theta}(\ell_1,\,\ell_2,\,\ldots,\,\ell_s)$ may be expressed as follows:

$$C_i^{\nu,\,\theta}(\ell_1,\,\ell_2,\,\ldots,\,\ell_s) = \frac{(2i+\lambda)}{(\theta+1)_i(\lambda+i)} \sum_{r_1=0}^{\ell_1}\cdots\sum_{r_s=0}^{\ell_s} \frac{(1)_{L_s-d_s}}{(1)_{L_s-i-d_s}}\, \frac{M^{L_s-d_s}(\theta+1)_{L_s-d_s}}{(\lambda+i+1)_{L_s-d_s}}$$

$$\times \prod_{j=1}^{s}(-1)^{\ell_j} \frac{(\theta_j+1)_{\ell_j}(\lambda_j+\ell_j)_{\ell_j-r_j}(-\ell_j)_{\ell_j-r_j}}{\ell_j!(\ell_j-r_j)!\,(\theta_j+1)_{\ell_j-r_j}} M^{-\ell_j+r_j}. \tag{72}$$

By the aid of Proposition 1, it is easy to see that formula (72) becomes

$$C_i^{\nu,\,\theta}(\ell_1,\,\ell_2,\,\ldots,\,\ell_s) = \frac{(-1)^i(2i+\lambda)(\theta+1)_{L_s}(-L_s)_i}{(\theta+1)_i(\lambda+i)(\lambda+i+1)_{L_s}} \prod_{j=1}^{s} \frac{(\lambda_j+\ell_j)_{\ell_j}}{\ell_j!}$$

$$\times \sum_{r_1=0}^{\ell_1}\cdots\sum_{r_s=0}^{\ell_s} \frac{(i-L_s)_{d_s}(-i-L_s-\lambda)_{d_s}}{(-L_s)_{d_s}(-L_s-\theta)_{d_s}} \prod_{j=1}^{s} \frac{(-\theta_j-\ell_j)_{r_j}(-\ell_j)_{r_j}}{r_j!(1-2\ell_j-\lambda_j)_{r_j}}, \tag{73}$$

which can be expressed as (71). □

As a direct consequence of Corollary 6 and relation (33), we obtain the following corollary.

Corollary 7 In the LP

$$\prod_{j=1}^{s} \mathscr{C}_{M,\,\ell_j}^{(\gamma_j)}(x) = \sum_{i=0}^{L_s} C_i^{\boldsymbol{\gamma},\nu}(\ell_1,\,\ell_2,\,\ldots,\,\ell_s)\ \mathscr{C}_{M,\,i}^{(\nu)}(x), \quad s \geq 1, \tag{74}$$

the expansion coefficients $C_i^{\gamma,\,\nu}(\ell_1,\,\ell_2,\,\ldots,\,\ell_s)$ can be expressed as

$$C_i^{\gamma,\,\nu}(\ell_1,\,\ell_2,\,\ldots,\,\ell_s)$$

$$=\frac{(-1)^i(2i+2\nu)(\nu+1/2)_{L_s}(-L_s)_i}{i!(2\nu+i)(2\nu+i+1)_{L_s}}\prod_{j=1}^{s}\frac{(2\gamma_j+\ell_j)_{\ell_j}}{(\gamma_j+1/2)_{\ell_j}}$$

$$\times\,{}^sF_{2,\,1}^{2,\,2}\left[\begin{array}{l} i-L_s,\,-i-L_s-2\nu;\,-\ell_1,\,-\gamma_1-\ell_1+1/2,\,\ldots,\,-\ell_s,\,-\gamma_s-\ell_s+1/2;\,1,\,\ldots,\,1 \\ -L_s,\,-L_s-\nu+1/2;\,1-2\ell_1-2\gamma_1,\,\ldots,\,1-2\ell_s-2\gamma_s \end{array}\right], \tag{75}$$

$$i=0,\,1,\,\ldots,\,L_s,$$

where γ denotes to the array $\gamma_1,\,\ldots,\,\gamma_s$.

Corollary 8 In the LP

$$\prod_{j=1}^{s}\mathscr{L}_{\ell_j}^{(\nu_j)}(x)=\sum_{i=0}^{L_s}C_i^{\nu,\,\theta}(\ell_1,\,\ell_2,\,\ldots,\,\ell_s)\;\mathscr{P}_{M,\,i}^{(\nu,\,\theta)}(x),\quad s\geq 1, \tag{76}$$

the expansion coefficients $C_i^{\nu,\,\theta}(\ell_1,\,\ell_2,\,\ldots,\,\ell_s)$ can be expressed as

$$C_i^{\nu,\,\theta}(\ell_1,\,\ell_2,\,\ldots,\,\ell_s)=\frac{(-1)^i(2i+\lambda)(\theta+1)_{L_s}(-L_s)_i(-M)^{L_s}}{(\theta+1)_i(\lambda+i)(\lambda+i+1)_{L_s}}\prod_{j=1}^{s}\frac{1}{\ell_j!}$$

$$\times\,{}^sF_{2,\,0}^{2,\,2}\left[\begin{array}{l} i-L_s,\,-i-L_s-\lambda;\,-\ell_1,\,-\nu_1-\ell_1,\,\ldots,\,-\ell_s,\,-\nu_s-\ell_s;\,-M,\,\ldots,\,-M \\ -L_s-\theta,\,-L_s;\,\emptyset,\,\ldots,\,\emptyset \end{array}\right], \tag{77}$$

$$i=0,\,1,\,\ldots,\,L_s,$$

where ν denotes to the array $\nu,\,\nu_1,\,\ldots,\,\nu_s$.

Proof. Based on Theorem 3 together with the expression of $D^r\mathscr{L}_\ell^{(\nu)}(0)$ (see Table A1) and following the same procedures in the proof of Corollary 6, yields formula (77).

Corollary 9 In the LP

$$\prod_{j=1}^{s}\mathscr{H}_{\ell_j}(x)=\sum_{i=0}^{L_s}C_i^{\nu,\,\theta}(\ell_1,\,\ell_2,\,\ldots,\,\ell_s)\;\mathscr{P}_{M,\,i}^{(\nu,\,\theta)}(x),\quad \ell\geq 1, \tag{78}$$

the expansion coefficients $C_i^{\nu,\,\theta}(\ell_1,\,\ell_2,\,\ldots,\,\ell_s)$ can be expressed as

$$
\begin{aligned}
&C_i^{\nu,\,\theta}(\ell_1,\,\ell_2,\,\dots,\,\ell_s)\\
&=(2M)^{L_s}\frac{(-1)^i(2i+\lambda)(\theta+1)_{L_s}(-L_s)_i}{(\theta+1)_i(\lambda+i)(\lambda+i+1)_{L_s}}\\
&\times\,{}^{s}F^{4,\,2}_{4,\,0}\left[\begin{matrix}-\frac{L_s-i}{2},\,-\frac{L_s-i-1}{2};\,-\frac{L_s+i+\lambda}{2},\,-\frac{L_s+i+\lambda-1}{2};\,-\frac{\ell_1-1}{2},\,-\frac{\ell_1}{2},\,\dots,\,-\frac{\ell_s-1}{2},\,-\frac{\ell_s}{2};\,-\frac{1}{M^2},\,\dots,\,-\frac{1}{M^2}\\ -\frac{L_s-1}{2},\,-\frac{L_s}{2},\,-\frac{L_s+\theta}{2},\,-\frac{L_s+\theta-1}{2};\,\emptyset,\,\dots,\,\emptyset\end{matrix}\right],\\
&\quad i=0,\,1,\,\dots,\,\ell_s.
\end{aligned}
\tag{79}
$$

Proof. Using Theorem 3 and formula $D^r\mathscr{H}_\ell(0)$ (see Table A1), one can get

$$
\begin{aligned}
C_i^{\nu,\,\theta}(\ell_1,\,\ell_2,\,\dots,\,\ell_s)&=\frac{(2i+\lambda)}{(\theta+1)_i(\lambda+i)}\\
&\times\sum_{k_1=0}^{[\ell_1/2]}\cdots\sum_{k_s=0}^{[\ell_s/2]}\frac{(1)_{L_s-2h_s}}{(1)_{L_s-i-2h_s}}\,\frac{M^{L_s-2h_s}(\theta+1)_{L_s-2h_s}}{(\lambda+i+1)_{L_s-2h_s}}\prod_{j=1}^{s}\frac{(-1)^{k_j}2^{\ell_j-2k_j}\,\ell_j!}{k_j!\,(\ell_j-2k_j)!},
\end{aligned}
\tag{80}
$$

where $h_s=k_1+\cdots+k_s$. By using Proposition 1, formula (80) takes the form

$$
\begin{aligned}
C_i^{\nu}(\ell_1,\,\ell_2,\,\dots,\,\ell_s)&=(2M)^{L_s}\frac{(-1)^i(2i+\lambda)(\theta+1)_{L_s}(-L_s)_i}{(\theta+1)_i(\lambda+i)(\lambda+i+1)_{L_s}}\times\sum_{k_1=0}^{[\ell_1/2]}\cdots\\
&\sum_{k_s=0}^{[\ell_s/2]}\frac{(-\frac{L_s-i}{2})_{h_s}(-\frac{L_s-i-1}{2})_{h_s}(-\frac{L_s+i+\lambda}{2})_{h_s}(-\frac{L_s+i+\lambda-1}{2})_{h_s}}{(-\frac{L_s-1}{2})_{h_s}(-\frac{L_s}{2})_{h_s}(-\frac{L_s+\theta}{2})_{h_s}(-\frac{L_s+\theta-1}{2})_{h_s}}\prod_{j=1}^{s}\frac{(-\frac{\ell_j-1}{2})_{k_j}(-\frac{\ell_j}{2})_{k_j}}{k_j!\,(-M^2)^{k_j}},
\end{aligned}
\tag{81}
$$

which can be expressed as (79). □

Corollary 10 In the LP

$$
\prod_{j=1}^{s}\mathscr{C}_{\ell_j}^{(\lambda_j)}(x)=\sum_{i=0}^{L_s}C_i^{\lambda,\,\nu,\,\theta}(\ell_1,\,\ell_2,\,\dots,\,\ell_s)\;\mathscr{P}_{M,\,i}^{(\nu,\,\theta)}(x),\;\ell\geq 1,
\tag{82}
$$

the expansion coefficients $C_i^{\lambda,\,\nu,\,\theta}(\ell_1,\,\ell_2,\,\dots,\,\ell_s)$ can be expressed as

$$C_i^{\lambda,\,\nu,\,\theta}(\ell_1,\,\ell_2,\,\ldots,\,\ell_s)$$

$$= (2M)^{L_s}\frac{(-1)^i(2i+\lambda)(\theta+1)_{L_s}(-L_s)_i}{(\theta+1)_i(\lambda+i)(\lambda+i+1)_{L_s}}$$

$$\times\, {}^sF^{4,\,2}_{4,\,1}\left[\begin{matrix} -\frac{L_s-i}{2},\, -\frac{L_s-i-1}{2};\, -\frac{L_s+i+\lambda}{2},\, -\frac{L_s+i+\lambda-1}{2};\, -\frac{\ell_1-1}{2},\, -\frac{\ell_1}{2},\, \ldots,\, -\frac{\ell_s-1}{2},\, -\frac{\ell_s}{2};\, \frac{1}{M^2},\, \ldots,\, \frac{1}{M^2} \\ -\frac{L_s-1}{2},\, -\frac{L_s}{2},\, -\frac{L_s+\theta}{2},\, -\frac{L_s+\theta-1}{2};\, 1-\ell_1-\lambda_1,\, \ldots,\, 1-\ell_s-\lambda_s \end{matrix}\right], \tag{83}$$

$$i = 0,\, 1,\, \ldots,\, L_s,$$

where λ denotes to the array $\lambda_1,\, \ldots,\, \lambda_s$.

Proof. Similar to the proof of Corollary 9. □

6. Results and discussions

The main results of the current article paper are Theorems 1, 2, and 3. These theorems enable us to compute the expansion coefficients that must be determined explicitly. As far as we know, most of the formulas in this article are novel. As expected in future work, the proposed approach may be extended straightforwardly to multivariable polynomials. In addition, we do believe that the presented approach can be followed to find other linearization formulas for other orthogonal polynomials. This will be an expected future work.

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Confilict of interest

The authors declare no competing financial interest.

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Appendix A

Table A1

$p_\ell(x)$	$D^r p_\ell(0)$
$\mathcal{P}_\ell^{(\gamma,\,\delta)}(x)=\sum_{k=0}^{\ell} c_{\ell,\,k}(\gamma,\,\delta)x^k$	$r!c_{\ell,\,r}(\gamma,\,\delta)$
$\mathcal{L}_\ell^{(\nu)}(x)=\frac{(\nu+1)_\ell}{\ell!}\sum_{s=0}^{\ell}\frac{(-\ell)_s}{(\nu+1)_s}\frac{x^s}{s!}$	$\frac{(-\ell)_r(\nu+1)_\ell}{\ell!(\nu+1)_r}$
$\mathcal{H}_\ell(x)=\ell!\sum_{s=0}^{[\ell/2]}\frac{(-1)^s 2^{\ell-2s}}{s!}\frac{x^{\ell-2s}}{(\ell-2s)!}$	$\begin{cases}\frac{(-1)^{\frac{\ell-r}{2}}2^r\ell!}{(\frac{\ell-r}{2})!}, & (\ell-r) \text{ even}\\ 0, & (\ell-r) \text{ odd}\end{cases}$
$\mathcal{C}_\ell^{(\lambda)}(x)=\frac{(-1)^\ell\ell!}{(2\lambda)_\ell}\sum_{s=0}^{[\ell/2]}\binom{-\lambda}{\ell-s}\binom{\ell-s}{s}(2x)^{\ell-2s}$	$\begin{cases}2^r\frac{(-1)^{(\frac{\ell-r}{2})}(\lambda)_{\frac{\ell+r}{2}}\ell!}{(\frac{\ell-r}{2})!(2\lambda)_\ell}, & (\ell-r) \text{ even}\\ 0, & (\ell-r) \text{ odd}\end{cases}$
$\mathcal{P}_\ell(x)=\frac{1}{2^\ell}\sum_{s=0}^{[\ell/2]}\frac{(-1)^s(2\ell-2s)!}{s!(\ell-s)!}\frac{x^{\ell-2s}}{(\ell-2s)!}$	$\begin{cases}\frac{1}{2^\ell}\frac{(-1)^{(\frac{\ell-r}{2})}(\ell+r)!}{(\frac{\ell-r}{2})!(\frac{\ell+r}{2})!}, & (\ell-r) \text{ even}\\ 0, & (\ell-r) \text{ odd}\end{cases}$
$\mathcal{T}_\ell(x)=\frac{\ell}{2}\sum_{s=0}^{[\ell/2]}\frac{(-1)^s(\ell-s-1)!}{s!}\frac{(2x)^{\ell-2s}}{(\ell-2s)!}$	$\begin{cases}n\frac{2^{r-1}(-1)^{(\frac{\ell-r}{2})}(\frac{\ell+r}{2}-1)!}{(\frac{\ell-r}{2})!}, & (\ell-r) \text{ even}\\ 0, & (\ell-r) \text{ odd}\end{cases}$
$\mathcal{U}_\ell(x)=\sum_{s=0}^{[\ell/2]}\frac{(-1)^s(\ell-s)!}{s!}\frac{(2x)^{\ell-2s}}{(\ell-2s)!}$	$\begin{cases}\frac{2^r(-1)^{(\frac{\ell-r}{2})}(\frac{\ell+r}{2})!}{(\frac{\ell-r}{2})!}, & (\ell-r) \text{ even}\\ 0, & (\ell-r) \text{ odd}\end{cases}$

where

$$c_{\ell,\,k}(\gamma,\,\delta)=2^{-k}\binom{\ell+\gamma}{\ell-k}\binom{\ell+k+\gamma+\delta}{k}\,{}_2F_1\left[\left.\begin{matrix}-(\ell-k),\ell+k+\gamma+\delta+1\\ k+\gamma+1\end{matrix}\right|\frac{1}{2}\right].$$