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Research Article

Metaheuristic Cost Optimization of $M^X/G_1, G_2/1$ Queue with Service Disruption, Working Breakdown, Balking, Catastrophe and Extended Server Vacation with Bernoulli Schedule

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Abstract: We presented a $M^X/G_1, G_2/1$ queueing system with a Bernoulli schedule that includes service disruption, working breakdown, balking, catastrophe and extended server vacations. All customers that arrive are served by a single server, with service time determined by general distribution. As soon as the system fails, the server keeps serving the current client at a reduced rate while repairs are made. It is believed that the vacation schedule is broad. Random disruptions to the server occur, with an exponential distribution in the length of the disruption. We assume that besides a Poisson stream of positive arrivals, there is a Poisson stream of negative arrivals, which we refer to as catastrophes. Using the supplementary variable technique, the Laplace transforms of the time-dependent probability of the system state are derived. We can infer steady-state outcomes from this. The average waiting time and queue size were also obtained. An additional topic of discussion is Adaptive Neuro-Fuzzy Inference system (ANFIS). Furthermore, this work uses Particle Swarm Optimisation (PSO), Artificial Bee Colonies (ABC), and Genetic Algorithms (GA) to swiftly find the system's optimal cost. Additionally, we examined the convergence of several graph-optimisation strategies.

***Keywords*:** batch arrival, service disruption, working breakdown, extended server vacation, ANFIS, cost optimization

MSC: 60K25, 90B22, 60K30

1. Introduction

An effective tool for simulating factories, networked transportation, systems of operation, communication networks is the queueing system. Computer networks and data transmission systems have experienced rapid technological growth in recent years, resulting in remarkable advancements across a wide range of applications. Consider the quick development of the internet, video and audio information flow, etc. G_1, G_2 refers to two stages of service. We also expect a batch arrival and use a minimum batch size of "a" and a maximal batch size of "b" in addition to providing a bulk service. For over 30 years, vacation queues have been studied as a highly helpful tool for modeling and analysing various systems, including industrial processing systems, computing devices, wireless networks, and a lot more.

Researchers Baskar et al. [1], Sundari and Srinivasan [2], Balamani [3], Khalaf [4], Ayyappan and Sathiya [5] have examined queues experiencing disruptions in service. Jain et al. [6] provide a comprehensive review and literature

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analysis on the performance modeling and evaluation of a single-server, general service queueing system incorporating service interruptions through the application of supplementary variable techniques.

Rajendiran and Kandaiyan [7], Nithya and Haridass [8], Rajadurai et al. [9] have all discussed queueing systems with balking. Ayyappan and Nirmala [10] as discussed investigated the impact of impatient customers in a bulk service queueing system featuring an unreliable server. Several investigations have explored queueing systems affected by server breaks, random failures, and subsequent repairs. These include comprehensive overviews provided by Ayyappan and Udayageetha [11], Thangaraj and Vanitha [12], Jain and Kumar [13]. A stochastic decomposition in the $M/G/1$ queue with generalised vacations was studied by Fuhrmann and Cooper [14]. Doshi [15] conducted an investigation into queueing systems with vacations. Karpagam and Ayyappan [16] conducted research on a bulk queueing system featuring a lone server. Their study delved into the influence of variables like server initiation failure and restoration, standby server, numerous break periods, shutdowns, and the adoption of an N-policy. Niranjan and Latha [17] suggested a queueing system with breakdown characterized by Two-Phase Heterogeneous and Batch Service. Ayyappan et al. [18] conducted an examination of a retrial system that includes features such as priority services, operational breakdowns, Bernoulli vacation, admission control, and balking. Recent years have seen a thorough investigation of vacation lines due to their theoretical framework and practical relevance in a range of real-world scenarios. A number of academics have recently become interested in the batch arrival queue with vacation. Prominent investigators include Laxmi Pikkala and Demie [19], Baba [20], Zadeh and Shahkar [21], Choi and Park [22], as well as Kalyanaraman and Suvitha [23].

Mathavavisakan and Indhira [24] have suggested a literature review focused on retrial queueing systems incorporating Bernoulli vacation. Ayyappan et al. [25] conducted an analysis of a single server fixed batch service queueing system under several vacation scenarios with a disaster. Ayyappan and Shyamala [26] presented the temporary queueing system solution, which is exposed to disasters, server outages, and maintenance. Vaishnawi et al. [27] researched the optimum cost estimation for a discrete-time recurrence queue with Bernoulli feedback and an emergency vacation. Yang and Wu CH [28] introduced the N-policy $M/M/1$ queueing systems with working vacations and server failures.

The outline of the paper is as follows: A detailed explanation of the mathematical framework is provided in section 2. Section 3 specifically determines the Probability generating function (PGF) for the queue length at each epoch and the system's SS performance. The relevant stability criterion has been identified in subsection 3.2. In subsection 3.3, we provide exact numbers for the mean queue size, average queue waiting time, and other metrics of performance for every system state. We present particular cases, practical application and a numerical evaluation and corresponding graphs in section 3.5, 4 and 5. An ANFIS is included in section 6. Section 7 provides Cost optimisation. Section 8 offers the conclusion.

2. Model description and analysis

Arrival process: In a compound Poisson process, different-sized batches of users join the system. The first-order probability that a set of "i" customers would join the system in a short amount of time $(\varsigma,\ \varsigma + d\varsigma]$ is represented by the expression $\varpi c_i d\varsigma (i = 1, 2, 3, ...)$. $\varpi > 0,\ 0 \leq c_i \leq 1$ and $\sum_{i=1}^{\infty} c_i = 1$ are the group average arriving rates. The FCFS system is used to provide individualised client service.

Service process: A single server attends to each arriving client, with a general distribution of service times. Let $A(\xi)$ and $a(\xi)$ stand for the distribution and density function, respectively, of the service time. Let $\mu(\xi)d\xi$ be the conditional probability of completion of a vacation inside the time interval $(\xi,\ \xi + d\xi]$ if the elapsed service time is ξ. Then

$$\mu_i(\xi) = \frac{a_i(\xi)}{1 - A_i(\xi)}$$

$$a_i(\delta) = \mu_i(\delta) e^{-\int_0^\delta \mu_i(\xi) d\xi}$$

Service disruption: Random disruptions to the server occur, and the length of each disruption is exponentially distributed. The rate at which the server disrupts attendees is denoted by α. In addition, we presume that the disrupted user moves back to the head of the line, where entries are handled at a mean rate of $\psi > 0$.

Extended server vacation process: After the service concludes, the server might choose to take a vacation with probability "q" or remain in the system with probability "1-q". When the original vacation is completed the server has the option to go on an extended vacation. Upon the end of vacation, the server returns to the system. A generic (arbitrary) distribution with a distribution function of $G_i(\delta)$ and a density function of $g_i(\delta)$ governs the servers vacation time. If the elapsed vacation time is ξ, then let $\Phi_i(\xi)d\xi$ be the probability that the conditions of finishing a vacation inside the time interval $(\xi,\ \xi + d\xi]$. This way,

$$\Phi_i(\xi) = \frac{g_i(\xi)}{1 - G_i(\xi)},\ i = 1, 2, 3$$

$$g_i(\delta) = \Phi_i(\delta)e^{-\int_0^\delta \Phi_i(\xi)d\xi}$$

When the server comes back from their vacation, they immediately start serving the person in front of the queue, provided there is someone waiting.

Balking rule: When a customer arrives and the system's server is busy, they have two choices: they can either decline the system with probability "1-d", or they can join the queue with probability "d".

Working breakdown process: Breakdowns in the system are predicted based on Poisson probability with parameter β and may occur throughout the regular service time. If there is a server failure, the current user's service will be completed at a slower rate of β_w. The server will be fixed after that.

Repair process: No clients are permitted while the repairs are being made. The primary server moves on to optional second-phase repair with probability ε and repair rate ω_2, following an exponential distribution with rate ω_1.

Let $R(\xi)$ and $r(\xi)$ stand for the distribution and density function, respectively, of the service time. If the elapsed vacation time is ξ, then let $\eta(\xi)d\xi$ be the probability that the conditions of finishing a vacation inside the time interval $(\xi,\ \xi + d\xi]$.

$$\eta_i(\xi) = \frac{r_i(\xi)}{1 - R_i(\xi)}$$

$$r_i(\delta) = \eta_i(\delta)e^{-\int_0^\delta \eta_i(\xi)d\xi}$$

Catastrophe: If the Poisson process with an average rate of Ω isn't empty, disasters might occur at the service facility. These disasters instantly eliminate all clients in the system, and the facility only resumes operation once it's been repaired.

2.1 *Definitions*

We define

(i) The probability that, at time "δ", the server is actively providing service and there are n $(n \geq 0)$ users in the queue, apart from the one currently being served, with elapsed service time ξ, is denoted as $P_n^{(i)}(\xi,\ \delta)$.

For this reason, $P_n^{(i)}(\delta) = \int_0^\infty P_n^{(i)}(\xi,\ \delta)d\xi$ is the likelihood that, regardless of ξ, at time "δ", there are n customers in the line, except a single client who is receiving service.

(ii) The probability that the server is on i^{th} vacation with elapsed vacation time ξ and n $(n \geq 0)$ customers in the queue at time "δ " is $G_n^{(i)}(\xi,\ \delta)$.

The probability of having n clients in the queue and the server being on its i^{th} break at time δ, irrespective of the duration of the break ξ for $(i = 1, 2, 3)$, is given by $G_n^{(i)}(\delta) = \int_0^{\infty} G_n^{(i)}(\xi,\ \delta) d\xi$.

(iii) The probability that the server will become inactive owing to a disruption arriving at time "δ" is $R_n(\delta)$.

(iv) $Q(\delta)$ represents the likelihood that at time δ, the server is available but idle, with no clients either waiting in the queue or being served.

Governing equations can be formulated as follows:

$$\frac{\partial}{\partial \xi} P_0^{(i)}(\xi, \delta) + \frac{\partial}{\partial \delta} P_0^{(i)}(\xi, \delta) + [\varpi + \beta + \psi + \mu_i(\xi) + \Omega] P_0^{(i)}(\xi, \delta) = \varpi(1-d) P_0^{(i)}(\xi, \delta), \quad i = 1, 2 \tag{1}$$

$$\begin{aligned} &\frac{\partial}{\partial \xi} P_n^{(i)}(\xi, \delta) + \frac{\partial}{\partial \delta} P_n^{(i)}(\xi, \delta) + [\varpi + \beta + \psi + \mu_i(\xi) + \Omega] P_n^{(i)}(\xi, \delta) \\ &= \varpi d \sum_{k=1}^{n} C_k P_{n-k}^{(i)}(\xi, \delta) + \varpi(1-d) P_n^{(i)}(\xi, \delta), \quad i = 1, 2, n \geq 1 \end{aligned} \tag{2}$$

$$\frac{\partial}{\partial \xi} G_0^{(1)}(\xi, \delta) + \frac{\partial}{\partial \delta} G_0^{(1)}(\xi, \delta) + [\varpi + \phi_1(\xi)] G_0^{(1)}(\xi, \delta) = \varpi(1-d) G_0^{(1)}(\xi, \delta) \tag{3}$$

$$\frac{\partial}{\partial \xi} G_n^{(1)}(\xi, \delta) + \frac{\partial}{\partial \delta} G_n^{(1)}(\xi, \delta) + [\varpi + \phi_1(\xi)] G_n^{(1)}(\xi, \delta) = \varpi d \sum_{k=1}^{n} C_k G_{n-k}^{(1)}(\xi, \delta) + \varpi(1-d) G_n^{(1)}(\xi, \delta), \quad n \geq 1 \tag{4}$$

$$\frac{\partial}{\partial \xi} G_0^{(2)}(\xi, \delta) + \frac{\partial}{\partial \delta} G_0^{(2)}(\xi, \delta) + [\varpi + \phi_2(\xi)] G_0^{(2)}(\xi, \delta) = \varpi(1-d) G_0^{(2)}(\xi, \delta) \tag{5}$$

$$\frac{\partial}{\partial \xi} G_n^{(2)}(\xi, \delta) + \frac{\partial}{\partial \delta} G_n^{(2)}(\xi, \delta) + [\varpi + \phi_2(\xi)] G_n^{(2)}(\xi, \delta) = \varpi d \sum_{k=1}^{n} C_k G_{n-k}^{(2)}(\xi, \delta) + \varpi(1-d) G_n^{(2)}(\xi, \delta), \quad n \geq 1 \tag{6}$$

$$\frac{\partial}{\partial \xi} G_0^{(3)}(\xi, \delta) + \frac{\partial}{\partial t} G_0^{(3)}(\xi, \delta) + [\varpi + \phi_3(\xi)] G_0^{(3)}(\xi, \delta) = \varpi(1-d) G_0^{(3)}(\xi, \delta) \tag{7}$$

$$\frac{\partial}{\partial \xi} G_n^{(3)}(\xi, \delta) + \frac{\partial}{\partial \delta} G_n^{(3)}(\xi, \delta) + [\varpi + \phi_3(\xi)] G_n^{(3)}(\xi, \delta) = \varpi d \sum_{k=1}^{n} C_k G_{n-k}^{(3)}(\xi, \delta) + \varpi(1-d) G_n^{(3)}(\xi, \delta), \quad n \geq 1 \tag{8}$$

$$\frac{\partial}{\partial \vartheta} R_0^{(i)}(\xi, \vartheta, \delta) + \frac{\partial}{\partial \delta} R_0^{(i)}(\xi, \vartheta, \delta) + [\varpi + \beta_w(\vartheta)] R_0^{(i)}(\xi, \vartheta, \delta) = \varpi(1-d) R_0^{(i)}(\xi, \vartheta, \delta) \tag{9}$$

$$\frac{\partial}{\partial \vartheta} R_n^{(i)}(\xi, \vartheta, \delta) + \frac{\partial}{\partial \varsigma} R_n^{(i)}(\xi, \vartheta, \delta) + [\varpi + \beta_w(\vartheta)] R_n^{(i)}(\xi, \vartheta, \delta) = \varpi d \sum_{k=1}^{n} C_k R_{n-k}^{(i)}(\delta) + \varpi(1-d) R_n^{(i)}(\xi, \vartheta, \delta) \tag{10}$$

$$\frac{d}{d\delta} S_0^{(i)}(\delta) + [\varpi + \alpha] S_0^{(i)}(\delta) = \varpi(1-d) S_0(\delta) \tag{11}$$

$$S_n^{(i)}(\delta)+[\varpi+\alpha]S_n^{(i)}(\delta)=\varpi d\sum_{k=1}^{n}C_kS_{n-k}^{(i)}(\delta)+(\Omega+\beta+\psi)\int_0^\infty P_{n-i}^{(i)}(\xi,\,\delta)d\xi+\varpi(1-d)S_n^{(i)}(\delta) \tag{12}$$

$$\frac{d}{d\delta}Q(\delta)=-\varpi Q(\delta)+\alpha S_0^{(i)}(\delta)+\int_0^\infty\phi_3(\xi)G_0^{(3)}(\xi,\,\delta)d\xi+\varpi(1-d)Q(\delta)+(1-q)\int_0^\infty\mu_i(\xi)P_0^{(i)}(\xi,\,\delta)d\xi \tag{13}$$

$$\frac{d}{d\delta}P_{00}^{(i)}(\delta)=-\varpi P_{00}^{(i)}(\delta)+\int_0^\infty\mu_i(\xi)P_0^{(i)}(\xi,\,\delta)d\xi+\theta Q_{00}(\delta) \tag{14}$$

$$\frac{d}{d\delta}Q_{00}(\delta)=-\theta Q_{00}(\delta)-\Omega P_{00}^{(i)}(\delta)+\Omega \tag{15}$$

Solving the above equation requires satisfying the following boundary scenarios:

$$\begin{aligned}P_n^{(1)}(0,\,\delta)=&\varpi dC_{n+1}Q(\delta)+(1-q)\int_0^\infty\mu_1(\xi)P_{n+1}(\xi,\,\delta)d\xi+\alpha S_{n+1}^{(1)}(\delta)\\&+\int_0^\infty\phi_3(\xi)G_{n+1}^{(3)}(\xi,\,\delta)+\omega_iR_n(\delta)\end{aligned} \tag{16}$$

$$P_n^{(2)}(0,\,\delta)=\int_0^\infty\mu_1(\xi)P_n^{(1)}(\xi,\,\delta)d\xi,\quad n\geq 1 \tag{17}$$

$$G_n^{(1)}(0,\,\delta)=q\int_0^\infty\mu_1(\xi)P_n^{(1)}(\xi,\,\delta)d\xi,\quad n\geq 0 \tag{18}$$

$$G_n^{(2)}(0,\,\delta)=\int_0^\infty\phi_1(\xi)G_n^{(1)}(\xi,\,\delta)d\xi,\quad n\geq 0 \tag{19}$$

$$G_n^{(3)}(0,\,\delta)=\int_0^\infty\phi_2(\xi)G_n^{(2)}(\xi,\,\delta)d\xi,\quad n\geq 0 \tag{20}$$

$$R_n^{(i)}(\xi,0)=\int_0^\infty\beta_iP_n^{(i)}(\xi,\,\vartheta)d\vartheta,\quad n\geq 1 \tag{21}$$

We assume that there were no users in the system initially and that the server was idle. Consequently, the initial conditions are

$$\begin{aligned}&G_0^{(i)}(0)=G_n^{(i)}(0);\quad Q(0)=1,\quad {}_n^{(i)}(0)=0\\&P_n^{(i)}(0)=0;\quad n\geq 0,\quad i=1,2,3\end{aligned} \tag{22}$$

Constructing equations relating to queue length:

$$P^{(i)}(\xi,\chi,\delta)=\sum_{n=0}^{\infty}\chi^n P_n^{(i)}(\xi,\delta);\quad P^{(i)}(\chi,\delta)=\sum_{n=0}^{\infty}\chi^n P_n^{(i)}(\delta),\quad i=1,2$$

$$G^{(i)}(\xi,\chi,\delta)=\sum_{n=0}^{\infty}\chi^n G_n^{(i)}(\xi,\delta);\quad G^{(i)}(\chi,\delta)=\sum_{n=0}^{\infty}\chi^n G_n^{(i)}(\delta),\quad i=1,2,3 \tag{23}$$

$$R^{(i)}(\chi,\delta)=\sum_{n=0}^{\infty}\chi^n R_n^{(i)}(\delta);\quad C(\chi)=\sum_{n=1}^{\infty}C_n\chi^n;\quad i=1,2$$

which, as a function $f(t)$ converges within the $\chi\le 1$ circle, describe the LT of that function.

$$\bar{f}(s)=\int_0^{\infty}e^{-s\delta}f(\delta)d\delta,\quad \mathscr{R}(s)\ge 0$$

Equations (1) to (21) are Laplace transformed, and by applying (22), we get,

$$\frac{\partial}{\partial\xi}\bar{P}_0^{(i)}(\xi,s)+[s+\varpi d+\beta+\psi+\mu_i(\xi)+\Omega]\bar{P}_0^{(i)}(\xi,s)=0 \tag{24}$$

$$\frac{\partial}{\partial\xi}\bar{P}_n^{(i)}(\xi,s)+[s+\varpi d+\beta+\psi+\mu_i(\xi)+\Omega]\bar{P}_n^{(i)}(\xi,s)=\varpi d\sum_{k=1}^{n}C_k\bar{P}_{n-k}^{(i)}(\xi,s) \tag{25}$$

$$\frac{\partial}{\partial\xi}\bar{G}_0^{(1)}(\xi,s)+[s+\varpi d+\phi_1(\xi)]\bar{G}_0^{(1)}(\xi,s)=0 \tag{26}$$

$$\frac{\partial}{\partial\xi}\bar{G}_n^{(1)}(\xi,s)+[s+\varpi d+\phi_1(\xi)]\bar{G}_n^{(1)}(\xi,s)=\varpi d\sum_{k=1}^{n}C_k\bar{G}_{n-k}^{(1)}(\xi,s),\quad n\ge 1 \tag{27}$$

$$\frac{\partial}{\partial\xi}\bar{G}_0^{(2)}(\xi,s)+[s+\varpi d+\phi_2(\xi)]\bar{G}_0^{(2)}(\xi,s)=0 \tag{28}$$

$$\frac{\partial}{\partial\xi}\bar{G}_n^{(2)}(\xi,s)+[s+\varpi d+\phi_2(\xi)]\bar{G}_n^{(2)}(\xi,s)=\varpi d\sum_{k=1}^{n}C_k\bar{G}_{n-k}^{(2)}(\xi,s),\quad n\ge 1 \tag{29}$$

$$\frac{\partial}{\partial\xi}\bar{G}_0^{(3)}(\xi,s)+[s+\varpi d+\phi_3(\xi)]\bar{G}_0^{(3)}(\xi,s)=0 \tag{30}$$

$$\frac{\partial}{\partial\xi}\bar{G}_n^{(3)}(\xi,s)+[s+\varpi d+\phi_3(\xi)]\bar{G}_n^{(3)}(\xi,s)=\varpi d\sum_{k=1}^{n}C_k\bar{G}_{n-k}^{(3)}(\xi,s),\quad n\ge 1 \tag{31}$$

$$\frac{\partial}{\partial \vartheta}\bar{R}_0^{(i)}(\xi,\vartheta,s)+[s+\varpi d+\beta_w(\vartheta)]\bar{R}_0^{(i)}(\xi,\vartheta,s)=0, \quad i=1,2 \tag{32}$$

$$\frac{\partial}{\partial \vartheta}\bar{R}_n^{(i)}(\xi,\vartheta,s)+[s+\varpi d+\beta_w(\vartheta)]\bar{R}_n^{(i)}(\xi,\vartheta,s)=\varpi d\sum_{k=1}^{n}C_k\bar{R}_{n-k}^{(i)}(s) \tag{33}$$

$$[s+\varpi d+\alpha]\bar{S}_0^{(i)}(s)=0 \tag{34}$$

$$[s+\varpi d+\alpha]\bar{S}_n^{(i)}(s)=\varpi d\sum_{k=1}^{n}C_k\bar{S}_{n-k}^{(i)}(s)+(\Omega+\beta+\psi)\int_0^\infty \bar{P}_{n-i}^{(i)}(\xi,s)d\xi \tag{35}$$

$$\begin{aligned}(s+\varpi d)\bar{Q}(s)=1+\alpha\bar{S}_0^{(i)}(s)+\int_0^\infty \phi_3(\xi)\bar{G}_0^{(3)}(\xi,s)d\xi+\varpi(1-d)\bar{Q}(t)\\ +(1-q)\int_0^\infty \mu_i(\xi)P_0^{(i)}(\xi,s)d\xi\end{aligned} \tag{36}$$

$$(s+\varpi)\bar{P}_{00}^{(i)}(s)=\int_0^\infty \mu_i(\xi)\bar{P}_0^{(i)}(\xi,s)d\xi+\theta\bar{Q}_{00}^{(i)}(s) \tag{37}$$

$$(s+\varpi)\bar{Q}_{00}^{(i)}(s)=\bar{\Omega}(1-P_{00}^{(i)}(s)) \tag{38}$$

For the resulting boundary conditions:

$$\begin{aligned}\bar{P}_n^{(1)}(0,s)=\varpi dC_{n+1}\bar{Q}(s)+(1-q)\int_0^\infty \mu_2(\xi)\bar{P}_{n+1}(\xi,s)d\xi+\alpha\bar{S}_{n+1}(s)\\ +\int_0^\infty \phi_3(\xi)\bar{G}_{n+1}^{(3)}(\xi,s)d\xi+\omega_i\bar{R}_n^{(i)}(s)\end{aligned} \tag{39}$$

$$\bar{P}_n^{(2)}(0,s)=\int_0^\infty \bar{P}_n^{(1)}(\xi,s)\mu_1(\xi)d\xi, \quad n\geq 1 \tag{40}$$

$$\bar{G}_n^{(1)}(0,s)=q\int_0^\infty \bar{P}_n^{(i)}(\xi,s)\mu_i(\xi)d\xi \tag{41}$$

$$\bar{G}_n^{(2)}(0,s)=\int_0^\infty \bar{G}_n^{(1)}(\xi,s)\phi_1(\xi)d\xi \tag{42}$$

$$\bar{G}_n^{(3)}(0,s)=\int_0^\infty \bar{G}_n^{(2)}(\xi,s)\phi_2(\xi)d\xi \tag{43}$$

$$\bar{R}_n^{(i)}(\xi, 0) = \int_0^\infty \beta_i \bar{P}_n^{(i)}(\xi, \vartheta) d\vartheta, \quad n \geq 1 \tag{44}$$

Add to equations (24), (28), (30), (32), (34), and multiply (25), (27), (29), (31), (33), (35), χ^n as well as adding up to n from 1 to ∞.

$$\frac{\partial}{\partial \xi} \bar{P}^{(i)}(\xi, \chi, s) + [s + \varpi d(1 - C(\chi)) + \beta + \psi + \mu_i(\xi) + \Omega] \bar{P}^{(i)}(\xi, \chi, s) = 0 \tag{45}$$

$$\frac{\partial}{\partial \xi} \bar{G}^{(1)}(\xi, \chi, s) + [s + \varpi d(1 - C(\chi)) + \phi_1(\xi)] \bar{G}^{(1)}(\xi, \chi, s) = 0 \tag{46}$$

$$\frac{\partial}{\partial \xi} \bar{G}^{(2)}(\xi, \chi, s) + [s + \varpi d(1 - C(\chi)) + \phi_2(\xi)] \bar{G}^{(2)}(\xi, \chi, s) = 0 \tag{47}$$

$$\frac{\partial}{\partial \xi} \bar{G}^{(3)}(\xi, \chi, s) + [s + \varpi d(1 - C(\chi)) + \phi_3(\xi)] \bar{G}^{(3)}(\xi, \chi, s) = 0 \tag{48}$$

$$\frac{\partial}{\partial \vartheta} \bar{R}^{(i)}(\xi, \vartheta, \chi, s) + [s + \varpi d(1 - C(\chi)) + \beta_w(\vartheta)] \bar{R}^{(i)}(\xi, \vartheta, \chi, s) = 0 \tag{49}$$

$$[s + \varpi d(1 - C(\chi)) + \alpha] \bar{S}^{(i)}(\chi, s) = (\Omega + \beta + \psi) \chi \int_0^\infty \bar{P}^{(i)}(\xi, \chi, s) d\xi \tag{50}$$

$$\begin{aligned} \chi \bar{P}^{(1)}(0, \chi, s) =& \varpi d C(\chi) \bar{Q}(s) + (1 - q) \int_0^\infty \mu_1(\xi) \bar{P}^{(1)}(\xi, \chi, s) d\xi + \alpha \bar{S}^{(1)}(\chi, s) \\ &+ \int_0^\infty \bar{G}^{(3)}(\xi, \chi, s) \phi_3(\xi) d\xi + \chi \omega_i \bar{R}^{(i)}(\chi, s) \end{aligned} \tag{51}$$

$$\bar{P}^{(2)}(0, \chi, s) = \int_0^\infty \bar{P}^{(1)}(\xi, \chi, s) \mu_1(\xi) d\xi, \quad n \geq 1 \tag{52}$$

$$\bar{G}^{(1)}(0, \chi, s) = q \int_0^\infty \bar{P}^{(1)}(\xi, \chi, s) \mu_1(\xi) d\xi \tag{53}$$

$$\bar{G}^{(2)}(0, \chi, s) = \int_0^\infty \bar{G}^{(1)}(\xi, \chi, s) \phi_1(\xi) d\xi \tag{54}$$

$$\bar{G}^{(3)}(0, \chi, s) = \int_0^\infty \bar{G}^{(2)}(\xi, \chi, s) \phi_2(\xi) d\xi \tag{55}$$

$$\bar{R}^{(i)}(\xi, 0, \chi, s) = \beta_i \int_0^{\infty} \bar{P}^{(i)}(\xi, \chi, \vartheta, s) d\vartheta, \quad n \geq 1 \tag{56}$$

Integrating (45) to (50) between 0 to ξ, we get

$$\bar{P}^{(1)}(\xi, \chi, s) = \bar{P}^{(1)}(0, \chi, s) e^{-[s+\varpi d(1-C(\chi))+\beta+\psi+\Omega]\xi - \int_0^{\xi} \mu_1(\xi) d\xi} \tag{57}$$

$$\bar{P}^{(2)}(\xi, \chi, s) = \bar{P}^{(2)}(0, \chi, s) e^{-[s+\varpi d(1-C(\chi))+\beta+\psi+\Omega]\xi - \int_0^{\xi} \mu_2(\xi) d\xi} \tag{58}$$

$$\bar{G}^{(1)}(\xi, \chi, s) = \bar{G}^{(1)}(0, \chi, s) e^{-[s+\varpi d(1-C(\chi))]\xi - \int_0^{\xi} \phi_1(\xi) d\xi} \tag{59}$$

$$\bar{G}^{(2)}(\xi, \chi, s) = \bar{G}^{(2)}(0, \chi, s) e^{-[s+\varpi d(1-C(\chi))]\xi - \int_0^{\xi} \phi_2(\xi) d\xi} \tag{60}$$

$$\bar{G}^{(3)}(\xi, \chi, s) = \bar{G}^{(3)}(0, \chi, s) e^{-[s+\varpi d(1-C(\chi))]\xi - \int_0^{\xi} \phi_3(\xi) d\xi} \tag{61}$$

$$\bar{R}^{(i)}(\xi, \vartheta, \chi, s) = \bar{R}^{(i)}(\xi, 0, \chi, s) e^{-[s+\varpi(1-C(\chi))]\vartheta - \int_0^{\vartheta} \beta_w(\vartheta) d\vartheta} \tag{62}$$

Once again, integrating (57) to (62) using fractions with respect to ξ, yiedls

$$\bar{P}^{(1)}(\chi, s) = \bar{P}^{(1)}(0, \chi, s) \left[\frac{1 - \bar{D}_1[s + \varpi d(1 - C(\chi)) + \beta + \psi + \Omega]}{(s + \varpi(1 - C(\chi)) + \beta + \psi + \Omega)} \right] \tag{63}$$

$$\bar{P}^{(2)}(\chi, s) = \bar{P}^{(2)}(0, \chi, s) \left[\frac{1 - \bar{D}_2[s + \varpi d(1 - C(\chi)) + \beta + \psi + \Omega]}{(s + \varpi(1 - C(\chi)) + \beta + \psi + \Omega)} \right] \tag{64}$$

$$\bar{G}^{(1)}(\chi, s) = \bar{G}^{(1)}(0, \chi, s) \left[\frac{1 - \bar{G}_1[s + \varpi d(1 - C(\chi))]}{(s + \varpi d(1 - C(\chi))} \right] \tag{65}$$

$$\bar{G}^{(2)}(\chi, s) = \bar{G}^{(2)}(0, \chi, s) \left[\frac{1 - \bar{G}_2[s + \varpi d(1 - C(\chi))]}{(s + \varpi d(1 - C(\chi))} \right] \tag{66}$$

$$\bar{G}^{(3)}(\chi, s) = \bar{G}^{(3)}(0, \chi, s) \left[\frac{1 - \bar{G}_3[s + \varpi d(1 - C(\chi))]}{(s + \varpi d(1 - C(\chi))} \right] \tag{67}$$

$$\bar{R}^{(i)}(\xi, \chi, s) = \bar{R}^{(i)}(\xi, 0, \chi, s) \left[\frac{1 - \bar{R}_i[s + \varpi(1 - C(\chi))]}{(s + \varpi(1 - C(\chi)))} \right] \tag{68}$$

Again integrating (68) by parts w.r.t ξ yields

$$\bar{R}^{(1)}(\chi, s) = \beta_1 \bar{P}^{(1)}(0, \chi, s)\left[\frac{1-\bar{D_1}[s+\varpi d(1-C(\chi))+\beta+\psi+\Omega]}{(s+\varpi(1-C(\chi))+\beta+\psi+\Omega)}\right]\left[\frac{1-\bar{R_1}[s+\varpi(1-C(\chi))]}{(s+\varpi(1-C(\chi)))}\right] \tag{69}$$

$$\bar{R}^{(2)}(\chi, s) = \beta_2 \bar{P}^{(2)}(0, \chi, s)\left[\frac{1-\bar{D_2}[s+\varpi d(1-C(\chi))+\beta+\psi+\Omega]}{(s+\varpi(1-C(\chi))+\beta+\psi+\Omega)}\right]\left[\frac{1-\bar{R_2}[s+\varpi(1-C(\chi))]}{(s+\varpi(1-C(\chi)))}\right] \tag{70}$$

where,

$$\left.\begin{aligned}
\bar{D_i}[s+\varpi d(1-C(\chi))+\beta+\psi+\Omega] &= \textstyle\int_0^\infty e^{-[s+\varpi(1-C(\chi))+\beta+\psi+\Omega]\xi} dD_i(\xi)\\
\bar{G_1}[s+\varpi d(1-C(\chi))] &= \textstyle\int_0^\infty e^{-[s+\varpi(1-C(\chi))]\xi} dG_1(\xi)\\
\bar{G_2}[s+\varpi d(1-C(\chi))] &= \textstyle\int_0^\infty e^{-[s+\varpi(1-C(\chi))]\xi} dG_2(\xi)\\
\bar{G_3}[s+\varpi d(1-C(\chi))] &= \textstyle\int_0^\infty e^{-[s+\varpi(1-C(\chi))]\xi} dG_3(\xi)\\
\bar{R_i}[s+\varpi d(1-C(\chi))] &= \textstyle\int_0^\infty e^{-[s+\varpi(1-C(\chi))]\vartheta} dR_i(\vartheta)
\end{aligned}\right\} \tag{71}$$

These are the Laplace-Stieltjes transforms of the following: service time $P_i(\xi)$; vacation time $G_1(\xi)$, $G_2(\xi)$, $G_3(\xi)$; first, second, and third stages; and repair time $R_i(\xi)$ respectively.

Now, multiplying both side of equations (57) to (62) by $\mu_1(\xi)$, $\mu_2(\xi)$, $\phi_1(\xi)$, $\phi_2(\xi)$, $\phi_3(\xi)$ and $\beta_w(\vartheta)$ and integrating over ξ, we get

$$\int_0^\infty \bar{P}^{(1)}(\xi, \chi, s)\mu_1(\xi)d\xi = \bar{P}^{(1)}(0, \chi, s)\bar{D_1}[s+\varpi(1-C(\chi))+\beta+\psi+\Omega] \tag{72}$$

$$\int_0^\infty \bar{P}^{(2)}(\xi, \chi, s)\mu_2(\xi)d\xi = \bar{P}^{(2)}(0, \chi, s)\bar{D_2}[s+\varpi(1-C(\chi))+\beta+\psi+\Omega] \tag{73}$$

$$\int_0^\infty \bar{G}^{(1)}(\xi, \chi, s)\phi_1(\xi)d\xi = \bar{G}^{(1)}(0, \chi, s)\bar{G_1}[s+\varpi d(1-C(\chi))] \tag{74}$$

$$\int_0^\infty \bar{G}^{(2)}(\xi, \chi, s)\phi_2(\xi)d\xi = \bar{G}^{(2)}(0, \chi, s)\bar{G_2}[s+\varpi d(1-C(\chi))] \tag{75}$$

$$\int_0^\infty \bar{G}^{(3)}(\xi, \chi, s)\phi_3(\xi)d\xi = \bar{G}^{(3)}(0, \chi, s)\bar{G_3}[s+\varpi d(1-C(\chi))] \tag{76}$$

$$\int_0^\infty \bar{R}^{(i)}(\xi, \vartheta, \chi, s)\beta_w(\vartheta)d\vartheta = \bar{R}^{(i)}(\xi, 0, \chi, s)\bar{R_i}[s+\varpi d(1-C(\chi))] \tag{77}$$

Using (72), (52) is reduced to

$$\bar{P}^{(2)}(0,\chi,s)=\bar{P}^{(1)}(0,\chi,s)\bar{D_1}[s+\varpi d(1-C(\chi))+\beta+\psi+\Omega] \tag{78}$$

$$\begin{aligned}\bar{R}^{(2)}(\chi,s)=&\beta_2\bar{P}^{(1)}(0,\chi,s)\bar{D_1}[s+\varpi d(1-C(\chi))+\beta+\psi+\Omega]\\&\left[\frac{1-\bar{D_2}[s+\varpi d(1-C(\chi))+\beta+\psi+\Omega]}{(s+\varpi(1-C(\chi))+\beta+\psi+\Omega)}\right]\left[\frac{1-\bar{R_2}[s+\varpi(1-C(\chi))]}{(s+\varpi(1-C(\chi)))}\right]\end{aligned} \tag{79}$$

Now using (72), (53) is reduced to

$$\bar{G}^{(1)}(0,\chi,s)=q\bar{P}^{(1)}(0,\chi,s)\bar{D_1}[s+\varpi d(1-C(\chi))+\beta+\psi+\Omega] \tag{80}$$

Using equation (74) & (80) in (54)

$$\bar{G}^{(2)}(0,\chi,s)=q\bar{P}^{(1)}(0,\chi,s)\bar{D_1}[s+\varpi d(1-C(\chi))+\beta+\psi+\Omega]\bar{G}_1[s+\varpi d(1-C(\chi))] \tag{81}$$

By using equations (75) & (81) in (55), we get

$$\bar{G}^{(3)}(0,\chi,s)=q\bar{P}^{(1)}(0,\chi,s)\bar{D_1}[s+\varpi d(1-C(\chi))+\beta+\psi+\Omega]\bar{G}_1[s+\varpi d(1-C(\chi))]\bar{G}_2[s+\varpi d(1-C(\chi))] \tag{82}$$

From (50) & (57), we get

$$\bar{S}^{(i)}(\chi,s)=\left[\frac{(\beta+\psi+\Omega)\chi}{(s+\varpi(1-C(\chi)))+\alpha}\right]\bar{P}^{(i)}(0,\chi,s)\left[\frac{1-\bar{D}_i[s+\varpi(1-C(\chi))+\beta+\psi+\Omega]}{(s+\varpi(1-C(\chi)))+\beta+\psi+\Omega}\right] \tag{83}$$

Using equation (36) in (51),

$$\bar{P}^{(1)}(0,\chi,s)=\frac{Nr(\chi)}{Dr(\chi)} \tag{84}$$

$$Nr(\chi)=\varpi d\bar{Q}(s)[C(\chi)-1]+1-s\bar{Q}(s)$$

$$Dr(\chi) = \chi - (1-q)\bar{D_1}[s+\varpi d(1-C(\chi))+\beta+\psi+\Omega] - q\bar{D_1}[s+\varpi d(1-C(\chi))+\beta+\psi+\Omega]\bar{G_1}[s+\varpi d(1-C(\chi))]$$

$$\bar{G_2}[s+\varpi d(1-C(\chi))]\bar{G_3}[s+\varpi d(1-C(\chi))] - \alpha\left[\frac{(\beta+\psi+\Omega)\chi}{(s+\varpi(1-C(\chi)))+\alpha}\right]$$

$$\left[\frac{1-\bar{D_1}[s+\varpi(1-C(\chi))+\beta+\psi+\Omega]}{(s+\varpi(1-C(\chi)))+\beta+\psi+\Omega}\right] - \chi\beta_1\omega_1\left[\frac{1-\bar{D_1}[s+\varpi(1-C(\chi))+\beta+\psi+\Omega]}{(s+\varpi(1-C(\chi)))+\beta+\psi+\Omega}\right]$$

$$\left[\frac{1-\bar{R_1}[s+\varpi(1-C(\chi))]}{(s+\varpi(1-C(\chi)))}\right] - \chi\beta_2\omega_2\left[\frac{1-\bar{D_2}[s+\varpi(1-C(\chi))+\beta+\psi+\Omega]}{(s+\varpi(1-C(\chi)))+\beta+\psi+\Omega}\right]$$

$$\bar{D_1}[s+\varpi(1-C(\chi))+\beta+\psi+\Omega]\left[\frac{1-\bar{R_2}[s+\varpi(1-C(\chi))]}{(s+\varpi(1-C(\chi)))}\right]$$

Substituting the values of $\bar{P}^{(1)}(0, \chi, s)$ from (63) to (67) and (69), (79), (83), we get

$$\left.\begin{aligned}
\bar{P}^{(1)}(\chi, s) &= \bar{P}^{(1)}(0, \chi, s)\left[\frac{1-\bar{D_1}[s+\varpi(1-C(\chi))+\beta+\psi+\Omega]}{(s+\varpi(1-C(\chi)))+\beta+\psi+\Omega}\right]\\
\bar{P}^{(2)}(\chi, s) &= \bar{P}^{(1)}(0, \chi, s)\bar{D_1}[s+\varpi(1-C(\chi))+\beta+\psi+\Omega]\left[\frac{1-\bar{D_2}[s+\varpi(1-C(\chi))+\beta+\psi+\Omega]}{(s+\varpi(1-C(\chi)))+\beta+\psi+\Omega}\right]\\
\bar{G}^{(1)}(\chi, s) &= q\bar{P}^{(1)}(0, \chi, s)\bar{D_1}(K_1(\chi, s))\left(\frac{1-\bar{G_1}(K_2(\chi, s))}{K_2(\chi, s)}\right)\\
\bar{G}^{(2)}(\chi, s) &= q\bar{P}^{(1)}(0, \chi, s)\bar{D_1}(K_1(\chi, s))\bar{G_1}(K_2(\chi, s))\left(\frac{1-\bar{G_2}(K_2(\chi, s))}{K_2(\chi, s)}\right)\\
\bar{G}^{(3)}(\chi, s) &= q\bar{P}^{(1)}(0, \chi, s)\bar{D_1}(K_1(\chi, s))\bar{G_1}(K_2(\chi, s))\bar{G_2}(K_2(\chi, s))\left(\frac{1-\bar{G_3}(K_2(\chi, s))}{K_2(\chi, s)}\right)\\
\bar{R}^{(1)}(\chi, s) &= \beta_1\bar{P}^{(1)}(0, \chi, s)\left(\frac{1-\bar{D_1}(K_1(\chi, s))}{K_1(\chi, s)}\right)\left(\frac{1-\bar{R_1}(K_2(\chi, s))}{K_2(\chi, s)}\right)\\
\bar{R}^{(2)}(\chi, s) &= \beta_2\bar{P}^{(1)}(0, \chi, s)\bar{D_1}(K_1(\chi, s))\left(\frac{1-\bar{D_2}(K_1(\chi, s))}{K_1(\chi, s)}\right)\left(\frac{1-\bar{R_2}(K_2(\chi, s))}{K_2(\chi, s)}\right)\\
\bar{S}^{(1)}(\chi, s) &= \left[\frac{(\beta+\psi+\Omega)\chi}{K_3(\chi, s)}\right]\bar{P}^{(1)}(0, \chi, s)\left(\frac{1-\bar{D_1}(K_1(\chi, s))}{K_1(\chi, s)}\right)\\
\bar{S}^{(2)}(\chi, s) &= \left[\frac{(\beta+\psi+\Omega)\chi}{K_3(\chi, s)}\right]\bar{P}^{(1)}(0, \chi, s)\bar{D_1}(K_1(\chi, s))\left(\frac{1-\bar{D_2}(K_1(\chi, s))}{K_1(\chi, s)}\right)
\end{aligned}\right\} \quad (85)$$

3. Steady state analysis

In this part, we are going to establish the steady-state probability (SS) distribution for our queueing system. To obtain the SS probability suppress the input "δ "whenever it appears in the time-dependent analysis. Using Tauberian property,

$$\lim_{s\to 0} s\bar{f}(s) = \lim_{\varsigma\to\infty} f(\delta)$$

$$\left.\begin{aligned}
\bar{P}^{(1)}(\chi) &= \bar{P}^{(1)}(0,\chi)\left[\frac{1-\bar{D}_1(K_1(\chi))}{K_1(\chi)}\right]\\
\bar{P}^{(2)}(\chi) &= \bar{P}^{(1)}(0,\chi)\bar{D}_1(K_1(\chi))\left[\frac{1-\bar{D}_2(K_1(\chi))}{(K_1(\chi))}\right]\\
\bar{G}^{(1)}(\chi) &= q\bar{P}^{(1)}(0,\chi)\bar{D}_1(K_1(\chi))\left(\frac{1-\bar{G}_1(K_2(\chi))}{K_2(\chi)}\right)\\
\bar{G}^{(2)}(\chi) &= q\bar{P}^{(1)}(0,\chi)\bar{D}_1(K_1(\chi))\bar{G}_1(K_2(\chi))\left(\frac{1-\bar{G}_2(K_2(\chi))}{K_2(\chi)}\right)\\
\bar{G}^{(3)}(\chi) &= q\bar{P}^{(1)}(0,\chi)\bar{D}_1(K_1(\chi))\bar{G}_1(K_2(\chi))\bar{G}_2(K_2(\chi))\left(\frac{1-\bar{G}_3(K_2(\chi))}{K_2(\chi)}\right)\\
\bar{R}^{(1)}(\chi) &= \beta_1\bar{P}^{(1)}(0,\chi)\left(\frac{1-\bar{D}_1(K_1(\chi))}{K_1(\chi)}\right)\left(\frac{1-\bar{R}_1(K_2(\chi))}{K_2(\chi)}\right)\\
\bar{R}^{(2)}(\chi) &= \beta_2\bar{P}^{(1)}(0,\chi)\bar{D}_1(K_1(\chi))\left(\frac{1-\bar{D}_2(K_1(\chi))}{K_1(\chi)}\right)\left(\frac{1-\bar{R}_2(K_2(\chi))}{K_2(\chi)}\right)\\
\bar{S}^{(1)}(\chi) &= \left[\frac{(\beta+\psi+\Omega)\chi}{K_3(\chi)}\right]\bar{P}^{(1)}(0,\chi)\left(\frac{1-\bar{D}_1(K_1(\chi))}{K_1(\chi)}\right)\\
\bar{S}^{(2)}(\chi) &= \left[\frac{(\beta+\psi+\Omega)\chi}{K_3(\chi)}\right]\bar{P}^{(1)}(0,\chi)\bar{D}_1(K_1(\chi))\left(\frac{1-\bar{D}_2(K_1(\chi))}{K_1(\chi)}\right)
\end{aligned}\right\} \tag{86}$$

where,

$$\bar{P}^{(1)}(0,\chi) = \frac{Nr(\chi)}{Dr(\chi)}$$

$$Nr(\chi) = \varpi d[C(\chi)-1]\bar{Q}$$

$$Dr(\chi) = \chi - (1-q)\bar{D_1}(K_1(\chi)) - q\bar{D_1}(K_1(\chi))\bar{G_1}(K_2(\chi))\bar{G_2}(K_2(\chi))\bar{G_3}(K_2(\chi)) - \alpha\left[\frac{(\beta+\psi+\Omega)\chi}{K_3(\chi)}\right]\left[\frac{1-\bar{D_1}(K_1(\chi))}{K_1(\chi)}\right]$$

$$-\chi\beta_1\omega_1\left[\frac{1-\bar{D_1}(K_1(\chi))}{K_1(\chi)}\right]\left[\frac{1-\bar{R_1}(K_2(\chi))}{K_2(\chi)}\right] - \chi\beta_2\omega_2\bar{D_1}(K_1(\chi))\left[\frac{1-\bar{D_2}(K_1(\chi))}{K_1(\chi)}\right]\left[\frac{1-\bar{R_2}(K_2(\chi))}{K_2(\chi)}\right]$$

where,

$$K_1(\chi) = [\varpi d(1-C(\chi)) + \beta + \psi + \Omega]$$

$$K_2(\chi) = [\varpi d(1-C(\chi))]$$

$$K_3(\chi) = [\varpi d(1-C(\chi)) + \alpha]$$

3.1 *Queue size distribution*

We may calculate the PGF of the queue size distribution at a random period by adding (86) with the idle term.

$$U(\chi) = P^{(1)}(\chi) + P^{(2)}(\chi) + G^{(1)}(\chi) + G^{(2)}(\chi) + G^{(3)}(\chi) + R^{(1)}(\chi) + R^{(2)}(\chi) + S^{(1)}(\chi) + S_2^{(2)}(\chi) + \bar{Q} \tag{87}$$

$$\bar{Q} = \frac{Nr(\chi)}{Dr(\chi)} \tag{88}$$

$$Nr(\chi) = 1 + \varpi dE(I)E(D_1) + q\varpi dE(I)\bar{D_1}(\beta+\psi+\Omega)[E(G_1)+E(G_2)+E(G_3)]$$

$$+\alpha\left[\frac{1-\bar{D_1}(\beta+\psi+\Omega)}{\varpi dE(I)}\right] + (\Omega+\beta+\psi)E(D_1) + \omega_1\beta_1\left[\frac{1-\bar{D_1}(\beta+\psi+\Omega)}{\beta+\psi+\Omega}\right]E(R_1)$$

$$+\omega_2\beta_2\bar{D_1}(\beta+\psi+\Omega)\left[\frac{1-\bar{D_2}(\beta+\psi+\Omega)}{\beta+\psi+\Omega}\right]E(R_2)$$

$$Dr(\chi) = -\varpi dE(I)\bar{D_1}(\beta+\psi+\Omega)E(D_2) - q\varpi dE(I)\bar{D_1}(\beta+\psi+\Omega)E(G_1)$$

$$+q[\varpi dE(I)]^2\bar{D_1}(\beta+\psi+\Omega)E(G_1)E(G_2) - q\bar{D_1}(\beta+\psi+\Omega)[\varpi dE(I)]^3E(G_1)E(G_2)E(G_3)$$

$$+\beta_1\varpi dE(I)E(D_1)E(R_1) + \beta_2\varpi dE(I)\bar{D_1}(\beta+\psi+\Omega)E(D_2)E(R_2) + (\beta+\psi+\Omega)E(D_1)$$

$$+(\beta+\psi+\Omega)\bar{D_1}(\beta+\psi+\Omega)E(D_2) + 1 + q\varpi dE(I)\bar{D_1}(\beta+\psi+\Omega)[E(G_2)+E(G_3)]$$

$$+\alpha\left[\frac{1-\bar{D}_1(\beta+\psi+\Omega)}{\varpi dE(I)}\right]+(\beta+\psi+\Omega)E(D_1)+\omega_1\beta_1\left[\frac{1-\bar{D}_1(\beta+\psi+\Omega)}{\beta+\psi+\Omega}\right]E(R_1)$$

$$+\omega_2\beta_2\bar{D}_1(\beta+\psi+\Omega)\left[\frac{1-\bar{D}_2(\beta+\psi+\Omega)}{\beta+\psi+\Omega}\right]E(R_2)$$

Substituting (86), (88) in (87)

$$U(\chi)=\frac{Nr(\chi)}{Dr(\chi)} \tag{89}$$

where,

$$\begin{aligned}
Nr(\chi)=\;&\bar{Q}[\{\varpi d(c(\chi)-1)\}\{(1-\bar{D}_1(K_1(\chi)))K_2(\chi)K_3(\chi)+\bar{D}_1(K_1(\chi))(1-\bar{D}_2(K_1(\chi)))K_2(\chi)K_3(\chi)\\
&+q\bar{D}_1(K_1(\chi))(1-\bar{G}_1(K_2(\chi)))K_1(\chi)K_3(\chi)+q\bar{D}_1(K_1(\chi))\bar{G}_1(K_2(\chi))(1-\bar{G}_2(K_2(\chi)))K_1(\chi)K_3(\chi)\\
&+q\bar{D}_1(K_1(\chi))\bar{G}_1(K_2(\chi))\bar{G}_2(K_2(\chi))(1-\bar{G}_3(K_2(\chi)))K_1(\chi)K_3(\chi)+\beta_1(1-\bar{D}_1(K_1(\chi)))(1-\bar{R}_1(K_2(\chi)))\\
&K_3(\chi)+\beta_2\bar{D}_1(K_1(\chi))(1-\bar{D}_2(K_1(\chi)))(1-\bar{R}_2(K_2(\chi)))K_3(\chi)+[\chi(\Omega+\beta+\psi)](1-\bar{D}_1(K_1(\chi)))K_2(\chi)\\
&+[\chi(\Omega+\beta+\psi)]\bar{D}_1(K_1(\chi))(1-\bar{D}_2(K_1(\chi)))K_2(\chi)\}+\{\chi K_1(\chi)K_2(\chi)K_3(\chi)-(1-q)(\bar{D}_1(K_1(\chi)))\\
&K_1(\chi)K_2(\chi)K_3(\chi)-q\bar{D}_1(K_1(\chi))\bar{G}_1(K_2(\chi))\bar{G}_2(K_2(\chi))\bar{G}_3(K_2(\chi))K_1(\chi)K_2(\chi)K_3(\chi)-\alpha[\chi(\Omega+\beta+\psi)]\\
&(1-\bar{D}_1(K_1(\chi)))K_2(\chi)-\chi\omega_1\beta_1(1-\bar{D}_1(K_1(\chi)))(1-\bar{R}_1(K_2(\chi)))K_3(\chi)-\chi\omega_2\beta_2\bar{D}_1(K_1(\chi))\\
&(1-\bar{D}_2(K_1(\chi)))(1-\bar{R}_2(K_2(\chi)))K_3(\chi)\}]\bar{D}_1(K_1(\chi))(1-\bar{R}_2(K_2(\chi)))K_3(\chi)\}]
\end{aligned}$$

$$\begin{aligned}
Dr(\chi)=\;&[\chi K_1(\chi)K_2(\chi)K_3(\chi)-(1-q)(\bar{D}_1(K_1(\chi)))K_1(\chi)K_2(\chi)K_3(\chi)-q\bar{D}_1(K_1(\chi))\bar{G}_1(K_2(\chi))\bar{G}_2(K_2(\chi))\bar{K}_1(\chi)\\
&G_3(K_2(\chi))K_2(\chi)K_3(\chi)-\alpha[\chi(\Omega+\beta+\psi)](1-\bar{D}_1(K_1(\chi)))K_2(\chi)-\chi\omega_1\beta_1(1-\bar{D}_1(K_1(\chi)))\\
&(1-\bar{R}_1(K_2(\chi)))K_3(\chi)-\chi\omega_2\beta_2\bar{D}_1(K_1(\chi))(1-\bar{D}_2(K_1(\chi)))(1-\bar{R}_2(K_2(\chi)))K_3(\chi)]
\end{aligned}$$

3.2 *Condition of stabilisation*

PGF has to get $P(1)=1$. This requirement is satisfied by the result obtained by equating the term to 1 and using the L' Hopital rules.

$$-\alpha\varpi dE(I)[\omega_1\beta_1(1-\bar{D}_1(\beta+\psi+\Omega))E(R_1)+\omega_2\beta_2 D_1(\beta+\psi+\Omega)(1-\bar{D}_2(\beta+\psi+\Omega))E(R_1)]$$

$$\rho = -\alpha\varpi dE(I)[\omega_1\beta_1(1-A_1)E(R_1)+\omega_2\beta_2 A_1(1-A_2)E(R_2)] \tag{90}$$

Then, $\rho < 1$ is the criterion that has to be satisfied in order for the SS to exist for the structure that is being studied.

3.3 *System performance evaluation*

- Let $P^{(i)}$ be the SS probability of the server being in busy state.

$$\bar{P}^{(i)}(\chi) = \lim_{\chi\to 1}\left(P^{(1)}(\chi)+P^{(2)}(\chi)\right)$$

$$= \left\{ \begin{array}{r} \dfrac{\varpi dE(I)\bar{Q}E(D_1)[-1+\varpi dE(I)E(D_2)]}{\begin{array}{r}(1+\varpi dE(I)E(D_1)+q\varpi dE(I)\bar{D}_1(\Omega+\beta+\psi))[E(G_1)+E(G_2)+E(G_3)] \\ +\alpha\left[\dfrac{1-\bar{D}_1(\beta+\psi+\Omega)}{\varpi dE(I)}\right]+(\beta+\psi+\Omega)E(D_1)+\omega_1\beta_1\left[\dfrac{1-\bar{D}_1(\beta+\psi+\Omega)}{\beta+\psi+\Omega}\right] \\ E(R_1)+\omega_2\beta_2\bar{D}_1(\beta+\psi+\Omega)\left[\dfrac{1-\bar{D}_2(\beta+\psi+\Omega)}{\beta+\psi+\Omega}\right]E(R_2)\end{array}} \end{array} \right\}$$

- Let $\bar{G}^{(i)}$ be the SS probability of the server being in vacation.

$$\bar{G}^{(i)}(\chi) = \lim_{\chi\to 1}\left(\bar{G}^{(1)}(\chi)+\bar{G}^{(2)}(\chi)+\bar{G}^{(3)}(\chi)\right)$$

$$= \left\{ \begin{array}{r} \dfrac{q[\varpi dE(I)]^2\bar{Q}E(D_1)E(G_1)(1-\varpi dE(I)E(G_2)+[\varpi dE(I))]^2E(G_2)E(G_3))}{\begin{array}{r}(1+\varpi dE(I)E(D_1)+q\varpi dE(I)\bar{D}_1(\Omega+\beta+\psi))[E(G_1)+E(G_2)+E(G_3)] \\ +\alpha\left[\dfrac{1-\bar{D}_1(\beta+\psi+\Omega)}{\varpi dE(I)}\right]+(\beta+\psi+\Omega)E(D_1)+\omega_1\beta_1\left[\dfrac{1-\bar{D}_1(\beta+\psi+\Omega)}{\beta+\psi+\Omega}\right] \\ E(R_1)+\omega_2\beta_2\bar{D}_1(\beta+\psi+\Omega)\left[\dfrac{1-\bar{D}_2(\beta+\psi+\Omega)}{\beta+\psi+\Omega}\right]E(R_2)\end{array}} \end{array} \right\}$$

- let $\bar{R}^{(i)}$ be the SS probability of the server being in repair state

$$\bar{R}^{(i)}(\chi) = \lim_{\chi\to 1}\left(\bar{R}^{(1)}(\chi)+\bar{R}^{(2)}(\chi)\right)$$

$$= \left\{ \begin{array}{r} \dfrac{\varpi dE(I)\bar{Q}E(D_1)[\beta_1 E(R_1)-\beta_2(\varpi dE(I))E(D_2)E(R_2)]}{\begin{array}{r}(1+\varpi dE(I)E(D_1)+q\varpi dE(I)\bar{D}_1(\Omega+\beta+\psi))[E(G_1)+E(G_2)+E(G_3)] \\ +\alpha\left[\dfrac{1-\bar{D}_1(\beta+\psi+\Omega)}{\varpi dE(I)}\right]+(\beta+\psi+\Omega)E(D_1)+\omega_1\beta_1\left[\dfrac{1-\bar{D}_1(\beta+\psi+\Omega)}{\beta+\psi+\Omega}\right] \\ E(R_1)+\omega_2\beta_2\bar{D}_1(\beta+\psi+\Omega)\left[\dfrac{1-\bar{D}_2(\beta+\psi+\Omega)}{\beta+\psi+\Omega}\right]E(R_2)\end{array}} \end{array} \right\}$$

- let $\bar{S}^{(i)}$ be the SS probability of the server being in service disruption

$$\bar{S}^{(i)}(\chi) = \lim_{\chi \to 1} \left(\bar{S}^{(1)}(\chi) + \bar{S}^{(2)}(\chi) \right)$$

$$= \left\{ \begin{array}{c} \dfrac{(\Omega+\psi+\beta)\bar{Q}E(D_1)[1-\varpi dE(I)E(D_2)]}{(1+\varpi dE(I)E(D_1)+q\varpi dE(I)\bar{D}_1(\Omega+\beta+\psi))[E(G_1)+E(G_2)+E(G_3)]} \\ +\alpha\left[\dfrac{1-\bar{D}_1(\beta+\psi+\Omega)}{\varpi dE(I)}\right]+(\beta+\psi+\Omega)E(D_1)+\omega_1\beta_1\left[\dfrac{1-\bar{D}_1(\beta+\psi+\Omega)}{\beta+\psi+\Omega}\right] \\ E(R_1)+\omega_2\beta_2\bar{D}_1(\beta+\psi+\Omega)\left[\dfrac{1-\bar{D}_2(\beta+\psi+\Omega)}{\beta+\psi+\Omega}\right]E(R_2) \end{array} \right\}$$

3.4 *Average queue length*

Differentiating (68) with respect to 'χ' and computing at $\chi = 1$ produces, under SS conditions, the average number of users in the line (L_q).

$$L_q = \lim_{\chi \to 1} \frac{d}{d\chi} P(\chi)$$

$$P'(1) = Q\left(\frac{Nr''(1)Dr'(1) - Dr''(1)Nr'(1)}{2(Dr'(1))^2} \right)$$

$$D'(1) = -\alpha\varpi dE(I)[\omega_1\beta_1(1-A_1)E(R_1)+\omega_2\beta_2A_1(1-A_2)E(R_2)]$$

$$D''(1) = A[\varpi dE(I)][\varpi dE(I)-\alpha]+A(1-A_1)\alpha\varpi dE(I)+(1-A_1)\omega_1\beta_1\varpi dE(I)E(R_1)\{-\alpha+\varpi dE(I)\}+A_1\omega_2$$

$$(1-A_2)\beta_2\varpi dE(I)E(R_2)\{-\alpha+\varpi dE(I)\}+\alpha[\varpi dE(I)]^2\{(1-A_1)+E(D_1)E(R_1)-\omega_2\beta_2(1-A_2)E(R_2)$$

$$E(D_1)-\omega_2\beta_2A_1E(R_2)E(D_2)+A(1-A_1)E(D_1)\}-AA_1[\varpi dE(I)]^2-\alpha AA_1[\varpi dE(I(I-1))](2-q)-\alpha[\varpi d$$

$$E(I(I-1))]\{\omega_1\beta_1E(R_1)(1-A_1)+\omega_2\beta_2E(R_2)A_1(1-A_2)\}-\alpha qAA_1[\varpi dE(I)]^2\{E(G_1)+E(G_2)+E(G_3)\}$$

$$N'(1) = -\alpha\varpi dE(I)[\omega_1\beta_1(1-A_1)E(R_1)+\omega_2\beta_2A_1(1-A_2)E(R_2)]$$

$$N''(1) = 2\left[(\varpi dE(I))^2\left\{\alpha(A_1A_2-1)+\alpha\left(\beta_1(1-A_1)E(R_1)+\beta_2A_1(1-A_2)E(R_2)\right)+(A_1-A+AA_1A_2)\right\}\right]$$

$$+\left[A(\varpi dE(I))[\varpi dE(I)-\alpha]+A(1-A_1)\alpha\varpi dE(I)+(1-A_1)\omega_1\beta_1\varpi dE(I)E(R_1)[\varpi dE(I)-\alpha]\right.$$

$$+A_1(1-A_2)\omega_2\beta_2\varpi dE(I)E(R_2)[\varpi dE(I)-\alpha]+\alpha[\varpi dE(I)]^2\{(1-A_1)+E(D_1)E(R_1)$$

$$-\omega_2\beta_2(1-A_2)E(R_2)E(D_1)-\omega_2\beta_2A_1E(R_2)E(D_2)+A(1-A_1)E(D_1)\}-3AA_1[\varpi dE(I)]^2$$

$$-\alpha AA_1(2-q)[\varpi dE(I(I-1))]-\alpha[\varpi dE(I(I-1))]\{\omega_1\beta_1E(R_1)(1-A_1)+\omega_2\beta_2E(R_2)A_1(1-A_2)\}$$

$$\left.+\alpha qAA_1(\varpi dE(I))^2[E(G_1)+E(G_2)+E(G_3)]\right]$$

where,

$$A=(\beta+\psi+\Omega);\quad A_1=D_1(\beta+\psi+\Omega);\quad A_2=D_2(\beta+\psi+\Omega)$$

• The Little's formula (W_q) is used to determine how long an average customer waits in line.

$$W_q=\frac{L_q}{\varpi E(I)}$$

3.5 *Particular cases*

Case 1 No working breakdown and no balking

Let $\beta_w=0$, $d=0$, our model is reduced to a $M^X/G_1, G_2/1$ queue with Bernoulli schedule, which coincides with [5].

Case 2 No Service disruption and no catastrophe

Let $\alpha=0$, $\Omega=0$, our model is reduced to a $M^X/G_1, G_2/1$ queue with Bernoulli schedule, which coincides with [11].

4. Practical application of the model

In this $M^X/G_1, G_2/1$ queueing model helps the large-scale internet service provider (ISP) understand and optimize its customer support operations by analyzing the impact of disruptions, breakdowns, balking behavior, catastrophic events, and agent availability on service delivery. By leveraging this model, the ISP can develop strategies to minimize downtime, improve response times, and enhance overall customer satisfaction with their broadband services.

• Service Disruption: Occasional technical issues, such as server downtimes, network outages, or software glitches, can disrupt the smooth provision of internet services. These disruptions can lead to delays in resolving customer issues and impact overall service quality.

• Working Breakdown: Technicians or support staff may encounter breakdowns in equipment (e.g., routers, modems) or face technical challenges while troubleshooting customer problems. These breakdowns can temporarily halt the resolution process until the issue is identified and rectified.

• Balking: Customers experiencing internet connectivity issues may hesitate to contact customer support if they anticipate long wait times or perceive previous experiences with support as unsatisfactory. They may resort to alternative methods for problem resolution or delay seeking assistance, resulting in increased frustration.

• Catastrophe: Catastrophic events such as natural disasters (e.g., hurricanes, earthquakes) or large-scale cyberattacks can severely disrupt internet infrastructure, leading to widespread service outages and impacting the ISP's ability to provide timely support to affected customers.

• Extended Server Vacation with Bernoulli Schedule: Customer support agents may take scheduled breaks, vacations, or training sessions, reducing the available workforce to handle support requests. The probability of an agent returning to handle customer inquiries after their break could depend on factors such as call volume, staffing levels, and operational priorities, following a Bernoulli schedule.

5. Numerical outcomes

This section utilises MATLAB to show how assessments of system behaviour are influenced by various parameters. In this section, arrivals occur in batches following a geometric distribution with an average size of 2. Subsequently, the service, vacation, and repair stages are characterized by exponential durations. By creating erroneous assumptions about the parameters, we make sure that the stability criterion is satisfied. Tables 1 to 3 present estimated values for our queueing system's utilisation factor (ρ), average queue length (L_q), and average waiting time (W_q).

Table 1. The influence of the arrival rate (ϖ) on ρ, L_q, W_q

Arrival rate (ϖ)	ρ	L_q	W_q
0.6	0.5210	7.96266	6.63555
0.7	0.6078	15.46139	11.04385
0.8	0.6947	27.68105	17.30065
0.9	0.7815	46.44279	25.80155
1.0	0.8683	73.91679	36.95839

$\alpha = 0.13$, $E = 1.6$, $\beta_1 = 0.6$, $\beta_2 = 1.1$ $R_1 = 2$, $R_2 = 1.2$, $D_1 = 1.5$, $D_2 = 1.7$, $d = 0.4$, $A = 1.2$, $I = 2.5$, $\omega_1 = 1.8$, $\omega_2 = 0.6$, $\Omega = 4.6$, $G_1 = 5.1$, $G_2 = 7.5$, $G_3 = 6$.

Table 2. The influence of the FPR rate $\eta_1(\xi)$ on ρ, L_q, W_q

FPR $\eta_1(\xi)$	ρ	L_q	W_q
2.0	0.1464	0.00958	0.00479
2.5	0.1474	0.00931	0.00465
3.0	0.1483	0.00901	0.00450
3.5	0.1493	0.00868	0.00434
4.0	0.1502	0.00832	0.00416

$\alpha = 0.13$, $E = 1.6$, $\beta_1 = 0.6$, $\beta_2 = 1.1$ $\varpi = 0.6$, $R_2 = 1.2$, $D_1 = 1.5$, $D_2 = 1.7$, $d = 0.4$, $A = 1.2$, $I = 2.5$, $\omega_1 = 1.8$, $\omega_2 = 0.6$, $\Omega = 4.6$, $G_1 = 5.1$, $G_2 = 7.5$, $G_3 = 6$.

Table 3. The influence of the SPR rate $\eta_2(\xi)$ on ρ, L_q, W_q

SPR $\eta_2(\xi)$	ρ	L_q	W_q
1.2	0.0913	0.00092	0.00463
1.3	0.0970	0.00091	0.00456
1.4	0.1028	0.00082	0.00411
1.5	0.1085	0.00063	0.00315
1.6	0.1143	0.00031	0.00156

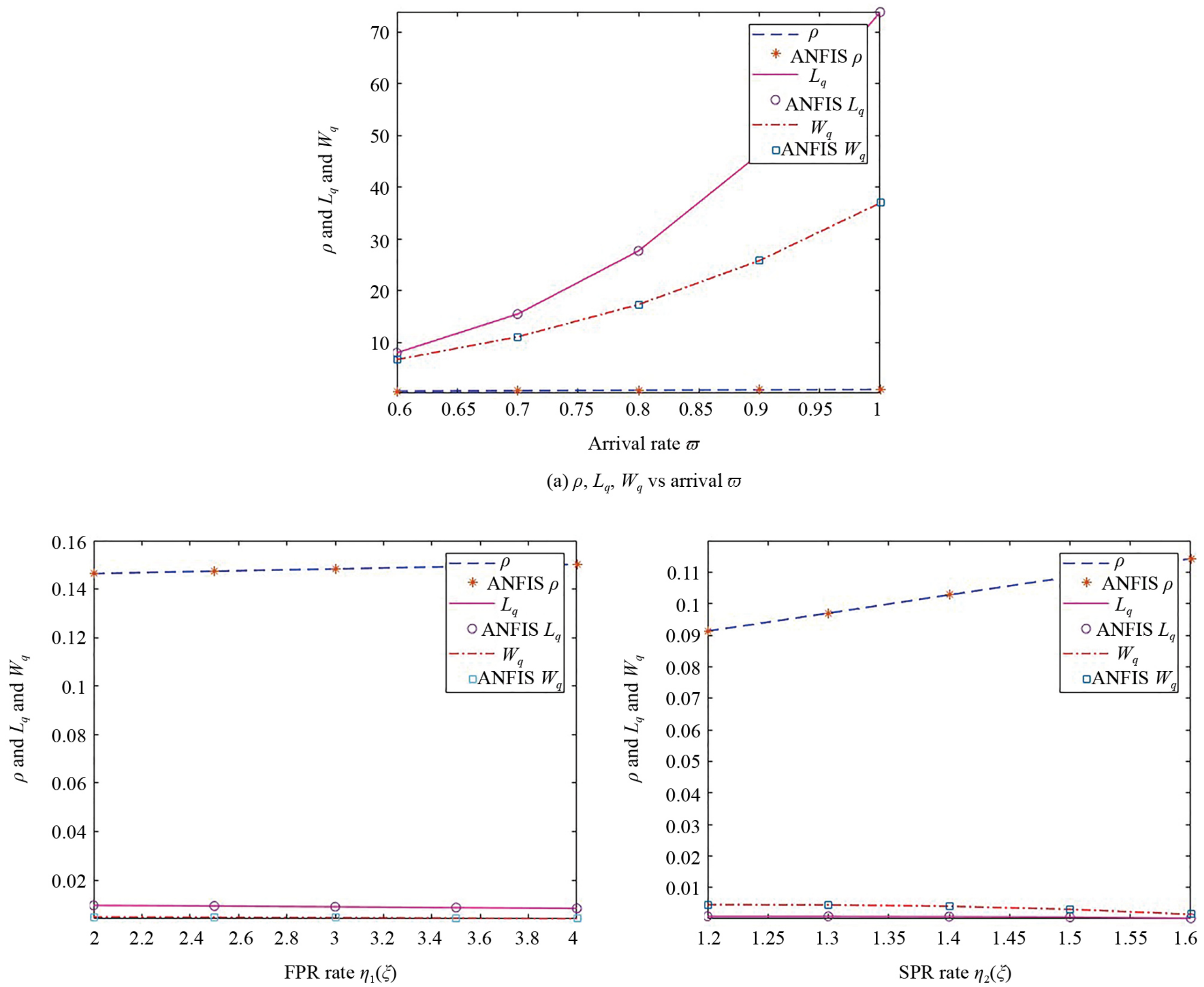
$\alpha = 0.13$, $E = 1.6$, $\beta_1 = 0.6$, $\beta_2 = 1.1$ $R_1 = 2$, $\varpi = 0.6$, $D_1 = 1.5$, $D_2 = 1.7$, $d = 0.4$, $A = 1.2$, $I = 2.5$, $\omega_1 = 1.8$, $\omega_2 = 0.6$, $\Omega = 4.6$, $G_1 = 5.1$, $G_2 = 7.5$, $G_3 = 6$.

In Figure 1(a)-1(c), the two-dimensional graph representing the system performance measurement is displayed.

• Figure 1(a) demonstrates how the utilisation factor (ρ), estimated queue length (L_q), and expected waiting time (W_q) all increase as the arrival rate (ϖ) does.

• The figure 1(b) shows that while the utilisation factor (ρ) increases, the FPR rate rises. Expected waiting time (W_q) and queue length (L_q) decrease.

• The figure 1 (c) shows that while the utilisation factor (ρ) increases, the SPR rate rises. Expected waiting time (W_q) and queue length (L_q) decrease.



(a) ρ, L_q, W_q vs arrival ϖ

(b) ρ, L_q, W_q vs FPR $\eta_1(\xi)$

(c) ρ, L_q, W_q vs SPR $\eta_2(\xi)$

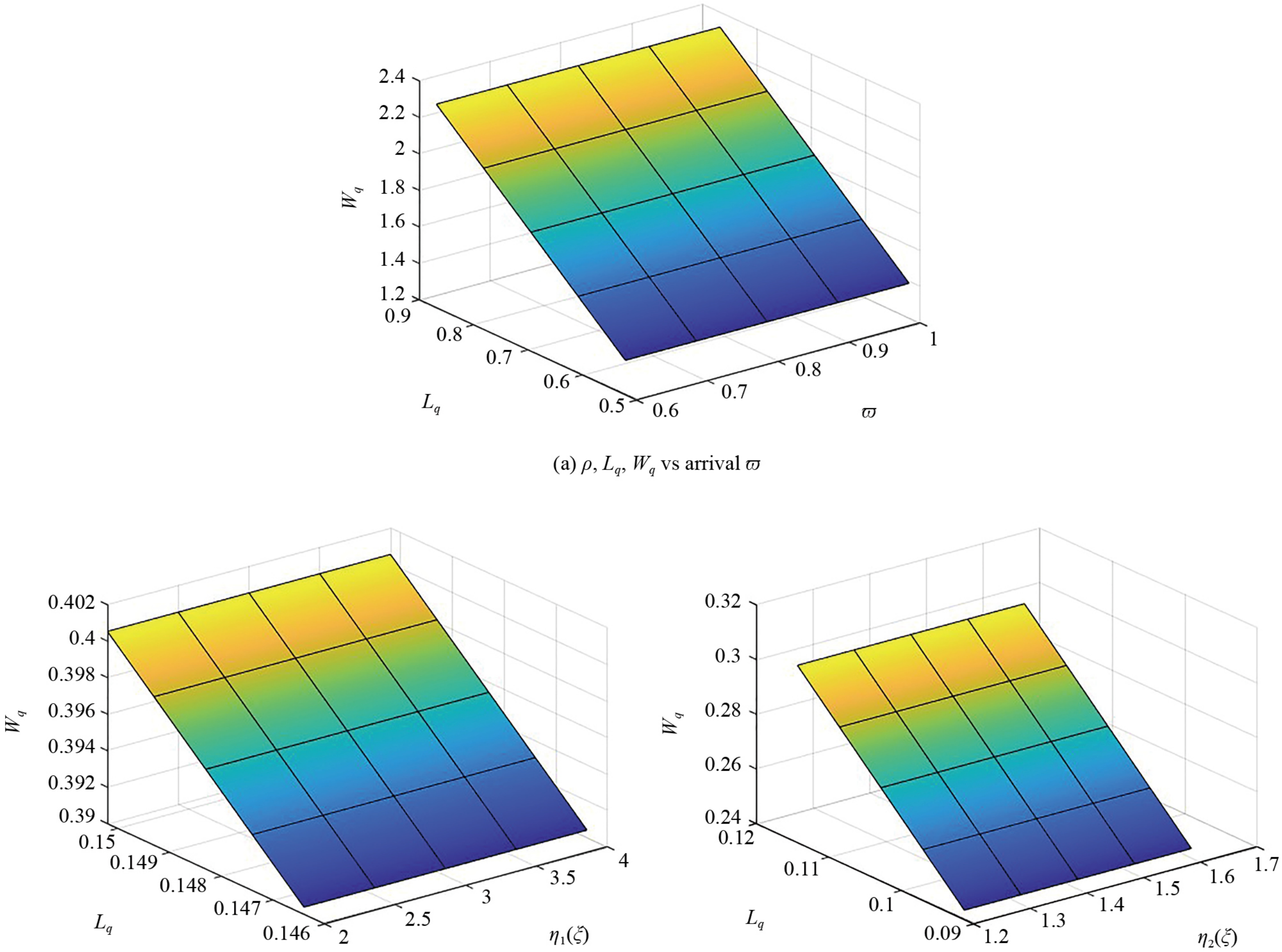
Figure 1. 2D representation effects

In Figure 2(a)-2(c), the three-dimensional graph representing the system performance measurement is displayed.

• The surface in Figure 2 (a) shows the growth of the arrival rate (ϖ), estimated length of the line (L_q), and estimated wait time (W_q).

• Figure 2 (b) shows that as the FPR rate rises, the estimated queue size (L_q) and waiting time (W_q) both decrease.

- Figure 2 (c) shows that as the SPR rises, expected queue lengths (L_q) and waiting times (W_q) both decrease.



(a) ρ, L_q, W_q vs arrival ϖ

(b) ρ, L_q, W_q vs FPR $\eta_1(\xi)$

(c) ρ, L_q, W_q vs SPR $\eta_2(\xi)$

Figure 2. 3D representation effects

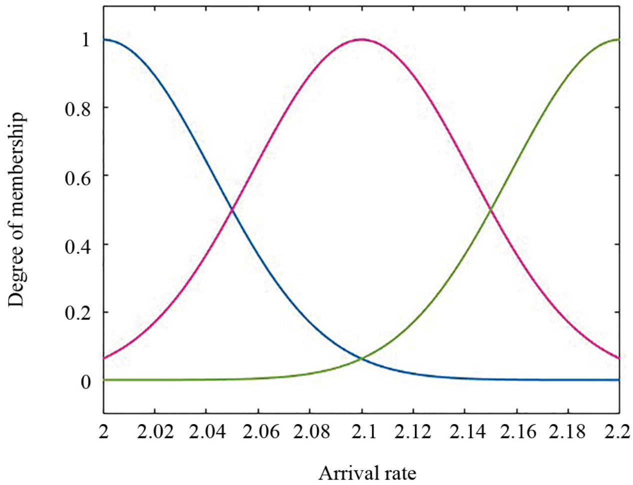
Based on the earlier numerical results, we can determine how attributes affect the system's evaluation criteria, and we can be sure that these findings are representative of real-world situations.

6. Adaptive neuro-fuzzy inference system

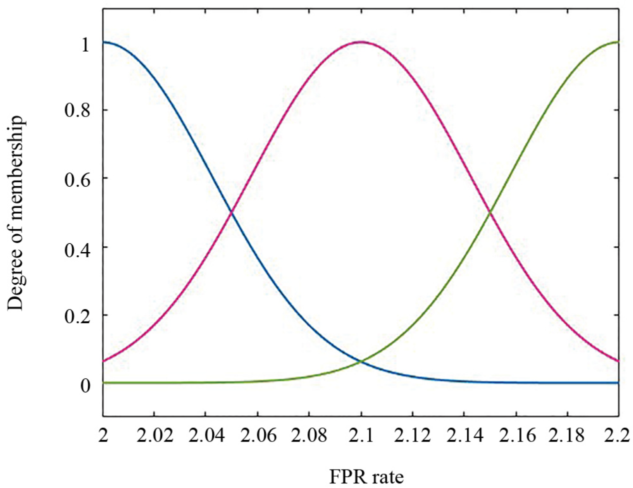
The ANFIS model is actually applicable in a variety of fields, such as modes of transport, congestion, telecommuting, atmospheric research, etc. Artificial neural networks are used in communications networks to accomplish a variety of goals, including an increase in customers, expense reduction, shorter wait times, etc. With variations in arrival rates while on vacation, service rates, repair rates, and repair to busy rates, the current model allows us to examine the impatience of the client while they wait for the service.

A very helpful approach for ANFIS is created by combining soft computing methods, artificial neural networks (ANNs), and fuzzy systems (FS). We are showing a simplified idea of the ANFIS architecture by using the fuzzy parameters. We can implement an ANFIS input-output function and input-output data pairs as fuzzy if-then logic. The fuzzy toolbox of MATLAB software can be utilised for contrasting the computational findings with the implementation of an ANFIS network.

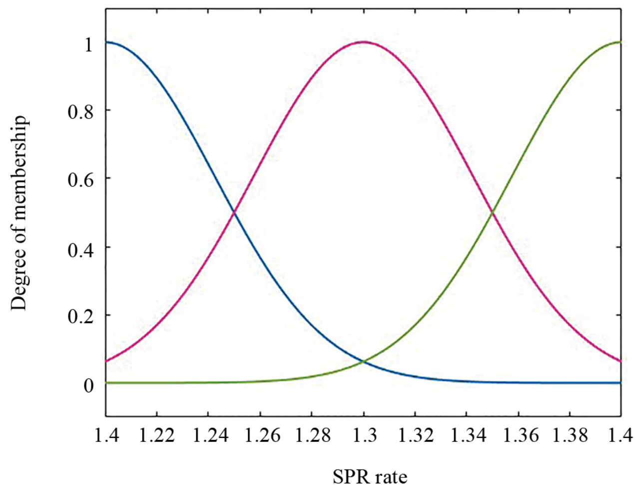
The input parameters and the membership function are assumed to be the ϖ, $\eta_1(\xi)$, and $\eta_2(\xi)$ Gaussian functions in order to produce computational results based on ANFIS. It is assumed that the linguistic values are low, moderate, or high. The graphical representations of the membership functions for the Arrival rate, FPR rate, and SPR rate are depicted in Figures 3(a), 3(b), and 3(c) respectively. Tick marks are placed over the curves made for the results obtained analytically in Figure 1(a)-1(c) to indicate the results produced by the ANFIS approach for the queue size. The figures show that the numerical outcomes produced using the Runge-Kutta method and the ANFIS results are nearly identical.



(a) ρ, L_q, W_q vs arrival ϖ



(b) ρ, L_q, W_q vs FPR $\eta_1(\xi)$



(c) ρ, L_q, W_q vs SPR $\eta_2(\xi)$

Figure 3. ANFIS membership function representation

7. Cost optimization

In a real-world context, a system's running cost and profit are closely related. Therefore, in order to increase the system's earnings, its creators or managers focus largely on lowering operating costs per unit of time. Finding the optimum average price per unit of time (TC) parameters is our goal. In order to accomplish this and make our created model more cost-effective, we will construct an expense capability in this area.

C_h represents the system's holding cost per unit time for each client.

C_b represents the cost per unit time spent on the server when it is on operation.

C_v represents the cost per unit of time spent on the server when it is on vacation.

C_r represents the cost per unit of time required to repair the server following a breakdown.

C_1 represents the cost per unit time in the first phase of busiest period.

C_2 represents the cost per unit time in the second phase of busiest period.

$$TC = C_h L_q + C_v(G_1 + G_2 + G_3) + C_b(P_1 + P_2) + C_r(R_1 + R_2) + C_1\mu_1 + C_2\mu_2$$

Numerous optimisation approaches have been created since the early 1960s. These algorithms have all demonstrated their ability to address a variety of optimisation problems. Because cost optimisation is so important, this study was conducted using three different global search optimisation algorithms: genetic algorithms (GA), artificial bee colonies (ABC), and particle swarm optimisation (PSO). Each of these algorithms is independently described in three different subsections of this section. As long as the algorithm's presumptions are accurate, local search approaches often provide the computing complexity needed to locate the global optimum. Table 4 displays the optimal values of three cost sets, which are determined through the application of PSO, ABC, and GA optimization algorithms using MATLAB programming.

Table 4. Cost sets for optimal policy

Cost sets	C_h	C_v	C_b	P_v	C_1	C_2	C_r
1	9	10	5	6	4	7	8
2	7	8	4	3	6	5	9
3	10	9	6	4	7	8	5

7.1 *Particle swarm optimization (PSO)*

PSO is an effective method for resolving challenging optimisation problems. For PSO, Kennedy and Eberhart developed an algorithmic rule. This tactic is employed to address the problems with cost-effectiveness measures. Many researchers have already investigated a variety of PSO adjustments in an effort to address the local minimum issues and boost the productivity of the PSO multimodel function method. We optimized the system settings by configuring the default parameters as follows: $\alpha = 0.13$, $\varpi = 0.6$, $\beta_1 = 0.6$, $\beta_2 = 1.1$, $R_1 = 2$, $R_2 = 1.2$, $D_1 = 1.5$, $D_2 = 1.7$. The minimum bound of $\mu_1(\xi)$ is set to 2, and the maximum bound of $\mu_2(\xi)$ is set to 10. Table 5 illustrates the impact of various cost elements, such as C_h, C_v, C_b, P_v, C_1, C_2, and C_r on the optimal service rates and optimal total cost for all three cost sets. Additionally, Algorithm 1 presents the pseudo code for the PSO algorithm.

Algorithm 1 Particle swarm optimization:

Step 1: Initialize Xi, Vi, iteration, pbest, gbest

Step 2: Generate random particle (p)

Step 3: For each particle (i)

Step 4: Calculate objective function (fi)

Step 5: Update pbest, gbest

Step 6: End for

Step 7: While iteration

Step 8: For each particle I

Step 9: Update Vi, Xi

Step 10: If $Xi >$ limit then Xi = limit
Step 11: Calculate objective function fi
Step 12: Update pbest, gbest
Step 13: End for
Step 14: end while

Table 5. Influence of ϖ, $\eta_1(\xi)$, $\eta_2(\xi)$ on TC^* using PSO

		TC^*		
Cost sets		Cost set 1	Cost set 2	Cost set 3
ϖ	0.6	27.5014	27.5012	37.5011
	0.7	27.5093	27.5077	37.5079
	0.8	27.5266	27.5214	37.5236
$\eta_1(\xi)$	2.0	27.5014	27.5012	37.5011
	2.1	27.5010	27.5009	37.5008
	2.2	27.5006	27.5006	37.5005
$\eta_2(\xi)$	1.2	27.5014	27.5012	37.5011
	1.3	27.4990	27.4991	37.4992
	1.4	27.4965	27.4970	37.4972

7.2 *Artificial bee colony(ABC)*

Dervis Karaboga's Artificial Bee Colony is one of the most recent algorithms he invented (in 2005), and it is inspired by the clever behaviour of honey bees. In essence, it uses only basic process variables like colonies and maximums. It is equally simple to comprehend as differential evolutionary techniques and PSO. A procedure called ABC is used to reduce the price of a suggested model. We optimized the system settings by configuring the default parameters as follows: $\alpha = 0.13$, $\varpi = 0.6$, $\beta_1 = 0.6$, $\beta_2 = 1.1$ $R_1 = 2$, $R_2 = 1.2$, $D_1 = 1.5$, $D_2 = 1.7$. The minimum bound of $\mu_1(\xi)$ is set to 2, and the maximum bound of $\mu_2(\xi)$ is set to 10. Table 6 illustrates the impact of various cost elements, such as C_h, C_v, C_b, P_v, C_1, C_2, and C_r on the optimal service rates and optimal total cost for all three cost sets. Additionally, Algorithm 2 presents the pseudo code for the ABC algorithm.

Algorithm 2 Artificial bee colony:
Step 1: begin
Step 2: Initial Pop ()
Step 3: While remain iterations do
Step 4: Site selection for local search
Step 5: assign bees for the selected sites
Step 6: and to compute objective
Step 7: find the bee with the best objective
Step 8: Allocate the remaining bees to look
Step 9: for randomly
Step 10: compute remaining bees objective
Step 11: Update optimum solution ()
Step 12: End while
Step 13: Return best solution
Step 14: end.

Table 6. Influence of $\varpi, \eta_1(\xi), \eta_2(\xi)$ on TC^* using ABC

		TC^*		
Cost sets		Cost set 1	Cost set 2	Cost set 3
	0.6	11.0014	11.0013	15.0011
ϖ	0.7	11.0093	11.0078	15.0079
	0.8	11.0266	11.0214	15.0235
	2.0	11.0014	11.0013	15.0011
$\eta_1(\xi)$	2.1	11.0011	11.0010	15.0008
	2.2	11.0007	11.0006	15.0005
	1.2	11.0014	11.0013	15.0011
$\eta_2(\xi)$	1.3	10.9990	10.9992	14.9992
	1.4	10.9965	10.9970	14.9972

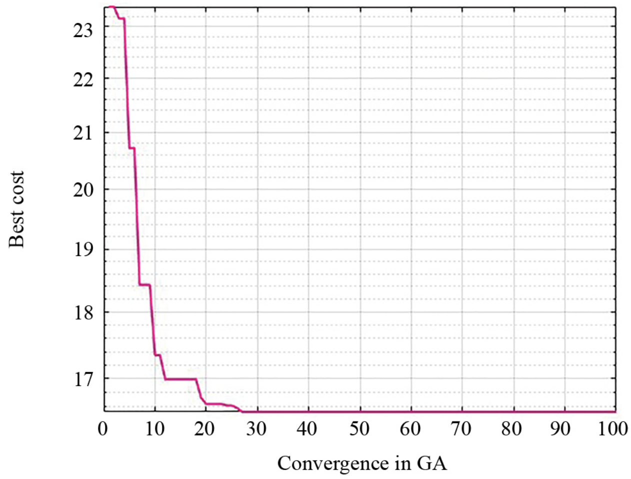
7.3 *Genetic algorithm (GA)*

Bremermann, Holland, and colleagues created the genetic algorithm in the 1960s and 1970s as a way to solve optimisation problems caused by natural selection, which is the mechanism that promotes the evolution of life. They are frequently used to provide excellent answers to stochastic search problems. The entire algorithm serves as an illustration of the selection criteria used to choose the most suitable individuals to have children in order to produce the next generation of humans. We optimized the system settings by configuring the default parameters as follows: $\alpha = 0.13$, $\varpi = 0.6$, $\beta_1 = 0.6$, $\beta_2 = 1.1$, $R_1 = 2$, $R_2 = 1.2$, $D_1 = 1.5$, $D_2 = 1.7$. The minimum bound of $\mu_1(\xi)$ is set to 2, and the maximum bound of $\mu_2(\xi)$ is set to 10. Table 7 illustrates the impact of various cost elements, such as C_h, C_v, C_b, P_v, C_1, C_2, and C_r on the optimal service rates and optimal total cost for all three cost sets. Additionally, Algorithm 3 presents the pseudo code for the GA algorithm.

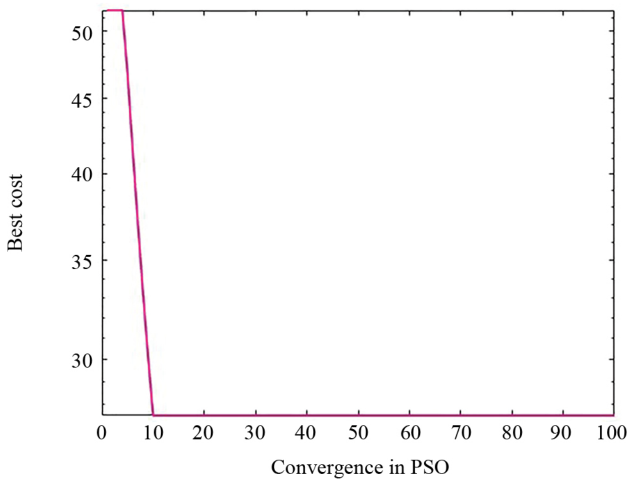
Algorithm 3 Genetic algorithm
Data: population of individuals;
Results: Best individuals;
Step 1: begin
Step 2: $t = 0$;
Step 3: Create an initial population-Pop(0);
Step 4: Evaluate individuals - calculate the value of the objective function for each individual in the population P_0;
Step 5: do
Step 6: select individuals for the new population Pop(t)-selection;
Step 7: perform crossover operation;
Step 8: perform the mutation of individuals;
Step 9: evaluate individuals;
Step 10: replace old population a new one;
Step 11: $t = t + 1$;
Step 12: While stop condition reached;
Step 13: end.

Table 7. Influence of $\varpi, \eta_1(\xi), \eta_2(\xi)$ on TC^* using GA

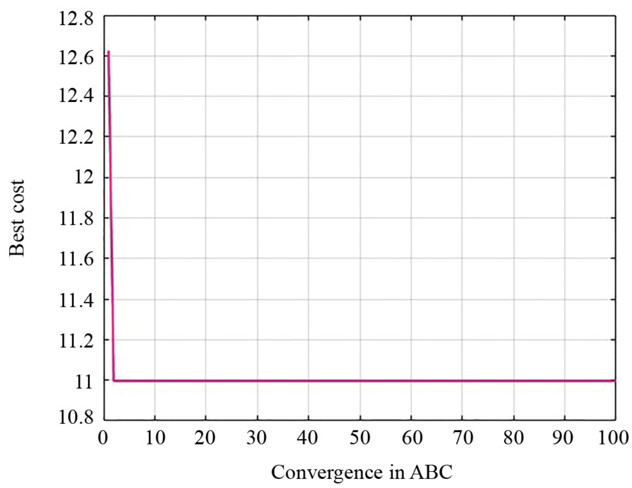
		TC^*		
Cost sets		Cost set 1	Cost set 2	Cost set 3
	0.6	16.5017	16.5015	22.5013
ϖ	0.7	16.5105	16.5087	22.5089
	0.8	16.5297	16.5239	22.5264
	2.0	16.5017	16.5015	22.5013
$\eta_1(\xi)$	2.1	16.5013	16.5012	22.5010
	2.2	16.5008	16.5008	22.5006
	1.2	16.5017	16.5015	22.5013
$\eta_2(\xi)$	1.3	16.4991	16.4993	22.4993
	1.4	16.4965	16.4970	22.4972



(a) TC vs Arrival (ϖ) in GA



(b) TC vs FPR $\eta_1(\xi)$ in PSO



(c) TC vs SPR $\eta_2(\xi)$ in ABC

Figure 4. Cost optimization effects

7.4 *Convergence in PSO, ABC and GA*

The three methods particle swarm optimization (PSO), artificial bee colony (ABC), and genetic algorithm (GA) are discussed in this section and compared to see which has the lowest cost using the associated MATLAB programs. The MATLAB programs for each of the above algorithms are then executed one by one. It is critical to understand whether a particle returns to normal or not after using an optimization methodology, such as PSO, ABC, or GA and whether it will wander around in search of a better answer. Consequently, convergence is a key component of cost analysis. A statistical summary of the results shows that ABC outperforms the PSO. The total number of functional evaluations in ABC was lower than PSO for the entire standard optimization. The results show that PSO converges more rapidly. If a quick result is desired for time-sensitive applications, ABC cannot be used. Figures 4(a)-4(c) depict graphical representations illustrating the convergence of PSO, ABC, and GA algorithms.

• The study demonstrates our model's compatibility with actual conditions. When analysts determine the total cost of the system, some of their financial problems will be somewhat resolved.

• The current circumstances, which demonstrate the logic of our approach and assist network managers and specialists in reducing the problem of communications services that specifically confront blocking, may heavily rely on the cost-benefit analysis that was generated.

8. Conclusion

We have examined batch arrival, interrupted essential services, working failure, balking, catastrophe, and extended server vacation in this study. The current research presents an explicit evaluation of our queueing system's steady state and transient solution results. The current investigation presents an overview of our queueing system's steady state and transient solution results. Through the use of generating functions, we have established the queue length distributions in this study, which are then used to determine important performance metrics. The analysis is supported by the use of supplementary variables and, in some instances, is validated by the literature already in existence. Additionally, PSO, ABC, and GA are employed to compute expenses. To find the best offer, these strategies contrast and compare the outcomes. The potential markets for the model that was created, such as contact centres, wireless networks, or telecom facilities, which might be powered by controlled precision test queueing systems to deliver excellent service at competitive costs, served as the study's main source of inspiration.

Authors contribution

All the authors made substantial contributions to the conception or design of the work.

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Conflicts of interest

The authors declare no competing financial interest.

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