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A New Model Making the Transition From Oil to Solar Energy

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Abstract: Today, the use of renewable energy sources such as sun, wind and hot water to generate electricity is increasingly becoming a global priority. In this paper, we propose a deterministic dynamical system reporting the transition from oil to solar energy, in which we then add the randomness of this phenomenon. First, we study the positivity of the deterministic oil-solar model and show that there is a maximum solution of the deterministic model and that the system is controllable at any time $T > 0$ from an initial point. We also prove the existence of an equilibrium state, study its nature and prove that there is no limit cycle between the quantity of oil available and the solar production capacity installed. Secondly, we formulate the stochastic solar energy model from the deterministic model, for which we proved the existence of a unique solution. Finally, using a model extracted from the model performed and examining the transition from the amount of available oil to the installed solar generation capacity, we associate a linear optimal control problem, from which we prove that the optimal control components are of bang-bang type, and we characterize them.

***Keywords*:** solar energy, oil energy, deterministic and stochastic models, Brownian motion, maximum Pontryagin's principle, limit cycles

MSC: 60H10, 49J15, 34C07

1. Introduction

The energy transition has been a highly topical issue since the 50s, and more specifically since the 1973 oil crisis, when new technologies enabled significant growth. Among all renewable sources, solar energy is one of the most important energy policy issues of our century, resulting from the transformation of sunlight into electrical energy by means of photovoltaics.

To describe this phenomenon, Tsur and Zemel proposed in [1] a study that models the process of capital accumulation in the production of solar energy in competition with non-renewable energies. In [2], Moser et al. treated a problem endogenously by applying learning-by-doing effects using ordinary differential equations. In this work, they used a log-linear curve to model the decrease in investment costs as a function of accumulated experience, while taking into account the seasonal fluctuation presented by solar radiation as an energy source (see [3, 4]).

Moreover, Amigues et al. introduced in [5] a more comprehensive model than the one of Moser et al. for the transition from oil to solar energy, with adjustment costs for both sources assumed to be constant. Assuming sufficient

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non-renewable resources at the outset, they chose a three-phase sequence. In [6], Noël Bonneuil and Raouf Boucekkine produced a work whose main objective is to resolve the trade-off between cheap fossil energy and more expensive renewable energy. This was done by maximizing the discounted cash flow of utilities from energy consumption and disutilities associated with pollution and the cost of investing in renewable energies.

The contribution of the present work is fourfold. In the second section, we extend the model established in [5] into a more general deterministic linear non-autonomous one. In the third section, after a mathematical analysis due to the intermittent nature of renewable energies, we formulate a stochastic model to take into account the random aspects disrupting the production of energy from the sun. In the fourth section, we treat an infinite-horizon optimal control problem for which we characterize the solution. Finally, in the fifth section, we prove the existence of an equilibrium state and that this equilibrium state is a saddle point, and that there is no cyclical behavior between the amount of oil available and the installed solar production capacity.

In the third section of this paper, we consider $(\Omega,\ \mathscr{F},\ \{\mathscr{F}_t\}_{t\geq 0},\ \mathbb{P})$ a complete probability space with a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ satisfying the usual conditions, i.e. it is increasing and right-continuous while $\mathscr{F}_0$ contains all P-null sets, and denote $(B_i(.))$, $i = 1,\ 2$, scalar Brownian motions on the probability space. We note det the determinant, $\mathbb{E}[.]$ the expectation and $a \vee b = \max(a,\ b)$, for all $a,\ b \in \mathbb{R}$.

In this paper, we express the amount of oil that must be exploited as a function of the amount of oil available at time t while assuming that a portion of it is not exploitable and therefore control the rate of extraction of the portion of oil available for exploitation. We use a depreciation factor for solar generation capacity rather than controlling the rate at which this capacity is scrapped as was done in previous work.

2. A deterministic solar-oil model

We consider an energetic sector wishing to have two sources of energy which are oil and solar radiation to generate electricity. The final objective is to have a high production of electricity based exclusively on solar energy while significantly decreasing the amount of oil used.

We consider an extension of the mathematical model from [5] in which we assume that a part of the available oil is not exploitable.

We consequently consider the following mathematical model:

$$\begin{cases} \dot{X}(t) = -x(t)X(t) + \alpha x(t), \\ \dot{K}(t) = k(t) - \theta K(t), \end{cases} \tag{1}$$

with the initial conditions $X(0) = X_0 > \alpha \geq 0$ and $K(0) = 0$, where:

- t is the time,
- $X(.)$ the amount of oil available,
- $K(.)$ the installed solar production capacity,
- α the non-extractable part of $X(.)$,
- $x(.)$ the extraction rate of the quantity of oil available for exploitation,
- $k(.)$ the purchase rate of solar equipment,
- $\theta \in [0,\ 1]$ the depreciation factor of installed solar production capacity.

We use a differential equation where $\dot{X}(.)$ is expressed as a function of $X(.)$ to model oil exploitation, unlike Amigues et al. in [5] who used a differential equation where $\dot{X}(.)$ is not expressed as a function of $X(.)$ $(\dot{X}(.) = -x(.))$. We don't control the solar panel scrap rate $\theta(.)$ as they did. Instead, we use it as a depreciation factor θ due to the solar production capacity $K(.)$.

Note that if there is no solar production capacity at the start, that is $K(0) = 0$, K remains identically zero. The model therefore does not represent the start of solar production capacity but only its evolution.

Proposition 1 The region $\mathscr{R} = \{(X,\ K) : X > 0,\ K > 0\}$ is positively invariant, that is, for any admissible control x and k, the solutions exist for all times and remain in $\mathscr{R}$.

Proof. Suppose $(X_0,\ K_0) \in \mathscr{R}$ and let x and k any Lebesgue measurable functions with values in $[0,\ x_{max}]$ and $[0,\ k_{max}]$ respectively. Integrating the equation

$$\dot{X}(t) = -x(t)X(t) + \alpha x(t), \tag{2}$$

we obtain

$$X(t) = (X_0 - \alpha)e^{-\int_0^t x(s)ds} + \alpha. \tag{3}$$

We have $X_0 - \alpha > 0$ and $\alpha \geq 0$ by assumption, therefore $X(t) > 0$ for all $t \geq 0$.

Similarly, the integration of

$$\dot{K}(t) = k(t) - \theta K(t) \tag{4}$$

between 0 and t gives

$$K(t) = C(t)e^{-\theta t}, \tag{5}$$

where $C(t)$ is a primitive of $k(t)e^{\theta t}$ and $K(0) = 0$. As $k(t)$ is a positive function for all t then $C(t)$ is a positive function as a primitive of the product of positive functions, then $K(t) > 0$ for all t.

We consider a part Ω of $\mathbb{R}$ for which we write the previous system in the following form, for all $t \geq 0$,

$$\begin{cases} \dot{Y}(t) = F(t,\ Y(t),\ u(t)), \\ Y(0) = Y_0, \end{cases} \tag{6}$$

with

$$Y = \begin{pmatrix} X \\ K \end{pmatrix},\ Y_0 = \begin{pmatrix} X_0 \\ 0 \end{pmatrix} \text{ and } u(t) = \begin{pmatrix} x(t) \\ k(t) \end{pmatrix}.$$

We have $(X_0,\ 0) \in \mathbb{R}_+^2$, $u(t) \in \Omega$ and

$$F : \mathbb{R} \times \mathbb{R}_+^2 \times \Omega \to \mathbb{R}_+^2$$

of class C^1 is given by

$$F(t,\, Y(t),\, u(t)) = \begin{pmatrix} -x(t)X(t) + \alpha x(t) \\ \\ k(t) - \theta K(t) \end{pmatrix}.$$

In order to properly pose the optimal control problem, we apply Carathéodory's theorem to prove that this Cauchy problem has a maximal solution while choosing the control $u(.)$ among the following functions $x(.)$ and $k(.)$. Once the control $u(.)$ is chosen, we fix the other functions so that the conditions of the theorem are checked. The following theorem proves the existence of the maximal solution of the problem.

Theorem 1 For any fixed control $u(.)$, there exists a maximal solution $Y_u(.)$ defined on $\mathbb{R}^2_+$ of problem (6).

Proof. We introduce a function G defined by:

$$G(t,\, Y(t)) = F(t,\, Y(t),\, u(t)). \tag{7}$$

The function G is measurable in t as a composite of measurable functions and continuous in Y as a composite of continuous functions. It is integrable in t, because the function F is integrable in time. The function G satisfies the conditions of Carathéodory's theorem, then there exists a maximal solution $Y_u(.)$ defined on $\mathbb{R}^2_+$ of the problem (6).

Theorem 2 The system (6) is controllable in time $T > 0$ from Y_0 if and only if $\alpha \neq 0$ and $x(t) \neq \theta$ for all $t \in [0,\, T]$.

Proof. The dynamical system (1) can be written in a matrix form:

$$\dot{Y}(t) = A(t)Y(t) + Bu(t), \tag{8}$$

where

$$A(t) = \begin{pmatrix} -x(t) & 0 \\ \\ 0 & -\theta \end{pmatrix},\ Y(t) = \begin{pmatrix} X(t) \\ \\ K(t) \end{pmatrix},\ B = \begin{pmatrix} \alpha \\ \\ 1 \end{pmatrix} \text{ and } u(t) = \begin{pmatrix} x(t) \\ \\ k(t) \end{pmatrix}.$$

The previous system is a non-autonomous linear system since the matrix A depends explicitly on the variable t. The study of the controllability of this system is to show that the controllability matrix K_T is invertible. The controllability matrix of the previous system is

$$K_T = \int_0^T R(s)^{-1} B(s) B(s)^t (R(s)^{-1})^t ds, \tag{9}$$

where $R(s)$ is the resolvent of this system. We have

$$K_T = \int_0^T R(s)^{-1} BB^t (R(s)^{-1})^t ds$$

$$= \int_0^T \begin{pmatrix} e^{\int_0^s x(\mu)d\mu} & 0 \\ 0 & e^{\theta s} \end{pmatrix} \begin{pmatrix} \alpha \\ 1 \end{pmatrix} \begin{pmatrix} \alpha \\ 1 \end{pmatrix}^t \begin{pmatrix} e^{\int_0^s x(\mu)d\mu} & 0 \\ 0 & e^{\theta s} \end{pmatrix}^t ds$$

$$= \begin{pmatrix} \alpha^2 \int_0^T e^{2\int_0^s x(\mu)d\mu} ds & \alpha \int_0^T e^{\theta s + \int_0^s x(\mu)d\mu} ds \\ \alpha \int_0^T e^{\theta s + \int_0^s x(\mu)d\mu} ds & \int_0^T e^{2\theta s} ds \end{pmatrix}$$

and

$$det(K_T) = \left(\alpha^2 \int_0^T e^{2\int_0^s x(\mu)d\mu} ds\right) \left(\int_0^T e^{2\theta s} ds\right) - \left(\alpha \int_0^T e^{\theta s + \int_0^s x(\mu)d\mu} ds\right)^2$$

$$= \alpha^2 \left[\left(\int_0^T e^{2\int_0^s x(\mu)d\mu} ds\right) \left(\int_0^T e^{2\theta s} ds\right) - \left(\int_0^T e^{\theta s + \int_0^s x(\mu)d\mu} ds\right)^2 \right].$$

We obtain $det(K_T) \neq 0$ if and only if $\alpha \neq 0$ and $x(t) \neq \theta$. The dynamical system is therefore controllable in $T > 0$ if and only if $\alpha \neq 0$ and $x(t) \neq \theta$ for all $t \in [0, T]$.

3. To a stochastic solar-oil model

When using renewable energy sources such as sun, to produce electricity, some phenomena can occur at any time, such as hurricane, etc. This is a major problem for the efficient production of renewable energies as they are intermittent. As a result, seasonal fluctuations frequently disrupt solar energy production. It is therefore interesting to approach it stochastically because daily and seasonal fluctuations are random.

In order to model the random effects on solar energy production, we will use a stochastic model (see [7–10]). To obtain the desired stochastic differential equation model, we add a diffusion term to the deterministic model to model the random perturbations of the parameters linked to seasonal fluctuations. This diffusion or volatility term will be the product of a measurable function $\gamma(K)$ which will depend on the installed solar production capacity $K(t)$ and a Brownian $B(t) = (0, B_2(t))$ defined in a filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ where $\{\mathscr{F}_t\}_{t\geq 0}$ on $\mathscr{F}$ satisfies the usual conditions (i.e. growth, continuity, while $\mathscr{F}_0$ contains all $\mathbb{P}$-null sets). We assume that this diffusion term can be written as $\gamma(K) = \gamma \times K$ where γ denotes the intensity of $B_2(t)$ (environmental noise intensity).

To do this, we consider the equation (4) modelling installed and maintained solar production capacity, for which we'd like to use a stochastic approach.

This stochastic model is constructed as in [11, 12] by adding to model (1) the diffusion $\gamma K(t) dB(t)$. We then obtain the new model.

We will therefore add the diffusion term to the deterministic model in order to formulate the following stochastic model:

$$\begin{cases} \dot{X}(t) = -x(t)(X(t) - \alpha), \\ \\ \dot{K}(t) = k(t) - \theta K(t) + \gamma K(t)dB_2(t), \end{cases} \tag{10}$$

where $X(0) = X_0 > \alpha \geq 0$ and $K(0) = 0$ are the initial conditions. This last model can be rewritten as:

$$\begin{cases} dX(t) = (-x(t)(X(t) - \alpha))dt + 0, \\ \\ dK(t) = (k(t) - \theta K(t))dt + \gamma K(t)dB_2(t), \end{cases} \tag{11}$$

here $X(0) = X_0 > \alpha \geq 0$ and $K(0) = 0$.

We then set

$$Y_t = \begin{pmatrix} X(t) \\ \\ K(t) \end{pmatrix}, \; F(t,\, Y_t) = \begin{pmatrix} -x(t)(X(t) - \alpha) \\ \\ k(t) - \theta K(t) \end{pmatrix}, \; M(t,\, Y_t) = \begin{pmatrix} 0 & 0 \\ \\ 0 & \gamma K(t) \end{pmatrix}.$$

The stochastic differential equation (11) is written as follows:

$$\begin{cases} dY_t = F(t,\, Y_t)dt + M(t,\, Y_t)dB(t), \\ \\ Y_0 = (X_0,\, 0) \in \mathbb{R}^2, \end{cases} \tag{12}$$

where $F(t,\, Y_t)$ is the drift term or deterministic coefficient and $M(t,\, Y_t)$ is the diffusion term or volatility coefficient. We'll show that the stochastic differential equation (11) has a unique solution.

Theorem 3 For any initial condition $Y(0) = (X_0,\, 0) \in \mathbb{R}^2$, the stochastic differential equation (11) has a unique solution.

Proof. The function $F(.,.)$ is composed of measurable functions and is therefore measurable. Similarly, the function $M(.,.)$ is measurable as a composite of measurable functions. The proof goes into two steps.

1. Let be $Y_t = (X(t),\, K(t))$ and $t \in [0,\, T]$, we have:

$$\|F(t,\, Y_t)\| + |M(t,\, Y_t)| = \sqrt{(\gamma K(t))^2} + \sqrt{(\alpha x(t) - x(t)X(t))^2 + (k(t) - \theta K(t))^2}.$$

Let's find a majoration for each of these terms. Since we're working with positive quantities, we use the inequality $(a-b)^2 \leqslant a^2 + b^2$, for all $a,\, b \geq 0$ to obtain the majoration of each terms. One has

$$(\alpha x(t) - x(t)X(t))^2 \leqslant (\alpha x(t))^2 + (x(t)X(t))^2$$

and

$$(k(t)-\theta K(t))^2 \leqslant (k(t))^2+(\theta K(t))^2.$$

So we have

$$\|F(t,\ Y_t)\|+|M(t,\ Y_t)| \leqslant \sqrt{(\gamma K(t))^2}+\sqrt{(\alpha x(t))^2+k(t)^2+(x(t)X(t))^2+(\theta K(t))^2}.$$

Since the functions $x(t)$ and $k(t)$ are respectively the instantaneous rate of extraction of oil and the instantaneous rate of purchase of solar panels, there are two positive constants x_{max} and k_{max} such as $x(t) \leq x_{max}$ et $k(t) \leq k_{max}$, for all $t \in [0,\ T]$ $\mathbb{P}-p.s$, which implies that

$$\|F(t,\ Y_t)\|+|M(t,\ Y_t)| \leq \sqrt{\gamma^2 K(t)^2}+\sqrt{\alpha^2 x_{max}^2+k_{max}^2+x_{max}^2 X(t)^2+\theta^2 K(t)^2}.$$

So

$$\|F(t,\ Y_t)\|+|M(t,\ Y_t)| \leq C(1+\|Y_t\|),$$

with

$$C=\sqrt{[x_{max}^2 \vee (\gamma^2+\theta^2)]} \vee \sqrt{\alpha^2 x_{max}^2+k_{max}^2}.$$

2. Let be $Y_1(t)=(X_1(t),\ K_1(t)), Y_2(t)=(X_2(t),\ K_2(t))$ and $t \in [0,\ T]$. Let's show that

$$\|F(t,\ Y_1(t))-F(t,\ Y_2(t))\|+|M(t,\ Y_1(t))-M(t,\ Y_2(t))| \leqslant D\|Y_1(t)-Y_2(t)\|.$$

For this we calculate $\|F(t,\ Y_1(t))-F(t,\ Y_2(t))\|$ and $|M(t,\ Y_1(t))-M(t,\ Y_2(t))|$.
One has

$$\|F(t,\ Y_1(t))-F(t,\ Y_2(t))\|=\left\|\begin{pmatrix}-x(t)(X_1(t)-\alpha)\\ \\ k(t)-\theta K_1(t)\end{pmatrix}-\begin{pmatrix}-x(t)(X_2(t)-\alpha)\\ \\ k(t)-\theta K_2(t)\end{pmatrix}\right\|$$

that is

$$\|F(t,\ Y_1(t))-F(t,\ Y_2(t))\|=\left\|\begin{pmatrix}x(t)(X_2(t)-X_1(t))\\ \\ \theta(K_2(t)-K_1(t))\end{pmatrix}\right\|$$

and

$$|M(t,\ Y_1(t)) - M(t,\ Y_2(t))| = \left| \begin{pmatrix} 0 & 0 \\ 0 & \gamma K_1(t) \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & \gamma K_2(t) \end{pmatrix} \right|$$

$$= \left| \begin{pmatrix} 0 & 0 \\ 0 & \gamma(K_1(t) - K_2(t)) \end{pmatrix} \right|.$$

Let's calculate

$$A = \|F(t,\ Y_1(t)) - F(t,\ Y_2(t))\| + |M(t,\ Y_1(t)) - M(t,\ Y_2(t))|.$$

One has

$$A = \sqrt{[x(t)(X_2(t) - X_1(t))]^2 + [\theta(K_2(t) - K_1(t))]^2} + \sqrt{[\gamma(K_1(t) - K_2(t))]^2}$$

$$\leqslant \sqrt{x_{max}^2 (X_2(t) - X_1(t))^2 + \theta^2 (K_2(t) - K_1(t))^2} + \sqrt{\gamma^2 (K_2(t) - K_1(t))^2}$$

because $x(t) \leq x_{max}$ for all $t \in [0,\ T]$ $\mathbb{P}-p.s.$ Therefore

$$A \leq D \|Y_1(t) - Y_2(t)\|$$

with $D = \sqrt{x_{max}^2 \vee (\theta^2 + \gamma^2)}$. Since the hypotheses of the existence and uniqueness theorem for the solution of a stochastic differential equation are satisfied, we can conclude that the equation (11) has a unique solution.

Lemma 1 Let be $Y^1(t)$ and $Y^2(t)$ two solutions of the stochastic differential equation (11), that is

$$Y^1(t) = y^1 + \int_0^t F(s,\ Y^1(s))ds + \int_0^t M(s,\ Y^1(s))dB_s$$

and

$$Y^2(t) = y^2 + \int_0^t F(s,\ Y^2(s))ds + \int_0^t M(s,\ Y^2(s))dB_s,$$

where $y^1 = Y^1(0)$ and $y^2 = Y^2(0)$. Then

$$\mathbb{E}\left[\sup_{[0,\ T]}\left|Y^1(t)-Y^2(t)\right|^2\right]\leq\beta_T(y^1-y^2)^2,$$

with β_T depending on T.

Proof. First, we calculate the difference between $Y^1(t)$ and $Y^2(t)$:

$$Y^1(t)-Y^2(t)=y^1-y^2+\int_0^t\left[F(s,\ Y^1(s))-F(s,\ Y^2(s))\right]ds+\int_0^t\left[M(s,\ Y^1(s))-M(s,\ Y^2(s))\right]dB_s.$$

Let's calculate $F(s,\ Y^1(s))-F(s,\ Y^2(s))$ and $M(s,\ Y^1(s))-M(s,\ Y^2(s))$.

We have

$$F(s,\ Y^1(s))-F(s,\ Y^2(s))=\begin{pmatrix}x(s)(X^2(s)-X^1(s))\\ \theta(K^2(s)-K^1(s))\end{pmatrix}$$

and

$$M(s,\ Y^1(s))-M(s,\ Y^2(s))=\begin{pmatrix}0 & 0\\ 0 & \gamma(K^1(s)-K^2(s))\end{pmatrix}.$$

If we set $A=Y^1(t)-Y^2(t)$, we have

$$\|A\|=\left\|y^1-y^2+\int_0^t\left[F(s,\ Y^1(s))-F(s,\ Y^2(s))\right]ds+\int_0^t\left[M(s,\ Y^1(s))-M(s,\ Y^2(s))\right]dB_s\right\|$$

that is

$$\|A\|=\left\|y^1-y^2+\int_0^t\begin{pmatrix}x(s)(X^2(s)-X^1(s))\\ \theta(K^2(s)-K^1(s))\end{pmatrix}ds+\int_0^t\begin{pmatrix}0 & 0\\ 0 & \gamma(K^1(s)-K^2(s))\end{pmatrix}dB_s\right\|.$$

Consequently,

$$\|A\|\leq\left\|y^1-y^2\right\|+\left\|\int_0^t\begin{pmatrix}x(s)(X^2(s)-X^1(s))\\ \theta(K^2(s)-K^1(s))\end{pmatrix}ds\right\|+\left\|\int_0^t\begin{pmatrix}0 & 0\\ 0 & \gamma(K^1(s)-K^2(s))\end{pmatrix}dB_s\right\|.$$

Let's take the squares of the two members of this inequality. The use of following inequality: $(a+b+c)^2 \leq 3a^2 + 3b^2 + 3c^2$ which is true for all a, b, $c \in \mathbb{R}$, implies

$$\|A\|^2 \leq 3\left\|y^1 - y^2\right\|^2 + 3\left\|\int_0^t \begin{pmatrix} x(s)(X^2(s) - X^1(s)) \\ \theta(K^2(s) - K^1(s)) \end{pmatrix} ds\right\|^2$$

$$+ 3\left\|\int_0^t \begin{pmatrix} 0 & 0 \\ 0 & \gamma(K^1(s) - K^2(s)) \end{pmatrix} dB_s\right\|^2.$$

By taking the sup on $[0,\ T]$ and the expectation, we have:

$$\mathbb{E}\left[\sup_{[0,\ T]} \|A\|^2\right] \leq 3(y^1 - y^2)^2 + 3\mathbb{E}\left[\sup_{[0,\ T]} \left\|\int_0^t \begin{pmatrix} x(s)(X^2(s) - X^1(s)) \\ \theta(K^2(s) - K^1(s)) \end{pmatrix} ds\right\|^2\right]$$

$$+ 3\mathbb{E}\left[\sup_{[0,\ T]} \left\|\int_0^t \begin{pmatrix} 0 & 0 \\ 0 & \gamma(K^1(s) - K^2(s)) \end{pmatrix} dB_s\right\|^2\right].$$

Let us set

$$B = \mathbb{E}\left[\sup_{[0,\ T]} \left\|\int_0^t \begin{pmatrix} x(s)(X^2(s) - X^1(s)) \\ \theta(K^2(s) - K^1(s)) \end{pmatrix} ds\right\|^2\right]$$

and

$$D = \mathbb{E}\left[\sup_{[0,\ T]} \left\|\int_0^t \begin{pmatrix} 0 & 0 \\ 0 & \gamma(K^1(s) - K^2(s)) \end{pmatrix} dB_s\right\|^2\right].$$

The use of Cauchy-Schwarz inequality gives:

$$B \leq \mathbb{E}\left[\sup_{[0,\, T]}\left(\left(\left(\int_0^t (1)^2 ds\right)^{\frac{1}{2}}\right)^2 \left(\left(\int_0^t \left\|\begin{pmatrix} x(s)(X^2(s)-X^1(s)) \\ \theta(K^2(s)-K^1(s)) \end{pmatrix}\right\|^2 ds\right)^{\frac{1}{2}}\right)^2\right]$$

$$\leq \mathbb{E}\left[\sup_{[0,\, T]} t \int_0^t \left\|\begin{pmatrix} x(s)(X^2(s)-X^1(s)) \\ \theta(K^2(s)-K^1(s)) \end{pmatrix}\right\|^2 ds\right]$$

$$\leq \mathbb{E}\left[T \int_0^t \left\|\begin{pmatrix} x(s)(X^2(s)-X^1(s)) \\ \theta(K^2(s)-K^1(s)) \end{pmatrix}\right\|^2 ds\right].$$

This gives us:

$$B \leq TC_1 \int_0^t \mathbb{E}\left[\left\|Y^1(s)-Y^2(s)\right\|^2\right] ds,$$

where $C_1 = x_{max} \vee \theta$.

Moreover, using Doob's inequality [13], we obtain:

$$D = \mathbb{E}\left[\sup_{[0,\, T]}\left\|\int_0^t \begin{pmatrix} 0 & 0 \\ 0 & \gamma(K^1(s)-K^2(s)) \end{pmatrix} dB_s\right\|^2\right]$$

$$\leq 4\mathbb{E}\left[\int_0^t \left\|\begin{pmatrix} 0 & 0 \\ 0 & \gamma(K^1(s)-K^2(s)) \end{pmatrix}\right\|^2 ds\right]$$

$$\leq 4\gamma^2 \int_0^t \mathbb{E}\left[\left\|Y^1(s)-Y^2(s)\right\|^2\right] ds.$$

Using the inequalities obtained for B and D, we have:

$$\mathbb{E}\left[\sup_{[0,\, T]} \|A\|^2\right] \leq 3(y^1-y^2)^2 + 3TC_1 \int_0^t \mathbb{E}\left[\left\|Y^1(s)-Y^2(s)\right\|^2\right] ds$$

$$+ 3 \times 4\gamma^2 \int_0^t \mathbb{E}\left[\left\|Y^1(s)-Y^2(s)\right\|^2\right] ds.$$

which can be rewritten as

$$\mathbb{E}\left[\sup_{[0,\ T]}\left\|Y^1(t)-Y^2(t)\right\|^2\right]\leq 3(y^1-y^2)^2+(3TC_1+12\gamma^2)\int_0^t\mathbb{E}\left[\left\|Y^1(s)-Y^2(s)\right\|^2\right]ds.$$

Let us set $h(t)=\mathbb{E}\left[\sup_{[0,\ T]}\left\|Y^1(t)-Y^2(t)\right\|^2\right]$. One has $h(.)$ is bounded on $[0,\ T]$ and

$$h(t)\leq 3(y^1-y^2)^2+3(TC_1+4\gamma^2)\int_0^t h(s)ds,\ \ \forall s\in[0,\ T].$$

Applying Gronwall's lemma, we then have:

$$h(t)\leq 3(y^1-y^2)^2\times exp\left[(3TC_1+12\gamma^2)T\right]\leq\beta_T(y^1-y^2)^2,$$

where $\beta_T=3exp\left[(3TC_1+12\gamma^2)T\right]$, which completes the proof.

As the dynamical system (1) is controllable, we now move on to the search for the optimal control, using the maximum Pontryagin's principle at infinite horizon.

4. Formulation of an optimal control problem

In this section, we characterize the optimal control of the problem (see [14–16]). To do so, we take an extension of the objective function introduced in [5] because the maintenance cost is applied to all installed solar production capacity:

$$f^0(u(t),\ X(t),\ K(t))=p(t)(x(t)+y(t))-c_x x(t)-c_y y(t)-P_K k(t)-C_K(1-\theta)K(t),\tag{13}$$

where $u(.)$ is a control that we choose among the functions $x(.)$, $y(.)$ and $k(.)$. The control functions are respectively the instantaneous oil extraction rate, the flow from solar production and the instantaneous purchase rate of the solar panels. It is necessary to take these control functions bounded. Therefore, we have $x(.)\in[0,\ x_{max}]$, $y(.)\in[0,\ y_{max}]$ and $k(.)\in[0,\ k_{max}]$ where x_{max}, y_{max} and k_{max} are strictly positive maximum values. We consider the following optimal control problem, where $r>0$ is the discount rate:

$$(\mathscr{P}):\begin{cases}\max_u\displaystyle\int_0^\infty e^{-rt}f^0(u(t),\ X(t),\ K(t))dt,\\ s.c\begin{cases}\dot X(t)=-x(t)(X(t)-\alpha), & X(0)=X_0, & X(t)\geq 0,\\ \dot K(t)=k(t)-\theta K(t), & K(0)=0, & K(t)\geq 0,\\ x(t),\ y(t)\geq 0,\ k(t)\geq 0, & K(t)-y(t)\geq 0.\end{cases}\end{cases}$$

Note that the $y(.)$ function is not in the state equations.

We use the maximum Pontryagin's principle at infinite horizon to characterize the optimal control. To do so, we first define the Hamiltonian $\mathscr{H}$, associated with the problem by means of the previously defined cost function and the scalar product between the adjoint vector (λ_X, λ_K) and the second members of the state equations:

$$\begin{aligned}\mathscr{H}(t, Y(t), u(t), \lambda_X(t), \lambda_K(t)) =& p(t)(x(t)+y(t)) - c_x x(t) - c_y y(t) - P_K(t)k(t) \\ & - C_K(1-\theta)K(t) - \lambda_X(t)x(t)(X(t)-\alpha) + \lambda_K(t)(k(t)-\theta K(t)).\end{aligned} \tag{14}$$

We define now the Lagrangian $\mathscr{L}$ in order to take into account all constraints:

$$\begin{aligned}\mathscr{L}(t, Y(t), u(t), \lambda_X(t), \lambda_K(t), \gamma_x, \gamma_y, \gamma_k, \gamma_K) =& p(t)(x(t)+y(t)) - c_x x(t) - c_y y(t) - P_K(t)k(t) \\ & - C_K(1-\theta)K(t) - \lambda_X(t)x(t)(X(t)-\alpha) + \lambda_K(t)(k(t)-\theta K(t)) \\ & + \gamma_x x(t) + \gamma_y y(t) + \gamma_k k(t) + \gamma_K(K(t)-y(t)).\end{aligned} \tag{15}$$

We have the following theorem.

Theorem 4 Let $(Y^*(.), u^*(.))$ be an optimal pair for the control problem $(\mathscr{P})$. Then there exists a continuously differentiable $(\lambda(.) = (\lambda_X(.), \lambda_K(.)) : \mathbb{R}_+ \to \mathbb{R}^2$ such that, for any $t \geq 0$,

$$\begin{cases} \dot{\lambda}_X(t) = r\lambda_X(t) - \dfrac{\partial \mathscr{L}}{\partial X}, \\ \dot{\lambda}_K(t) = r\lambda_K(t) - \dfrac{\partial \mathscr{L}}{\partial K}, \end{cases}$$

with the transversality conditions

$$\lim_{t \to +\infty} e^{-rt} X(t)\lambda_X(t) = 0$$

and

$$\lim_{t \to +\infty} e^{-rt} K(t)\lambda_K(t) = 0.$$

Moreover, we have the maximization condition:

$$\mathscr{H}(t, X^*(t), K^*(t), u^*(t), \lambda^*(t)) = \max_u \mathscr{H}(t, X(t), K(t), u(t), \lambda(t)).$$

Proof. For the proof of this theorem, we apply the maximum Pontryagin's principle to an infinite horizon. Therefore, the system of adjoint equations becomes:

$$\begin{cases} \dot{\lambda}_X(t) = (r + x(t))\lambda_X(t), \\ \\ \dot{\lambda}_K(t) = (r+\theta)\lambda_K(t) + C_K(1-\theta) - \gamma_K, \end{cases}$$

which is equivalent to

$$\begin{cases} \lambda_X(t) = \lambda_X(0)e^{rt + \int_0^t x(s)ds}, \\ \\ \lambda_K(t) = \lambda_K(0)e^{(r+\theta)t} - \dfrac{C_K(1-\theta) - \gamma_K}{r+\theta}, \end{cases}$$

where $\lambda_X(0)$ et $\lambda_K(0)$ are two non-zero constants. This gives us the existence of $\lambda(.) = (\lambda_K(.),\ \lambda_K(.))$.

In our case, the Lagrangian is linear with respect to each component of the optimal control $u^*(.) = (x^*(.),\ y^*(.),\ k^*(.))$, thus we have the following proposition:

Proposition 2 The components of the optimal control $u^*(.) = (x^*(.),\ y^*(.),\ k^*(.))$ for the control problem $(\mathscr{P})$ are of bang-bang type.

Proof. Using the Lagrangian which is linear with respect to x, y and k, the optimality conditions give

$$\begin{cases} \dfrac{\partial \mathscr{L}}{\partial x} = p(t) - c_x - \lambda_X(t)(X(t) - \alpha) + \gamma_x, \\ \\ \dfrac{\partial \mathscr{L}}{\partial y} = p(t) - c_y + \gamma_y - \gamma_K, \\ \\ \dfrac{\partial \mathscr{L}}{\partial k} = -P_K(t) + \lambda_K(t) + \gamma_k, \end{cases} \tag{16}$$

which are independent of x, y and k, so the controls $x^*(.)$, $y^*(.)$ and $k^*(.)$ are bang-bang.

We have just proved that the components of the optimal control are of bang-bang type. We are going to state and prove a theorem in order to characterize each component of this optimal control $u^*(.) = (x^*(.),\ y^*(.),\ k^*(.))$.

Theorem 5 Let $u^*(.) = (x^*(.),\ y^*(.),\ k^*(.))$ be the optimal control of the control problem $(\mathscr{P})$. The optimal controls $u_1^*(.) = x^*(.)$, $u_2^*(.) = y^*(.)$ and $u_3^*(.) = k^*(.)$ are characterized as follows:

$$u_1^*(t) = x^*(t) = \begin{cases} 0 & if \quad t > \dfrac{1}{r}\ln\left[\dfrac{p(t) + \gamma_x - c_x}{(X_0 - \alpha)\,\lambda_X(0)}\right], \\ \\ ? & if \quad t = \dfrac{1}{r}\ln\left[\dfrac{p(t) + \gamma_x - c_x}{(X_0 - \alpha)\,\lambda_X(0)}\right], \\ \\ x_{max} & if \quad t < \dfrac{1}{r}\ln\left[\dfrac{p(t) + \gamma_x - c_x}{(X_0 - \alpha)\,\lambda_X(0)}\right], \end{cases}$$

$$u_2^*(t) = y^*(t) = \begin{cases} 0 & if \quad p(t) < c_y + \gamma_K - \gamma_y, \\ ? & if \quad p(t) = c_y + \gamma_K - \gamma_y, \\ y_{max} & if \quad p(t) > c_y + \gamma_K - \gamma_y, \end{cases}$$

and

$$k^*(t) = \begin{cases} 0 & if \quad t < \frac{1}{(r+\theta)} \ln\left(\frac{(r+\theta)(P_K(t) - \gamma_k) + C_K(1-\theta) - \gamma_K}{\lambda_K(0)(r+\theta)}\right), \\ ? & if \quad t = \frac{1}{(r+\theta)} \ln\left(\frac{(r+\theta)(P_K(t) - \gamma_k) + C_K(1-\theta) - \gamma_K}{\lambda_K(0)(r+\theta)}\right), \\ k_{max} & if \quad t > \frac{1}{(r+\theta)} \ln\left(\frac{(r+\theta)(P_K(t) - \gamma_k) + C_K(1-\theta) - \gamma_K}{\lambda_K(0)(r+\theta)}\right). \end{cases}$$

Proof. The optimality conditions are given by equation (16). According to the previous proposition, the components of the optimal control $u^*(.)$ i.e. $x^*(.)$, $y^*(.)$ and $k^*(.)$ are of bang-bang type. Let's define φ_i the switching functions for u_i^*, $i \in \{1,\ 2,\ 3\}$, where $u_1 = x$, $u_2 = y$ and $u_3 = k$:

$$\varphi_1(t) = \frac{\partial \mathscr{L}}{\partial u_1} = \frac{\partial \mathscr{L}}{\partial x} = p(t) - c_x - \lambda_X(t)(X(t) - \alpha) + \gamma_x,$$

$$\varphi_2(t) = \frac{\partial \mathscr{L}}{\partial u_2} = \frac{\partial \mathscr{L}}{\partial y} = p(t) - c_y + \gamma_y - \gamma_K,$$

$$\varphi_3(t) = \frac{\partial \mathscr{L}}{\partial u_3} = \frac{\partial \mathscr{L}}{\partial k} = -P_K(t) + \lambda_K(t) + \gamma_k.$$

We have $u_1^*(t) = x^*(t) = 0$ if and only if $\varphi_1(t) < 0$, that is

$$p(t) - c_x - \lambda_X(t)(X(t) - \alpha) + \gamma_x < 0.$$

Replacing $\lambda_K(t)$ by its expression, we obtain:

$$e^{rt} > \frac{p(t) + \gamma_x - c_x}{(X_0 - \alpha)\lambda_X(0)}.$$

This is equivalent to writing:

$$rt > \ln\left[\frac{p(t)+\gamma_x - c_x}{(X_0-\alpha)\,\lambda_X(0)}\right],$$

since $p(t)+\gamma_x > c_x,\ \forall t \in [0,\ T]$ and $(X_0-\alpha)\,\lambda_X(0) > 0$.

That is

$$t > \frac{1}{r}\ln\left[\frac{p(t)+\gamma_x - c_x}{(X_0-\alpha)\,\lambda_X(0)}\right].$$

Moreover, we have

$$u_1^*(t) = x^*(t) = \begin{cases} 0 & if \quad t > \dfrac{1}{r}\ln\left[\dfrac{p(t)+\gamma_x - c_x}{(X_0-\alpha)\,\lambda_X(0)}\right], \\ ? & if \quad t = \dfrac{1}{r}\ln\left[\dfrac{p(t)+\gamma_x - c_x}{(X_0-\alpha)\,\lambda_X(0)}\right], \\ x_{max} & if \quad t < \dfrac{1}{r}\ln\left[\dfrac{p(t)+\gamma_x - c_x}{(X_0-\alpha)\,\lambda_X(0)}\right]. \end{cases}$$

But $u_2^*(t) = y^*(t) = 0$ if and only if $\varphi_2(t) < 0$ that is $p(t) < c_y + \gamma_K - \gamma_y$. So we have

$$u_2^*(t) = y^*(t) = \begin{cases} 0 & if \quad p(t) < c_y + \gamma_K - \gamma_y, \\ ? & if \quad p(t) = c_y + \gamma_K - \gamma_y, \\ y_{max} & if \quad p(t) > c_y + \gamma_K - \gamma_y. \end{cases}$$

Finally, we have $u_3^*(t) = k^*(t) = 0$ if and only if $\varphi_3(t) < 0$ that is

$$-P_K(t) + \lambda_K(t) + \gamma_k < 0.$$

After replacing $\lambda_K(t)$ with its expression, we obtain

$$e^{(r+\theta)t} < \frac{(r+\theta)\,(P_K(t)-\gamma_k) + C_K(1-\theta) - \gamma_K}{\lambda_K(0)\,(r+\theta)},$$

that is

$$t < \frac{1}{(r+\theta)}\ln\left(\frac{(r+\theta)\,(P_K(t)-\gamma_k) + C_K(1-\theta) - \gamma_K}{\lambda_K(0)\,(r+\theta)}\right)$$

and hence

$$k^*(t)=\begin{cases} 0 & if \quad t<\dfrac{1}{(r+\theta)}\ln\left(\dfrac{(r+\theta)\,(P_K(t)-\gamma_k)+C_K(1-\theta)-\gamma_K}{\lambda_K(0)\,(r+\theta)}\right), \\ ? & if \quad t=\dfrac{1}{(r+\theta)}\ln\left(\dfrac{(r+\theta)\,(P_K(t)-\gamma_k)+C_K(1-\theta)-\gamma_K}{\lambda_K(0)\,(r+\theta)}\right), \\ k_{max} & if \quad t>\dfrac{1}{(r+\theta)}\ln\left(\dfrac{(r+\theta)\,(P_K(t)-\gamma_k)+C_K(1-\theta)-\gamma_K}{\lambda_K(0)\,(r+\theta)}\right). \end{cases}$$

5. Existence of the equilibrium state

In this section, we prove the existence of the equilibrium state by considering the canonical system composed with the two state equations and the two adjoint equations. We assume that the oil source is being exploited, that is $x(t)>0$ for all $t\in[0,\ T]$. We have

Theorem 6 The following system

$$\begin{cases} \dot{X}=-xX+\alpha x, \\ \dot{K}=k-\theta K, \\ \dot{\lambda}_X=(r+x)\lambda_X, \\ \dot{\lambda}_K=(r+\theta)\lambda_K+C_K(1-\theta)-\gamma_K, \end{cases} \tag{17}$$

has an equilibrium state.

Proof. This system has an equilibrium state $(X^*,\ K^*,\ \lambda_X^*,\ \lambda_K^*)$ if and only if $(X^*,\ K^*,\ \lambda_X^*,\ \lambda_K^*)$ is solution of

$$\begin{cases} -xX+\alpha x=0, \\ k-\theta K=0, \\ (r+x)\lambda_X=0, \\ (r+\theta)\lambda_K+C_K(1-\theta)-\gamma_k=0. \end{cases}$$

Let J be the associated Jacobian matrix defined by:

$$J=\begin{pmatrix} -x(t) & 0 & 0 & 0 \\ 0 & -\theta & 0 & 0 \\ 0 & 0 & r+x(t) & 0 \\ 0 & 0 & 0 & r+\theta \end{pmatrix}.$$

We have

$$det(J) = x(t) \times \theta \times (r + x(t)) \times (r + \theta). \quad (18)$$

One has $det(J) > 0$ because r, θ and $x(t)$ for all $t \in [0,\ T]$ are strictly positive. So there is a unique equilibrium state. Now we study the stability of this equilibrium state which we have previously proved the existence.

Theorem 7 The equilibrium state of the System (17) is a saddle point.

Proof. The proof is based on a theorem of [17]. To do so, we compute and study the sign of the sum of all diagonal minors of order 2 of the Jacobian matrix which is denoted by Z:

$$Z = \begin{vmatrix} -x(t) & 0 \\ 0 & r + x(t) \end{vmatrix} + \begin{vmatrix} -\theta & 0 \\ 0 & r + \theta \end{vmatrix} + 2 \times \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix},$$

that is

$$Z = -[x(t)(r + x(t)) + \theta(r + \theta)]. \quad (19)$$

Thus $Z < 0$ because r, θ and $x(t)$, for all $t \in [0,\ T]$, are strictly positive.

Let us now compute $det(J) - \left(\frac{Z}{2}\right)^2$. After calculations, we obtain

$$det(J) - \left(\frac{Z}{2}\right)^2 = \frac{-(x(t)(r + x(t)) - \theta(r + \theta))^2}{4} \leq 0$$

where

$$det(J) = x(t)\theta(r + x(t))(r + \theta) > 0.$$

Therefore, $0 < det(J) \leq \left(\frac{Z}{2}\right)^2$ and we can conclude that this equilibrium point is a saddle point.

5.1 *Absence of limit cycle*

We state and prove the following theorem to show that there is no cyclical behavior (see [17–19]) between the amount of available pre-oil and the installed solar production capacity.

Theorem 8 For any time $t \geq 0$, there is no cyclical behavior between the available oil stock $X(t)$ and the installed solar generation capacity $K(t)$.

Proof. To prove the existence of a limit cycle between the quantity of available oil $X(t)$ and the solar production capacity $K(t)$, it is necessary to know the signs of the determinant of the Jacobian matrix J and of the sum of all the diagonal minors of order 2 of the Jacobian matrix Z computed at the equilibrium point. In the previous section, we found that the determinant of the Jacobian matrix given by equation (18) is strictly positive and that the sum of all diagonal minors of order 2 of the Jacobian matrix Z given by equation (19) is strictly negative. We can conclude that there is no cyclic behavior between the amount of available oil $X(t)$ and the installed solar generation capacity $K(t)$.

6. Conclusion

In this work, we shown that by controlling the oil extraction rate and the instantaneous purchase rate of solar panels, we can increase solar production capacity while considerably reducing the amount of oil used in order to maximize the expected benefits. Furthermore, we proved that the dynamical system admits a maximal solution that it is controllable and we formulated the stochastic model for which we proved the existence of a unique solution. We characterized the optimal control components $u^*(.) = (x^*(.),\ y^*(.),\ k^*(.))$. We proved the existence of the equilibrium state and that it is a saddle point. Finally, we have shown that at any time t, there is no cyclical behavior between the amount of oil available and the installed solar production capacity. Following on from the work carried out in this paper, it would be interesting to study a model taking into account oil, sun, wind, and its sensitivity. The random aspect of wind power generation should be taken into account and this could provide a new case study. In this paper, we focused on an optimal control problem for a stochastic model. Another approach could be done about stability analysis and/or resource limitation settings, as it has been done in [20, 21] for COVID-19.

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Conflict of Interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

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