High Convergence Order Q-Step Methods for Solving Equations and Systems of Equations

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Abstract: The local convergence analysis of iterative methods is important since it demonstrates the degree of difficulty for choosing initial points. In the present study, we introduce generalized multi-step high order methods for solving nonlinear equations. The local convergence analysis is given using hypotheses only on the first derivative which actually appears in the methods in contrast to earlier works using hypotheses on higher order derivatives. This way we extend the applicability of these methods. The analysis includes the computable radius of convergence as well as error bounds based on Lipschitz-type conditions not given in earlier studies. Numerical examples conclude this study.

***Keywords*:** multi step method, local convergence, Fréchet derivative, system of equations, Banach space

1. Introduction

Iterative regularization models are changing the face of the world by offering the scientists and mathematicians the opportunity to examine many real life problems, with a far greater generality and precision. To make use of the full power of the iterative methods, they must have a firm grip on numerical techniques developed for various mathematical models and their analysis. Application of the iterative schemes is found in any scientific field, where real world problems are modeled into mathematical equations.

Iterative schemes/methods are general terminology used for certain classes of numerical schemes where the solution procedure starts with an approximate value/function and then apply the method repeatedly to obtain a better approximation. Many mathematical equations are in the form (or are reduced to),

$$F(x) = 0 \tag{1}$$

where $F : D \subseteq B_1 \to B_2$ is a Fréchet-differentiable operator, B_1 and B_2 are Banach spaces and D is a nonempty open convex subset of B_1. Such equations can be linear or nonlinear in nature and there are various iterative schemes used to obtain the solution. Also, these iterative schemes are useful in solving many optimization problems from different disciplines. Many of these methods are firmly based on various calculus and functional analysis concepts and they can be effectively implemented by taking the advantage of the speed and the power of modern computer technologies. In particular three step methods have been introduced in the special case when $B_1 = B_2 = \mathbb{R}^i$ (i a natural number) to solve nonlinear systems [1-3, 5-7, 12, 13, 16, 18-21, 25-34]. We introduce in a Banach space setting multi-step method consisting of q+2 steps, $q \in \mathbb{N}$ defined for each n = 0, 1, 2, . . . by for $x_0 \in D$ an initial point

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$$
\begin{aligned}
y_n^{(1)} &= \varphi_0(x_n) = x_n - F'(x_n)^{-1} F(x_n) \\
z_n &= \varphi_1(x_n, y_n) \\
z_n^{(1)} &= z_n - \varphi(x_n, y_n) F(z_n) \\
&\;\;\vdots \\
z_n(q-1) &= z_n^{(q-2)} - \varphi(x_n, y_n) F(z_n^{(q-2)}) \\
x_{n+1} &= z_n^{(q-1)} - \varphi(x_n, y_n) F(z_n^{(q-1)}) \\
\varphi(x_n, y_n) &= \frac{1}{2}\left(F'(x_n)^{-1} F'(x_n) F'(y_n)^{-1} + F'(x_n)^{-1}\right)
\end{aligned}
\tag{2}
$$

Here functions are defined: $\varphi_0 : D \to B_1, \varphi_1 : D^2 \to B_1$ are iteration operators. Usually φ_1 is an iteration operator of convergence order $p \geq 2$. The order of convergence was shown to be $p+2q$ [32]. Numerous popular iterative methods are special cases of method (2) [2–11,14,15,17–20,22–28,30–34] (see Section 3).

The local convergence analysis usually involves Taylor expansions and con-ditions on higher order derivatives not appearing in these methods. Moreover, these approaches do not provide a computable radius of convergence and error estimates on the distances $\|x_n - x^*\|$. Therefore the initial point is a shot in the dark. These problems limit the usage of these methods. That is why in the present study using only conditions on the first derivative, we address the preceding problems in the more general setting of methods (2) and Banach space.

We find computable radii of convergence as well as error bounds on the distances based on Lipschitz-type conditions. The order of convergence is found using computable order of convergence (COC) or approximate computational order of convergence (ACOC) [31] (see Remark 4) that do not require usage of higher order derivatives. This way we expand the applicability of three step method (2) under weak conditions.

The rest of the study is organized as follows: Section 2 contains the local convergence of the method (2), wherein the concluding Section 3 applications and numerical examples can be found.

2. Local convergence analysis

The local convergence analysis of the method (2) is based on some parameters and scalar functions that appear in the proof.

We shall adopt the notation for $x \in B_1$ and $\mu > 0, U(x,\mu) = \{y \in \|x-y\| < \mu\}$ and $\bar{U}(x,\mu) = \{y \in B_1 : \|x-y\| \leq \mu\}$. The local convergence analysis of method (2) is based on conditions (A):

(A_1) There exist a continuous and increasing function $w_0 : (0,\infty) \to (0,\infty)$, such that equation

$$w_0(t) - 1=0 \tag{3}$$

has a least positive zero denoted by ρ_{-2}. Define functions g_{-2} and h_{-2} on the interval $(0,\rho_{-2})$ with values in $(0,\infty)$ as

$$g_{-2}(t) = \frac{\int_0^1 w\big((1-\theta)t\big)d\theta}{1-w_0(t)} \text{ and } h_{-2}(t) = g_{-2}(t) - 1$$

where w is a given continuous and increasing function defined on $(0,\rho_{-2})$ with values in $(0,\infty)$. Equation $h_{-2}(t)=0$ has a least zero in $(0,\rho_{-2})$ denoted by r_{-2};

(A_2) There exists a continuous and increasing function g_{-1} on $(0,\rho_{-1})$, $(p_{-1} \leq p_{-2})$ with values in $(0,\infty)$ such that equation $h_{-1}(t) = g_{-1}(t) - 1 = 0$ has a least zero in $(0,\rho_{-1})$ denoted by r_{-1}.

(A_3) Equations $w_0(g_{i-1}(t)t) - 1 = 0$, $i = 0, 1, 2, \ldots q-1$ have least solutions $p_i \in (0,\rho_{i-1}), p_i \leq \rho_{i-1}$, where the g_i functions are defined as

$$g_i(t)=[g_{i-2}\left(g_{i-1}(t)t\right)+\left(\frac{w_0(t)+w_0\left(g_{i-1}(t)t\right)}{(1-w_0(t))\left(1-w_0\left(g_{i-1}(t)t\right)\right)}+\frac{1}{2}\frac{w_0(t)+w_0\left(g_{i-2}(t)t\right)}{(1-w_0(t))\left(1-w_0\left(g_{i-2}(t)t\right)\right)}\right)\times\int_0^1 w_1\left(\theta g_{i-1}(t)t\right)d\theta]g_{i-1}(t)$$

and w_1 is a continuous and increasing function defined on $(0,\rho_{-1})$ with values in $(0,\rho_{-1})$. Moreover, define functions $h_i(t) = g_i(t) - 1$. Suppose equations $h_i(t) = 0$ have least solutions on $(0, p_i)$ denoted by r_i. Define a radius of convergence r by

$$r=\min\{p_j\},\ j=-2,-1,0,1,\ldots,q-1 \tag{4}$$

Then, for each $t\in[0,r)$

$$0 < w_0(t) < 1 \tag{5}$$

$$0 < w_0(g_{j-2}(t)t) < 1 \tag{6}$$

and

$$0 < g_j(t) < 1 \tag{7}$$

(A_4) $F: D\subseteq B_1\to B_2$ is a continuously Fréchet differentiable operator.

(A_5) There exists $x^*\in D$ such that $F(x^*)=0$ and $F'(x^*)^{-1}\in L(B_2,B_1)$.

(A_6) There exists a continuous and increasing function w_0 defined on the interval $(0,\infty)$ with values in itself such that for $x\in D$

$$\left\|F'(x^*)^{-1}\left(F'(x)-F'(x^*)\right)\right\|\le w_0\left(\left\|x-x^*\right\|\right)$$

Set $D_0=D\cap U(x^*,\rho_{-2})$

(A_7) There exists a continuous and increasing function g_{-1} on $(0,\rho_{-2})$ with values in $(0,\infty)$, and an iteration function $\varphi_1: D_0\times D_0\to L(B_1,B_2)$ such that for all $x\in D_0$

$$\left\|\varphi_1(x,y)-x^*\right\|\le g_{-2}(\left\|x-x^*\right\|)\left\|x-x^*\right\|$$

where $y=x-F'(x)^{-1}F(x)$

Set $D_1=D\cap U(x^*,\rho_{-1})$

(A_8) There exist continuous and increasing functions w and w_1 defined on $(0,\rho_{-1})$ with values in $(0,\infty)$ such that for all $x,y\in D_1$

$$\left\|F'(x^*)^{-1}\left(F'(y)-F'(x)\right)\right\|\le w\left(\left\|y-x\right\|\right)$$

and

$$\left\|F'(x^*)^{-1}F'(x)\right\|\le w_1\left(\left\|x-x^*\right\|\right)$$

(A_9) $\bar{U}(x^*,r)\subset D$, where r is defined by (4).

(A_{10}) There exists $R\ge r$ such that $\int_0^1 w_0(\theta R)d\theta<1$. Set $D_2=D\cap\bar{U}(x^*,R)$.

Next, we present the local convergence analysis of the method (2) under the conditions (A), the classical Banach lemma on invertible operators [17], and the preceding notation.

Theorem 2.1 Suppose that the "(A)" conditions hold. Then sequence $\{x_n\}$ generated for $x_0\in U(x^*,r)$ by method (2) is well defined in $U(x^*, r)$, remains in $U(x^*, r)$ for each $n = 0, 1, 2, \ldots$ and converged to x^*. Moreover, the following error bounds hold

$$\|y_n^{(1)} - x^*\| \le g_{-2}\left(\|x_n - x^*\|\right)\|x_n - x^*\| \le \|x_n - x^*\| < r \tag{8}$$

$$\|z_n - x^*\| \le g_{-1}\left(\|x_n - x^*\|\right)\|x_n - x^*\| \le \|x_n - x^*\| < r \tag{9}$$

and

$$\|z_n^{(j)} - x^*\| \le g_i\left(\|x_n - x^*\|\right)\|x_n - x^*\| \le \|x_n - x^*\| \tag{10}$$

where the iteration functions are defined previously.

Proof. Using induction, condition $x_0 \in U(x^*,r)$ and conditions (A), we obtain that estimates (8)-(9) hold for $n = 0$. By (4) and (A_6), we have

$$\left\|F'(x^*)^{-1}\left(F'(x) - F'(x^*)\right)\right\| \le w_0\left(\|x - x^*\|\right) \le w_0(r) < 1$$

which together with the classical Banach lemma on invertible operators [?, ?] shows $F'(x)$ is invertible with

$$\|F'(x)^{-1}F'(x_*)\| \le \frac{1}{1 - w_0\left(\|x - x^*\|\right)} \tag{11}$$

and y_0 is well defined by method (2) for $n = 0$. Then, by (4), (A_4), (A_5) and (11), we get

$$\begin{aligned}
\|y_0 - x^*\| &= \|x_0 - x^* - F'(x_0)^{-1}F(x_0)\| \\
&= \left\|F'(x_0)^{-1}F'(x^*)[F'(x^*)^{-1}]\int_0^1 F'(x^* + \theta(x_0 - x^*)) - F'(x_0))(x_0 - x^*)d\theta\right. \\
&\le \frac{\int_0^1 w((1-\theta)\|x_0 - x^*\|d\theta}{1 - w_0\|x_0 - x^*\|} \\
&\le g_{-2}\left(\|x_0 - x^*\|\right)\|x_0 - x^*\| \le \|x_0 - x^*\| < r
\end{aligned} \tag{12}$$

so (8) holds for $n = 0$ and $y_0 \in U(x^*,r)$. Then, z_0 is also well defined by the second substep of method (2) for $n = 0$. So, by (4) and (A_7)

$$\|z_0 - x^*\| = \|\varphi_1(x_0, y_0) - x^*\| \le g_{-1}\left(\|x_0 - x^*\|\right)\|x_0 - x^*\| \le \|x_0 - x^*\| \tag{13}$$

so (9) holds for $n = 0$ and $z_0 \in U(x^*,r)$. Then, we obtain by (4), (6), (7)(for $j = 0$), (11)-(??) and the third substep of the method (2) for $n = 0$ that

$$\left\|z_0^{(1)}-x^*\right\|=\left\|(z_0-x^*-F'(z_0)^{-1}F(z_0))+F'(z_0)^{-1}-\varphi(x_0,y_0)))F(z_0)\right\|$$

$$\leq\left\|z_0-x^*-F'(z_0)^{-1}F(z_0)\right\|+E_0$$

$$\leq\left[\frac{\int_0^1 w((1-\theta)\left\|z-x^*\right\|d\theta}{1-w_0\left(\left\|z_0-x^*\right\|\right)}+\left(\frac{w_0\left(\left\|x_0-x^*\right\|\right)+w_0\left(\left\|z_0-x^*\right\|\right)}{\left(1-w_0\left(\left\|x_0-x^*\right\|\right)\right)\left(1-w_0\left(\left\|z_0-x^*\right\|\right)\right)}\right.\right.$$

$$\left.\left.+\frac{1}{2}\left(\frac{w_0\left(\left\|x_0-x^*\right\|\right)+w_0\left(\left\|y_0-x^*\right\|\right)}{\left(1-w_0\left(\left\|x_0-x^*\right\|\right)\right)\left(1-w_0\left(\left\|y_0-x^*\right\|\right)\right)}\right)\int_0^1 w_1(\theta\left\|z_0-x^*\right\|d\theta\right]\left\|z_0-x^*\right\|$$

$$\leq g_0\left(\left\|x_0-x^*\right\|\right)\left\|x_0-x^*\right\|\leq\left\|x_0-x^*\right\| \tag{14}$$

where $E_0=\left\|F'(z_0)^{-1}-\varphi(x_0,y_0)F(z_0)\right\|$, so (10) holds for $i = 0$ and $z_0^{(0)} \in U(x^*,r)$, where we also used the estimates

$$F(x)=F(x)-F(x^*)=\int_0^1 w_1(x^*+\theta(x-x^*))d\theta(x-x^*) \tag{15}$$

$$\left\|F'(x^*)^{-1}F'(x)\right\|\leq\int_0^1 w_1\left(\theta\left\|x-x^*\right\|\right)d\theta\left\|x-x^*\right\| \tag{16}$$

and

$$F'(z_0)^{-1}-\varphi(x_0,y_0)=F'(z_0)^{-1}-\frac{1}{2}[F'(x_0)^{-1}F'(x_0)F'(y_0)^{-1}+F'(x_0)^{-1}]$$

$$=F'(z_0)^{-1}-F'(x_0)^{-1}-\frac{1}{2}[F'(x_0)^{-1}F'(x_0)F'(y_0)^{-1}-F'(x_0)^{-1}]$$

$$=F'(z_0)^{-1}(F'(x_0)-F'(z_0))F'(x_0)^{-1}-\frac{1}{2}F'(x_0)^{-1}(F'(x_0)-F'(y_0))F'(y_0)^{-1}$$

leading to

$$\left\|E_0F_0(z_0)\right\|=\left\|(E_0F'(x^*))(F'(x^*)^{-1}F(z_0))\right\|$$

$$\leq\left(\frac{w_0\left(\left\|x_0-x^*\right\|\right)+w_0\left(\left\|z_0-x^*\right\|\right)}{\left(1-w_0\left(\left\|x_0-x^*\right\|\right)\right)\left(1-w_0\left(\left\|z_0-x^*\right\|\right)\right)}\frac{1}{2}\frac{w_0\left(\left\|x_0-x^*\right\|\right)+w_0\left(\left\|y_0-x^*\right\|\right)}{\left(1-w_0\left(\left\|x_0-x^*\right\|\right)\right)\left(1-w_0\left(\left\|y_0-x^*\right\|\right)\right)}\right)$$

$$\times\int_0^1 w_1\left(\theta\left\|z_0-x^*\right\|\right)d\theta\left\|z_0-x^*\right\| \tag{17}$$

Then, replacing $z_0^{(1)}$ by $z_0^{(2)}, \ldots, z_0^{(3)}, z_0^{(q)}$, we get

$$\left\|z_0^{(i)}-x^*\right\|\leq g_i\left(\left\|x_0-x^*\right\|\right)\left\|x_0-x^*\right\| \tag{18}$$

In particular, we have

$$\left\|x_1-x^*\right\|\leq c\left\|x_0-x^*\right\|, c=g_q\left(\left\|x_0-x^*\right\|\right)\in[0,1) \tag{19}$$

Hence, items (8)-(10) hold for $n = 0$. By simply replacing x_0 , y_0 , z_0 , $z_0^{(j)}$ by x_k , y_k , z_k , $z_k^{(j)}$, respectively in the preceding computations, we obtain estimates (8)-(10). Then, from the estimate

$$\left\|x_{k+1}-x^*\right\| = \left\|z_n^{(q)}-x^*\right\| \le c\left\|x_n-x^*\right\| < \left\|x_n-x^*\right\| \tag{20}$$

we obtain $\lim_{k\to\infty} x_k = x^*$ and $x_{k+1} \in U(x^*, r)$. The uniqueness part is obtained form the estimate

$$\left\|F'(x^*)^{-1}\left(\int_0^1 F'(a+\theta(x^*-a)) - F'(x^*)\right)\right\| \le \int_0^1 w_0(\theta\left\|x^*-a\right\|)d\theta \le \int_0^1 w_0(\theta R)d\theta < 1 \tag{21}$$

where $a \in D_2$ with $F(a) = 0$. Moreover, from (21) $Q = \int_0^1 F'(a+\theta(x^*-a))d\theta$ is invertible. Then, by $0 = F(x^*) - F(a) = Q(x^*-a)$, we deduve x^*-a.

Remark 2.2 We can compute the computational order of convergence (COC)[31] defined by

$$\xi = \ln\left(\frac{\left\|x_{n+1}-x^*\right\|}{\left\|x_n-x^*\right\|}\right) / \ln\left(\frac{\left\|x_n-x^*\right\|}{\left\|x_{n-1}-x^*\right\|}\right)$$

or the approximate computational order of convergence

$$\xi = \ln\left(\frac{\left\|x_{n+1}-x_n\right\|}{\left\|x_n-x_{n-1}\right\|}\right) / \ln\left(\frac{\left\|x_n-x_{n-1}\right\|}{\left\|x_{n-1}-x_{n-2}\right\|}\right)$$

This way we obtain in practice the order of convergence without resorting to the computation of higher order derivatives appearing in the method or in the sufficient convergence criteria usually appearing in the Taylor expansions for the proofs of those results [5, 6, 14, 20, 25–28, 30–34]. It is worth noticing that the computation of ξ and ξ_1 uses the method (2) and does not depend on Theorem 2.1 which simply guarantees convergence to x^*. In particular, the computation of ξ_1 does not even require knowledge of x^*. Indeed notice that we rely on the iterates x_n picked from the method (2) which in turn rely on the iteration operators. Moreover, in the case of ACOC not even knowledge of the solution x^* is required.

3. Numerical examples

Let us consider a specialization of method (2) to test the convergence criteria defined as

$$y_n = x_n - F'(x_n)^{-1} F(x_n)$$

$$x_{n+1} = y_n - F'(y_n)^{-1} F(y_n)$$

Then, we have

$$g_{-2}(t) = \frac{\int_0^1 w((1-\theta)t)d\theta}{1-w_0(t)}$$

and

$$g_{-1}(t) = \frac{\int_0^1 w((1-\theta)g_{-2}(t)t)d\theta g_{-2}(t)}{1-w_0(g_{-2}(t)t)}$$

Example 3.1 Let us consider a system of differential equations governing the motion of an object and given by

$$F_1'(x) = e^x,\ F_2'(y) = (\mathrm{e}-1)y + 1,\ f_3(z) = 1$$

with initial conditions $F_1(0) = F_2(0) = F_3(0) = 0$. Let $F = (F_1, F, F_3)$. Let $B_1 = B_2 = \mathbb{R}^3, D = \bar{U}(0,1), x^* = (0,0,0)^T$. Define function F on D for $w = (x, y, z)^T$ by

$$F(w) = (e^x - 1, \frac{e-1}{2} y^2 + y, z)^T$$

The Fréchet-derivative is defined by

$$F'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Notice that using the (A) conditions for $x^* = (0, 0, 0)^T$, we get $w_0(t) = (e-1)t, w(t) = e^{\frac{1}{e-1}} t, w_1(t) = e^{\frac{1}{e-1}}$. The radii are

$r_{-2} = 0.382691912232385744729867838032 08 = r$, $r_{-1} = 0.382691912232385966774472763063 39$.

Example 3.2 Let $B_1 = B_2 = C[0,1]$, the space of continuous functions defined on [0, 1] be equipped with the max norm. Let $D = \bar{U}(0,1)$. Define function
F on D by

$$F(\varphi)(x) = \varphi(x) - 5\int_0^1 x\theta\varphi(\theta)^3 d\theta \tag{22}$$

We have that

$$F'(\varphi(\xi))(x) = \xi(x) - 15\int_0^1 x\theta\varphi(\theta)^2 \xi(\theta) d\theta \text{, for each } \xi \in D.$$

Then, we get for $x^* = 0$, that $w_0(t) = 7.5t$, $w(t) = 15t$ and $w_1(t) = 2$. Then the radii are

$r_{-2} = 0.0666666666666666666666666666666667 = r_{-1} = r$

Examlpe 3.3 Let $B_1 = B_2 = \mathbb{R}$, and $D = [-\frac{1}{2}, \frac{3}{2}]$. Define F on D by

$$F(x) = x^3 \log x^2 + x^5 - x^4$$

Then, we get $x_* = 1$, $w_0(t) = w(t) = 96.6629073t$ and $w_1(t) = 2$. Then the radii are

$r_{-2} = 0.00689681994146545528784342238282 08 = r$, $r_{-1} = 0.00689681994146545615520516037122 43$.

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