Research Article

Normalized Laplacian Energy and Distance Based Energy for Cross Monic Zero Divisor Graphs Associated with Commutative Ring

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Abstract: Cross monic zero divisor graph for a commutative ring $\mathcal{R}$ is a connected graph, denoted by $CMZ\mathcal{G}(\mathbb{Z}/\langle n \rangle \times \mathbb{Z}_m[x]/\langle f(x) \rangle)$ with order $\xi$, whose vertices are non-zero zero divisors $\mathbb{Z}(\mathcal{R})/\{0\}$ of commutative ring, and two vertices $u, v$ are connected by an edge if and only if $uv = 0$. In this paper, we discuss energy, Laplacian energy, distance energy and distance signless Laplacian energy for $CMZ\mathcal{G}(\mathbb{Z}_2 \times \mathbb{Z}_p \langle x \rangle / \langle x^2 \rangle)$ and $CMZ\mathcal{G}(\mathbb{Z}_p \times \mathbb{Z}_p \langle x \rangle / \langle x^2 \rangle)$. Also, we determine the normalized Laplacian energy.

Keywords: commutative ring, laplacian energy, distance signless laplacian energy, normalized laplacian energy

MSC: 13A70, 05C50, 05E40

1. Introduction

Let $\mathcal{R}$ be a commutative ring with multiplicative identity $1 \neq 0$. If there exists $x_2 \in \mathcal{R}$ ($x_2 \neq 0$) such that $x_1 x_2 = 0$ for some $x_1 \in \mathcal{R}$ ($x_1 \neq 0$), then $x_1$ is referred to as a zero divisor of $\mathcal{R}$. The collection of zero divisors is symbolized by $\mathbb{Z}(\mathcal{R})$, while $\mathbb{Z}(\mathcal{R})/\{0\} = \mathbb{Z}(\mathcal{R})^*$ is the collection of nonzero zero divisors of $\mathcal{R}$. The zero divisor graph $\Gamma(\mathcal{R})$ of $\mathcal{R}$ is a graph, where $\mathbb{Z}(\mathcal{R})$ is its node set and two different nodes $y, z \in \mathbb{Z}(\mathcal{R})$ are connected if $yz = 0$. Beck [1] established such graphs over commutative rings in his concept, he incorporated the identity and was primarily concerned with the coloring of a commutative ring. Following that, Anderson et al. [2] updated the concept of $\Gamma(\mathcal{R})$ by omitting the identity of $\mathcal{R}$. The finite field of order $n$ is represented by $\mathbb{F}_n$ and a ring of integers modulo $n$ by $\mathbb{Z}_n$. The order of $\Gamma(\mathbb{Z}_n)$ is $n - 1 - \phi(n)$, where as $\phi$ is Euler’s phi function. The graph theoretic characteristics of $\Gamma(\mathbb{Z}_n)$ are widely investigated [3–5]. Shang [6] focuses on the commutativity aspects within prime near-rings, providing valuable insights that enrich the broader understanding of ring theory. Investigation of the spectral properties of matrices associated with graphs is always interesting and challenging. We note that the graphs associated with different algebraic structures, for instance, power graphs [7], annihilator monic prime graph [8] and commuting graphs of groups [9, 10] have helped to solve several problems both in algebra and combinatorics. Alali et al. [11], implies a study of algebraic structures within $\mathbb{Z}_n$ and their connections with topological indices and entropies, underscoring the interdisciplinary intersection of algebra and graph theory. The adjacency matrix of $\mathcal{G}$ is the $n \times n$ matrix $\mathcal{A} = (a_{ij})$, where $a_{ij} = 1$ if there is an edge between vertex $i$ and vertex $j$, otherwise $a_{ij} = 0$. For an $n$-vertex graph $G$ with adjacency matrix $\mathcal{A}$ having eigenvalues
The structure of this paper is outlined as follows: In Section 2, we explore the energy and Laplacian energy of cross monic zero divisor graphs within the commutative rings \( \mathbb{Z}_2 \times \mathbb{Z}_p[x]/\langle x^2 \rangle \) and \( \mathbb{Z}_p \times \mathbb{Z}_p[x]/\langle x^2 \rangle \). Section 3 is dedicated to the examination of the distance energy and distance signless Laplacian energy of cross monic zero divisor graphs. Furthermore, in Section 4, we delve into the discussion of the normalized Laplacian eigenvalues and their energy in the context of cross monic zero divisor graphs.
$$\mathcal{M}(\mathbb{Z}_2 \times \mathbb{Z}_3[x]/(x^2))$$

Figure 1.

$$\mathcal{M}(\mathbb{Z}_3 \times \mathbb{Z}_4[x]/(x^2))$$

Figure 2.
2. Energy and Laplacian energy of cross monic zero divisor graphs of commutative ring

**Theorem 1** Energy of cross monic zero divisor graph of commutative ring \( \mathbb{Z}_2 \times \mathbb{Z}_p[x]/(x^2) \) is

\[
\mathcal{E}(\mathcal{C} \cup \mathcal{G} \cup \mathcal{E} \cup \mathcal{G} \cup \mathcal{D} \cup \mathcal{G}) = \frac{1}{4} \left( 6 \sqrt{p(p-1)} + 3p^2 + 4p + 2 \right)
\]

where \( p \) is prime number greater than 2.

**Proof.** Let the cross monic zero divisor graph of \( \mathbb{Z}_2 \times \mathbb{Z}_p[x]/(x^2) \) be a simple graph, then the adjacent matrix is

**Table 1. Characteristic polynomial of cross monic zero divisor of \( \mathbb{Z}_3 \times \mathbb{Z}_4[x]/(x^2) \)**

<table>
<thead>
<tr>
<th>Characteristic Polynomial</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P_0(\lambda) )</td>
</tr>
<tr>
<td>( (\lambda^2 - \lambda - 2)^2 )</td>
</tr>
<tr>
<td>( P_1(\lambda) )</td>
</tr>
<tr>
<td>( (\lambda^2 - 4\lambda + 2)^2 )</td>
</tr>
<tr>
<td>( P_2(\lambda) )</td>
</tr>
<tr>
<td>( (\lambda^2 - 4\lambda + 2)^2 )</td>
</tr>
</tbody>
</table>

**Table 2. Eigenvalues of cross monic zero divisor of \( \mathbb{Z}_3 \times \mathbb{Z}_4[x]/(x^2) \)**

<table>
<thead>
<tr>
<th>Matrix 1</th>
<th>Eigenvalues</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathcal{A} )</td>
<td>(-6.152(1)^1, -3.250(1)^1, -2.391(4)^1, -2.373(2)^2, -1(3)^3, 0(1)^8, 0.720(3)^1 )</td>
</tr>
<tr>
<td>( \mathcal{L} )</td>
<td>(0(1)^1, 1.145(1)^1, 1.876(9)^2, 2(7)^2, 3(9)^5, 5.186(3)^1, 7(1)^7 )</td>
</tr>
<tr>
<td>( \mathcal{D} )</td>
<td>(10.123(2)^1, 10.944(3)^1, 11(3)^3, 15(1)^1, 17.642(5)^1, 23.081(4)^1 )</td>
</tr>
<tr>
<td>( \mathcal{G} )</td>
<td>(-13.731(6)^1, -5.373(2)^2, -2.961(3)^2, -2(18)^{-1}, -1(3)^{-1}, -0.629(2)^1, 0.373(2)^2 )</td>
</tr>
<tr>
<td>( \mathcal{G} )</td>
<td>(0.565(9)^1, 2.770(1)^1, 62.986(4)^1 )</td>
</tr>
<tr>
<td>( \mathcal{G} )</td>
<td>(-0.337(1)^1, 34.570(7)^1, 35(1)^1, 46.828(2)^1, 51(3)^1, 51.566(5)^1, 52.450(2)^2 )</td>
</tr>
<tr>
<td>( \mathcal{G} )</td>
<td>(63(1)^1, 64.053(5)^1, 67(9)^9, 67.549(6)^2, 76(7)^7, 83.318(2)^1 )</td>
</tr>
<tr>
<td>( \mathcal{G} )</td>
<td>(31(1)^1, 32.845(3)^1, 45.253(9)^1, 46.514(7)^2, 49(3)^9, 52.697(1)^1, 57.991(7)^1 )</td>
</tr>
<tr>
<td>( \mathcal{G} )</td>
<td>(59(1)^1, 60.395(8)^1, 63(9)^9, 63.485(3)^2, 72(7)^7, 128.816(1)^1 )</td>
</tr>
</tbody>
</table>
The characteristic polynomial is $\lambda^2 - 3\lambda + 1 = 0$. Then the eigenvalues satisfying $p(p - 1) \leq \lambda_i \leq \left(\frac{p}{2}\right)^2 + 1$. Then

$$\text{Spec}_{CMZG}(\mathbb{Z}_2 \times \mathbb{Z}_p[x]/\langle x^2 \rangle) = \left(\begin{array}{cccc}
-1 & 0 & \sqrt{p(p-1)} & \frac{\sqrt{p(p-1)} + 1}{2} \\
p - 1 & p^2 - 3 & 1 & 1 & 1 & 1
\end{array}\right)$$

$$\mathcal{E}(CMZG(\mathbb{Z}_2 \times \mathbb{Z}_p[x]/\langle x^2 \rangle)) = \sum_{i=1}^{p^2 - 1} |\lambda_i|$$

$$= (p - 1) + (p^2 - 3)(0) + \sqrt{p(p-1)} + \frac{\sqrt{p(p-1)}}{2} + \frac{\sqrt{p(p-1)} + 1}{2} + \sqrt{p(p-1)} + 1$$

$$= \frac{3}{2} \sqrt{p(p-1)} + p - 1 + \frac{1}{4} \left(p^2 + 2\right) + \frac{1}{2} \left(p^2 + 2\right)$$

$$= \frac{3}{2} \sqrt{p(p-1)} + \frac{3p^2}{4} + p + \frac{1}{2}$$

$$= \frac{1}{4} \left(6\sqrt{p(p-1)} + 3p^2 + 4p + 2\right) .$$

**Theorem 2** Let $CMZG$ be a commutative ring $\mathbb{Z}_p \times \mathbb{Z}_p[x]/\langle x^2 \rangle$ of order $2p^2 - p - 1$ and size $\frac{1}{2} \left(4p^3 - 7p^2 + p + 2\right)$, then

$$\mathcal{E}(CMZG(\mathbb{Z}_p \times \mathbb{Z}_p[x]/\langle x^2 \rangle)) \leq \frac{4p^3 - 7p^2 + p + 2}{2p^2 - p - 1} + \sqrt{2p^2 - p - 2 \left[4p^3 - 7p^2 + p + 2 - \left(\frac{4p^3 - 7p^2 + p + 2}{2p^2 - p - 1}\right)^2\right] .}$$

where $p$ is odd prime.
Proof. Let the cross monic zero divisor graph of $\mathbb{Z}_p \times \mathbb{Z}_p[x]/\langle x^2 \rangle$ be a simple graph with 

$$|V| = 2p^2 - p - 1,$$

$$|E| = \frac{1}{2}(4p^3 - 7p^2 + p + 2),$$

then adjacent matrix is 

$$A(\mathcal{CMZ}(Z_p \times Z_p[x]/\langle x^2 \rangle)) = \begin{pmatrix} I - I & I & I & 0 \\ I & 0 & 0 & I \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{pmatrix}$$

Suppose that $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_{2p^2-p-1}$ are the eigenvalues of $\mathcal{CMZ}(Z_p \times Z_p[x]/\langle x^2 \rangle)$. Then, as is well known, we have 

$$\lambda_1 \geq \frac{4p^3 - 7p^2 + p + 2}{2p^2 - p - 1}$$

(see [21], for example). Moreover, since 

$$\sum_{i=1}^{2p^2-p-1} \lambda_i^2 = 4p^3 - 7p^2 + p + 2$$

must hold (for example, see [22]), we have 

$$\sum_{i=2}^{2p^2-p-1} \lambda_i^2 = 4p^3 - 7p^2 + p + 2 - \lambda_1^2.$$ 

Using this together with the Cauchy-Schwartz inequality, applied to the vectors $(|\lambda_2|, |\lambda_3|, \ldots, |\lambda_{2p^2-p-1}|)$ and $(1, 1, 1, \ldots, 1)$ with $2p^2 - p - 2$ entries, we obtain the inequality 

$$\sum_{i=2}^{2p^2-p-1} |\lambda_i| \leq \sqrt{(2p^2 - p - 2)(4p^3 - 7p^2 + p + 2 - \lambda_1^2)}$$

Thus, we must have
\[ E(\mathcal{CMZ}G) \leq \lambda_1 + \sqrt{(2p^2 - p - 2)(4p^3 - 7p^2 + p + 2 - \lambda_1^2)} \]

Now, since the function \( f(y) = y + \sqrt{(2p^2 - p - 2)(4p^3 - 7p^2 + p + 2 - y^2)} \) decreases on the interval

\[ \sqrt{\frac{4p^3 - 7p^2 + p + 2}{2p^2 - p - 1}} < y \leq \sqrt{4p^3 - 7p^2 + p + 2} \]

in view of \( 4p^3 - 7p^2 + p + 2 \geq 2p^2 - p - 1 \), we see that \( \sqrt{4p^3 - 7p^2 + p + 2} \leq \frac{4p^3 - 7p^2 + p + 2}{2p^2 - p - 1} \leq \lambda_1 \) must hold, and hence \( f(\lambda_1) \leq f\left(\frac{4p^3 - 7p^2 + p + 2}{2p^2 - p - 1}\right) \) must hold as well. From this fact, and inequality \( E(\mathcal{CMZ}G) \leq \lambda_1 + \sqrt{(2p^2 - p - 2)(4p^3 - 7p^2 + p + 2 - \lambda_1^2)} \), it immediately follows that inequality \( E(\mathcal{CMZ}G) \leq \frac{4p^3 - 7p^2 + p + 2}{2p^2 - p - 1} + \sqrt{2p^2 - p - 2 \left[ 4p^3 - 7p^2 + p + 2 - \left(\frac{4p^3 - 7p^2 + p + 2}{2p^2 - p - 1}\right)^2 \right]} \) holds. Hence the proof.

**Example 1** For cross monic zero divisor graph of commutative ring \( \mathbb{Z}_3 \times \mathbb{Z}_3 [x]/\langle x^2 \rangle \) with order 14 and size 25, we have

\[ E(\mathcal{CMZ}G(\mathbb{Z}_3 \times \mathbb{Z}_3 [x]/\langle x^2 \rangle)) \leq 25.5755 \]

**Solution** We consider cross monic zero divisor graph of commutative ring \( \mathbb{Z}_3 \times \mathbb{Z}_3 [x]/\langle x^2 \rangle \) (Figure 3),

![Figure 3. \( \mathcal{CMZ}G(\mathbb{Z}_3 \times \mathbb{Z}_3 [x]/\langle x^2 \rangle) \)](image)

The adjacent matrix is
\[ \mathfrak{A}(\mathcal{M} \mathcal{Z} \mathcal{G}(\mathbb{Z}_3 \times \mathbb{Z}_3[x]/\langle x^2 \rangle)) = \begin{pmatrix} \mathcal{O}_{4 \times 4} & \mathcal{I}_{4 \times 2} & \mathcal{O}_{4 \times 2} & \mathcal{O}_{4 \times 6} \\ \mathcal{I}_{2 \times 4} & \mathcal{I} - \mathcal{I}_{2 \times 2} & \mathcal{I}_{2 \times 2} & \mathcal{O}_{2 \times 6} \\ \mathcal{I}_{2 \times 4} & \mathcal{I}_{2 \times 2} & \mathcal{O}_{2 \times 2} & \mathcal{I}_{2 \times 6} \\ \mathcal{O}_{6 \times 4} & \mathcal{O}_{6 \times 2} & \mathcal{I}_{6 \times 2} & \mathcal{O}_{6 \times 6} \end{pmatrix} \]

Now \(|\mathfrak{A}(\mathcal{M} \mathcal{Z} \mathcal{G}(\mathbb{Z}_3 \times \mathbb{Z}_3[x]/\langle x^2 \rangle)) - \lambda, \mathcal{I}| = 0.

Then the characteristic polynomial is \(\lambda^{14} - 25\lambda^{12} - 12\lambda^{11} + 108\lambda^{10} + 96\lambda^9 = 0\). The spectrum of the graph is

\[
\text{Spec}_{\lambda}(\mathcal{M} \mathcal{Z} \mathcal{G}(\mathbb{Z}_3 \times \mathbb{Z}_3[x]/\langle x^2 \rangle)) = \begin{pmatrix} -4.1425 & -2 & -1 & 0 & 2.4913 & 4.6512 \\ 1 & 1 & 1 & 9 & 1 & 1 \end{pmatrix}
\]

Therefore

\[
\mathfrak{E}(\mathcal{M} \mathcal{Z} \mathcal{G}(\mathbb{Z}_3 \times \mathbb{Z}_3[x]/\langle x^2 \rangle)) = \sum_{i=1}^{14} |\lambda_i|
\]

\[
= | -4.1425 | + | -2 | + | -1 | + 0 + 2.4913 + 4.6512
\]

\[= 14.285\]

**Theorem 3** Let the cross monic zero divisor graph of \(\mathbb{Z}_2 \times \mathbb{Z}_p[x]/\langle x^2 \rangle\) have order \(p^2 + p - 1\), \(\eta = \frac{1}{2}(5p^2 - 7p + 2)\) and \(\Delta = p^2 - 1\). Then

\[
\mathfrak{L}\mathfrak{E}(\mathcal{M} \mathcal{Z} \mathcal{G}(\mathbb{Z}_2 \times \mathbb{Z}_p[x]/\langle x^2 \rangle)) < \frac{2\eta}{p^2 + p - 1} + \sqrt{\eta \left( \eta \left( \frac{4}{(p^2 + p - 1)^2} - 2 \right) + p^4 + 2p^3 - 2p^2 - 3p + 2 \right)}
\]

**Proof.** For the cross monic zero divisor graph of \(\mathbb{Z}_2 \times \mathbb{Z}_p[x]/\langle x^2 \rangle\) with order \(p^2 + p - 1\), \(\eta = \frac{1}{2}(5p^2 - 7p + 2)\) and \(\Delta = p^2 - 1\), we have
\[
\mathcal{L}(\mathcal{M}(\mathbb{Z}_2 \times \mathbb{Z}_p[x]/(x^2))) = \begin{pmatrix}
\mathcal{J} & -\mathcal{J} & 0 & 0 \\
-\mathcal{J} & p^2 - 1 & -\mathcal{J} & 0 \\
0 & -\mathcal{J} & \mathcal{M}_1 & -\mathcal{J} \\
0 & 0 & -\mathcal{J} & \mathcal{M}_2
\end{pmatrix}
\]

where \( \mathcal{M}_1 = 2(p - 1)\mathcal{J} + \mathcal{J} - \mathcal{J} \) and \( \mathcal{M}_2 = (p - 1)\mathcal{J} \). Then 
\[
|\mathcal{L}(\mathcal{M}(\mathbb{Z}_2 \times \mathbb{Z}_p[x]/(x^2))) - \lambda \mathcal{J}| = 0.
\]
Eigenvalues of \( \mathcal{L} \) satisfies the inequality \( 0 \leq \mu_a \leq \triangle + 2 \). Now

\[
\text{Spec}_{\mathcal{L}}(\mathcal{M}(\mathbb{Z}_2 \times \mathbb{Z}_p[x]/(x^2))) = \begin{pmatrix}
0 & \tau_1 & 1 & p - 1 & \tau_2 & 2p - 1 & \tau_3 \\
1 & 1 & p(p - 1) - 1 & p - 2 & 1 & p - 2 & 1
\end{pmatrix}
\]

where \( \tau_1 > 0, \tau_2 > 2(p - 1), \tau_3 > p^2 \).

\[
\mathcal{L}(\mathcal{M}(\mathbb{Z}_2 \times \mathbb{Z}_p[x]/(x^2))) = \sum_{\alpha=1}^{\nu} |\mu_{\alpha} - \frac{2\eta}{p^2 + p - 1}|
\]

\[
= \left| -\frac{2(p^2 + p - 1)}{\frac{1}{2}(5p^2 - 7p + 2)} + (p(p - 1) - 1) \right| - \frac{2(p^2 + p - 1)}{\frac{1}{2}(5p^2 - 7p + 2)}
\]

\[
+ \cdots + (p - 2) \left| (p - 1) - \frac{4(p^2 + p - 1)}{(5p - 2)(p - 1)} \right|
\]

\[
= \frac{2(p^2 + p - 1)}{\frac{1}{2}(5p^2 - 7p + 2)} + (p(p - 1) - 1) \left( \frac{p^2 - 11p + 6}{5p^2 - 7p + 2} \right)
\]

\[
+ \cdots + (p - 2) \left( -\frac{4(p^2 + p - 1)}{5p^2 - 7p + 2} + p - 1 \right)
\]

\[
= \frac{2(p^2 + p - 1)}{\frac{1}{2}(5p^2 - 7p + 2)} + (-1) \left( \frac{(p^2 - 11p + 6)(p(p - 1) - 1)}{5p^2 - 7p + 2} \right)
\]

\[
+ \cdots + (-1) \left( \frac{(p - 2)(5p^3 - 16p^2 + 5p + 2)}{(p - 1)(5p - 2)} \right)
\]
\[ \mathcal{L}(\mathcal{M} \mathcal{Z} G(Z_2 \times Z_p[x]/\langle x^2 \rangle)) < \frac{2\eta}{p^2 + p - 1} + \sqrt{\eta \left( \eta \left( \frac{4}{(p^2 + p - 1)^2} - 2 \right) + p^4 + 2p^3 - 2p^2 - 3p + 2 \right)} \]

Laplacian matrix and spectrum of cross monic zero divisor graph of \( Z_p \times Z_{p^2}[x]/\langle x^2 \rangle \) are

\[
\begin{pmatrix}
(p^2 - 1)I - J & -J & -J & 0 \\
-J & (p^2 - 1)I - J & 0 & -J \\
-J & 0 & (p - 1)J & 0 \\
0 & -J & 0 & (p - 1)J
\end{pmatrix}
\]

\[
\text{Spec}_{\mathcal{L}(\mathcal{M} \mathcal{Z} G(Z_p \times Z_p[x]/\langle x^2 \rangle))} = \begin{pmatrix} 0 & \tau_1 & p - 1 & \tau_2 & (p - 1) + (p(p - 1)) & \tau_3 \\ 1 & 1 & 2p^2 - 3p - 1 & 1 & 2p - 4 & 1 \end{pmatrix}
\]

respectively.

3. Distance based energy of cross monic zero divisor graphs

**Theorem 4** Upper and lower bounds of distance energy of cross monic zero divisor graph of \( Z_2 \times Z_p[x]/\langle x^2 \rangle \) is

\[
\sqrt{2 \sum_{\eta_1 < \eta_2} (d_{\eta_1 \eta_2})^2 + (p^2 + p - 1)(p^2 + p) \rho^{\frac{2}{p^2 + p - 1}}} \leq E_{\mathcal{Z} G}
\]

\[
\leq \sqrt{2(p^2 + p) \sum_{\eta_1 < \eta_2} (d_{\eta_1 \eta_2})^2 + (p^2 + p - 1) \rho^{\frac{2}{p^2 + p - 1}}}
\]

**Proof.** Distance matrix and spectrum of the graph is
\[ D(\mathcal{CMZ} G(\mathbb{Z}_2 \times \mathbb{Z}_p / \langle x^2 \rangle)) = \begin{pmatrix} 2(\mathcal{J} - \mathcal{I}) & \mathcal{J} & 2\mathcal{J} & 3\mathcal{J} \\ \mathcal{J} & 0 & \mathcal{J} & 2\mathcal{J} \\ 2\mathcal{J} & 2\mathcal{J} & 0 & 2\mathcal{J} - \mathcal{I} \\ 3\mathcal{J} & 2\mathcal{J} & \mathcal{J} & 2\mathcal{J} - \mathcal{I} \end{pmatrix} \]

\[ \text{Spec}_D(\mathcal{CMZ} G(\mathbb{Z}_2 \times \mathbb{Z}_p / \langle x^2 \rangle)) = \begin{pmatrix} -2 & -1 & \sqrt{p(p-1)} + 2p - 2 & \sqrt{p-\mathcal{I}} - 1 & \sqrt{p} \frac{\mathcal{J}}{\mathcal{I}} \\ p^2 - 3 & p - 2 & 1 & 1 & 1 & 1 \end{pmatrix} \]

Then

\[ \mathcal{M} \leq p^2 + p - 1 \sum_{\eta_1} \mu_{\eta_1}^2 - \left( \sum_{\eta_1} |\mu_{\eta_1}| \right)^2 \leq (p^2 + p) \mathcal{M} \]

\[ \mathcal{M} \leq 2p^2 + 2p - 2 \sum_{\eta_1 < \eta_2} (d_{\eta_1 \eta_2})^2 - (E_D)^2 \leq (p^2 + p) \mathcal{M} \]

where

\[ \mathcal{M} = p^2 + p - 1 \left[ 2 \sum_{\eta_1 \eta_2} (d_{\eta_1 \eta_2})^2 - \prod_{\eta_1} |\mu_{\eta_1}| \frac{\mathcal{J}}{\mathcal{I} + p - 1} \right] \]

\[ = p^2 + p - 1 \left[ 2 \sum_{\eta_1 < \eta_2} (d_{\eta_1 \eta_2})^2 - \prod_{\eta_1} |\mu_{\eta_1}| \frac{\mathcal{J}}{\mathcal{I} + p - 1} \right] \]

\[ = 2 \sum_{\eta_1 < \eta_2} (d_{\eta_1 \eta_2})^2 - (p^2 + p - 1) \rho \frac{2}{p^{\mathcal{I} + p - 1}} \]

**Theorem 5** If the transmission degree sequence of \( \mathcal{CMZ} G \) is \( \{\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_\xi\} \) and \( \triangle = \left| D^j(\mathcal{CMZ} G) - \frac{1}{\xi} \sum_{a=1}^{\xi} \mathcal{T}_a \mathcal{J}_a \right| \), then
\[
\sqrt{2 \sum_{1 \leq a < b \leq \xi} (\text{dis}_{ab})^2 + \sum_{a=1}^{\xi} \text{Tr}_a^2 - \frac{4\sigma_0^2(\mathcal{C.M.2^d})}{\xi}} + \xi (\frac{\xi}{\xi} - 1) \Delta \frac{\xi}{\xi}
\]

\[
\leq \xi \xi (\mathcal{C.M.2^d}) \leq \sqrt{(\xi - 1) \left( 2 \sum_{1 \leq a < b \leq \xi} (\text{dis}_{ab})^2 + \sum_{a=1}^{\xi} \text{Tr}_a^2 - \frac{4\sigma_0^2(\mathcal{C.M.2^d})}{\xi} \right) + \xi \Delta \frac{\xi}{\xi}}
\]

\textbf{Proof.} Let us choose \( s_a = \alpha_a^2 \), for \( a = 1, 2, 3, \ldots, \xi \). We obtain

\[
M \leq \xi \xi \left( \sum_{a=1}^{\xi} \alpha_a^2 - \left( \sum_{a=1}^{\xi} |\alpha_a| \right)^2 \right)
\]

\[
\leq (\xi - 1) M
\]

i.e.,

\[
M \leq \xi \left( 2 \sum_{1 \leq a < b \leq \xi} (\text{dis}_{ab})^2 + \sum_{a=1}^{\xi} \text{Tr}_a^2 - \frac{4\sigma_0^2(\mathcal{C.M.2^d})}{\xi} \right) - E_{\rho\xi}(\mathcal{C.M.2^d})
\]

\[
\leq (\xi - 1) M,
\]

where

\[
M = \xi \left( \frac{1}{\xi} \sum_{a=1}^{\xi} \alpha_a^2 - \left( \prod_{a=1}^{\xi} \alpha_a^2 \right)^{\frac{1}{\xi}} \right)
\]

\[
= \xi \left( \frac{1}{\xi} \left( 2 \sum_{1 \leq a < b \leq \xi} (\text{dis}_{ab})^2 + \sum_{a=1}^{\xi} \text{Tr}_a^2 - \frac{4\sigma_0^2(\mathcal{C.M.2^d})}{\xi} \right) - \left( \prod_{a=1}^{\xi} \alpha_a^2 \right)^{\frac{1}{\xi}} \right)
\]

\[
= 2 \sum_{1 \leq a < b \leq \xi} (\text{dis}_{ab})^2 + \sum_{a=1}^{\xi} \text{Tr}_a^2 - \frac{4\sigma_0^2(\mathcal{C.M.2^d})}{\xi} + \xi \Delta \frac{\xi}{\xi}
\]

Hence, we get the required bounds.

Distance Laplacian matrix and Distance signless Laplacian matrix of cross monic zero divisor graph is shown in Table 3.
Table 3. Block matrix of distance (signless Laplacian) of cross monic zero divisor graph

<table>
<thead>
<tr>
<th></th>
<th>(D_L(CMZ(G(Z_2 \times Z_{p^2} &gt; Z_2[x]/\langle x^2 \rangle)))</th>
<th>(D_Q(CMZ(G(Z_2 \times Z_{p^2} &gt; Z_2[x]/\langle x^2 \rangle)))</th>
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<tr>
<td>(A_1)</td>
<td>(-J - 2J - 3J)</td>
<td>(B_1 J 2J 3J)</td>
</tr>
<tr>
<td>(-J)</td>
<td>(A_2 - J - 2J)</td>
<td>(J B_2 J 2J)</td>
</tr>
<tr>
<td>(-2J)</td>
<td>(-J A_3 - J)</td>
<td>(2J J B_3 J)</td>
</tr>
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<td>(-3J)</td>
<td>(-2J - J A_4)</td>
<td>(3J 2J J B_4)</td>
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<tr>
<td>(A_2)</td>
<td>(p^2 + 2p - 3)</td>
<td>(B_2 = p^2 + 2p - 3)</td>
</tr>
<tr>
<td>(A_3)</td>
<td>(-J + (2p^2 - 1)I)</td>
<td>(B_3 = J + (2p^2 - 3)I)</td>
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<tr>
<td>(A_4)</td>
<td>(-2J + (3p^2 - 1)I)</td>
<td>(B_4 = 2J + (3p^2 - 5)I)</td>
</tr>
<tr>
<td>(A_1)</td>
<td>(-2J + (2p^2 + 3p - 4)I)</td>
<td>(B_1 = 2J + (2p^2 + 3p - 8)I)</td>
</tr>
</tbody>
</table>

**Theorem 6.** If \(CMZ(G)\) is a connected graph with order \(\xi\) and diameter \(\beta\), then

\[
\xi \sqrt{(\xi - 1) \left( \beta^2 + \frac{\beta^2(\beta - 1)}{4} - \xi + 1 \right)} \geq \mathcal{E}_{g,2} \geq \sqrt{\xi}(\xi - 1).
\]

**Proof.** Since \(dis_{ab} \geq 1\) for \(a \neq b\) and there are \(\frac{\xi(\xi - 1)}{2}\) pairs of vertices in \(CMZ(G)\), then we get

\[
\mathcal{E}_{g,2}(CMZ(G)) \geq \sqrt{\frac{2 \sum_{1 \leq a < b \leq \xi} (dis_{ab})^2 + \sum_{a=1}^{\xi} \text{Tr}_a^2 - \frac{4\sigma_0^2(CMZ(G))}{\xi}}{\xi}}
\]

\[
\geq \sqrt{\frac{2 \frac{\xi(\xi - 1)}{2} + \sum_{a=1}^{\xi} \text{Tr}_a^2 - \sum_{a=1}^{\xi} \text{Tr}_a^2}{\xi}}
\]

\[
= \sqrt{\xi}(\xi - 1).
\]

Again, \(dis_{ab} \leq \beta\) for \(a \neq b\) and there are \(\frac{\xi(\xi - 1)}{2}\) pairs of vertices in \(CMZ(G)\), then we get

\[
\mathcal{E}_{g,2}(CMZ(G)) \leq \sqrt{\xi \left( 2 \sum_{1 \leq a < b \leq \xi} (dis_{ab})^2 + \sum_{a=1}^{\xi} \text{Tr}_a^2 - \frac{4\sigma_0^2(CMZ(G))}{\xi} \right)}
\]

\[
\leq \sqrt{\xi \left( 2 \frac{\xi(\xi - 1)}{2} \beta^2 + \frac{\xi^3(\xi - 1)^2}{4} - \xi(\xi - 1)^2 \right)}
\]

\[
= \xi \sqrt{(\xi - 1) \left( \beta^2 + \frac{\xi^2(\xi - 1)}{4} - \xi + 1 \right)}.
\]
Hence the result.

4. Normalized Laplacian energy of graphs

**Theorem 7** Normalized Laplacian energy of cross monic zero divisor graph is

\[ E_{NL}(CMZG(Z_2 \times Z_p[\langle x^2 \rangle])) < \frac{p^2 + p - 1}{2} \]

**Proof.** Normalized Laplacian matrix of \( NL(CMZG) \) is

\[
NL(CMZG) = \begin{pmatrix}
\mathcal{I} & -\frac{1}{\sqrt{\triangle(CMZG)}} & 0 & 0 \\
-\frac{1}{\sqrt{\triangle(CMZG)}} & \mathcal{I} & -\frac{1}{\sqrt{\triangle(CMZG)(2(p-1)}} & 0 \\
0 & -\frac{1}{\sqrt{\triangle(CMZG)(2(p-1)}} & \mathcal{A} & -\frac{1}{\sqrt{2(p-1)^2}} \\
0 & 0 & -\frac{1}{\sqrt{2(p-1)^2}} & \mathcal{I}
\end{pmatrix}
\]

where \( \mathcal{A} = \mathcal{I} + \mathcal{J} \left[-\frac{1}{2(p-1)}\right] + \mathcal{J} \left[\frac{1}{2(p-1)}\right] \), whose entries are lies \(-1 < a_{ij} \leq 1.5\). If \( p = 3 \), then \( P_{NL} = ((-1 + \delta)^2(301678245 + 2477350454279\delta - 13997313671875\delta^2 + 21056283984375\delta^3 - 12207031250000\delta^4 + 24414062500000\delta^5))) / 24414062500000, \( \delta \) denotes the eigenvalues. Then

\[
\text{Spec}_{NL}(CMZG(Z_2 \times Z_3[\langle x^2 \rangle])) = \begin{pmatrix}
0 & 0.2714 & 1 & 1.25 & 1.5609 & 1.9177 \\
1 & 1 & 6 & 1 & 1 & 1
\end{pmatrix}
\]

\[
E_{NL}(CMZG(Z_2 \times Z_3[\langle x^2 \rangle])) = \sum_{i=1}^{11} \delta_i(\text{CMZG}(Z_2 \times Z_3[\langle x^2 \rangle])) - 1
\]

\[ = 3.4572 \]

If \( p = 5 \), then \( P_{NL} = ((-1 + \delta)^2(-9 + 8\delta)^3(-240380053 - 7487301171875\delta + 55013759375000\delta^2 - 56640625000000\delta^3 + 156250000000000\delta^4))/8000000000000000. \) Then

\[
\text{Spec}_{NL}(CMZG(Z_2 \times Z_5[\langle x^2 \rangle])) = \begin{pmatrix}
0 & 0.1619 & 1 & 1.125 & 1.5365 & 1.9266 \\
1 & 1 & 22 & 3 & 1 & 1
\end{pmatrix}
\]
If $p = 7$, then $P_{N L} = ((-1 + \delta)^46(-10833 + 10000\delta)^3(-12307899999 - 429953071962500\delta + 420638415000000\delta^2 - 447937500000000000\delta^3 + 1250000000000000\delta^4))/1250000000000000000000. Then

$$\text{Spec}_{N L}(CMZ(G(Z_2 \times Z_7[x]/(x^2)))) = \begin{pmatrix} 0 & 0.116 & 1 & 1.083 & 1.5256 & 1.9417 \\ 1 & 1 & 46 & 5 & 1 & 1 \end{pmatrix}$$

$$E_{N L}(CMZ(G(Z_2 \times Z_7[x]/(x^2)))) = 3.6762$$

Generalized matrix of normalized Laplacian of family $(p, p)$ is

$$NLE(CMZ(G)) = \begin{pmatrix} \mathcal{A} & -\frac{1}{\sqrt{p^4-3p^3+2}} & -\frac{1}{\sqrt{(p^2-2)(p-1)}} & 0 \\ -\frac{1}{\sqrt{p^4-3p^3+2}} & \mathcal{J} & \mathcal{O} & -\frac{1}{\sqrt{(p^2-1)(p-1)}} \\ -\frac{1}{\sqrt{(p^2-2)(p-1)}} & \mathcal{O} & \mathcal{J} & \mathcal{O} \\ \mathcal{O} & -\frac{1}{\sqrt{(p^2-1)(p-1)}} & \mathcal{O} & \mathcal{J} \end{pmatrix}$$

where $\mathcal{A} = \mathcal{J} + \mathcal{J} \left[ -\frac{1}{p^2-2} \right] + \mathcal{J} \left[ -\frac{1}{p^2-2} \right].$

**Theorem 8** Suppose $CMZ(G)(Z_2 \times Z_7[x]/(x^2))$ is a connected graph of order $\xi$. Then

$$E_{N L}(CMZ(G)) \geq 1 + \frac{\xi}{\Delta(CMZ(G))} + 2\left(\frac{\xi-1}{2}\right)^{\xi-1} \sqrt[\xi]{\phi(CMZ(G), 1)^2}$$

where $\Delta(CMZ(G))$ is the maximal degree in $CMZ(G)$.

**Proof.** Let $\delta_1 = 0$, and hence

$$E_{N L}(CMZ(G)) = \sum_{k=1}^{\xi} |\delta_k - 1|$$

$$= 1 + \sum_{k=2}^{\xi} |\delta_k - 1|.$$
Now applying arithmetic geometric inequality, we get

\[
\left( \sum_{k=2}^{\xi} |\delta_k - 1| \right)^2 = \sum_{k=2}^{\xi} |\delta_k - 1|^2 + \sum_{2 \leq k \neq f \leq \xi} |\delta_k - 1||\delta_f - 1|
\]

\[
\geq 2R_{-1}(\mathcal{M}^2 \mathcal{L}) - 1 + (\xi - 1)(\xi - 2) \left( \prod_{2 \leq k \neq f \leq \xi} |\delta_k - 1||\delta_f - 1| \right)^{\frac{1}{2}}
\]

\[
= 2R_{-1}(\mathcal{M}^2 \mathcal{L}) - 1 + 2 \left( \frac{\xi - 1}{2} \right) \left( \prod_{k=2}^{\xi} (\delta_k - 1)^2(\xi - 2) \right)^{\frac{1}{2}}
\]

\[
= 2R_{-1}(\mathcal{M}^2 \mathcal{L}) - 1 + 2 \left( \frac{\xi - 1}{2} \right) \sqrt{\left( \prod_{k=2}^{\xi} (\delta_k - 1) \right)^2}
\]

\[
= 2R_{-1}(\mathcal{M}^2 \mathcal{L}) - 1 + 2 \left( \frac{\xi - 1}{2} \right) \sqrt{\varphi(\mathcal{M}^2 \mathcal{L}, 1)^2}.
\]

We know that \( R_{-1} \geq \frac{\xi}{2\Delta(\mathcal{M}^2 \mathcal{L})} \), therefore

\[
E_{(\mathcal{M}^2 \mathcal{L})} \geq 1 + \sqrt{\frac{\xi}{\Delta(\mathcal{M}^2 \mathcal{L})}} + 2 \left( \frac{\xi - 1}{2} \right) \sqrt{\varphi(\mathcal{M}^2 \mathcal{L}, 1)^2}.
\]

\[\square\]

**Theorem 9** Let \( \mathbb{Z}_\alpha \times \mathbb{Z}_\beta[x]/(x^2) \) be connected, \( \mathcal{M}^2 \mathcal{L} \) with smallest \( \delta_s \) and largest \( \delta_\xi \) non-negative normalized Laplacian eigenvalues. Then

(i) \( \delta_\xi - \delta_s \geq \frac{2}{\xi - 1} \sqrt{(\xi - 1)(\xi + 2R_{-1}(\mathcal{M}^2 \mathcal{L}))-\xi^2} \)

(ii) \( \sqrt{\frac{\delta_\xi}{\delta_s}} + \sqrt{\frac{\delta_s}{\delta_\xi}} \geq \frac{2}{\xi} \sqrt{(\xi - 1)(\xi + 2R_{-1}(\mathcal{M}^2 \mathcal{L}))} \)

**Proof.** Consider \( \xi > 2 \), recall the Ozekis inequality [23], stating that \( p_k \) and \( q_k \), \( 1 \leq k \leq \xi \), are positive real numbers, then

\[
\sum_{k=1}^{\xi} p_k^2 \sum_{k=1}^{\xi} q_k^2 - \left( \sum_{k=1}^{\xi} p_k q_k \right)^2 \leq \frac{\xi^2}{4} (N_1N_2 - n_1n_2)^2,
\]

where \( N_1 = \max_{1 \leq k \leq \xi} p_k \), \( N_2 = \max_{1 \leq k \leq \xi} q_k \), \( n_1 = \min_{1 \leq k \leq \xi} p_k \) and \( n_2 = \min_{1 \leq k \leq \xi} q_k \). An application of Ozekis inequality with \( p_k = 1 \) and \( q_k = \delta_k \), \( 2 \leq k \leq \xi \), yields
\[(\xi - 1) \sum_{k=2}^{\xi} \delta_k^2 - \left( \sum_{k=2}^{\xi} \delta_k \right)^2 \leq \frac{(\xi - 1)^2}{4} (\delta_\xi - \delta_2)^2.\]

In view of [24], it is easy to see that
\[
\delta_\xi - \delta_2 \geq \frac{2}{\xi - 1} \sqrt{(\xi - 1)(\xi + 2R_{-1}(CMZG)) - \xi^2}
\]
yielding the assertion (i). To prove assertion (ii), we recall the Polya-Szego inequality, stating that if \(p_k, q_k, N_k, n_k, 1 \leq k \leq \xi\) are as in part (i), then we have
\[
\sum_{k=1}^{\xi} p_k^2 \sum_{k=1}^{\xi} q_k^2 \leq \frac{1}{4} \left( \frac{N_1N_2}{n_1n_2} \frac{n_1n_2}{N_1N_2} \right)^2 \left( \sum_{k=1}^{\xi} p_kq_k \right)^2.
\]

Applying the last inequality \(p_k = 1\) and \(q_k = \delta_k, 2 \leq k \leq \xi\), we get
\[
(\xi - 1) \sum_{k=2}^{\xi} \delta_k^2 \leq \frac{1}{4} \left( \left( \delta_\xi^2 + \delta_2^2 \right) \left( \sum_{k=2}^{\xi} \delta_k \right)^2 \right),
\]
\[
(\xi - 1)(\xi + 2R_{-1}(CMZG)) \leq \frac{\xi^2}{4} \left( \left( \delta_\xi^2 + \delta_2^2 \right) \right)^2.
\]
Therefore
\[
\sqrt{\frac{\delta_\xi^2}{\delta_2^2}} + \sqrt{\frac{\delta_2^2}{\delta_\xi^2}} \geq \frac{2}{\xi} \sqrt{(\xi - 1)(\xi + 2R_{-1}(CMZG))}.
\]

5. Conclusion

The primary emphasis of this paper is the exploration of Laplacian energy, distance based energy, and normalized Laplacian energy concerning the cross monic zero divisor graph within a commutative ring, denoted as \(CMZ(G, Z^m[x]/(x^2))\). The paper also includes visual representations of the concepts discussed. In essence, the contribution of this paper lies in enhancing our comprehension of the graph properties linked to the cross monic zero divisor graph within the framework of commutative ring.
Conflict of interest

The authors declare no competing financial interest.

References