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Research Article

Normalized Laplacian Energy and Distance Based Energy for Cross Monic Zero Divisor Graphs Associated with Commutative Ring

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Abstract: Cross monic zero divisor graph for a commutative ring $\mathscr{R}$ is a connected graph, denoted by $\mathscr{CMZG}(\mathbb{Z}_n \times \mathbb{Z}_m[x]/\langle f(x)\rangle)$ with order ξ, whose vertices are non-zero zero divisors $\mathbb{Z}(\mathscr{R})/\{0\}$ of commutative ring, and two vertices u, v are connected by an edge if and only if $uv = 0$. In this paper, we discuss energy, Laplacian energy, distance energy and distance signless Laplacian energy for $\mathscr{CMZG}(\mathbb{Z}_2 \times \mathbb{Z}_{p>2}[x]/\langle x^2\rangle)$ and $\mathscr{CMZG}(\mathbb{Z}_p \times \mathbb{Z}_p[x]/\langle x^2\rangle)$. Also, we determine the normalized Laplacian energy.

***Keywords*:** commutative ring, laplacian energy, distance signless laplacian energy, normalized laplacian energy

MSC: 13A70, 05C50, 05E40

1. Introduction

Let $\mathscr{R}$ be a commutative ring with multiplicative identity $1 \neq 0$. If there exists $x_2 \in \mathscr{R}$ $(x_2 \neq 0)$ such that $x_1x_2 = 0$ for some $x_1 \in \mathscr{R}$ $(x_1 \neq 0)$, then x_1 is referred to as a zero divisor of $\mathscr{R}$. The collection of zero divisors is symbolized by $\mathbb{Z}(\mathscr{R})$, while $\mathbb{Z}(\mathscr{R})/\{0\} = \mathbb{Z}(\mathscr{R})^*$ is the collection of nonzero zero divisors of $\mathscr{R}$. The zero divisor graph $\Gamma(\mathscr{R})$ of $\mathscr{R}$ is a graph, where $\mathbb{Z}(\mathscr{R})$ is its node set and two different nodes $y,\ z \in \mathbb{Z}(\mathscr{R})$ are connected if $yz = zy = 0$. Beck [1] established such graphs over commutative rings in his concept, he incorporated the identity and was primarily concerned with the coloring of a commutative ring. Following that, Anderson et al. [2] updated the concept of $\Gamma(\mathscr{R})$ by omitting the identity of $\mathscr{R}$. The finite field of order n is represented by $\mathbb{F}_n$ and a ring of integers modulo n by $\mathbb{Z}_n$. The order of $\Gamma(\mathbb{Z}_n)$ is $n-1-\phi(n)$, where as ϕ is Euler's phi function. The graph theoretic characteristics of $\Gamma(\mathbb{Z}_n)$ are widely investigated [3–5]. Shang [6] focuses on the commutativity aspects within prime near-rings, providing valuable insights that enrich the broader understanding of ring theory. Investigation of the spectral properties of matrices associated with graphs is always interesting and challenging. We note that the graphs associated with different algebraic structures, for instance, power graphs [7], annihilator monic prime graph [8] and commuting graphs of groups [9, 10] have helped to solve several problems both in algebra and combinatorics. Alali et al. [11], implies a study of algebraic structures within $\mathbb{Z}_n$ and their connections with topological indices and entropies, underscoring the interdisciplinary intersection of algebra and graph theory. The adjacency matrix of $\mathscr{G}$ is the $n \times n$ matrix $\mathscr{A} = (a_{ij})$, where $a_{ij} = 1$ if there is an edge between vertex i and vertex j, otherwise $a_{ij} = 0$. For an n-vertex graph G with adjacency matrix $\mathscr{A}$ having eigenvalues

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$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n$, the energy $\mathscr{E}(G)$ is defined as $\mathscr{E}(G) = \sum_{i=1}^{n} |\lambda_i|$. This energy encapsulates essential information about the graph's structural properties and connectivity. Specifically, the eigenvalues serve as indicators of the graph's spectral characteristics, offering insights into the algebraic features of the underlying ring. Adjacency eigenvalues of zero-divisor graphs discussed by Young [12]. In addition to this, the Laplacian energy $\mathscr{L}\mathscr{E}(G) = \sum_i \left| \mu_i - \frac{2|E|}{|V|} \right|$ of a zero divisor graph is ascertained through the eigenvalues of its Laplacian matrix. This Laplacian energy imparts supplementary insights by concentrating on the connections between vertices, capturing the inherent algebraic structure of the graph. Investigating energy and Laplacian energy in zero divisor graphs entails a detailed examination of spectral properties, adding to a more profound comprehension of how algebraic structures and graph-theoretic characteristics interact within this specific context. The distance energy of a graph is denoted by $\mathscr{E}_{\mathscr{D}}$ and a quantitative measure that reflects the structural properties of the graph based on distances between its vertices. It is defined as the sum of the absolute values of the eigenvalues of the distance matrix of the graph. The distance matrix represents the pairwise distances between vertices in the graph. Rather et al [13], probably extensively explores Laplacian eigenvalues and their ramifications for the zero divisor graph in the domain of modular arithmetic. Similarly, distance energy [14] and distance Laplacian energy (distance signless Laplacian energy) [15, 16], linked to the distance Laplacian matrix (distance signless Laplacian matrix) respectively, focus on capturing the relationships between vertices while incorporating distance information (Let $\mathscr{D}^{\mathscr{L}}(G) = Diag(Tr) - \mathscr{D}(G)$ and $\mathscr{D}^{\mathscr{Q}}(G) = Diag(Tr) - \mathscr{D}(G)$ be respectively, the distance Laplacian matrix and the distance signless Laplacian matrix, where $Diag(Tr)$ is diagonal matrix of vertex transmissions. Eigenvalues of $\mathscr{D}^{\mathscr{L}}(G)$ and $\mathscr{D}^{\mathscr{Q}}(G)$ denoted by $\partial_i^{\mathscr{L}}$ and $\partial_i^{\mathscr{Q}}$. Then $\mathscr{D}\mathscr{L}\mathscr{E} = \sum_i |\partial_i^{\mathscr{L}} - t(G)|$ and $\mathscr{D}\mathscr{S}\mathscr{L}\mathscr{E} = \sum_i |\partial_i^{\mathscr{Q}} - t(G)|$ where $t(G) = \frac{1}{n}\sum_i Tr(v_i)$). Alhevaz et al. [17] discusses distance signless Laplacian Estrada index combines distance information with the graph's signless Laplacian matrix, offering a comprehensive perspective on the graph's structure, ultimately contributing to the advancement of graph theory and its applications in various fields. This note aims to explore the implications of these distance based energy measures in the context of zero divisor graphs, shedding light on their applications and significance in algebraic graph theory. The normalized Laplacian energy is computed from the eigenvalues of this matrix and serves as a measure of the graph's, how efficiently information can propagate through the networks. Research on normalized Laplacian energy explores its applications in diverse fields, including computer science, physics, and biology. Entries of the normalized Laplacian matrix are 1 if $i = j$ and $-\frac{1}{\sqrt{d(v_i)d(v_j)}}$ if $v_iv_j \in E$, otherwise 0, $\mathscr{N}\mathscr{L}\mathscr{E} = \sum_{i=1} |\delta_i - 1|$, some recent work on the normalized Laplacian see [18–20].

Motivated by the above articles, we investigate Laplacian energy, distance based energy, and normalized Laplacian energy for cross monic zero divisor graph of a commutative ring. Cross monic zero divisor graph of a commutative ring, denoted $\mathscr{C}\mathscr{M}\mathscr{Z}\mathscr{G}(\mathbb{Z}_n \times \mathbb{Z}_m[x]/\langle f(x)\rangle)$, whose vertices are the non-zero zero divisors of the commutative ring, and whose two vertices u, v are connected by an edge if and only if $uv = 0$. For example, cross monic zero divisor graph of $\mathbb{Z}_2 \times \mathbb{Z}_3[x]/\langle x^2\rangle$ and $\mathbb{Z}_3 \times \mathbb{Z}_4[x]/\langle x^2\rangle$ is shown in Figure 1 and Figure 2. Characteristic polynomial and eigenvalues of adjacency matrix, Laplacian matrix, distance matrix, distance Laplacian matrix and distance signless Laplacian matrix of Figure 2 is shown in Table 1 and Table 2.

The structure of this paper is outlined as follows: In Section 2, we explore the energy and Laplacian energy of cross monic zero divisor graphs within the commutative rings $\mathbb{Z}_2 \times \mathbb{Z}_p[x]/\langle x^2\rangle$ and $\mathbb{Z}_p \times \mathbb{Z}_p[x]/\langle x^2\rangle$. Section 3 is dedicated to the examination of the distance energy and distance signless Laplacian energy of cross monic zero divisor graphs. Furthermore, in Section 4, we delve into the discussion of the normalized Laplacian eigenvalues and their energy in the context of cross monic zero divisor graphs.

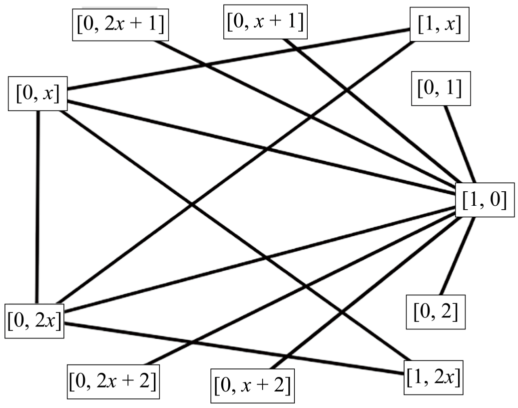

Figure 1. $\mathscr{CMZG}(\mathbb{Z}_2 \times \mathbb{Z}_3[x]/\langle x^2 \rangle)$

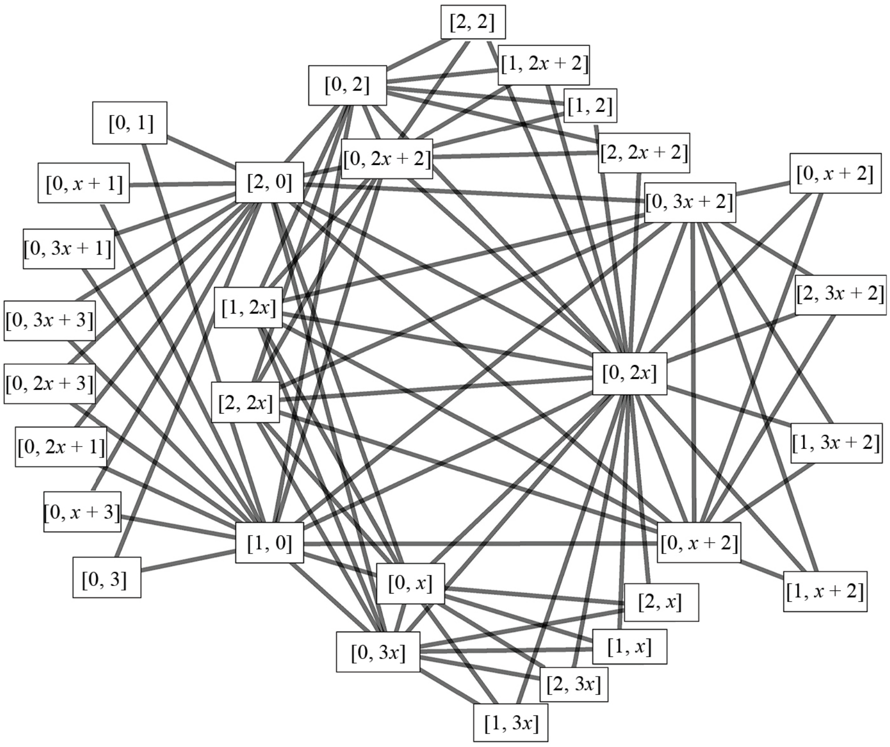

Figure 2. $\mathscr{CMZG}(\mathbb{Z}_3 \times \mathbb{Z}_4[x]/\langle x^2 \rangle)$

Table 1. Characteristic polynomial of cross monic zero divisor of $\mathbb{Z}_3 \times \mathbb{Z}_4[x]/\langle x^2 \rangle$

	Characteristic Polynomial
$P_{\mathscr{A}}(\lambda)$	$\lambda^{18}(\lambda+1)^3(\lambda^2-\lambda-8)^2(\lambda^6-\lambda^5-70\lambda^4-64\lambda^3+768\lambda^2+928\lambda-1024)$
$P_{\mathscr{D}}(\partial)$	$(\partial+1)^3(\partial+2)^{18}(\partial^2+5\partial-2)^2(\partial^6-49\partial^5-886\partial^4+200\partial^3+7424\partial^2+368\partial-2528)$
$P_{\mathscr{L}}(\mu)$	$(\mu-11)^3(\mu-3)^9(\mu-2)^7\mu(\mu^2-12\mu+10)^2(\mu^7-80\mu^6+2567\mu^5-42346\mu^4+381423\mu^3$ $-1825964\mu^2+4077353\mu-2779770)$
$P_{\mathscr{D}\mathscr{L}}(\partial^L)$	$(\partial^L-76)^7(\partial^L-67)^9(\partial^L-51)^3((\partial^L)^2-120\partial^L+3543)^2((\partial^L)^8-378(\partial^L)^7$ $+60299(\partial^L)^6-5261384(\partial^L)^5+271088615(\partial^L)^4-8240009810(\partial^L)^3+136424567565(\partial^L)^2$ $-935438945868(\partial^L)-331111761120)$
$P_{\mathscr{D}\mathscr{Q}}(\partial^Q)$	$(\partial^Q-72)^7(\partial^Q-63)^9(\partial^Q-49)^3((\partial^Q)^2-110\partial^Q+2953)^2((\partial^Q)^8-468(\partial^Q)^7+92537(\partial^Q)^6$ $-10160396(\partial^Q)^5+680149555(\partial^Q)^4-28492198500(\partial^Q)^3+730496306515(\partial^Q)^2$ $-10489822459516(\partial^Q)+64635829556032)$

Table 2. Eigenvalues of cross monic zero divisor of $\mathbb{Z}_3 \times \mathbb{Z}_4[x]/\langle x^2 \rangle$

Matrix 1	Eigenvalues					
$\mathscr{A}$	$-6.1528^{(1)}$	$-3.2505^{(1)}$	$-2.3914^{(1)}$	$-2.3723^{(2)}$	$-1^{(3)}$	$0^{(18)}$	$0.7203^{(1)}$
	$3.3723^{(2)}$	$3.4446^{(1)}$	$8.6299^{(1)}$				
$\mathscr{L}$	$0^{(1)}$	$1.1454^{(1)}$	$1.8769^{(2)}$	$2^{(7)}$	$3^{(9)}$	$5.1863^{(1)}$	$7^{(1)}$
	$10.1231^{(2)}$	$10.9443^{(1)}$	$11^{(3)}$	$15^{(1)}$	$17.6425^{(1)}$	$23.0814^{(1)}$	
$\mathscr{D}$	$-13.7316^{(1)}$	$-5.3723^{(2)}$	$-2.9610^{(1)}$	$-2^{(18)}$	$-1^{(3)}$	$-0.6292^{(1)}$	$0.3723^{(2)}$
	$0.5659^{(1)}$	$2.7701^{(1)}$	$62.9864^{(1)}$				
$\mathscr{D}^{\mathscr{L}}$	$-0.3371^{(1)}$	$34.5707^{(1)}$	$35^{(1)}$	$46.8282^{(1)}$	$51^{(3)}$	$51.5665^{(1)}$	$52.4502^{(2)}$
	$63^{(1)}$	$64.0535^{(1)}$	$67^{(9)}$	$67.5498^{(2)}$	$76^{(7)}$	$83.3182^{(1)}$	
$\mathscr{D}^{\mathscr{Q}}$	$31^{(1)}$	$32.8453^{(1)}$	$45.2539^{(1)}$	$46.5147^{(2)}$	$49^{(3)}$	$52.6971^{(1)}$	$57.9917^{(1)}$
	$59^{(1)}$	$60.3958^{(1)}$	$63^{(9)}$	$63.4853^{(2)}$	$72^{(7)}$	$128.8161^{(1)}$	

2. Energy and Laplacian energy of cross monic zero divisor graphs of commutative ring

Theorem 1 Energy of cross monic zero divisor graph of commutative ring $\mathbb{Z}_2 \times \mathbb{Z}_p[x]/\langle x^2 \rangle$ is

$$\mathscr{E}(\mathscr{C}\mathscr{M}\mathscr{Z}\mathscr{G}(\mathbb{Z}_2 \times \mathbb{Z}_p[x]/\langle x^2 \rangle)) = \frac{1}{4}\left(6\sqrt{p(p-1)}+3p^{\frac{3}{2}}+4p+2\right)$$

where p is prime number greater than 2.

Proof. Let the cross monic zero divisor graph of $\mathbb{Z}_2 \times \mathbb{Z}_p[x]/\langle x^2 \rangle$ be a simple graph, then the adjacent matrix is

$$\mathscr{A}(\mathscr{CMZG}) = \begin{pmatrix} \mathscr{O} & \mathscr{J} & \mathscr{O} & \mathscr{O} \\ \mathscr{J} & \mathscr{O} & \mathscr{J} & \mathscr{O} \\ \mathscr{O} & \mathscr{J} & \mathscr{J}-\mathscr{I} & \mathscr{J} \\ \mathscr{O} & \mathscr{O} & \mathscr{J} & \mathscr{O} \end{pmatrix}$$

Now $|\mathscr{A}(\mathscr{CMZG}) - \lambda\mathscr{I}| = 0$. Then the characteristic polynomial is $\lambda^{p^2-3}(\lambda+1)^{p-2}(\lambda^4-(p-2)\lambda^3-4p\tau_p\lambda^2+p(p-1)(p-2)\lambda)$. The eigenvalues satistifying $\sqrt{p(p-1)} \leq \lambda_i \leq (\frac{p}{2}\sqrt{p})+1$. Then

$$Spec_{\mathscr{A}}(\mathscr{CMZG}(\mathbb{Z}_2\times\mathbb{Z}_p[x]/\langle x^2\rangle)) = \begin{pmatrix} -1 & 0 & \sqrt{p(p-1)} & \frac{\sqrt{p(p-1)}}{2} & \frac{\sqrt{p}(\frac{p}{2})+1}{2} & \sqrt{p}(\frac{p}{2})+1 \\ p-1 & p^2-3 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$\mathscr{E}(\mathscr{CMZG}(\mathbb{Z}_2\times\mathbb{Z}_p[x]/\langle x^2\rangle)) = \sum_{i=1}^{p^2+p-1}|\lambda_i|$$

$$= (p-1)+(p^2-3)(0)+\sqrt{p(p-1)}+\frac{\sqrt{p(p-1)}}{2}+\frac{\sqrt{p}(\frac{p}{2})+1}{2}+\sqrt{p}\left(\frac{p}{2}\right)+1$$

$$= \frac{3}{2}\sqrt{p(p-1)}+p-1+\frac{1}{4}\left(p^{\frac{3}{2}}+2\right)+\frac{1}{2}\left(p^{\frac{3}{2}}+2\right)$$

$$= \frac{3}{2}\sqrt{p(p-1)}+\frac{3p^{\frac{3}{2}}}{4}+p+\frac{1}{2}$$

$$= \frac{1}{4}\left(6\sqrt{p(p-1)}+3p^{\frac{3}{2}}+4p+2\right).$$

□

Theorem 2 Let $\mathscr{CMZG}$ be a commutative ring $\mathbb{Z}_p\times\mathbb{Z}_p[x]/\langle x^2\rangle$ of order $2p^2-p-1$ and size $\frac{1}{2}\left(4p^3-7p^2+p+2\right)$, then

$$\mathscr{E}(\mathscr{CMZG}(\mathbb{Z}_p\times\mathbb{Z}_p[x]/\langle x^2\rangle)) \leq \frac{4p^3-7p^2+p+2}{2p^2-p-1}+\sqrt{2p^2-p-2\left[4p^3-7p^2+p+2-\left[\frac{4p^3-7p^2+p+2}{2p^2-p-1}\right]^2\right]}$$

where p is odd prime.

Proof. Let the cross monic zero divisor graph of $\mathbb{Z}_p \times \mathbb{Z}_p[x]/\langle x^2 \rangle$ be a simple graph with

$$|V| = 2p^2 - p - 1,$$

$$|E| = \frac{1}{2}(4p^3 - 7p^2 + p + 2),$$

then adjacent matrix is

$$\mathscr{A}(\mathscr{C}\mathscr{M}\mathscr{Z}\mathscr{G}(\mathbb{Z}_p \times \mathbb{Z}_p[x]/\langle x^2 \rangle)) = \begin{pmatrix} \mathscr{J}-\mathscr{I} & \mathscr{J} & \mathscr{J} & \mathscr{O} \\ \mathscr{J} & \mathscr{O} & \mathscr{O} & \mathscr{J} \\ \mathscr{J} & \mathscr{O} & \mathscr{O} & \mathscr{O} \\ \mathscr{O} & \mathscr{J} & \mathscr{O} & \mathscr{O} \end{pmatrix}$$

Suppose that $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_{2p^2-p-1}$ are the eigenvalues of $\mathscr{C}\mathscr{M}\mathscr{Z}\mathscr{G}(\mathbb{Z}_p \times \mathbb{Z}_p[x]/\langle x^2 \rangle)$. Then, as is well known, we have

$$\lambda_1 \geq \frac{4p^3 - 7p^2 + p + 2}{2p^2 - p - 1}$$

(see [21], for example). Moreover, since

$$\sum_{i=1}^{2p^2-p-1} \lambda_i^2 = 4p^3 - 7p^2 + p + 2$$

must hold (for example, see [22]), we have

$$\sum_{i=2}^{2p^2-p-1} \lambda_i^2 = 4p^3 - 7p^2 + p + 2 - \lambda_1^2.$$

Using this together with the Cauchy-Schwartz inequality, applied to the vectors $(|\lambda_2|,\ |\lambda_3|,\ \ldots,\ |\lambda_{2p^2-p-1}|)$ and $(1,\ 1,\ 1,\ \ldots,\ 1)$ with $2p^2 - p - 2$ entries, we obtain the inequality

$$\sum_{i=2}^{2p^2-p-1} |\lambda_i| \leq \sqrt{(2p^2 - p - 2)(4p^3 - 7p^2 + p + 2 - \lambda_1^2)}$$

Thus, we must have

$$\mathscr{E}(\mathscr{CMZG}) \leq \lambda_1 + \sqrt{(2p^2-p-2)(4p^3-7p^2+p+2-\lambda_1^2)}$$

Now, since the function $f(y) = y + \sqrt{(2p^2-p-2)(4p^3-7p^2+p+2-y^2)}$ decreases on the interval

$$\sqrt{\frac{4p^3-7p^2+p+2}{2p^2-p-1}} < y \leq \sqrt{4p^3-7p^2+p+2}$$

in veiw of $4p^3-7p^2+p+2 \geq 2p^2-p-1$, we see that $\sqrt{\frac{4p^3-7p^2+p+2}{2p^2-p-1}} \leq \frac{4p^3-7p^2+p+2}{2p^2-p-1} \leq \lambda_1$ must hold, and hence $f(\lambda_1) \leq f(\frac{4p^3-7p^2+p+2}{2p^2-p-1})$ must hold as well. From this fact, and inequality $\mathscr{E}(\mathscr{CMZG}) \leq \lambda_1 + \sqrt{(2p^2-p-2)(4p^3-7p^2+p+2-\lambda_1^2)}$, it immediately follows that inequality $\mathscr{E}(\mathscr{CMZG}) \leq \frac{4p^3-7p^2+p+2}{2p^2-p-1} + \sqrt{2p^2-p-2\left[4p^3-7p^2+p+2-\left[\frac{4p^3-7p^2+p+2}{2p^2-p-1}\right]^2\right]}$ holds. Hence the proof. □

Example 1 For cross monic zero divisor graph of commutative ring $\mathbb{Z}_3 \times \mathbb{Z}_3[x]/\langle x^2 \rangle$ with order 14 and size 25, we have

$$\mathscr{E}(\mathscr{CMZG}(\mathbb{Z}_3 \times \mathbb{Z}_3[x]/\langle x^2 \rangle)) \leq 25.5755$$

Solution We consider cross monic zero divisor graph of commutative ring $\mathbb{Z}_3 \times \mathbb{Z}_3[x]/\langle x^2 \rangle$ (Figure 3),

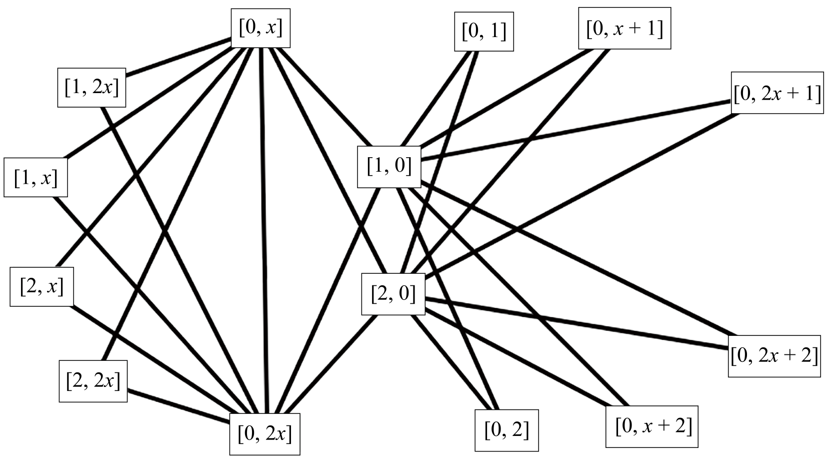

Figure 3. $\mathscr{CMZG}(\mathbb{Z}_3 \times \mathbb{Z}_3[x]/\langle x^2 \rangle)$

The adjacent matrix is

$$\mathscr{A}(\mathscr{CMZG}(\mathbb{Z}_3 \times \mathbb{Z}_3[x]/\langle x^2\rangle)) = \begin{pmatrix} \mathscr{O}_{4\times4} & \mathscr{I}_{4\times2} & \mathscr{O}_{4\times2} & \mathscr{O}_{4\times6} \\ \mathscr{I}_{2\times4} & \mathscr{J}-\mathscr{I}_{2\times2} & \mathscr{I}_{2\times2} & \mathscr{O}_{2\times6} \\ \mathscr{O}_{2\times4} & \mathscr{I}_{2\times2} & \mathscr{O}_{2\times2} & \mathscr{I}_{2\times6} \\ \mathscr{O}_{6\times4} & \mathscr{O}_{6\times2} & \mathscr{I}_{6\times2} & \mathscr{O}_{6\times6} \end{pmatrix}$$

Now $|\mathscr{A}(\mathscr{CMZG}(\mathbb{Z}_3 \times \mathbb{Z}_3[x]/\langle x^2\rangle)) - \lambda\mathscr{I}| = 0.$
Then the characteristic polynomial is $\lambda^{14} - 25\lambda^{12} - 12\lambda^{11} + 108\lambda^{10} + 96\lambda^9 = 0.$ The spectrum of the graph is

$$Spec_{\mathscr{A}}(\mathscr{CMZG}(\mathbb{Z}_3 \times \mathbb{Z}_3[x]/\langle x^2\rangle)) = \begin{pmatrix} -4.1425 & -2 & -1 & 0 & 2.4913 & 4.6512 \\ 1 & 1 & 1 & 9 & 1 & 1 \end{pmatrix}$$

Therefore

$$\mathscr{E}(\mathscr{CMZG}(\mathbb{Z}_3 \times \mathbb{Z}_3[x]/\langle x^2\rangle)) = \sum_{i=1}^{14} |\lambda_i|$$

$$= |-4.1425| + |-2| + |-1| + 0 + 2.4913 + 4.6512$$

$$= 14.285$$

Theorem 3 Let the cross monic zero divisor graph of $\mathbb{Z}_2 \times \mathbb{Z}_p[x]/\langle x^2\rangle$ have order p^2+p-1, $\eta = \frac{1}{2}(5p^2-7p+2)$ and $\triangle = p^2-1$. Then

$$\mathscr{LE}(\mathscr{CMZG}(\mathbb{Z}_2 \times \mathbb{Z}_p[x]/\langle x^2\rangle)) < \frac{2\eta}{p^2+p-1} + \sqrt{\eta\left(\eta\left(\frac{4}{(p^2+p-1)^2} - 2\right) + p^4 + 2p^3 - 2p^2 - 3p + 2\right)}$$

Proof. For the cross monic zero divisor graph of $\mathbb{Z}_2 \times \mathbb{Z}_p[x]/\langle x^2\rangle)$ with order p^2+p-1, $\eta = \frac{1}{2}(5p^2-7p+2)$ and $\triangle = p^2-1$, we have

$$\mathscr{LM}(\mathscr{CMZG}(\mathbb{Z}_2\times\mathbb{Z}_p[x]/\langle x^2\rangle))=\begin{pmatrix}\mathscr{I} & -\mathscr{J} & \mathscr{O} & \mathscr{O}\\ -\mathscr{J} & p^2-1 & -\mathscr{J} & \mathscr{O}\\ \mathscr{O} & -\mathscr{J} & \mathscr{M}_1 & -\mathscr{J}\\ \mathscr{O} & \mathscr{O} & -\mathscr{J} & \mathscr{M}_2\end{pmatrix}$$

where $\mathscr{M}_1=2(p-1)\mathscr{I}+\mathscr{I}-\mathscr{J}$ and $\mathscr{M}_2=(p-1)\mathscr{I}$. Then $|\mathscr{LM}(\mathscr{CMZG}(\mathbb{Z}_2\times\mathbb{Z}_p[x]/\langle x^2\rangle))-\lambda\mathscr{I}|=0$. Eigenvalues of $\mathscr{LM}$ satisfies the inequality $0\leq\mu_\alpha\leq\triangle+2$. Now

$$Spec_{\mathscr{LM}}(\mathscr{CMZG}(\mathbb{Z}_2\times\mathbb{Z}_p[x]/\langle x^2\rangle))=\begin{pmatrix}0 & \tau_1 & 1 & p-1 & \tau_2 & 2p-1 & \tau_3\\ 1 & 1 & p(p-1)-1 & p-2 & 1 & p-2 & 1\end{pmatrix}$$

where $\tau_1>0,\tau_2>2(p-1),\tau_3>p^2$.

$$\begin{aligned}\mathscr{LM}(\mathscr{CMZG}(\mathbb{Z}_2\times\mathbb{Z}_p[x]/\langle x^2\rangle)) &= \sum_{\alpha=1}^{|V|}\left|\mu_\alpha-\frac{2\eta}{p^2+p-1}\right|\\
&= \left|-\frac{2(p^2+p-1)}{\frac{1}{2}(5p^2-7p+2)}\right|+(p(p-1)-1)\left|1-\frac{2(p^2+p-1)}{\frac{1}{2}(5p^2-7p+2)}\right|\\
&\quad+\cdots+(p-2)\left|(p-1)-\frac{4(p^2+p-1)}{(5p-2)(p-1)}\right|\\
&= \frac{2(p^2+p-1)}{\frac{1}{2}(5p^2-7p+2)}+(p(p-1)-1)\left(\frac{p^2-11p+6}{5p^2-7p+2}\right)\\
&\quad+\cdots+(p-2)\left(-\frac{4(p^2+p-1)}{5p^2-7p+2}+p-1\right)\\
&= \frac{2(p^2+p-1)}{\frac{1}{2}(5p^2-7p+2)}+(-1)\left(\frac{(p^2-11p+6)(p(p-1)-1)}{5p^2-7p+2}\right)\\
&\quad+\cdots+(-1)\left(\frac{(p-2)(5p^3-16p^2+5p+2)}{(p-1)(5p-2)}\right)\end{aligned}$$

$$\mathscr{LE}(\mathscr{CMZG}(\mathbb{Z}_2\times\mathbb{Z}_p[x]/\langle x^2\rangle)) < \frac{2\eta}{p^2+p-1}+\sqrt{\eta\left(\eta\left(\frac{4}{(p^2+p-1)^2}-2\right)+p^4+2p^3-2p^2-3p+2\right)}$$

□

Laplacian matrix and spectrum of cross monic zero divisor graph of $\mathbb{Z}_p\times\mathbb{Z}_{p>2}[x]/\langle x^2\rangle$ are

$$\mathscr{LM}(\mathscr{CMZG}(\mathbb{Z}_p\times\mathbb{Z}_p[x]/\langle x^2\rangle)) = \begin{pmatrix} (p^2-1)\mathscr{I}-\mathscr{J} & -\mathscr{J} & -\mathscr{J} & \mathscr{O} \\ -\mathscr{J} & (p^2-1)\mathscr{I} & \mathscr{O} & -\mathscr{J} \\ -\mathscr{J} & \mathscr{O} & (p-1)\mathscr{I} & \mathscr{O} \\ \mathscr{O} & -\mathscr{J} & \mathscr{O} & (p-1)\mathscr{I} \end{pmatrix}$$

$$Spec_{\mathscr{LM}}(\mathscr{CMZG}(\mathbb{Z}_p\times\mathbb{Z}_p[x]/\langle x^2\rangle)) = \begin{pmatrix} 0 & \tau_1 & p-1 & \tau_2 & (p-1)+(p(p-1)) & \tau_3 \\ 1 & 1 & 2p^2-3p-1 & 1 & 2p-4 & 1 \end{pmatrix}$$

respectively.

3. Distance based energy of cross monic zero divisor graphs

Theorem 4 Upper and lower bounds of distance energy of cross monic zero divisor graph of $\mathbb{Z}_2\times\mathbb{Z}_p[x]/\langle x^2\rangle$ is

$$\sqrt{2\sum_{\eta_1<\eta_2}(d_{\eta_1\eta_2})^2+(p^2+p-1)(p^2+p)\rho^{\frac{2}{p^2+p-1}}} \leq \mathscr{E}_{\mathscr{D}}$$

$$\leq \sqrt{2(p^2+p)\sum_{\eta_1<\eta_2}(d_{\eta_1\eta_2})^2+(p^2+p-1)\rho^{\frac{2}{p^2+p-1}}}$$

Proof. Distance matrix and spectrum of the graph is

$$\mathscr{D}(\mathscr{CMZG}(\mathbb{Z}_2\times\mathbb{Z}_p[x]/\langle x^2\rangle))=\begin{pmatrix}2(\mathscr{J}-\mathscr{I}) & \mathscr{J} & 2\mathscr{J} & 3\mathscr{J}\\ \mathscr{J} & \mathscr{O} & \mathscr{J} & 2\mathscr{J}\\ 2\mathscr{J} & \mathscr{J} & \mathscr{J}-\mathscr{I} & \mathscr{J}\\ 3\mathscr{J} & 2\mathscr{J} & \mathscr{J} & 2\mathscr{J}-\mathscr{I}\end{pmatrix}$$

$$Spec_{\mathscr{D}}(\mathscr{CMZG}(\mathbb{Z}_2\times\mathbb{Z}_p[x]/\langle x^2\rangle))=\begin{pmatrix}-2 & -1 & \sqrt{p(p-1)}+2p-2 & \sqrt{p-2}-1 & \frac{\sqrt{p}}{\lceil\frac{p}{2}\rceil} & \tau_\alpha\\ p^2-3 & p-2 & 1 & 1 & 1 & 1\end{pmatrix}$$

Then

$$\mathscr{M}\leq p^2+p-1\sum_{\eta_1}\mu_{\eta_1}^2-\left(\sum_{\eta_1}|\mu_{\eta_1}|\right)^2\leq(p^2+p)\mathscr{M}$$

$$\mathscr{M}\leq 2p^2+2p-2\sum_{\eta_1<\eta_2}(d_{\eta_1\eta_2})^2-(E_D)^2\leq(p^2+p)\mathscr{M}$$

where

$$\mathscr{M}=p^2+p-1\left[\frac{1}{p^2+p-1}\sum_{\eta_1}\mu_{\eta_1}^2-\left[\prod_{\eta_1}\mu_{\eta_1}^2\right]^{\frac{1}{p^2+p-1}}\right]$$

$$=p^2+p-1\left[\frac{2}{p^2+p-1}\sum_{\eta_1<\eta_2}(d_{\eta_1\eta_2})^2-\left[\prod_{\eta_1}|\mu_{\eta_1}|\right]^{\frac{2}{p^2+p-1}}\right]$$

$$=2\sum_{\eta_1<\eta_2}(d_{\eta_1\eta_2})^2-(p^2+p-1)\rho^{\frac{2}{p^2+p-1}}$$

Theorem 5 If the transmission degree sequence of $\mathscr{CMZG}$ is $\{Tr_1, Tr_2, \ldots, Tr_\xi\}$ and $\triangle=\left|\mathscr{D}^{\mathscr{Q}}(\mathscr{CMZG})-\frac{1}{\xi}\sum_{a=1}^{\xi}Tr_aI_\xi\right|$, then

$$\sqrt{2\sum_{1\leq a<b\leq\xi}(dis_{ab})^2+\sum_{a=1}^{\xi}Tr_a^2-\frac{4\sigma_0^2(\mathscr{CMZG})}{\xi}+\xi(\xi-1)\triangle^{\frac{2}{\xi}}}$$

$$\leq\mathscr{E}_{\mathscr{D}\mathscr{Q}}(\mathscr{CMZG})\leq\sqrt{(\xi-1)\left(2\sum_{1\leq a<b\leq\xi}(dis_{ab})^2+\sum_{a=1}^{\xi}Tr_a^2-\frac{4\sigma_0^2(\mathscr{CMZG})}{\xi}\right)+\xi\triangle^{\frac{2}{\xi}}}$$

Proof. Let us choose $s_a=\alpha_a^2$, for $a=1,\ 2,\ 3,\ \ldots,\ \xi$. We obtain

$$M\leq\xi\sum_{a=1}^{\xi}\alpha_a^2-\left(\sum_{a=1}^{\xi}|\alpha_a|\right)^2$$

$$\leq(\xi-1)M$$

i.e.,

$$M\leq\xi\left(2\sum_{1\leq a<b\leq\xi}(dis_{ab})^2+\sum_{a=1}^{\xi}Tr_a^2-\frac{4\sigma_0^2(\mathscr{CMZG})}{\xi}\right)-E_{D\mathscr{Q}}^2(\mathscr{CMZG})$$

$$\leq(\xi-1)M,$$

where

$$M=\xi\left(\frac{1}{\xi}\sum_{a=1}^{\xi}\alpha_a^2-\left(\prod_{a=1}^{\xi}\alpha_a^2\right)^{\frac{1}{\xi}}\right)$$

$$=\xi\left(\frac{1}{\xi}\left(2\sum_{1\leq a<b\leq\xi}(dis_{ab})^2+\sum_{a=1}^{\xi}Tr_a^2-\frac{4\sigma_0^2(\mathscr{CMZG})}{\xi}\right)-\left(\prod_{a=1}^{\xi}\alpha_a^2\right)^{\frac{1}{\xi}}\right)$$

$$=2\sum_{1\leq a<b\leq\xi}(dis_{ab})^2+\sum_{a=1}^{\xi}Tr_a^2-\frac{4\sigma_0^2(\mathscr{CMZG})}{\xi}+\xi\triangle^{\frac{2}{\xi}}$$

Hence, we get the required bounds. □

Distance Laplacian matrix and Distance signless Laplacian matrix of cross monic zero divisor graph is shown in Table 3.

Table 3. Block matrix of distance (signless Laplacian) of cross monic zero divisor graph

$\mathscr{D}_{\mathscr{L}}(\mathscr{CMZG}(\mathbb{Z}_2\times\mathbb{Z}_{p>2}[x]/\langle x^2\rangle))$	$\mathscr{D}_{\mathscr{Q}}(\mathscr{CMZG}(\mathbb{Z}_2\times\mathbb{Z}_{p>2}[x]/\langle x^2\rangle))$
$\begin{bmatrix} A_1 & -J & -2J & -3J \\ -J & A_2 & -J & -2J \\ -2J & -J & A_3 & -J \\ -3J & -2J & -J & A_4 \end{bmatrix}$	$\begin{bmatrix} B_1 & J & 2J & 3J \\ J & B_2 & J & 2J \\ 2J & J & B_3 & J \\ 3J & 2J & J & B_4 \end{bmatrix}$
$A_2 = p^2+2p-3$	$B_2 = p^2+2p-3$
$A_3 = -J+(2p^2-1)I$	$B_3 = J+(2p^2-3)I$
$A_4 = -2J+(3p^2-1)I$	$B_4 = 2J+(3p^2-5)I$
$A_1 = -2J+(2p^2+3p-4)I$	$B_1 = 2J+(2p^2+3p-8)I$

Theorem 6 If $\mathscr{CMZG}$ is a connected graph with order ξ and diameter β, then

$$\xi\sqrt{(\xi-1)\left(\beta^2+\frac{\beta^2(\beta-1)}{4}-\xi+1\right)} \geq \mathscr{E}_{\mathscr{DQ}} \geq \sqrt{\xi(\xi-1)}.$$

Proof. Since $dis_{ab}\geq 1$ for $a\neq b$ and there are $\dfrac{\xi(\xi-1)}{2}$ pairs of vertices in $\mathscr{CMZG}$, then we get

$$\mathscr{E}_{\mathscr{DQ}}(\mathscr{CMZG}) \geq \sqrt{2\sum_{1\leq a<b\leq\xi}(dis_{ab})^2+\sum_{a=1}^{\xi}Tr_a^2-\frac{4\sigma_0^2(\mathscr{CMZG})}{\xi}}$$

$$\geq \sqrt{2\frac{\xi(\xi-1)}{2}+\sum_{a=1}^{\xi}Tr_a^2-\sum_{a=1}^{\xi}Tr_a^2}$$

$$= \sqrt{\xi(\xi-1)}.$$

Again, $dis_{ab}\leq\beta$ for $a\neq b$ and there are $\dfrac{\xi(\xi-1)}{2}$ pairs of vertices in $\mathscr{CMZG}$, then we get

$$\mathscr{E}_{\mathscr{DQ}}(\mathscr{CMZG}) \leq \sqrt{\xi\left(2\sum_{1\leq a<b\leq\xi}(dis_{ab})^2+\sum_{a=1}^{\xi}Tr_a^2-\frac{4\sigma_0^2(\mathscr{CMZG})}{\xi}\right)}$$

$$\leq \sqrt{\xi\left(2\frac{\xi(\xi-1)}{2}\beta^2+\frac{\xi^3(\xi-1)^2}{4}-\xi(\xi-1)^2\right)}$$

$$= \xi\sqrt{(\xi-1)\left(\beta^2+\frac{\xi^2(\xi-1)}{4}-\xi+1\right)}.$$

Hence the result. □

4. Normalized Laplacian energy of graphs

Theorem 7 Normalized Laplacian energy of cross monic zero divisor graph is

$$\mathscr{E}_{\mathscr{N}\mathscr{L}}(\mathscr{C}\mathscr{M}\mathscr{Z}\mathscr{G}(\mathbb{Z}_2 \times \mathbb{Z}_p[x]/\langle x^2\rangle)) < \frac{p^2+p-1}{2}$$

Proof. Normalized Laplacian matrix of $\mathscr{N}\mathscr{L}(\mathscr{C}\mathscr{M}\mathscr{Z}\mathscr{G})$ is

$$\mathscr{N}\mathscr{L}(\mathscr{C}\mathscr{M}\mathscr{Z}\mathscr{G}) = \begin{pmatrix} \mathscr{I} & -\frac{1}{\sqrt{\triangle(\mathscr{C}\mathscr{M}\mathscr{Z}\mathscr{G})}} & \mathscr{O} & \mathscr{O} \\ -\frac{1}{\sqrt{\triangle(\mathscr{C}\mathscr{M}\mathscr{Z}\mathscr{G})}} & \mathscr{J} & -\frac{1}{\sqrt{(\triangle(\mathscr{C}\mathscr{M}\mathscr{Z}\mathscr{G}))(2(p-1))}} & \mathscr{O} \\ \mathscr{O} & -\frac{1}{\sqrt{(\triangle(\mathscr{C}\mathscr{M}\mathscr{Z}\mathscr{G}))(2(p-1))}} & \mathscr{A} & -\frac{1}{\sqrt{2(p-1)^2}} \\ \mathscr{O} & \mathscr{O} & -\frac{1}{\sqrt{2(p-1)^2}} & \mathscr{I} \end{pmatrix}$$

where $\mathscr{A} = \mathscr{I} + \mathscr{J}\left[-\frac{1}{2(p-1)}\right] + \mathscr{I}\left[\frac{1}{2(p-1)}\right]$, whose entries are lies $-1 < a_{ij} \le 1.5$. If $p = 3$, then $P_{\mathscr{N}\mathscr{L}} = ((-1+\delta)^6(301678245 + 24773504542790\delta - 139973136718750\delta^2 + 210562839843750\delta^3 - 122070312500000\delta^4 + 2441406250000x^5))$ $/2441406250000$, δ_i denotes the eigenvalues. Then

$$Spec_{\mathscr{N}\mathscr{L}}(\mathscr{C}\mathscr{M}\mathscr{Z}\mathscr{G}(\mathbb{Z}_2 \times \mathbb{Z}_3[x]/\langle x^2\rangle)) = \begin{pmatrix} 0 & 0.2714 & 1 & 1.25 & 1.5609 & 1.9177 \\ 1 & 1 & 6 & 1 & 1 & 1 \end{pmatrix}$$

$$\mathscr{E}_{\mathscr{N}\mathscr{L}}(\mathscr{C}\mathscr{M}\mathscr{Z}\mathscr{G}(\mathbb{Z}_2 \times \mathbb{Z}_3[x]/\langle x^2\rangle)) = \sum_{i=1}^{11} |\delta_i(CMZG(\mathbb{Z}_2 \times \mathbb{Z}_3[x]/\langle x^2\rangle)) - 1|$$

$$= 3.4572$$

If $p = 5$, then $P_{\mathscr{N}\mathscr{L}} = ((-1+\delta)^2 2(-9+8\delta)^3(-240380053 - 74873011718750\delta + 550137593750000\delta^2 - 566406250000000\delta^3 + 156250000000000\delta^4))/8000000000000000$. Then

$$Spec_{\mathscr{N}\mathscr{L}}(\mathscr{C}\mathscr{M}\mathscr{Z}\mathscr{G}(\mathbb{Z}_2 \times \mathbb{Z}_5[x]/\langle x^2\rangle)) = \begin{pmatrix} 0 & 0.1619 & 1 & 1.125 & 1.5365 & 1.9266 \\ 1 & 1 & 22 & 3 & 1 & 1 \end{pmatrix}$$

$$\mathscr{E}_{\mathscr{N}\mathscr{L}}(\mathscr{C}\mathscr{M}\mathscr{Z}\mathscr{G}(\mathbb{Z}_2 \times \mathbb{Z}_5[x]/\langle x^2 \rangle)) = 3.6762$$

If $p = 7$, then $P_{\mathscr{N}\mathscr{L}} = ((-1+\delta)^4 6(-10833+10000\delta)^5(-12307899999-429953071962500\delta+4206384150000000\delta^2-4479375000000000\delta^3+1250000000000000\delta^4))/125000000000000000000$. Then

$$Spec_{\mathscr{N}\mathscr{L}}(\mathscr{C}\mathscr{M}\mathscr{Z}\mathscr{G}(\mathbb{Z}_2 \times \mathbb{Z}_7[x]/\langle x^2 \rangle)) = \begin{pmatrix} 0 & 0.116 & 1 & 1.083 & 1.5256 & 1.9417 \\ 1 & 1 & 46 & 5 & 1 & 1 \end{pmatrix}$$

$$\mathscr{E}_{\mathscr{N}\mathscr{L}}(\mathscr{C}\mathscr{M}\mathscr{Z}\mathscr{G}(\mathbb{Z}_2 \times \mathbb{Z}_7[x]/\langle x^2 \rangle)) = 3.7678$$

□

Generalized matrix of normalized Laplacian of family $(p,\ p)$ is

$$\mathscr{N}\mathscr{L}\mathscr{E}(\mathscr{C}\mathscr{M}\mathscr{Z}\mathscr{G}) = \begin{pmatrix} \mathscr{A} & -\frac{1}{\sqrt{p^4-3p^3+2}} & -\frac{1}{\sqrt{(p^2-2)(p-1)}} & \mathscr{O} \\ -\frac{1}{\sqrt{p^4-3p^3+2}} & \mathscr{I} & \mathscr{O} & -\frac{1}{\sqrt{(p^2-1)(p-1)}} \\ -\frac{1}{\sqrt{(p^2-2)(p-1)}} & \mathscr{O} & \mathscr{I} & \mathscr{O} \\ \mathscr{O} & -\frac{1}{\sqrt{(p^2-1)(p-1)}} & \mathscr{O} & \mathscr{I} \end{pmatrix}$$

where $\mathscr{A} = \mathscr{I} + \mathscr{J}\left[-\frac{1}{p^2-2}\right] + \mathscr{I}\left[\frac{1}{p^2-2}\right]$.

Theorem 8 Suppose $\mathscr{C}\mathscr{M}\mathscr{Z}\mathscr{G}(\mathbb{Z}_s \times \mathbb{Z}_t[x]/\langle x^2 \rangle)$ is a connected graph of order ξ. Then

$$\mathscr{E}_{\mathscr{N}\mathscr{L}}(\mathscr{C}\mathscr{M}\mathscr{Z}\mathscr{G}) \geq 1 + \sqrt{\frac{\xi}{\triangle(\mathscr{C}\mathscr{M}\mathscr{Z}\mathscr{G})} + 2\binom{\xi-1}{2}\sqrt[\xi-1]{\varphi(\mathscr{C}\mathscr{M}\mathscr{Z}\mathscr{G},1)^2}}$$

where $\triangle(\mathscr{C}\mathscr{M}\mathscr{Z}\mathscr{G})$ is the maximal degree in $\mathscr{C}\mathscr{M}\mathscr{Z}\mathscr{G}$.

Proof. Let $\delta_1 = 0$, and hence

$$\mathscr{E}_{\mathscr{N}\mathscr{L}}(\mathscr{C}\mathscr{M}\mathscr{Z}\mathscr{G}) = \sum_{k=1}^{\xi} |\delta_k - 1|$$

$$= 1 + \sum_{k=2}^{\xi} |\delta_k - 1|.$$

Now applying arithmetic geometric inequality, we get

$$\left(\sum_{k=2}^{\xi}|\delta_k-1|\right)^2=\sum_{k=2}^{\xi}|\delta_k-1|^2+\sum_{2\leq k\neq f\leq \xi}|\delta_k-1||\delta_f-1|$$

$$\geq 2R_{-1}(\mathscr{C}\mathscr{M}\mathscr{Z}\mathscr{G})-1+(\xi-1)(\xi-2)\left(\prod_{2\leq k\neq f\leq \xi}|\delta_k-1||\delta_f-1|\right)^{\frac{1}{(\xi-1)(\xi-2)}}$$

$$=2R_{-1}(\mathscr{C}\mathscr{M}\mathscr{Z}\mathscr{G})-1+2\binom{\xi-1}{2}\left(\prod_{k=2}^{\xi}(\delta_k-1)^{2(\xi-2)}\right)^{\frac{1}{(\xi-1)(\xi-2)}}$$

$$=2R_{-1}(\mathscr{C}\mathscr{M}\mathscr{Z}\mathscr{G})-1+2\binom{\xi-1}{2}\sqrt[\xi-1]{\left(\prod_{k=2}^{\xi}(\delta_k-1)\right)^2}$$

$$=2R_{-1}(\mathscr{C}\mathscr{M}\mathscr{Z}\mathscr{G})-1+2\binom{\xi-1}{2}\sqrt[\xi-1]{\varphi(\mathscr{C}\mathscr{M}\mathscr{Z}\mathscr{G},1)^2}.$$

We know that $R_{-1}\geq \frac{\xi}{2\triangle(\mathscr{C}\mathscr{M}\mathscr{Z}\mathscr{G})}$, therefore

$$\mathscr{E}_{\mathscr{N}\mathscr{L}}(\mathscr{C}\mathscr{M}\mathscr{Z}\mathscr{G})\geq 1+\sqrt{\frac{\xi}{\triangle(\mathscr{C}\mathscr{M}\mathscr{Z}\mathscr{G})}+2\binom{\xi-1}{2}\sqrt[\xi-1]{\varphi(\mathscr{C}\mathscr{M}\mathscr{Z}\mathscr{G},1)^2}}.$$

□

Theorem 9 Let $\mathbb{Z}_\alpha\times\mathbb{Z}_\beta[x]/\langle x^2\rangle$ be connected, $\mathscr{C}\mathscr{M}\mathscr{Z}\mathscr{G}$ with smallest δ_s and largest δ_ξ non-negative normalized Laplacian eigenvalues. Then

(i) $\delta_\xi-\delta_s\geq \frac{2}{\xi-1}\sqrt{(\xi-1)(\xi+2R_{-1}(\mathscr{C}\mathscr{M}\mathscr{Z}\mathscr{G}))-\xi^2}$

(ii) $\sqrt{\frac{\delta_\xi}{\delta_s}}+\sqrt{\frac{\delta_\xi}{\delta_s}}\geq \frac{2}{\xi}\sqrt{(\xi-1)(\xi+2R_{-1}(\mathscr{C}\mathscr{M}\mathscr{Z}\mathscr{G}))}$

Proof. Consider $\xi>2$, recall the Ozekis inequality [23], stating that p_k and q_k, $1\leq k\leq\xi$, are positive real numbers, then

$$\sum_{k=1}^{\xi}p_k^2\sum_{k=1}^{\xi}q_k^2-\left(\sum_{k=1}^{\xi}p_kq_k\right)^2\leq\frac{\xi^2}{4}(N_1N_2-n_1n_2)^2,$$

where $N_1=max_{1\leq k\leq\xi}p_k$, $N_2=max_{1\leq k\leq\xi}q_k$, $n_1=min_{1\leq k\leq\xi}p_k$ and $n_2=min_{1\leq k\leq\xi}q_k$. An application of Ozekis inequality with $p_k=1$ and $q_k=\delta_k$, $2\leq k\leq\xi$, yields

$$(\xi-1)\sum_{k=2}^{\xi}\delta_k^2-\left(\sum_{k=2}^{\xi}\delta_k\right)^2\leq\frac{(\xi-1)^2}{4}(\delta_\xi-\delta_s)^2.$$

In view of [24], it is easy to see that

$$\delta_\xi-\delta_s\geq\frac{2}{\xi-1}\sqrt{(\xi-1)(\xi+2R_{-1}(CMZG))-\xi^2}$$

yielding the assertion (i). To prove assertion (ii), we recall the Polya-Szego inequality, stating that if $p_k,\ q_k,\ N_k,\ n_k$, $1\leq k\leq\xi$ are as in part (i), then we have

$$\sum_{k=1}^{\xi}p_k^2\sum_{k=1}^{\xi}q_k^2\leq\frac{1}{4}\left(\sqrt{\frac{N_1N_2}{n_1n_2}}\sqrt{\frac{n_1n_2}{N_1N_2}}\right)^2\left(\sum_{k=1}^{\xi}p_kq_k\right)^2.$$

Applying the last inequality $p_k=1$ and $q_k=\delta_k$, $2\leq k\leq\xi$, we get

$$(\xi-1)\sum_{k=2}^{\xi}\delta_k^2\leq\frac{1}{4}\left(\sqrt{\frac{\delta_\xi}{\delta_s}}+\sqrt{\frac{\delta_s}{\delta_\xi}}\right)^2\left(\sum_{k=2}^{\xi}\delta_k\right)^2,$$

$$(\xi-1)(\xi+2R_{-1}(\mathscr{CMZG}))\leq\frac{\xi^2}{4}\left(\sqrt{\frac{\delta_\xi}{\delta_s}}+\sqrt{\frac{\delta_s}{\delta_\xi}}\right)^2.$$

Therefore

$$\sqrt{\frac{\delta_\xi}{\delta_s}}+\sqrt{\frac{\delta_\xi}{\delta_s}}\geq\frac{2}{\xi}\sqrt{(\xi-1)(\xi+2R_{-1}(\mathscr{CMZG}))}.$$

□

5. Conclusion

The primary emphasis of this paper is the exploration of Laplacian energy, distance based energy, and normalized Laplacian energy concerning the cross monic zero divisor graph within a commutative ring, denoted as $\mathscr{CMZG}(\mathbb{Z}_n\times\mathbb{Z}_m[x]/\langle x^2\rangle)$. The paper also includes visual representations of the concepts discussed. In essence, the contribution of this paper lies in enhancing our comprehension of the graph properties linked to the cross monic zero divisor graph within the framework of commutative ring.

Confilict of interest

The authors declare no competing financial interest.

References

[1] Beck I. Coloring of commutative rings. *Journal of Algebra*. 1988; 116(1): 208-226.

[2] Anderson DF, Livingston PS. The zero divisor graph of a commutative ring. *Journal of Algebra*. 1999; 217(4): 434-447.

[3] Sharma S, Bhat VK. Fault-tolerant metric dimension of zero-divisor graphs of commutative rings. *AKCE International Journal of Graphs and Combinatorics*. 2022; 19(1): 24-30.

[4] Pirzada S, Altaf A. Cliques in the extended zero-divisor graph of finite commutative rings. *Communications in Combinatorics and Optimization*. 2023. Available from: https://doi.org/10.22049/cco.2023.28740.1693.

[5] Singh P, Bhat VK. Graph invariants of the line graph of zero divisor graph of $\mathbb{Z}_n$. *Journal of Applied Mathematics and Computing*. 2022; 68(2): 1271-1287.

[6] Shang Y. A note on the commutativity of prime near-rings. *Algebra Colloquium*. 2015; 22(3): 361-366.

[7] Bera S, Bhuniya AK. On enhanced power graphs of finite groups. *Journal of Algebra and its Applications*. 2018; 17(8): 1850146.

[8] Sarathy R, Sankar JR. Applications on color (distance) signless laplacian energy of annihilator monic prime graph of commutative rings. *Ain Shams Engineering Journal*. 2024; 15(3): 102469.

[9] Mahmoudifar A, Moghaddamfar AR. Commuting graphs of groups and related numerical parameters. *Communications in Algebra*. 2017; 45(7): 3159-3165.

[10] Sarathy R, Ravi Sankar J. Coloring of graphs associated with commutative rings. *Journal of Applied Mathematics and Computing*. 2024: 1-18. Available from: https://doi.org/10.1007/s12190-024-02055-x.

[11] Alali AS, Ali S, Hassan N, Mahnashi AM, Shang Y, Assiry A. Algebraic structure graphs over the commutative ring $\mathbb{Z}_m$: exploring topological indices and entropies using $\mathbb{M}$-polynomials. *Mathematics*. 2023; 11(18): 3833.

[12] Young M. Adjacency matrices of zero-divisor graphs of integers modulo n. *Involve, a Journal of Mathematics*. 2015; 8(5): 753-761.

[13] Rather BA, Pirzada S, Naikoo TA, Shang Y. On Laplacian eigenvalues of the zero-divisor graph associated to the ring of integers modulo n. *Mathematics*. 2021; 9(5): 482.

[14] Gutman I, Indulal G, Vijayakumar A. On distance energy of graphs. *MATCH Communications in Mathematical and in Computer Chemistry*. 2008; 60: 461-472.

[15] Banerjee S. Distance Laplacian spectra of various graph operations and its application to graphs on algebraic structures. *Journal of Algebra and Its Applications*. 2023; 22(1): 2350022.

[16] Das KC, Aouchiche M, Hansen P. On (distance) Laplacian energy and (distance) signless Laplacian energy of graphs. *Discrete Applied Mathematics*. 2018; 243: 172-185.

[17] Alhevaz A, Baghipur M, Shang Y. Merging the spectral theories of distance Estrada and distance signless Laplacian Estrada indices of graphs. *Mathematics*. 2019; 7(10): 995.

[18] Das K, Sun S. Extremal graph on normalized Laplacian spectral radius and energy. *The Electronic Journal of Linear Algebra*. 2015; 29: 237-253.

[19] Hakimi-Nezhaad M, Hua H, Ashrafi AR, Qian S. The normalized Laplacian Estrada index of graphs. *Journal of Applied Mathematics Informatics*. 2014; 32(1-2): 227-245.

[20] Milovanović E, Matejić MM, Milovanović I. On the normalized Laplacian spectral radius, Laplacian incidence energy and Kemeny's constant. *Linear Algebra and its Applications*. 2019; 582: 181-196.

[21] Cvetkovic D, Doob M, Sachs H. *Spectra of Graphs: Theory and Application*. Heidelberg, Leipzig: Johann Ambrosius Barth Verlag; 1995.

[22] Biggs N. *Algebraic Graph Theory*. Cambridge university press; 1993.

[23] Hardy GH, Littlewood JE, George P. *Inequalities*. Cambridge university press; 1988.

[24] Cavers M, Fallat S, Kirkland S. On the normalized Laplacian energy and general Randić index R-1 of graphs. *Linear Algebra and its Applications*. 2010; 433(1): 172-190.