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Research Article

Inclusion Results on Hypergeometric Functions in a Class of Analytic Functions Associated with Linear Operators

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Abstract: This study focuses on analyzing the inclusion properties of a certain subclass of analytic functions. This study examines the Hohlov integral transform, which is associated with the normalized Gaussian hypergeometric function, and the Komatu integral operator, which is associated with the generalized polylogarithm. The investigation utilizes Taylor coefficients to analyze these analytic functions. Various results are obtained for these operators for specific values of the parameters. The results presented here include several previously known results as their special cases.

***Keywords*:** Gaussian hypergeometric functions, starlike functions, convex functions, Hohlov integral transform, Komatu integral operator

MSC: 30C45, 30C50, 33C90

1. Introduction

The study of geometric properties of certain subclasses, which includes hypergeometric functions and their representation in the form of hypergeometric series. The ability to express a mathematical function in terms of a hypergeometric function is a highly helpful analytical tool for quickly comprehending many of its aspects. The hypergeometric representation of the analytic normalized function defined in the unit open disc is frequently employed in univalent function theory to determine if the function belongs to a certain class.

Let $\hbar(z)$ be a normalized analytic function with the form,

$$\hbar(z) = z + \sum_{m=2}^{\infty} x_m z^m \tag{1}$$

and class $\mathcal{A}$ define collection of such functions in $\mathcal{D}$. Let $\mathcal{S}$ be the class of univalent function in $\mathcal{D}$ and $\mathcal{S}^*$, $\mathcal{K}$ are the subclasses of $\mathcal{S}$, which are known as starlike functions and convex functions respectively. That are,

$$\mathcal{S}^* = \{\hbar(z) \in \mathcal{S} : \hbar(z) \text{ is a starlike function}\}$$

and

$$\mathcal{K} = \{\hbar(z) \in \mathcal{S} : \hbar(z) \text{ is a convex function}\}.$$

A function $\hbar(z) \in \mathcal{A}$ is said to be close-to-convex if and only if $\operatorname{Re}\left\{e^{i\lambda}\dfrac{z\hbar'(z)}{g(z)}\right\} > 0$, $z \in \mathcal{D}$ where $g(z)$ is a fixed starlike function and $\lambda \in \mathbb{R}$. For $\lambda = 0$ the class of close-to-convex functions is denoted by $\mathcal{C}$. There are several generalizations of these classes in the literature, as well as many subclasses of $\mathcal{S}$ (see [1–5]) and further generalizations (see [6, 7]). For $0 \leq \delta < 1$, $\mathcal{S}^*(\delta)$ and $\mathcal{K}(\delta)$ are the subclasses of $\mathcal{S}$, known as starlike of order δ and convex of order δ respectively. The subclasses $\mathcal{S}^*(\delta)$ and $\mathcal{K}(\delta)$ are defined as

$$\mathcal{S}^*(\delta) = \left\{\hbar(z) \in \mathcal{A} : \operatorname{Re}\left\{\frac{z\hbar'(z)}{\hbar(z)}\right\} > \delta,\ z \in \mathcal{D}\right\}$$

and

$$\mathcal{K}(\delta) = \left\{\hbar(z) \in \mathcal{A} : \operatorname{Re}\left\{1 + \frac{z\hbar''(z)}{\hbar'(z)}\right\} > \delta,\ z \in \mathcal{D}\right\}.$$

For $0 \leq \delta < 1$, $\mathcal{S}^*(\delta) \subset \mathcal{S}^*(0) \equiv \mathcal{S}^* \subset \mathcal{S}$ but for $\delta < 0$ the functions in $\mathcal{S}^*(\delta)$ need not be univalent. There are several subclasses and the generalization of $\mathcal{S}^*(\delta)$ has been defined and studied over the years. Let $\hbar(z) \in \mathcal{A}$ and $\lambda > 0$ then $\mathcal{S}^*_\lambda$ is defined as [8]

$$\mathcal{S}^*_\lambda = \left\{\hbar(z) \in \mathcal{A} : \left|\frac{z\hbar'(z)}{\hbar(z)} - 1\right| < \lambda,\ z \in \mathcal{D}\right\}.$$

Note that $\mathcal{S}^*_\lambda \subset \mathcal{S}^*(1-\lambda)$. Recent research has focused on the study of geometric properties for various integral transforms using the duality technique and negative coefficients [9–11] and references therein, but it is difficult to determine because the results include multiple integrals, and it is challenging to identify the prerequisites for the inclusion property of the integral transforms to exist. It is possible to explore the characteristics of a certain special function that will be included in analytic subclasses. In this work, we are interested in finding the inclusion properties for the Hohlov integral operator (see [12])

$$\mathcal{H}_{p,q,\tau}(\hbar)(z) = z\mathcal{F}(p, q;\, \tau;\, z) * \hbar(z)$$

and Komatu integral operator

$$\mathcal{K}_a^\ell(\hbar)(z) = \mathcal{K}_a^\ell(z) * \hbar(z)$$

where $*$ is the Hadamard product and $\hbar(z)$ belongs to a certain subclass of $\mathcal{S}$. The geometric properties of different integral transforms of the type

$$\mathcal{V}_{\lambda}(\hbar) = \int_0^1 \lambda(t)\frac{\hbar(tz)}{t}dt,\ \ \hbar(z) \in \mathscr{M}^{\tau}_{\zeta,\psi}(\beta)$$

is discussed by several authors [4, 9, 10, 13] with the suitable restrictions on $\lambda(t)$. For a particular choice of $\lambda(t)$ it reduces to various integral operators. If

$$\lambda(t) = \frac{\Gamma(\mathfrak{r})}{\Gamma(q)\Gamma(\mathfrak{r}-q)}t^{q-1}(1-t)^{\mathfrak{r}-q-1}$$

then $\mathcal{V}_{\lambda}(\hbar) = \mathcal{L}(q,\mathfrak{r})(\hbar)(z)$ is the Carlson-Schaffer operator and

$$\mathcal{H}_{1.q.\mathfrak{r}}(\hbar)(z) = \mathcal{L}(q,\mathfrak{r})(\hbar)(z).$$

Let $\hbar(z) \in \mathcal{A}$ and $\mathfrak{r} > p+q-1 > 0$, $p > 0$ and $q > 0$, then $\mathcal{V}_{\lambda}(\hbar)$ is the Hohlov integral operators, i.e.,

$$\mathcal{V}_{\lambda}(\hbar)(z) = \mathcal{H}_{p,q,\mathfrak{r}}(\hbar)(z)$$

where

$$\mathcal{H}_{p,q,\mathfrak{r}}(\hbar)(z) = \frac{\Gamma(\mathfrak{r})}{\Gamma(p)\Gamma(q)}\int_0^1 \frac{(1-t)^{\mathfrak{r}-p-q}}{\Gamma(\mathfrak{r}-p-q+1)}t^{q-2}\mathcal{F}(\mathfrak{r}-p,1-p;\ \mathfrak{r}-p-q+1;\ 1-t)\hbar(tz)dt.$$

The Komatu operator [14], $\mathcal{K}^{\ell}_{a} : \mathcal{A} \to \mathcal{A}$ is defined as

$$\mathcal{K}^{\ell}_{a}(\hbar)(z) = \frac{(1+a)^{\ell}}{\Gamma(\ell)}\int_0^1 \left(\log\left(\frac{1}{t}\right)\right)^{\ell-1} t^{a-1}\hbar(tz)dt$$

where $a > 1$ and $\ell \geq 0$. It is represented by a series as

$$\mathcal{K}^{\ell}_{a}(\hbar)(z) = z + \sum_{m=2}^{\infty}\frac{(1+a)^{\ell}}{(m+a)^{\ell}}x_m z^m$$

and concerning convolution, we can write

$$\mathcal{K}^{\ell}_{a}(\hbar)(z) = \mathcal{K}^{\ell}_{a}(z) * \hbar(z)$$

where

$$\mathcal{K}_a^{\ell}(z) = z + \sum_{m=2}^{\infty} \frac{(1+a)^{\ell}}{(m+a)^{\ell}} z^m.$$

Many authors used different operators to obtain various formulas for the transformation of hypergeometric functions of higher-order that help study the geometric properties of subclasses (see [15, 16]).

In this work, we define a new subclass $\mathscr{M}_{\zeta,\psi}^{\tau}(\beta)$, for $\beta < 1$ and $\tau \in \mathbb{C} \setminus \{0\}$

$$\mathscr{M}_{\zeta,\psi}^{\tau}(\beta) := \left\{ \hbar \in \mathcal{A} : \left| \frac{\hbar' + \psi z \hbar'' + \zeta z^2 \hbar''' - 1}{2\tau(1-\beta) + \hbar' + \psi z \hbar'' + \zeta z^2 \hbar''' - 1} \right| < 1, \ z \in \mathcal{D} \right\},$$

where $0 \leq \zeta < 1$ and $0 \leq \psi < 1$.

Some particular cases of $\mathscr{M}_{\zeta,\psi}^{\tau}(\beta)$, studied in the literature.

• For $\zeta = 0$, the class $\mathscr{M}_{0,\psi}^{\tau}(\beta)$ is considered in [17], concerning the certain conditions of the parameters to be in the class.

• For $\zeta = 0$, $\psi = 1$ and $\tau = e^{i\eta} cos\eta$, $|\eta| < \frac{\pi}{2}$, the class $\mathscr{M}_{0,1}^{\tau}(0)$ is considered in [5] and discussed the properties with certain integral operators.

Let $p, q, \mathfrak{r} \in \mathbb{C}$ then the Gaussian hypergeometric function $\mathcal{F}(p, q; \mathfrak{r}; z)$ is defined as

$$\mathcal{F}(p, q; \mathfrak{r}; z) = \sum_{m=0}^{\infty} \frac{(p)_m (q)_m}{(\mathfrak{r})_m m!} z^m, \quad z \in \mathcal{D}$$

where $\mathfrak{r} \neq 0, -1, -2, -3, \cdots$ and $(\delta)_m$ is Pochhammer symbol. The Pochhammer symbol is defined as

$$(\delta)_m = \frac{\Gamma(\delta+m)}{\Gamma(\delta)}, \ (m \in \mathbb{N})$$

and $\Gamma(\delta+1) = \delta\Gamma(\delta)$, where $\Gamma(\delta)$ is Gamma function. The generalized hypergeometric function was studied by many authors (see [3, 18, 19]).

Definition 1 [20] A function $\hbar(z) \in \mathcal{S}^*$ is said to be uniformly starlike in unit disk $\mathcal{D}$ if for every circular arc γ, with center z_0 contained in $\mathcal{D}$ then $\hbar(\gamma)$ is also starlike with respect to $\hbar(z_0)$ in $\mathcal{D}$. We denote the class of uniformly starlike functions with $\mathcal{UST}$.

Definition 2 [21] A function $\hbar(z) \in \mathcal{K}$ is said to be uniformly convex in $\mathcal{D}$ if for every circular arc γ with center z_0 in $\mathcal{D}$ then $\hbar(\gamma)$ is also convex in $\mathcal{D}$. We use $\mathcal{UCV}$ to denote the class of uniformly convex functions.

Definition 3 [22] Let $\hbar(z) \in \mathcal{A}$ is said to be parabolic starlike functions if

$$\text{Re}\left(\frac{z\hbar'(z)}{\hbar(z)}\right) > \left|\frac{z\hbar'(z)}{\hbar(z)} - 1\right| + \sigma, \ -1 \leq \sigma < 1, \ z \in \mathcal{D}$$

the class of such functions denoted by $\mathcal{S}_p(\sigma)$. In other words, the class $\mathcal{S}_p$ consists of functions $\hbar(z) = zf'(z)$ where $f(z) \in \mathcal{UCV}$.

Definition 4 [11] Let $\hbar(z)$ is of from (1) and $z \in \mathcal{D}$ then $\mathcal{CP}(\alpha)$ is defined as

$$\mathcal{CP}(\alpha)=\left\{\hbar(z)\in\mathcal{S}:\left|\frac{z\hbar''(z)}{\hbar'(z)}+1-\alpha\right|\leq \text{Re}\left(\frac{z\hbar''(z)}{\hbar'(z)}\right)+1+\alpha,\ 0<\alpha<\infty\right\}.$$

Definition 5 [23, 24] Let $\ell-\mathcal{UCV}$ and $\ell-\mathcal{ST}$ be the subclasses of $\mathcal{S}$, known as ℓ-uniformly convex and ℓ-starlike respectively in $\mathcal{D}$, which are defined as

$$\ell-\mathcal{UCV}=\left\{\hbar\in\mathcal{S}:\text{Re}\left(1+\frac{z\hbar''(z)}{\hbar'(z)}\right)>\ell\left|\frac{z\hbar''(z)}{\hbar'(z)}\right|,\ 0\leq\ell<\infty,\ z\in\mathcal{D}\right\}$$

and

$$\ell-\mathcal{ST}=\left\{\hbar\in\mathcal{S}:\text{Re}\left(\frac{z\hbar'(z)}{\hbar(z)}\right)>\ell\left|\frac{z\hbar'(z)}{\hbar(z)}-1\right|,\ 0\leq\ell<\infty,\ z\in\mathcal{D}\right\}.$$

Definition 6 [25] Let $\hbar(z)\in\mathcal{A}$, for $0\leq\ell<\infty$, $0\leq\sigma<1$ then the function $\hbar(z)$ to be in $\ell-\mathcal{UCV}(\sigma)$ if and only if

$$\text{Re}\left(1+\frac{z\hbar''(z)}{\hbar'(z)}\right)\geq\ell\left|\frac{z\hbar''(z)}{\hbar'(z)}\right|+\sigma,\ 0\leq\ell<\infty,\ 0\leq\sigma<1.$$

When $\sigma=0$ then $\ell-\mathcal{UCV}(\sigma)=\ell-\mathcal{UCV}$.

The class $\ell-\mathcal{UCV}(\sigma)$ generalizes many other classes. The domain of values for the expression $p(z)=1+\dfrac{z\hbar''(z)}{\hbar'(z)}$, $z\in\mathcal{D}$ is geometrically described by the class $1-\mathcal{UCV}(0)=\mathcal{UCV}$ [21]. A related class $\ell-\mathcal{S}_p(\sigma)$ is created by using the Alexander transform as $\hbar(z)\in\ell-\mathcal{UCV}(\sigma)$ if and only if $z\hbar'\in\ell-\mathcal{S}_p(\sigma)$ [23]. There are results in the literature for the condition of these classes' Taylor coefficients of functions.

2. Preliminary results

Lemma 1 [17, 26] Consider the following results of the Gaussian hypergeometric function $\mathcal{F}(p,q;\,\mathfrak{r};\,z)$, that are useful in proving our main results:

i. Let $\mathfrak{r}$ be a non-negative integer and is not zero, then

$$\mathcal{F}(p,q;\,\mathfrak{r};\,1)=\frac{\Gamma(\mathfrak{r})\Gamma(\mathfrak{r}-q-p)}{\Gamma(\mathfrak{r}-q)\Gamma(\mathfrak{r}-p)} \tag{2}$$

where $\text{Re}(\mathfrak{r}-q-p)>0$.

ii. For $p,q>0$ and $\mathfrak{r}>1+q+p$ then

$$\sum_{m=0}^{\infty}(m+1)\frac{(p)_m(q)_m}{(\mathfrak{r})_m(1)_m}=\left[\frac{pq}{\mathfrak{r}-1-q-p}+1\right]\mathcal{F}(p,q;\,\mathfrak{r};\,1).$$

iii. For $\mathfrak{r}>\max\{q+p-1,0\}$ and $p,q,\mathfrak{r}\neq1$ then

$$\sum_{m=0}^{\infty}\frac{(p)_m(q)_m}{(\imath)_m(1)_{m+1}}=\frac{(\imath-1)}{(q-1)(p-1)}\left[\mathcal{F}(p-1,\,q-1;\,\imath-1;\,1)-1\right]. \tag{3}$$

iv. For $\imath>\max\{2\mathrm{Re}(p)-1,\,0\}$ and $p,\,\imath\neq 1$ then

$$\sum_{m=0}^{\infty}\frac{|(p)_m|^2}{(\imath)_m(1)_{m+1}}=\frac{(\imath-1)}{|p-1|^2}\left[\mathcal{F}(p-1,\,\bar{p}-1;\,\imath-1;\,1)-1\right].$$

Lemma 2 [27] Let $\hbar(z)\in\mathcal{A}$ is of from (1). If

$$\sum_{m=2}^{\infty}m|x_m|\leq 1$$

then $\hbar(z)$ is univalent in $\mathcal{D}$ and maps that region to a starlike region with center at the origin. If

$$\sum_{m=2}^{\infty}m^2|x_m|\leq 1$$

then $\hbar(z)$ is univalent in $\mathcal{D}$ and maps that region on to a convex region.

Lemma 3 [8, 28] Let $\hbar(z)\in\mathcal{A}$ is of from (1). If

$$\sum_{m=2}^{\infty}(m+\lambda-1)|x_m|\leq\lambda$$

then $\hbar(z)\in\mathcal{S}^*_\lambda$.

Lemma 4 [8] Let $\hbar(z)\in\mathcal{A}$ is of from (1). If

$$\sum_{m=2}^{\infty}m|x_m|\leq\frac{1}{\ell}$$

where $\ell=\sqrt{K}\approx 1.2557$ and $K\approx 1.5770$, then $\hbar(z)\in\mathcal{UST}$. The bound $\frac{1}{\ell}$ is sharp.

Lemma 5 [29] Let $\hbar(z)\in\mathcal{A}$ is of from (1). If

$$\sum_{m=2}^{\infty}m(2m-1)|x_m|\leq 1$$

then $\hbar(z)\in\mathcal{UCV}$. This is the best possible for 1 on the right-hand side.

Lemma 6 [23] Let $\hbar(z)\in\mathcal{A}$ is of the form (1). If the inequality

$$\sum_{m=2}^{\infty} m(m-1)|x_m| \leq \frac{1}{\hbar+2},\ 0 \leq \hbar < \infty$$

holds for some value of $\hbar$ then $\hbar(z) \in \hbar - \mathcal{UCV}$. The number $\dfrac{1}{\hbar+2}$ cannot be made larger.

Lemma 7 [24] Let $\hbar(z) \in \mathcal{A}$ is of the form (1). If the inequality

$$\sum_{m=2}^{\infty} \big(m+\hbar(m-1)\big)|x_m| < 1,\ 0 \leq \hbar < \infty$$

holds for some value of $\hbar$ then $\hbar(z) \in \hbar - \mathcal{ST}$.

Lemma 8 [11] A function $\hbar(z) \in \mathcal{A}$ is satisfies the following condition that is

$$\sum_{m=2}^{\infty} m\big(m(1+\hbar)-(\hbar+\sigma)\big)|x_m| \leq 1-\sigma$$

then the function $\hbar(z) \in \hbar - \mathcal{UCV}(\sigma)$. Again, a condition

$$\sum_{m=2}^{\infty} \big(m(1+\hbar)-(\hbar+\sigma)\big)|x_m| \leq 1-\sigma$$

is sufficient for $\hbar(z) \in \hbar - \mathcal{S}_p(\sigma)$ and necessary if $x_m < 0$ for $\hbar(z) \in \mathcal{A}$.

Lemma 9 [11] Let $\hbar(z) \in \mathcal{A}$ and it is of the form (1), if

$$\sum_{m=2}^{\infty} \big(m+2(\alpha-1)\big)m|x_m| \leq 2\alpha-1,\ 0 < \alpha < \infty$$

then $\hbar(z) \in \mathcal{CP}(\alpha)$.

3. Main results

The conditions on the Taylor coefficients of $\hbar(z) \in \mathcal{M}^{\tau}_{\zeta,\psi}(\beta)$, which are provided in the subsequent results, are necessary to achieve the objective.

Lemma 10 Let a univalent function $\hbar(z)$ is of the form (1) and if $\hbar(z) \in \mathcal{M}^{\tau}_{\zeta,\psi}(\beta)$ then

$$|x_m| \leq \frac{2|\tau|(1-\beta)}{m+m(2\zeta-\psi)+m^2(\psi-3\zeta)+m^3\zeta}.$$

proof. As $\hbar(z) \in \mathcal{M}^{\tau}_{\zeta,\psi}(\beta)$, it is equivalent to write

$$1+\frac{1}{\tau}(\hbar'+\psi z\hbar''+\zeta z^2\hbar'''-1)=\frac{1+(1-2\beta)\varphi(z)}{1-\varphi(z)},\ z\in\mathcal{D}$$

and the function $\varphi(z)$ is analytic in $\mathcal{D}$ have the conditions $\varphi(0)=0$, $|\varphi(z)|<1$.

Hence

$$\frac{1}{\tau}(\hbar'+\psi z\hbar''+\zeta z^2\hbar'''-1)=\varphi(z)\left[2(1-\beta)+\frac{1}{\tau}(\hbar'+\psi z\hbar''+\zeta z^2\hbar'''-1)\right].$$

We have

$$\hbar'+\psi z\hbar''+\zeta z^2\hbar'''-1=\sum_{m=2}^{\infty}(m+(m-1)m\psi+(m-2)(m-1)m\zeta)x_m z^{m-1}. \tag{4}$$

Using (4) and $\varphi(z)=\sum_{m=1}^{\infty}p_m z^m$ then

$$\left(2(1-\beta)+\frac{1}{\tau}\sum_{m=2}^{\infty}m(m^2\zeta+m(\psi-3\zeta)+2\zeta-\psi+1)x_m z^{m-1}\right)\left(\sum_{m=1}^{\infty}p_m z^m\right)$$

$$=\frac{1}{\tau}\sum_{m=2}^{\infty}m(m^2\zeta+m(\psi-3\zeta)+2\zeta-\psi+1)x_m z^{m-1}.$$

Now equating the coefficient of z^{m-1} in the above equation, it is simple to see that the coefficient x_m on both sides of the above equation depends only on $x_2, x_3, \cdots x_{m-1}$. So, for $m\geq 2$

$$\left(2(1-\beta)+\frac{1}{\tau}\sum_{m=2}^{j-1}m\big(m^2\zeta+m(\psi-3\zeta)+2\zeta-\psi+1\big)x_m z^{m-1}\right)\varphi(z)$$

$$=\frac{1}{\tau}\sum_{m=2}^{j}m\big(m^2\zeta+m(\psi-3\zeta)+2\zeta-\psi+1\big)x_m z^{m-1}+\sum_{m=j+1}^{\infty}q_m z^{m-1}.$$

With the use of $|\varphi(z)|<1$, it reduces to

$$\left|2(1-\beta)+\frac{1}{\tau}\sum_{m=2}^{j-1}m\big(m^2\zeta+m(\psi-3\zeta)+2\zeta-\psi+1\big)x_m z^{m-1}\right|$$

$$>\left|\frac{1}{\tau}\sum_{m=2}^{j}m\big(m^2\zeta+m(\psi-3\zeta)+2\zeta-\psi+1\big)x_m z^{m-1}+\sum_{m=j+1}^{\infty}q_m z^{m-1}\right|.$$

By squaring the inequality mentioned above and integrating towards $|z| = u$, $0 < u < 1$ then we get

$$4(1-\beta)^2 + \frac{1}{|\tau|^2}\sum_{m=2}^{j-1} m^2\left(m^2\zeta + m(\psi - 3\zeta) + 2\zeta - \psi + 1\right)^2 |x_m|^2 u^{2(m-1)}$$

$$> \frac{1}{|\tau|^2}\sum_{m=2}^{j} m^2\left(m^2\zeta + m(\psi - 3\zeta) + 2\zeta - \psi + 1\right)^2 |x_m|^2 u^{2(m-1)} + \sum_{m=j+1}^{\infty} |q_m|^2 u^{2(m-1)}.$$

Taking $u \to 1$, then

$$4(1-\beta)^2 \geq \frac{1}{|\tau|^2} m^2\left(m^2\zeta + m(\psi - 3\zeta) + 2\zeta - \psi + 1\right)^2 |x_m|^2$$

which gives the desired result.

Note that when $\zeta = 0$ in Lemma 10, it is equivalent to Theorem 2.1 of [17] and it is also equivalent to Lemma 2.2 of [25] when $\alpha - 2\gamma = 1$.

Remark 1

The coefficient inequality in Lemma 10 is equivalently expressed as for m = 2, 3, $\cdots$

$$|x_m| \leq \frac{2|\tau|(1-\beta)}{1 + (2\psi + 1)(m-1) + (3\zeta + \psi)(m-2)(m-1) + \zeta(m-3)(m-2)(m-1)}. \tag{5}$$

Lemma 11

Let $\hbar(z) \in \mathcal{A}$ is of the form (1). A sufficient condition for $\hbar(z) \in \mathscr{M}^{\tau}_{\zeta,\psi}(\beta)$

$$\sum_{m=2}^{\infty} m(1 + (m-1)\psi + (m-2)(m-1)\zeta)|x_m| \leq |\tau|(1-\beta).$$

Proof. As $\hbar(z) \in \mathcal{A}$ and have the form (1) so

$$\mathrm{Re}(e^{i\eta})\left(\hbar' + \psi z\hbar'' + \zeta z^2\hbar''' - \beta\right)$$

$$= (1-\beta)cos\eta + \mathrm{Re}(e^{i\eta})\sum_{m=2}^{\infty} m(1 - \psi + 2\zeta + m(\psi - 3\zeta) + m^2\zeta)x_m z^{m-1}.$$

Since $z \in \mathcal{D}$ so $|z| < 1$ and the above series is convergent, then

$$|(1-\beta)cos\eta| + |\mathrm{Re}(e^{i\eta})| \sum_{m=2}^{\infty} \left| m(1-\psi+2\zeta+m(\psi-3\zeta)+m^2\zeta) \right| |x_m||z^{m-1}|$$

$$\leq (1-\beta)|\tau| + |\mathrm{Re}(e^{i\eta})| \sum_{m=2}^{\infty} \left| m(1-\psi+2\zeta+m(\psi-3\zeta)+m^2\zeta) \right| |x_m|.$$

From the Lemma 10 we get

$$\frac{1}{2}\left[m+m(2\zeta-\psi)+m^2(\psi-3\zeta)+m^3\zeta\right]|x_m| \leq |\tau|(1-\beta).$$

So, we conclude that

$$(1-\beta)cos\eta - \sum_{m=2}^{\infty} \left| m(1-\psi+2\zeta+m(\psi-3\zeta)+m^2\zeta) \right| |x_m| \geq 0.$$

Then the proof is completed by using the hypothesis and it is equivalent to the analytical description of $\hbar(z)$ in $\mathscr{M}^{\tau}_{\zeta,\psi}(\beta)$.

Note that when $\zeta = 0$ in Lemma 11, it is equivalent to Theorem 2.2 of [17] and it is also equivalent to Lemma 2.3 of [25] when $\alpha - 2\gamma = 1$.

Theorem 1 Let a function $\hbar(z) \in \mathscr{A}$ has the form (1). For q, $p \neq 1$ and $\mathfrak{r} > q + p$ it satisfies the condition,

$$\mathscr{F}(p,\, q;\, \mathfrak{r};\, 1)\frac{(\mathfrak{r}-q-p)}{(q-1)(p-1)} \leq 1 + \frac{(\mathfrak{r}-1)}{(q-1)(p-1)} + \frac{(\psi-3\zeta)}{2|\tau|(1-\beta)}.$$

Then for $\hbar(z) \in \mathscr{M}^{\tau}_{\zeta,\psi}(\beta)$ where $0 \leq \psi \leq 1$, $0 \leq \zeta \leq 1$ and $0 \leq \beta < 1$ we have $\mathscr{H}_{p,q,\mathfrak{r}}(\hbar)(z) \in \mathscr{S}^*$.

Proof. For $\hbar(z) \in \mathscr{M}^{\tau}_{\zeta,\psi}(\beta)$, then from Lemma 10

$$|x_m| \leq \frac{2|\tau|(1-\beta)}{m+m(2\zeta-\psi)+m^2(\psi-3\zeta)+m^3\zeta}.$$

Let

$$m+m(2\zeta-\psi)+m^2(\psi-3\zeta)+m^3\zeta = m(1-\psi)+m\zeta(2+m^2)+m^2(\psi-3\zeta)$$

$$\geq m^2(\psi-3\zeta).$$

So

$$|x_m| \leq \frac{2|\tau|(1-\beta)}{m^2(\psi-3\zeta)}. \tag{6}$$

From Lemma 2

$$\sum_{m=2}^{\infty} m\,|\mathcal{X}_m| \leq 1$$

where

$$\mathcal{X}_m = \frac{(p)_{m-1}(q)_{m-1}}{(\mathfrak{r})_{m-1}(1)_{m-1}} x_m \tag{7}$$

Then

$$\sum_{m=2}^{\infty} \frac{1}{m} \frac{(p)_{m-1}(q)_{m-1}}{(\mathfrak{r})_{m-1}(1)_{m-1}} \leq \frac{(\psi - 3\zeta)}{2|\tau|(1-\beta)}.$$

Then

$$\sum_{m=0}^{\infty} \frac{(p)_m(q)_m}{(\mathfrak{r})_m(1)_{m+1}} - 1 \leq \frac{(\psi - 3\zeta)}{2|\tau|(1-\beta)}.$$

Now applying (3), we get

$$\frac{(\mathfrak{r}-1)}{(q-1)(p-1)} \left[\mathcal{F}(p-1, q-1;\ \mathfrak{r}-1;\ 1) - 1\right] \leq 1 + \frac{(\psi - 3\zeta)}{2|\tau|(1-\beta)}.$$

From Gaussian hypergeometric function

$$\mathcal{F}(p-1, q-1;\ \mathfrak{r}-1;\ 1) = \frac{(\mathfrak{r}-q-p)}{(\mathfrak{r}-1)} \mathcal{F}(p, q;\ \mathfrak{r};\ 1). \tag{8}$$

Now using the hypothesis of the theorem and (8) we will get the required result.
Theorem 2 Let $\hbar(z) \in \mathcal{A}$ is of the form (1) and satisfy the condition, that is for $\mathfrak{r} > q + p$

$$\mathcal{F}(p, q;\ \mathfrak{r};\ 1) \leq 1 + \frac{(\psi - 3\zeta)}{2|\tau|(1-\beta)}.$$

Then for $\hbar(z) \in \mathcal{M}^{\tau}_{\zeta,\psi}(\beta)$ where $0 \leq \psi \leq 1$, $0 \leq \zeta \leq 1$ and $0 \leq \beta < 1$ we have $\mathcal{H}_{p,q,\mathfrak{r}}(\hbar)(z) \in \mathcal{K}$.
Proof. As $\hbar(z) \in \mathcal{M}^{\tau}_{\zeta,\psi}(\beta)$, then from (6) and Lemma 2

$$\sum_{m=0}^{\infty} \frac{(p)_m(q)_m}{(\mathfrak{r})_m(1)_m} - 1 \leq \frac{(\psi - 3\zeta)}{2|\tau|(1-\beta)}.$$

Now, using the hypothesis of the theorem, we will get the required result.
Theorem 3 Let $\hbar(z) \in \mathcal{A}$ has the form (1), for $q, p \neq 1$ and $\mathfrak{r} > q + p$ is satisfies the following condition:

$$\mathcal{F}(p, q; \mathfrak{r}; 1)\left(2 - \frac{(\mathfrak{r} - q - p)}{(q-1)(p-1)}\right) + \frac{(\mathfrak{r} - 1)}{(q-1)(p-1)} \leq 1 + \frac{(\psi - 3\zeta)}{2|\tau|(1-\beta)}.$$

Then for $\hbar(z) \in \mathcal{M}_{\zeta,\psi}^{\tau}(\beta)$, $0 \leq \zeta \leq 1$, $0 \leq \psi \leq 1$ and $0 \leq \beta < 1$ we have $\mathcal{H}_{p,q,\mathfrak{r}}(\hbar)(z) \in \mathcal{UCV}$.
Proof. As $\hbar(z) \in \mathcal{M}_{\zeta,\psi}^{\tau}(\beta)$, then from (6) and Lemma 5

$$\sum_{m=2}^{\infty}(2m^2 - m)\frac{1}{m^2}\frac{(p)_{m-1}(q)_{m-1}}{(\mathfrak{r})_{m-1}(1)_{m-1}} \leq \frac{(\psi - 3\zeta)}{2|\tau|(1-\beta)}.$$

Then

$$2\sum_{m=2}^{\infty}\frac{(p)_{m-1}(q)_{m-1}}{(\mathfrak{r})_{m-1}(1)_{m-1}} - \sum_{m=2}^{\infty}\frac{1}{m}\frac{(p)_{m-1}(q)_{m-1}}{(\mathfrak{r})_{m-1}(1)_{m-1}} \leq \frac{(\psi - 3\zeta)}{2|\tau|(1-\beta)}.$$

Now using (3), (8) and the hypothesis of the theorem then we get the required results.
Theorem 4 Let $\hbar(z) \in \mathcal{A}$ is of the form (1) and satisfy the condition, that is for $p, q \neq 1$ and $\mathfrak{r} > q + p$

$$\mathcal{F}(p, q; \mathfrak{r}; 1)\left(1 + \frac{2(\alpha - 1)(\mathfrak{r} - q - p)}{(q-1)(p-1)}\right) + (1 - 2\alpha) \leq \frac{2(\alpha - 1)(\mathfrak{r} - 1)}{(q-1)(p-1)} + \frac{(2\alpha - 1)(\psi - 3\zeta)}{2|\tau|(1-\beta)}.$$

Then for $\hbar(z) \in \mathcal{M}_{\zeta,\psi}^{\tau}(\beta)$, $0 \leq \zeta \leq 1$, $0 \leq \psi \leq 1$ and $0 \leq \beta < 1$ we have $\mathcal{H}_{p,q,\mathfrak{r}}(\hbar)(z) \in \mathcal{CP}(\alpha)$ where $0 < \alpha < \infty$.
proof. As $\hbar(z) \in \mathcal{M}_{\zeta,\psi}^{\tau}(\beta)$, then from (6) and Lemma 9

$$\sum_{m=2}^{\infty}(m^2 + 2m(\alpha - 1))\frac{1}{m^2}\frac{(p)_{m-1}(q)_{m-1}}{(\mathfrak{r})_{m-1}(1)_{m-1}} \leq \frac{(2\alpha - 1)(\psi - 3\zeta)}{2|\tau|(1-\beta)}.$$

Then

$$\left(\sum_{m=0}^{\infty}\frac{(p)_m(q)_m}{(\mathfrak{r})_m(1)_m} - 1\right) + 2(\alpha - 1)\left(\sum_{m=0}^{\infty}\frac{(p)_m(q)_m}{(\mathfrak{r})_m(1)_{m+1}} - 1\right) \leq \frac{(2\alpha - 1)(\psi - 3\zeta)}{2|\tau|(1-\beta)}.$$

Now using (3), (8) and the hypothesis of the theorem then we get the required results.
Theorem 5 Let $\hbar(z) \in \mathcal{A}$ is of the form (1). For $\hbar \geq 0$, $0 \leq \sigma < 1$ it satisfies the following conditions:
i. $|p|, |q| \neq 1$ and $\mathfrak{r} > |q| + |p|$

$$\mathcal{F}(|p|-1,\,|q|-1;\,\mathfrak{r}-1;\,1)\frac{\big((|q|-1)(|p|-1)+\sigma(|q|+|p|-\mathfrak{r})+\hbar(|pq|+1-\mathfrak{r})\big)(\mathfrak{r}-1)}{(\mathfrak{r}-|q|-|p|)(|q|-1)(|p|-1)}$$

$$+\frac{(\mathfrak{r}-1)(\hbar+\sigma)}{(|q|-1)(|p|-1)}\leq\left(\frac{(\psi-3\zeta)}{2|\tau|(1-\beta)}+1\right)(1-\sigma).$$

ii. $|p|$, $|q|\neq 1$ and $\mathfrak{r}\geq 0$

$$\mathcal{F}(|p|,\,|q|;\,\mathfrak{r};\,1)\frac{(|q|-1)(|p|-1)+\sigma(|q|+|p|-\mathfrak{r})+\hbar(|pq|+1-\mathfrak{r})}{(|q|-1)(|p|-1)}$$

$$+\frac{(\mathfrak{r}-1)(\hbar+\sigma)}{(|q|-1)(|p|-1)}\leq\left(\frac{(\psi-3\zeta)}{2|\tau|(1-\beta)}+1\right)(1-\sigma).$$

Then for $\hbar(z)\in\mathscr{M}^{\tau}_{\zeta,\psi}(\beta)$, $0\leq\zeta\leq 1$, $0\leq\psi\leq 1$ and $0\leq\beta<1$ we have $\mathcal{H}_{p,q,\mathfrak{r}}(\hbar)(z)\in\hbar-\mathcal{UCV}(\sigma)$.
proof. As $\hbar(z)\in\mathscr{M}^{\tau}_{\zeta,\psi}(\beta)$, then from (6)

$$|x_m|\leq\frac{2|\tau|(1-\beta)}{m^2(\psi-3\zeta)}.$$

From the Lemma 8

$$\sum_{m=2}^{\infty}\big((1+\hbar)m-(\hbar+\sigma)\big)m|\mathcal{X}_m|\leq 1-\sigma$$

where

$$\mathcal{X}_m=\frac{(p)_{m-1}(q)_{m-1}}{(\mathfrak{r})_{m-1}(1)_{m-1}}x_m.$$

Then

$$\sum_{m=2}^{\infty}\big((1+\hbar)m-(\hbar+\sigma)\big)m\frac{(|p|)_{m-1}(|q|)_{m-1}}{(\mathfrak{r})_{m-1}(1)_{m-1}}|x_m|\leq(1-\sigma).$$

Now applying (6), we get

$$\sum_{m=2}^{\infty}\frac{\big((1+\hbar)m-(\hbar+\sigma)\big)}{m}\frac{(|p|)_{m-1}(|q|)_{m-1}}{(\mathfrak{r})_{m-1}(1)_{m-1}}\leq\frac{(\psi-3\zeta)}{2|\tau|(1-\beta)}(1-\sigma).$$

Let

$$\mathcal{M}=\sum_{m=2}^{\infty}(1+\hbar)\frac{(|p|)_{m-1}(|q|)_{m-1}}{(\mathfrak{r})_{m-1}(1)_{m-1}}-\frac{(\hbar+\sigma)}{m}\frac{(|p|)_{m-1}(|q|)_{m-1}}{(\mathfrak{r})_{m-1}(1)_{m-1}}$$

$$=(1+\hbar)\left(\sum_{m=0}^{\infty}\frac{(|p|)_{m}(|q|)_{m}}{(\mathfrak{r})_{m}(1)_{m}}-1\right)-(\hbar+\sigma)\left(\sum_{m=0}^{\infty}\frac{(|p|)_{m}(|q|)_{m}}{(\mathfrak{r})_{m}(1)_{m+1}}-1\right).$$

Using (3) in the second sum, we obtain

$$\mathcal{M}=(1+\hbar)\mathcal{F}(|p|,\,|q|;\,\mathfrak{r};\,1)-(\hbar+\sigma)\frac{(\mathfrak{r}-1)}{(|q|-1)(|p|-1)}\mathcal{F}(|p|-1,\,|q|-1;\,\mathfrak{r}-1;\,1)$$

$$+(\hbar+\sigma)\frac{(\mathfrak{r}-1)}{(|q|-1)(|p|-1)}-(1-\sigma).$$

With the use of (2), we ge

i.

$$\mathcal{M}=(1+\hbar)\frac{\Gamma(\mathfrak{r})\Gamma(\mathfrak{r}-|q|-|p|)}{\Gamma(\mathfrak{r}-|q|)\Gamma(\mathfrak{r}-|p|)}-(\hbar+\sigma)\frac{(\mathfrak{r}-1)}{(|q|-1)(|p|-1)}\mathcal{F}(|p|-1,\,|q|-1;\,\mathfrak{r}-1;\,1)$$

$$+(\hbar+\sigma)\frac{(\mathfrak{r}-1)}{(|q|-1)(|p|-1)}-(1-\sigma)$$

$$=(1+\hbar)\frac{(\mathfrak{r}-1)}{(\mathfrak{r}-|p|-|q|)}\mathcal{F}(|p|-1,\,|q|-1;\,\mathfrak{r}-1;\,1)+(\hbar+\sigma)\frac{(\mathfrak{r}-1)}{(|q|-1)(|p|-1)}$$

$$-(\hbar+\sigma)\frac{(\mathfrak{r}-1)}{(|q|-1)(|p|-1)}\mathcal{F}(|p|-1,\,|q|-1;\,\mathfrak{r}-1;\,1)-(1-\sigma).$$

ii.

$$\mathcal{M} = (1+\mathcal{k})\mathcal{F}(|p|, |q|;\ \mathcal{r};\ 1) - \frac{(\mathcal{k}+\sigma)(\mathcal{r}-1)}{(|q|-1)(|p|-1)} \frac{(\mathcal{r}-|q|-|p|)}{(\mathcal{r}-1)} \frac{\Gamma(\mathcal{r})\Gamma(\mathcal{r}-|q|-|p|)}{\Gamma(\mathcal{r}-|q|)\Gamma(\mathcal{r}-|p|)}$$

$$+ (\mathcal{k}+\sigma)\frac{(\mathcal{r}-1)}{(|q|-1)(|p|-1)} - (1-\sigma)$$

$$= \mathcal{F}(|p|, |q|;\ \mathcal{r};\ 1)\left[(1+\mathcal{k}) - (\mathcal{k}+\sigma)\frac{(\mathcal{r}-|q|-|p|)}{(|q|-1)(|p|-1)}\right]$$

$$+ (\mathcal{k}+\sigma)\frac{(\mathcal{r}-1)}{(|q|-1)(|p|-1)} - (1-\sigma).$$

Using the hypothesis of the theorem, we conclude the required results.

If we choose $p = \bar{q}$ in $\mathcal{F}(p, q;\ \mathcal{r};\ z)$, where q is some negative integer, we have a polynomial with positive coefficients. Therefore, the aforementioned theorem is beneficial in characterizing complex polynomials and we present the related findings separately.

Coeollary 1 Let $\mathcal{k}(z) \in \mathcal{A}$ is of the form (1). For $\mathcal{k} \geq 0$, $0 \leq \sigma < 1$, it satisfies the following conditions:

i. $q \neq 1$ and $\mathcal{r} > 2\mathrm{Re}(q)$

$$\mathcal{F}(\bar{q}-1, q-1;\ \mathcal{r}-1;\ 1)\frac{\left(|q-1|^2 + \mathcal{k}(|q|^2 - \mathcal{r}+1) + \sigma(2\mathrm{Re}(q) - \mathcal{r})\right)(\mathcal{r}-1)}{|q-1|^2(\mathcal{r}-2\mathrm{Re}(q))}$$

$$+ \frac{(\mathcal{r}-1)(\mathcal{k}+\sigma)}{|q-1|^2} \leq (1-\sigma)\left(1 + \frac{(\psi-3\zeta)}{2|\tau|(1-\beta)}\right).$$

ii. $q \neq 1$ and $r > 0$

$$\mathcal{F}(\bar{q}, q;\ \mathcal{r};\ 1)\frac{|q-1|^2 + \sigma(2\mathrm{Re}(q) - \mathcal{r}) + \mathcal{k}(|q|^2 - \mathcal{r}+1)}{|q-1|^2} + \frac{(\mathcal{r}-1)(\mathcal{k}+\sigma)}{|q-1|^2}$$

$$\leq (1-\sigma)\left(1 + \frac{(\psi-3\zeta)}{2|\tau|(1-\beta)}\right).$$

Then for $\mathcal{k}(z) \in \mathcal{M}^{\tau}_{\zeta,\psi}(\beta)$, $0 \leq \psi \leq 1$ and $0 \leq \beta < 1$ we have $\mathcal{H}_{\bar{q},q,\mathcal{r}}\mathcal{k}(z) \in \mathcal{k} - \mathcal{UCV}(\sigma)$.

Theorem 6 Let $\mathcal{k}(z) \in \mathcal{A}$ is of the form (1). For $\mathcal{k} \geq 0$, $0 \leq \zeta < 1$ it satisfies the following conditions, that are

i. $|p|, |q| \neq 1$ and $\mathcal{r} > |q| + |p|$

$$\frac{(|pq| - \mathcal{r}+1)(\mathcal{r}-1)}{(\mathcal{r}-|q|-|p|)(|q|-1)(|p|-1)}\mathcal{F}(|p|-1, |q|-1;\ \mathcal{r}-1;\ 1) - \frac{(\mathcal{r}-1)}{(|q|-1)(|p|-1)}$$

$$\leq \frac{(\psi-3\zeta)}{2|\tau|(1-\beta)(k+2)}.$$

ii. $|p|$, $|q| \neq 1$ and $\mathfrak{r} \geq 0$

$$\frac{(|pq|-\mathfrak{r}+1)}{(|q|-1)(|p|-1)}\mathcal{F}(|p|,\,|q|;\;\mathfrak{r};\;1)+\frac{(\mathfrak{r}-1)}{(|q|-1)(|p|-1)}\quad\leq\frac{(\psi-3\zeta)}{2|\tau|(1-\beta)(k+2)}.$$

Then for $\hbar(z) \in \mathscr{M}^{\tau}_{\zeta,\psi}(\beta)$, $0 \leq \psi \leq 1$ and $0 \leq \beta < 1$, we have $\mathcal{H}_{p,q,\mathfrak{r}}\hbar(z) \in \hbar - \mathcal{UCV}$.
proof. Since $\hbar(z) \in \mathscr{M}^{\tau}_{\zeta,\psi}(\beta)$ so from (6) and Lemma 6

$$\sum_{m=2}^{\infty}(m-1)\frac{1}{m}\frac{(|p|)_{m-1}(|q|)_{m-1}}{(\mathfrak{r})_{m-1}(1)_{m-1}}\leq\frac{(\psi-3\zeta)}{2|\tau|(\hbar+2)(1-\beta)}.$$

Then

$$\left(\sum_{m=0}^{\infty}\frac{(|p|)_m(|q|)_m}{(\mathfrak{r})_m(1)_m}-1\right)-\left(\sum_{m=0}^{\infty}\frac{(|p|)_m(|q|)_m}{(\mathfrak{r})_m(1)_{m+1}}-1\right)\leq\frac{(\psi-3\zeta)}{2|\tau|(\hbar+2)(1-\beta)}.$$

Now using (3), (8) and the hypothesis of the theorem then we get the required results.
If we take $p=\bar{q}$ in, $\mathcal{F}(p,\,q;\;\mathfrak{r};\;z)$ then the following results directly.
Corollary 2 Let $\hbar(z) \in \mathcal{A}$ is of the form (1). For $\hbar \geq 0$, $0 \leq \zeta < 1$ it satisfies the following conditions:
i. $q \neq 1$ and $\mathfrak{r} > 2\mathrm{Re}(|q|)$

$$\frac{(|q|^2-\mathfrak{r}+1)(\mathfrak{r}-1)}{|q-1|^2(\mathfrak{r}-2\mathcal{R}e(q))}\mathcal{F}(\bar{q}-1,\,q-1;\;\mathfrak{r}-1;\;1)-\frac{(\mathfrak{r}-1)}{|q-1|^2}\leq\frac{(\psi-3\zeta)}{2|\tau|(\hbar+2)(1-\beta)}.$$

ii. $q \neq 1$, and $\mathfrak{r} \geq 0$

$$\frac{(|q|^2-\mathfrak{r}+1)}{|q-1|^2}\mathcal{F}(\bar{q},\,q;\;\mathfrak{r};\;1)+\frac{(\mathfrak{r}-1)}{|q-1|^2}\leq\frac{(\psi-3\zeta)}{2|\tau|(\hbar+2)(1-\beta)}.$$

Then for $\hbar(z) \in \mathscr{M}^{\tau}_{\zeta,\psi}(\beta)$, we have $\mathcal{H}_{\bar{q},q,\mathfrak{r}}(\hbar)(z) \in \hbar - \mathcal{UCV}$.
Theorem 7 Let $\hbar(z) \in \mathcal{A}$ is of the form (1). Suppose $a > 1$, $\ell \geq 0$ and

$$\sum_{m=2}^{\infty}\left(\frac{1}{m}+\frac{\hbar(m-1)}{m^2}\right)\mathcal{B}_m(a,\,\ell)<\frac{(\psi-3\zeta)}{2|\tau|(1-\beta)}$$

where

$$\mathcal{B}_m(a,\,\ell)=\frac{(1+a)^{\ell}}{(m+a)^{\ell}}.$$

Then for $\hbar(z) \in \mathscr{M}^{\tau}_{\zeta,\psi}(\beta)$, $0 \leq \zeta \leq 1$, $0 \leq \psi \leq 1$ and $0 \leq \beta < 1$ we have $\mathcal{K}^{\ell}_{a}(\hbar)(z) \in \hbar - \mathcal{ST}$.
proof. As $\hbar(z) \in \mathscr{M}^{\tau}_{\zeta,\psi}(\beta)$, so from (6)

$$|x_m| \leq \frac{2|\tau|(1-\beta)}{m^2(\psi-3\zeta)}.$$

Now using the Lemma 7, that is

$$\sum_{m=2}^{\infty}(m+\hbar(m-1))|\mathcal{X}_m| < 1$$

and $|\mathcal{X}_m| = \mathcal{B}_m(a, \ell)|x_m|$, we get

$$\sum_{m=2}^{\infty}\big(m+\hbar(m-1)\big)\frac{\mathcal{B}_m(a, \ell)}{m^2} < \frac{(\psi-3\zeta)}{2|\tau|(1-\beta)}.$$

From the above inequality, we will get the required result.

Theorem 8 Let $\hbar(z) \in \mathcal{A}$ is of the form (1), with $a > 1$, $\ell \geq 0$ and

$$\sum_{m=2}^{\infty}\left((1+\hbar)-\frac{(\hbar+\sigma)}{m}\right)\mathcal{B}_m(a, \ell) \leq \frac{(1-\sigma)(\psi-3\zeta)}{2|\tau|(1-\beta)}$$

where

$$\mathcal{B}_m(a, \ell) = \frac{(1+a)^{\ell}}{(m+a)^{\ell}}.$$

Then for $\hbar(z) \in \mathcal{M}^{\tau}_{\zeta,\psi}(\beta)$ and $0 \leq \zeta \leq 1$, $0 \leq \psi \leq 1$ and $0 \leq \beta < 1$ we have $\mathcal{K}^{\ell}_{a}(\hbar)(z) \in \hbar - \mathcal{UCV}(\sigma)$.

proof. Since $\hbar(z) \in \mathcal{M}^{\tau}_{\zeta,\psi}(\beta)$ then from (6) and Lemma 8, that is

$$\sum_{m=2}^{\infty}\big((1+\hbar)m-(\hbar+\sigma)\big)m|\mathcal{X}_m| \leq 1-\sigma$$

and $|\mathcal{X}_m| = \mathcal{B}_m(a, \ell)|x_m|$, we get

$$\sum_{m=2}^{\infty}\big((1+\hbar)m-(\hbar+\sigma)\big)\frac{\mathcal{B}_m(a, \ell)}{m} \leq \frac{(1-\sigma)(\psi-3\zeta)}{2|\tau|(1-\beta)}.$$

From the above inequality, we will get the required result.

Theorem 9 Let $\hbar(z) \in \mathcal{A}$ is of the form (1), with $a > 1$, $\ell \geq 0$ and

$$\sum_{m=2}^{\infty}\left(\frac{(1+\hbar)}{m}-\frac{(\hbar+\sigma)}{m^2}\right)\mathcal{B}_m(a, \ell) \leq \frac{(1-\sigma)(\psi-3\zeta)}{2|\tau|(1-\beta)}$$

where $\mathcal{B}_m(a,\ell)=\dfrac{(1+a)^\ell}{(m+a)^\ell}$. Then for $\hbar(z)\in\mathscr{M}^{\tau}_{\zeta,\psi}(\beta)$, $0\le\zeta\le1$, $0\le\psi\le1$ and $0\le\beta<1$ we have $\mathcal{K}^{\ell}_{a}(\hbar)(z)\in \hbar-\mathcal{S}_p(\sigma)$.

proof. Using Lemma 8 the proof is the same as the above theorem.

It is easy to see, for $m\ge2$

$$\mathcal{B}_m(a,\ell)=\frac{(1+a)^\ell}{(m+a)^\ell}<1,\ a>1,\ \ell\ge0$$

which leads to the next results.

Corollary 3 Let $\hbar(z)\in\mathcal{A}$ is of the form (1). Suppose $a>1$, $\ell\ge0$ and

$$\sum_{m=2}^{\infty}\left(\frac{1}{m}+\frac{\hbar(m-1)}{m^2}\right)<\frac{(\psi-3\zeta)}{2|\tau|(1-\beta)},\ m\ge2.$$

Then for $\hbar(z)\in\mathscr{M}^{\tau}_{\zeta,\psi}(\beta)$, $0\le\beta<1$, $0\le\zeta\le1$ and $0\le\psi\le1$, we have $\mathcal{K}^{\ell}_{a}(\hbar)(z)\in\hbar-\mathcal{ST}$.

Corollary 4 Let $\hbar(z)\in\mathcal{A}$ is of the form (1). Suppose $a>-1$, $\ell\ge0$ and

$$\sum_{m=2}^{\infty}\left((1+\hbar)-\frac{(\hbar+\sigma)}{m}\right)\le\frac{(1-\sigma)(\psi-3\zeta)}{2|\tau|(1-\beta)},\ m\ge2.$$

Then for $\hbar(z)\in\mathscr{M}^{\tau}_{\zeta,\psi}(\beta)$ we have $\mathcal{K}^{\ell}_{a}(\hbar)(z)\in\hbar-\mathcal{UCV}(\sigma)$.

Corollary 5 Let $\hbar(z)\in\mathcal{A}$ is of the form (1). Suppose $a>1$, $\ell\ge0$ and

$$\sum_{m=2}^{\infty}\left(\frac{(1+\hbar)}{m}-\frac{(\hbar+\sigma)}{m^2}\right)\le\frac{(1-\sigma)(\psi-3\zeta)}{2|\tau|(1-\beta)},\ m\ge2.$$

Then for $\hbar(z)\in\mathscr{M}^{\tau}_{\zeta,\psi}(\beta)$ we have $\mathcal{K}^{\ell}_{a}(\hbar)(z)\in\hbar-\mathcal{S}_p(\sigma)$.

Theorem 10 Let $\hbar(z)\in\mathcal{A}$ is of the form (1). Suppose $a>1$, $\ell\le0$ and

$$\sum_{m=2}^{\infty}(m+\lambda-1)\frac{\mathcal{B}_m(a,\ell)}{m^2}\le\frac{\lambda(-3\zeta)}{2|\tau|(1-\beta)}$$

where $\mathcal{B}_m(a,\ell)=\dfrac{(1+a)^\ell}{(m+a)^\ell}$. Then for $\hbar(z)\in\mathscr{M}^{\tau}_{\zeta,\psi}(\beta)$, $0\le\beta<1$, $0\le\zeta\le1$ and $0\le\psi\le1$, we have $\mathcal{K}^{\ell}_{a}(\hbar)(z)\in\mathcal{S}^{*}_{\lambda}$.

proof. As $\hbar(z)\in\mathscr{M}^{\tau}_{\zeta,\psi}(\beta)$ then from (6) and Lemma 3, that is

$$\sum_{m=2}^{\infty}(m+\lambda-1)|\mathcal{X}_m|\le\lambda.$$

Using the hypothesis of the theorem, we conclude the required results.

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Conflict of interest

The authors declare that they have no competing interests.

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