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Research Article

Fejér-Quadrature Collocation Algorithm for Solving Fractional Integro-Differential Equations via Fibonacci Polynomials

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Abstract: In this article, we introduce a novel spectral algorithm utilizing Fibonacci polynomials to numerically solve both linear and nonlinear integro-differential equations with fractional-order derivatives. Our approach employs a quadrature-collocation method, transforming complex equations and associated conditions into systems of linear or nonlinear algebraic equations. The solutions to these equations, involving unknown coefficients, provide accurate numerical approximations for the original fractional-order equations. To validate the method, we present numerical examples illustrating its robustness and versatility. Comparative analyses with available analytical solutions affirm the reliability and accuracy of our algorithm, establishing its practical utility in addressing fractional-order integro-differential equations. This research contributes to computational mathematics and spectral methods, offering a promising tool for diverse scientific and engineering challenges.

***Keywords*:** Fibonacci polynomials, Fejér quadrature, collocation method, fractional-order integro-differential equations

MSC: 65L05, 34K06, 34K28

1. Introduction

The rich tapestry of mathematical research over the past century has extensively delved into various sequences of polynomials, among which the renowned Fibonacci and Lucas polynomials occupy a prominent place. These polynomials have been subjects of profound investigation, revealing intricate interrelations that have captivated mathematicians and researchers alike. The exploration of Fibonacci and Lucas polynomials transcends disciplinary boundaries, finding applications across a spectrum of mathematical domains. In the realm of algebra, these polynomials play a crucial role in understanding algebraic structures and patterns. In geometry, they contribute to geometric constructions and shape analyses. Combinatorics benefits from the combinatorial properties embedded in these polynomials, offering insights into counting and arrangement problems. Additionally, the influence of Fibonacci and Lucas polynomials extends to approximation theory, providing powerful tools for approximating functions and solving numerical problems. In statistics, their properties are harnessed for probabilistic modeling, while in number theory, they reveal fascinating connections with the distribution of prime numbers and the properties of integers. Notably, Fibonacci polynomials, considered as special cases of Chebyshev polynomials, have attracted the attention of numerous

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mathematicians who have delved into their properties and behaviors at an advanced level. This comprehensive exploration underscores the versatility and significance of Fibonacci and Lucas polynomials, demonstrating their enduring impact on the diverse landscapes of mathematical research and applications [1-2].

Simultaneously, the advent of fractional calculus represents a transformative extension beyond conventional derivatives and integrals, introducing a mathematical framework adept at handling non-integer orders. This innovative paradigm has become instrumental in addressing and modeling complex scientific and engineering phenomena, with applications spanning a diverse array of disciplines. Fractional differential equations, in particular, have assumed pivotal roles across various domains, captivating the interest of researchers from both theoretical and practical standpoints. The significance of fractional calculus becomes evident in its broad applicability to a spectrum of scientific and engineering problems. In the realm of acoustics, fractional differential equations provide a nuanced understanding of wave propagation and signal dynamics, offering insights into the behavior of sound in heterogeneous media. In the study of damping laws, fractional calculus enables the formulation of more accurate models for describing viscoelastic materials, contributing to advancements in structural engineering and material science [3].

Moreover, fractional calculus, as illuminated by Diethelm [4], plays a crucial role in the realm of electroanalytical chemistry, where complex electrochemical processes are intricately described by fractional differential equations. This mathematical tool proves indispensable for characterizing phenomena such as electrode kinetics and diffusion-limited reactions [4]. Brunner, Pedas, and Vainikko [5] demonstrate the efficacy of piecewise polynomial collocation methods for linear Volterra integro-differential equations with weakly singular kernels, enhancing our ability to design and optimize electrochemical systems. In the domain of neuroscience, as Kilbas, Srivastava, and Trujillo illustrate [6], neuron modeling benefits substantially from fractional calculus, allowing for a more nuanced representation of the intricate dynamics and interactions within neural networks. Fractional-order models prove advantageous in capturing the long-range dependencies and memory effects that are characteristic of neuronal systems.

Additionally, fractional calculus finds application in elucidating diffusion processes, offering refined models that accurately depict the non-local and non-Markovian nature of particle movement in heterogeneous environments [7]. The versatile and powerful nature of fractional calculus positions it as a cornerstone in the mathematical toolbox for modeling and understanding diverse scientific and engineering challenges, ranging from the microscopic intricacies of electroanalytical chemistry to the macroscopic dynamics of material sciences. The continued exploration and refinement of fractional calculus methodologies promise to unlock new dimensions in our ability to comprehend and engineer complex systems across a multitude of disciplines [8-10]. Sadek et al. [11] propose a numerical approach based on the Bernstein collocation method, applying it to differential Lyapunov and Sylvester matrix equations.

Spectral methods have played a pivotal role in solving differential equations across various scientific disciplines, offering efficient and accurate numerical solutions. Researchers have extensively utilized spectral algorithms to tackle a wide range of differential equation problems. The work of Hafez and Youssri [12] introduce a fully Jacobi-Galerkin algorithm for two-dimensional time-dependent partial differential equations in physics, showcasing the versatility of spectral methods in handling complex physical phenomena. Youssri and Atta [13] present a modal spectral Tchebyshev Petrov-Galerkin stratagem specifically designed for the time-fractional nonlinear Burgers' equation, providing insights into the application of spectral methods in addressing nonlinear partial differential equations. Additionally, Hafez, Youssri, and Atta [14] propose a Jacobi Rational Operational Approach for solving time-fractional sub-diffusion equations on a semi-infinite domain, demonstrating the applicability of spectral methods to fractional differential equations. These works, along with the contributions of Magdy et al. [15] and Abdelhakem et al. [16], collectively underscore the potency and efficacy of spectral methods in solving differential equations of various complexities. For more studies, please see [17-18].

In the annals of mathematical history, the year 1933 marked a seminal moment with the introduction of an influential method for evaluating definite integrals by the Hungarian mathematician Lipót Fejér. This pioneering technique, now celebrated as the Fejér quadrature formula, represents a sophisticated approach to numerical integration that has stood the test of time. The core principle of this method involves the expansion of the integrand in a finite Chebyshev series, a strategy that facilitates a comprehensive understanding of the function's behavior over a specified interval. Subsequently, each term in this series is individually integrated, culminating in an approximation of the original definite integral.

The Fejér quadrature formula [19-20] has garnered widespread recognition and adoption due to its remarkable efficiency, accuracy, and the inherent simplicity that characterizes its implementation. One of its key advantages lies in the ease with which error estimates can be derived, providing practitioners with valuable insights into the reliability of the obtained results. The formula, is expressed as:

$$\int_0^1 f(x)dx \cong \sum_{k=0}^{n} \omega_k f\left(\frac{1+x_k}{2}\right), \tag{1}$$

where

$$x_k = \cos v_k, \ \omega_k = \frac{2\sin(v_k)}{n} \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{\sin\left((2j-1)v_k\right)}{2j-1}, \tag{2}$$

and

$$v_k = \frac{k\pi}{n}, \ k = 0, 1, \ldots, n. \tag{3}$$

Fibonacci polynomials, initially studied in 1883 by Eugene Charles Catalan and E. Jacobsthal, have been a subject of ongoing exploration. These polynomials, denoted as $F_n(x)$, satisfy the recurrence relation

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x),$$

with initials $F_0(x) = 1$ and $F_1(x) = x$. Notably, these polynomials gained further attention in 1966 through the work of Swamy [21] at the University of Saskatchewan in Canada.

Motivated by the prevalence of fractional differential equations in diverse physical phenomena, this paper introduces an algorithm for solving fractional-order integro-differential equations. The proposed method leverages the Fejér quadrature formula and the collocation spectral method. In this context, our paper introduces a novel algorithm for solving fractional-order integro-differential equations, leveraging the well-established Fejér quadrature formula and the collocation spectral method. The Fejér quadrature formula, introduced in 1933, offers a sophisticated numerical integration approach with efficiency, accuracy, and simplicity. Our algorithm builds on this foundation, incorporating Fibonacci polynomials and fractional calculus to address complex mathematical challenges. To establish the foundation for our results, Section 2 presents mathematical preliminaries, including key definitions in fractional calculus and relevant properties of Fibonacci polynomials. Section 3 outlines and implements the algorithm, Section 4 for the study of the convergence analysis, while Section 5 provides numerical examples to demonstrate its efficiency, simplicity, and applicability. The paper concludes in Section 6 with a summary of findings and potential avenues for future research.

2. Essential preliminaries

2.1 *Some definitions and properties of fractional calculus*

We present some notations, definitions and preliminary facts of the fractional calculus theory which will be useful throughout this article.

Definition 1. The Rieman-Liouville fractional integral operator I^{α} of order α on the usual Lebesgue space $L_1[0, 1]$ is defined as

$$I^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau & \alpha > 0, \\ f(t), & \alpha = 0. \end{cases} \tag{4}$$

The operator I^{α} has the following properties:

(i) $I^{\alpha} I^{\beta} = I^{\alpha+\beta}$,

(ii) $(I^{\alpha} I^{\beta} f)(t) = (I^{\beta} I^{\alpha} f)(t)$,

(iii) $I^{\alpha} (t-a)^{\nu} = \frac{\Gamma(\nu+1)}{\Gamma(\nu+\Gamma+1)} (t-a)^{\nu+\alpha}$,

where $f \in L_1[0, 1]$, $\alpha, \beta \geq 0$, and $\nu > -1$. Also, $\Gamma(\alpha)$ is the gamma function.

Definition 2. The Rieman-Liouville fractional derivative of order $\alpha > 0$ is defined by

$$(D^{\alpha} f)(t) = \left(\frac{d}{dt}\right)^n (I^{n-\alpha} f)(t),\ n-1 \leqslant \alpha < n,\ n \in \mathbb{N}, \tag{5}$$

where n is an integer. However, its derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, we will present the definition proposed by Caputo of the fractional differential operator D^{α}.

Definition 3. Let $f(t) \in C^n[0, \ell]$, $\ell > 0$. Then the Caputo definition of fractional differential operator is given by

$$(D_*^{\alpha} f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau,\ \alpha > 0,\ t > 0, \tag{6}$$

where $n - 1 \leq \alpha < n$, $n \in \mathbb{N}$.

The operator D_*^{α} satisfies the following properties: for $n - 1 \leq \alpha < n$,

$$(D_*^{\alpha} I^{\alpha} f)(t) = f(t),$$

$$(I^{\alpha} D_*^{\alpha} f)(t) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0^+)}{k!} t^k,\ t > 0,$$

$$D_*^{\alpha} t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\beta)} t^{\beta-\alpha}. \tag{7}$$

For more details on the mathematical properties of fractional derivatives and integrals, see for example, [22-24].

2.2 *Some properties of Fibonacci polynomials*

Fibonacci polynomials satisfying the recurrence relation [25, 26]

$$F_r(t) = t F_{r-1}(t) + F_{r-2}(t),\ r = 2,\ 3,\ \ldots,$$

starting with $F_0(t) = 0$, $F_1(t) = 1$.

Fibonacci polynomials have the analytic form

$$F_n(t) = \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \frac{(n-k-1)!}{k!(n-2k-1)!} t^{n-2k-1}. \tag{8}$$

Theorem 1. The Caputo fractional derivative of $F_n(t)$ is given by

$$D_*^{\alpha} F_n(t) = \sum_{k=0}^{\left\lfloor \frac{n-2\alpha-1}{2} \right\rfloor} \frac{(n-k-1)!}{k!\Gamma(n-2k-\alpha)} t^{n-2k-\alpha-1}. \tag{9}$$

Proof. Apply property (7) to (8), we get the result. □

3. Numerical solution of fractional integro-differential equation

In this section, using the collocation method with the aid of Fejér-quadrature formula, we solve the fractional integro-differential equations (FIDE) of the form:

$$F\left(t,\ u(t),\ D_*^{\sigma_1} u(t),\ \ldots,\ D_*^{\sigma_\ell} u(t)\right) = G\left(t,\ u(t),\ \int_0^t H\left(\tau,\ u(\tau)\right) d\tau\right), \tag{10}$$

subject to the initial conditions

$$u^{(i)}(0) = \xi_i,\ i = 0,\ 1,\ \ldots,\ \ell - 1, \tag{11}$$

where, $r - 1 < \sigma_r \le r$, $r = 1, 2, \ldots, \ell$. First we approximate $u(t)$ in terms of Fibonacci expansion as follows:

$$u(t) \approx \sum_{k=1}^{N+1} a_k F_k(t) = U(t), \tag{12}$$

substitution of Eq. (12) into Eq. (10), yields,

$$F\left(t,\ U(t),\ D_*^{\sigma_1} U(t),\ \ldots,\ D_*^{\sigma_\ell} U(t)\right) \approx G\left(t,\ U(t),\ \int_0^t H\left(\tau,\ U(\tau)\right) d\tau\right). \tag{13}$$

We apply the transformation $x \mapsto tx$ for Eq. (1), to get

$$\int_0^t f(x)dx \cong \frac{t}{2}\sum_{k=0}^{n} \omega_k f\left(\frac{t}{2}(1+x_k)\right), \tag{14}$$

where, x_k, ω_k are given in Eq. (2). Now we apply the quadrature formula (14), to the integral on the right hand side of Eq. (13), to get

$$F\left(t, U(t), D_*^{\sigma_1}U(t), \ldots, D_*^{\sigma_\ell}U(t)\right) \approx G\left(t, U(t), \frac{t}{2}\sum_{k=0}^{n} \omega_k H\left(\frac{t}{2}(1+x_k), U\left(\frac{t}{2}(1+x_k)\right)\right)\right), \tag{15}$$

finally, we collocate Eq. (15) at $z_j = \dfrac{1+\cos\left(\dfrac{2j+1}{2N+2}\pi\right)}{2}$, $j = 0, 1, \ldots, N - \ell$, the first distinct $N - \ell + 1$ roots of the shifted Chebyshev polynomials $T^*_{N+1}(x)$, to get,

$$F\left(z_j, U(z_j), D_*^{\sigma_1}U(z_j), \ldots, D_*^{\sigma_\ell}U(z_j)\right)$$

$$\approx G\left(z_j, U(z_j), \frac{z_j}{2}\sum_{k=0}^{n} \omega_k H\left(\frac{z_j}{2}(1+x_k), U\left(\frac{z_j}{2}(1+x_k)\right)\right)\right), j = 0, 1, \ldots, N-\ell, \tag{16}$$

moreover the use of the initial conditions

$$U^{(j)}(0) = \xi_j, j = 0, 1, \ldots, \ell-1, \tag{17}$$

Eqs (16)-(17), generates a system of N equations in the unknown expansion coefficients $\{a_k : k = 1, \ldots, N + 1\}$, which we solve using the well-known Newton's iterative scheme, ultimately we get $U(t)$.

Algorithm 1	Coding algorithm for the proposed technique
Input	n, N, ℓ, G, H and σ_r, $r = 1, 2, \ldots, \ell$.
Step 1.	Assume an approximate solution $U(t)$ as in (12).
Step 2.	Apply the a quadrature-collocation method to obtain the system in (16)-(17).
Step 3.	Use *FindRoot* command with initial guess $\{a_k = 10^{-k}, k : 1, 2, \ldots, N+1\}$, to to solve the system in (16)-(17) to get a_k.
Output	$U(t)$

4. Convergence analysis

We ascertain the convergence of the Fibonacci approximate solutions by reporting the following two theorems, providing a rigorous foundation for the reliability and accuracy of our proposed algorithm.

The first Theorem establishes the convergence of the Fibonnaci expansion, validating its effectiveness in numerically approximating the solution of differential/integral problems. The second Theorem delves into the convergence of the approximate spectral solution when the number of retained modes be large. Together, these theorems bolster the confidence in the convergence properties of our proposed algorithm, establishing a solid theoretical underpinning. As we navigate the complexities of fractional-order integro-differential equations, these theorems assure researchers and practitioners that the Fibonacci approximate solutions converge reliably and accurately to the true solutions as computational parameters are refined. This assurance is paramount for the algorithm's widespread applicability across diverse scientific and engineering domains.

Theorem 2. [27] If $\phi(z)$ is defined on $[0, \ell]$ and $|\phi^{(i)}(r)| \leq Q^i$, $i \geq 0$, where r is any point in $(0, \ell)$, Q is a positive constant, and if $\phi(z)$ has the expansion $\phi(z) = \sum_{k=0}^{\infty} c_k F_{k+1}(z)$, then one has:

$$|c_k| < \frac{\sigma Q^{k+1}}{k!},$$

where $\sigma = \frac{6\Omega}{Qr^2}\cosh(2Q)$ and $\Omega = Li_6\left(\frac{r^2}{3}\right)$, and, $Li_n(z)$ is the well-known polylogarithm function.

Theorem 3. [27] If $f(x)$ obeys the assumptions of Theorem 2, and if we let $E_N = \max_{x\in(0,1)} |e_N(x)|$, where, $e_N(x) = \sum_{k=N+1}^{\infty} c_k F_{k+1}(x)$, then the following estimation is satisfied by the global error

$$E_N < \tilde{C}\frac{\zeta^N}{(N-1)!},$$

where $\tilde{C} = \sigma Q e^{\zeta}$, $\zeta = Q\rho$ and $\rho = \sqrt{\ell^2 + 4}$.

5. Numerical results and comparisons

In this section we check the applicability of our proposed algorithm by exhibiting two numerical test problems.

Example 1. Consider the following linear FIDEs (see, [28]):

$$D_*^{\frac{1}{2}} u(t) = \left(\cos t - \sin t\right)u(t) + t\int_0^t \sin\tau\, u(\tau)d\tau + f(t),\ t \in [0, 1], \tag{18}$$

where

$$f(t) = \frac{8t^{3/2}}{3\sqrt{\pi}} + 2t + \frac{2\sqrt{t}}{\sqrt{\pi}} + t^3\cos t - t^2\sin t - 3t\cos t,$$

subject to

$$u(0) = 0.$$

The exact solution of (18) is $u(t) = t + t^2$. We apply the Fejér-quadrature collocation method (FQCM) presented in Section 3, to Eq. (18) for the case corresponding to $n = 6$, $N = 2$. In such case, we get

$$a_1 = -1,\ a_2 = 1,\ a_3 = 1,$$

and hence

$$U(t) = -F_1(t) + F_2(t) + F_3(t) = t + t^2,$$

which is the exact solution.

Example 2. Consider the following nonlinear FIDE (see, [28]):

$$D_*^{\frac{1}{2}} u(t) = f(t)u(t) + g(t) + \sqrt{t}\int_0^t (u(\tau))^2\, d\tau,\ t \in [0, 1], \tag{19}$$

where

$$f(t) = 2\sqrt{t} + 2t^{\frac{3}{2}} - \sqrt{t}(1+t)\ln(1+t),$$

$$g(t) = \frac{2\sinh^{-1}\left(\sqrt{t}\right)}{\sqrt{\pi}\sqrt{t+1}} - 2t^{3/2},$$

subject to

$$u(0) = 0.$$

The exact solution of Eq. (19) is $u(t) = \ln(1 + t)$. In Table 1, the maximum absolute error E is listed for various values of n and N, while in Table 2 we give a comparison between the best errors obtained by the method developed in [28] and FQCM. Figure 1 illustrates the absolute errors at different values of N.

Table 1. Maximum absolute error E for Example 2

N	$n = 6$			$n = 10$		
	4	10	14	6	8	10
E	2.65×10^{-6}	1.27×10^{-9}	2.58×10^{-10}	5.27×10^{-7}	6.54×10^{-10}	1.41×10^{-12}

Table 2. Best errors for Example 2

Method in [28]	FQCM at $N = 16$, $n = 10$	Our CPU time
9.0178×10^{-6}	1.4105×10^{-12}	25.328

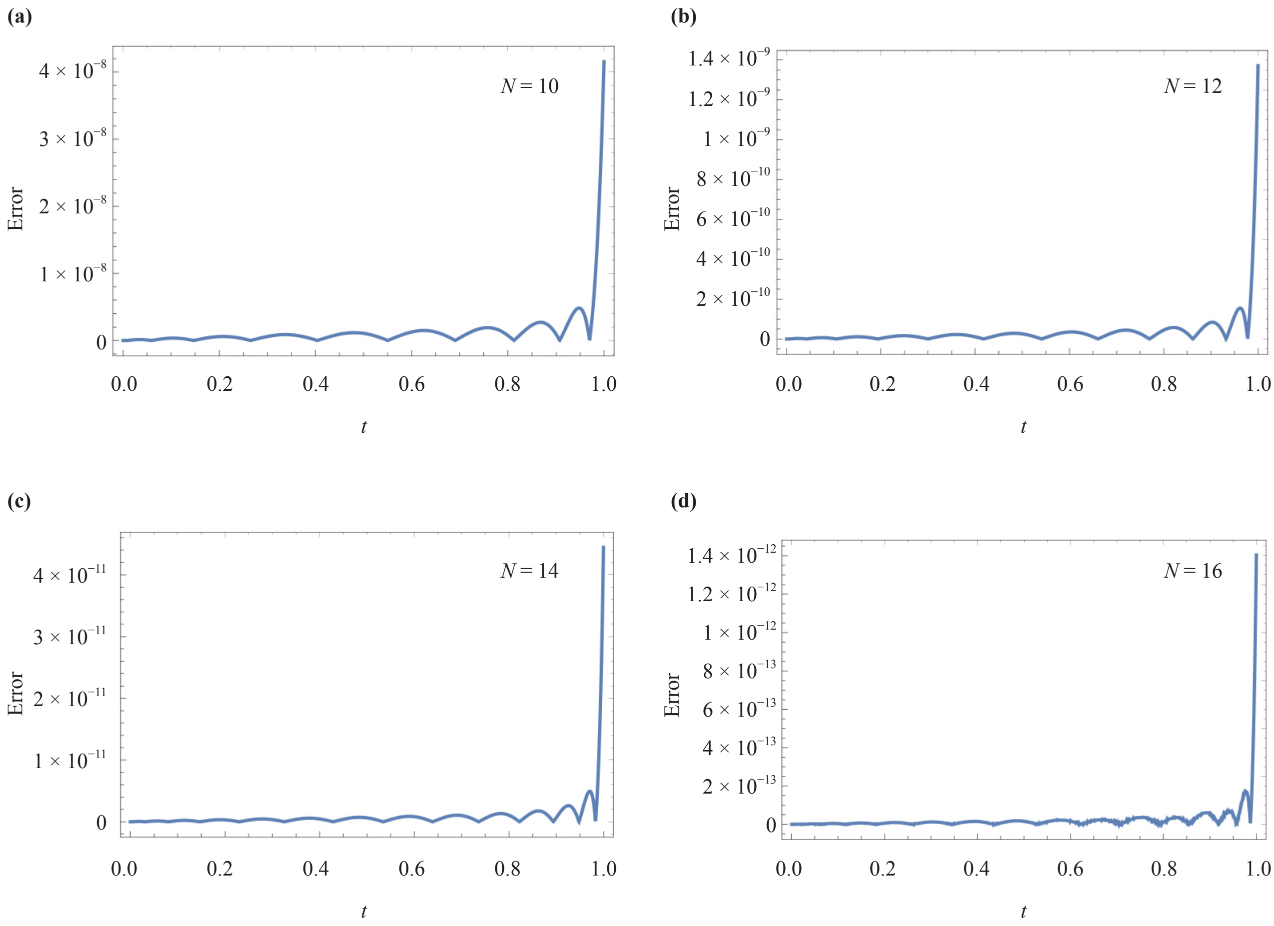

Figure 1. The absolute errors of Example 2

Example 3. Consider the following nonlinear FIDE (see, [29-30]):

$$D_{*}^{\alpha} u(t) = f(t) - \int_{0}^{t} \left(u(\tau)\right)^{2} d\tau, \ \alpha \in (1, 2], \ t \in [0, 1], \tag{20}$$

where

$$f(t) = -\frac{t}{2} + \sinh(t) + \frac{1}{4}\sinh(2t),$$

subject to

$$u(0) = 0, \ u'(0) = 1.$$

The exact solution of Eq. (19), for the case corresponds to $\alpha = 2$, is $u(t) = \sinh t$. In Table 3, the maximum absolute error E is listed for $n = 12$, various values of α and N, while in Table 4 we give a comparison between the best errors obtained by the methods developed in [29-30] and FQCM. Figure 2 illustrates the absolute errors at different values of N.

Table 3. Maximum absolute error E for Example 3

N	$\alpha = 1.25$			$\alpha = 1.5$			$\alpha = 1.75$			$\alpha = 2$		
	10	14	18	8	10	12	10	14	18	8	10	12
E	6.31×10^{-7}	2.64×10^{-11}	1.57×10^{-13}	6.38×10^{-10}	2.51×10^{-13}	2.22×10^{-16}	7.88×10^{-7}	3.91×10^{-11}	2.22×10^{-13}	3.07×10^{-10}	5.28×10^{-13}	6.17×10^{-16}

Table 4. Best errors for Example 3

Method in [29]	Method in [30]	FQCM at $N = 16$, $n = 6$	Our CPU time
1.94×10^{-8}	1.05×10^{-10}	6.17×10^{-16}	25.134

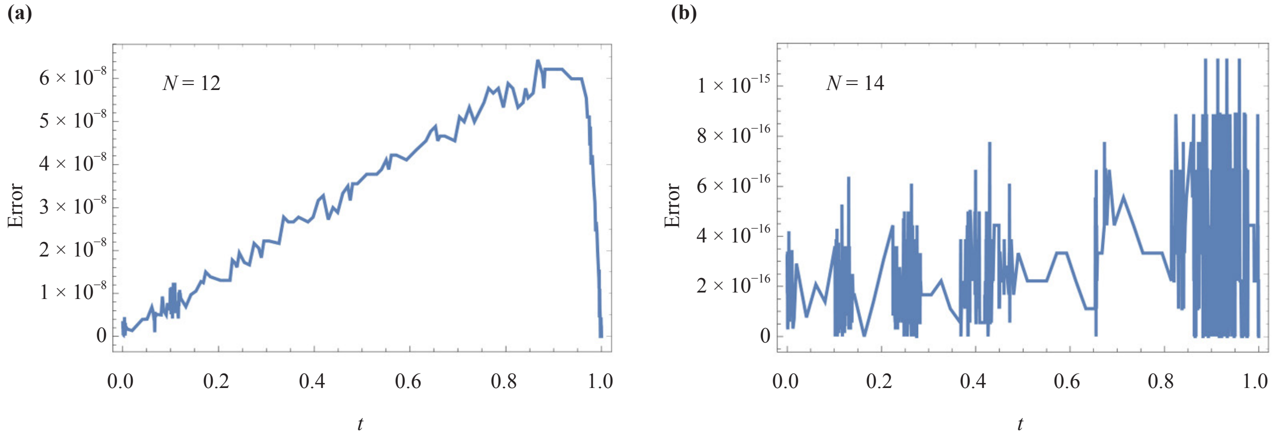

Figure 2. The absolute errors of Example 3

Example 4. Consider the following nonlinear FIDE:

$$D_*^{\frac{1}{2}} u(t) + u(t) = 2\int_0^t (t - 2\tau) u(\tau) d\tau,\ t \in [0, 1], \tag{21}$$

subject to

$$u(0) = 0.$$

Since the exact solution of (21) is not available, so let's define the following absolute residual error norm

$$RE = \max_{t\in(0,1)} \left| D_*^{\frac{1}{2}} U(t) + U(t) - 2\int_0^t (t-2\tau)U(\tau)d\tau \right|. \tag{22}$$

and applying our method at $N = 18$, $n = 7$ and $\alpha = 0.5$ to get Table 5, which illustrates the RE at different values of t.

Table 5. The *RE* of Example 4

t	0.1	0.3	0.5	0.7	0.9	CPU time
RE	1.5992×10^{-18}	3.03577×10^{-18}	1.13841×10^{-18}	3.41524×10^{-18}	3.17671×10^{-17}	19.75

6. Concluding remarks

Within the pages of this paper, we unveil a novel algorithm designed to procure numerical spectral solutions for FIDE. The genesis of this algorithm lies in a meticulous combination of the Fejér-quadrature method and the spectral collocation method, providing a robust and versatile framework for tackling the challenges posed by FIDE. One noteworthy feature of the presented algorithm is its applicability to a broad spectrum of FIDE, encompassing both linear and nonlinear instances. This adaptability renders the algorithm a valuable tool for researchers and practitioners confronting a diverse array of mathematical problems.

An inherent strength of the developed algorithm lies in its ability to yield highly accurate approximate solutions while requiring only a modest number of retained modes within the Fibonacci expansion. This efficiency is particularly advantageous, as it not only expedites the computational process but also mitigates the computational burden associated with solving complex FIDE. The algorithm's capacity to achieve substantial accuracy with a limited number of modes underscores its efficacy in delivering precise numerical solutions, offering a practical advantage in scenarios where computational resources may be constrained.

Moreover, the algorithm's versatility positions it as a versatile tool across various scientific and engineering domains, enabling researchers to explore and understand the dynamics of complex systems described by FIDE. The incorporation of the Fejér-quadrature method ensures a robust numerical foundation, while the spectral collocation method enhances the algorithm's precision in capturing intricate details of the underlying mathematical models.

In essence, this algorithm contributes to the evolving landscape of numerical methods for FIDE, providing a potent and accessible tool for researchers seeking efficient and accurate solutions to a broad class of problems. As we delve into the details of the algorithm and its applications, the potential for advancing our understanding of complex fractional systems becomes evident, opening avenues for further exploration and refinement in the realm of numerical analysis and computational mathematics. All codes were written and debugged by Mathematica 11 on HP Z420 Workstation, Processor: Intel (R) Xeon(R) CPU E5-1620-3.6 GHz, 16 GB Ram DDR3, and 512 GB storage.

Conflict of interest

The authors declare no competing financial interest.

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