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Introduction to Homological Algebra for Quantaloids: Semisimple Modules over a Quantaloid

Arij Benkhadra[1*], **Dirar Benkhadra**[2]

[1] Modal'X Laboratory, Paris Nanterre University, 200 BC. Republic, 92100 Nanterre, France
[2] Department of Mathematics, Mohammed University 5, PO Box 1014, Agdal PO Box 554 Rabat Chellah, 3 Michlifen Street, Rabat, Morocco
E-mail: dirar.benkhadra@hotmail.fr

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Abstract: This paper is a step towards a homological classification of quantales and quantaloids by characterizing semisimple quantales and quantaloids via their modules. We prove that semisimple quantales and quantaloids are quite similar to semisimple rings via their homological dimensions. More precisely, we study simple and semisimple modules and we connect them to the notions of artianity and noetherianity. We proved that simple modules are cyclic, establishing a connection with maximal congruences. Furthermore, we demonstrated that the structure of semisimple quantaloid modules closely resembles that of semisimple modules over a ring. Specifically, the ascending and descending chain conditions coincide, along with the occurrence of split short exact sequences and other related properties.

***Keywords*:** semisimple modules, modules over quantales, simple modules

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1. Introduction

In homological algebra, characterizations of rings through appropriate categories of modules holds considerable significant [1–3]. This idea has led to similar results in both non-commutative and non-additive contexts over the past few decades: for example the homological classification of monoids [4], or the homological classification of distributive lattices [5].

In the last few decades, there has been an increasing interest in the theory of modules over quantales and quantaloids, because of its connections with various fields such as mathematics, computer science, quantum physics among other other scientifis domains [6–12]. As algebraic objects, quantales are, by definition, monoids in the monoidal category $\mathsf{S}l$ of sup-lattices, in just the same way that rings are monoids in the monoidal category $\mathsf{A}b$ of abelian groups. Quantaloids are the "many object" generalization of quantales (precisely like groupoids generalize groups), that is, quantaloids are exactly categories enriched in $\mathsf{S}l$. Modules over quantales or quantaloids can therefore be defined in much the same way as one does for rings.

It is thus natural, When studying quantaloids and their modules, to leverage the similarities in methodologies and techniques drawn from ring theory. In the theory of modules over rings, homological dimension emerges as a pivotal concept in providing a measure of a module's projectivity or injectivity within a category of modules. Central

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to the definition of homological dimension are projective and injective resolutions, which play a fundamental role in characterizing the length of the longest projective resolution, such as the global dimension of a ring. Moreover, homological dimensions facilitate comparisons across various classes of modules. A comparison of the projective dimension of modules over quantaloids, for instance, can unveil similarities or distinctions in their algebraic properties. These comparative approaches can contribute to the advancement of representation theory in the context of quantaloids through the exploration of module homology.

Another example that fuels our motivation is the ready simulation quantale [13]. In the domain of logic semantics, this quantale captures the behavior of one system in relation to another. Its application lies in defining a concept of behavioral equivalence or refinement between concurrent systems. This notion is frequently utilized in the investigation of formal methods and process algebras, particularly within the framework of labeled transition systems. The underlying principle is to establish a correspondence between states in different systems, accounting for their potential transitions. The quantale of ready simulation, denoted as $\mathscr{Q}_{RS}$, was thoroughly examined in [13]. The study demonstrated a striking similarity between this quantale and the algebraic structure of the ring of polynomials. Moreover, they share an analogous homological characterization: $\mathscr{Q}_{RS}$ is not semisimple but is coherent, mirroring the characteristics of the ring of polynomials. This aligns with our perspective, where non-semisimple quantaloids exhibit a higher dimension, with semisimple quantaloids corresponding to zero projective/injective dimensions and coherent ones having projective/injective dimensions equal to 1.

In this paper, our interest goes specifically to the homological characterization of semisimple quantaloids. This provides a promising context for advancing homological algebra. While the results are intriguing on their own, they might also offer a fresh perspective on the "classical" homological algebra applied to rings.

For the reader's ease, Section 2 covers the preliminaries on quantales, quantaloids, and their modules. Moving on to Section 3, we introduce the notion of simplicity for quantaloid-modules, and give some algebraic and homological characterizations thereof. These results will help us in establishing our main results, presented in Section 4, on the semisimplicity of quantaloid-modules. In Section 5 we make several relations between semisimple quantaloids and other special quantaloids such as Noetherian, Artinian quantaloids.

2. Preliminaries

For general facts on categories enriched in a symmetric monoidal closed category $\mathscr{V}$ (such as $\mathscr{V} = \mathsf{A}b$ or $\mathscr{V} = \mathsf{S}l$), we refer to [14, 15] (or [16] for a short introduction); specifically for quantaloids and their modules we refer to [7], but see also [8, 12, 17–21].

2.1 *Quantaloids, modules*

We denote by $\mathsf{S}l$ the symmetric monoidal closed category of complete lattices and supremum-preserving maps ("sup-lattices and sup-morphisms"), the monoidal structure on $\mathsf{S}l$ being given by the usual tensor product of complete lattices. A quantaloid is defined as an $\mathsf{S}l$-enriched category. More specificaly, a quantaloid is a category $\mathscr{Q}$ in which, for any given pair of objects A and B, the hom-set $\mathscr{Q}(A, B)$ is a sup-lattice, and the composition of morphisms in $\mathscr{Q}(A, B)$ is distributive over suprema:

$$f(\bigvee g_i) = \bigvee_i fg_i \quad \text{and} \quad (\bigvee f_i)g = \bigvee_i f_i g.$$

A quantale is a quantaloid with a single object, i.e. it is a monoid in $\mathsf{S}l$. In simpler terms, a quantale $Q = (Q, \bigvee, \cdot, 1)$ is a sup-lattice $(Q, \bigvee)$ equipped with a monoid structure $(Q, \cdot, 1)$ satisfying the distributivity law mentioned above.

By the same token, a morphism of quantaloids is an $\mathsf{S}l$-enriched functor $F\colon \mathscr{Q} \to \mathscr{R}$; explicitly, it is a functor that preserves suprema of morphisms:

$$F(\bigvee_i f_i) = \bigvee_i F f_i.$$

When considering quantales (i.e. one-object quantaloids), this corresponds with the expected notion of (homo)morphism, namely a sup-preserving map that preserves the monoid structure. Finally, given two parallel quantaloid morphisms $F\colon \mathscr{Q} \to \mathscr{R}$ and $G\colon \mathscr{Q} \to \mathscr{R}$, natural transformations $\alpha\colon F \to G$ are defined as for ordinary categories (the $\mathsf{S}l$-enrichment does not play any role here).

Throughout the remainder, we denote by$\mathscr{Q}$, a *small* quantaloid, i.e. a quantaloid whose collection $\mathscr{Q}_0$ of objects forms a set. Let us remark that $\mathsf{S}l$ itself is a quantaloid, but a *large* one (its objects form a proper class).

A (left, covariant) $\mathscr{Q}$-module M is a morphism of quantaloids $M\colon \mathscr{Q} \to \mathsf{S}l$; so in particular is every object A of $\mathscr{Q}$ mapped to a complete lattice MA, and every morphism $f\colon A \to B$ in $\mathscr{Q}$ is mapped to a sup-morphism $Mf\colon MA \to MB$. These modules are the objects of the category $\mathscr{Q}$-$\mathsf{M}od$, which has natural transformations as morphisms. Explicitly, a morphism $\alpha\colon M \to N$ in $\mathscr{Q}$-$\mathsf{M}od$ is a family

$$\alpha = \{\alpha_A \in \mathsf{S}l(MA, NA) \mid A \in \mathscr{Q}\}$$

of morphisms in $\mathsf{S}l$ such that, for every $f \in \mathscr{Q}(A, B)$, the following diagram in $\mathsf{S}l$ commutes:

$$\begin{array}{ccc} MA & \xrightarrow{\alpha_A} & NA \\ {\scriptstyle Mf}\downarrow & & \downarrow{\scriptstyle Nf} \\ MB & \xrightarrow{\alpha_B} & NB \end{array}$$

The pointwise supremum of $\mathscr{Q}$-morphisms gives rise to a new $\mathscr{Q}$-morphism; moreover the composition of $\mathscr{Q}$-morphisms is distributive over these suprema on both sides: consequently, $\mathscr{Q}$-$\mathsf{M}od$ is a (large) quantaloid.

As per usual, any object $A \in \mathscr{Q}$ represents a $\mathscr{Q}$-module, which we shall write as $h_A\colon \mathscr{Q} \to \mathsf{S}l$, acting thusly:

$$\Big(f\colon X \to Y\Big) \mapsto \Big(f \circ -\colon \mathscr{Q}(A, X) \to \mathscr{Q}(A, Y)\Big).$$

Similarly, any morphism $g\colon A \to B$ in $\mathscr{Q}$ represents a $\mathscr{Q}$-module morphism $h_g\colon h_B \to h_A$ (note the contravariancy), whose components are given by pre-composition with g. The ($\mathsf{S}l$-enriched) Yoneda Lemma establishes, for any $\mathscr{Q}$-module M and any object $A \in \mathscr{Q}$, that

$$\mathscr{Q}\text{-}\mathsf{M}od(h_A, M) \to MA\colon \alpha \mapsto \alpha_A(1_A)$$

is an isomorphism in $\mathsf{S}l$. As a consequence, this makes for a fully faithful ($\mathsf{S}l$-enriched) Yoneda embedding $\mathscr{Q}^{op} \to \mathscr{Q}$-$\mathsf{M}od$.

For any quantaloid, the opposite category (obtained by reversing the morphisms, but not the lattice order in the homs) is also a quantaloid. In particular, a *right* (*contravariant*) module on $\mathscr{Q}$ is, by definition, a (left) module on $\mathscr{Q}^{op}$. In fact, the module category $\mathscr{Q}^{op}$-$\mathsf{M}od$ is isomorphic to $(\mathscr{Q}\text{-}\mathsf{M}od)^{op}$; this follows easily from the self-duality $\mathsf{S}l \cong \mathsf{S}l^{op}$ (where a sup-morphism $f\colon A \to B$ is identified with the order-opposite of its right (Galois) adjoint inf-morphism $f^*\colon B \to A$).

Hence, any valid statement in a category of $\mathcal{Q}$-modules has a a corresponding dual statement. In this paper we shall work with *left* Q-modules (calling them just $\mathcal{Q}$-modules), but it is clear that all results remain true for right modules.

In every quantaloid, categorical products and sums coincide, often referred to as direct sums; this is thus in particular the case in $\mathcal{Q}$-$\mathsf{M}od$. Epimorphisms in $\mathcal{Q}$-$\mathsf{M}od$ are precisely the module morphisms that have surjective components; analogously, monomorphisms are exactly the module morphisms with injective components. Further, $\mathcal{Q}$-$\mathsf{M}od$ is a Barr-exact category. Since we have considered $\mathcal{Q}$ to be a small quantaloid, it follows from the Yoneda embedding that the representable $\mathcal{Q}$-modules constitute a small separating set of objects in $\mathcal{Q}$-$\mathsf{M}od$ and that $\mathcal{Q}$-$\mathsf{M}od$ is wellpowered and co-wellpowered. Thus, strongly complete and strongly cocomplete. All the categorical properties above underline the similarity between the categories of quantaloid-modules and those of ring-modules.

It follows from general principles in enriched category theory, but can easily be verified *ad hoc* too, that if we write **2** for the Boolean algebra $\{0, A\}$, which we view as a quantale with conjunction as multiplication, then **2**-$\mathsf{M}od$ is *precisely* the category $\mathsf{S}l$. Therefore, general $\mathcal{Q}$-module theory is a ("many-typed, many-valued") generalization of the theory of complete lattices and sup-morphisms. This perspective will be already be useful in the following few paragraphs.

2.2 *Quotients and cyclicity*

Quotients of $\mathcal{Q}$-modules, i.e. epimorphisms in $\mathcal{Q}$-$\mathsf{M}od$, can conveniently be described by means of suitable closure operators.

First, a closure operator on a poset P is a monotone map $\Gamma\colon P \to P$ that is expansive and idempotent; it thus co-restricts to a residuated surjection $\Gamma\colon P \twoheadrightarrow P_\Gamma$ onto the poset of its fixpoints $P_\Gamma = \{x \in P \mid \Gamma x = x\}$, its right (Galois) adjoint being simply the inclusion $P_\Gamma \hookrightarrow P$. This correspondence between closure operators and residuated surjections is (essentially) bijective: any residuated surjection $\rho\colon P \twoheadrightarrow P'$ (with right (Galois) adjoint ρ^*) defines a closure operator $\Gamma = \rho^* \circ \rho$, which comes with a unique isomorphism between its poset of fixpoints P_Γ and P' that commutes with the surjections $P \to P_\Gamma$ and $P \twoheadrightarrow P'$. Specifically, if P is a complete lattice then P_Γ is also a complete lattice and $\Gamma\colon P \twoheadrightarrow P_\Gamma$ is a surjective sup-morphism (but note that the supremum in P_Γ is given by the closure of the supremum in P): so quotients in $\mathsf{S}l$ arise precisely from closure operators.

Now, let M be a $\mathcal{Q}$-module, we define a closure operator Γ on M is to be a family of closure operators on complete lattices

$$\Gamma = \{\Gamma_A\colon MA \to MA \mid A \in \mathcal{Q}\}$$

such that, for every $f \in \mathcal{Q}(A, B)$,

$$Mf \circ \Gamma_A \leq \Gamma_B \circ Mf.$$

This so-called "structural inequality" is precisely required to allow for the construction of the fixpoint module $M_\Gamma\colon \mathcal{Q} \to \mathsf{S}l$, mapping a morphism $f\colon A \to B$ in $\mathcal{Q}$ to the composite

$$(MA)_{\Gamma_A} \hookrightarrow MA \xrightarrow{Mf} MB \twoheadrightarrow (MB)_{\Gamma_B}.$$

Moreover, the co-restrictions $\Gamma_A\colon MA \twoheadrightarrow M_\Gamma A$ are the components of an epimorphism $\Gamma\colon M \twoheadrightarrow M_\Gamma$ in $\mathcal{Q}$-$\mathsf{M}od$. Conversely, if $\rho\colon M \twoheadrightarrow N$ is an epimorphic $\mathcal{Q}$-module morphism, then each $\rho_A\colon MA \twoheadrightarrow NA$ is a surjective sup-morphism, so has an injective right (Galois) adjoint inf-morphism $\rho^*\colon NA \to MA$, defining a closure $\Gamma\colon = \rho^* \circ \rho$ on M; and there is a unique isomorphism $N \cong M_\Gamma$ commuting with the epimorphisms $M \twoheadrightarrow M_\Gamma$ and $M \twoheadrightarrow N$.

So, generalizing the case of complete lattices, quotients of a $\mathscr{Q}$-module M are essentially the same thing as closure operators on M.

By the Yoneda Lemma, given a $\mathscr{Q}$-module M, any element $a \in MA$ uniquely corresponds with a $\mathscr{Q}$-module morphism $\mu_a\colon h_A \to M$ whose components are

$$(\mu_a)_B\colon \mathscr{Q}(A, B) \to MB\colon f \mapsto Mf(a).$$

We say that M is generated by $a \in MA$ if $\mu_a\colon h_A \to M$ is epimorphic; in that case, we write $\langle A, a\rangle$ for that generator of M. This thus precisely means that, for any $b \in MB$, there exists a (not necessarily unique) morphism $f_b\colon A \to B$ in $\mathscr{Q}$ such that $M(f_b)(a) = b$. A $\mathscr{Q}$-module M is cyclic if it has a generator; in other words, M is cyclic if and only if M is a quotient of a representable module, if and only if it is the fixpoint module of a closure operator on a representable module.

2.3 *Submodules and ideals*

Let M and N be $\mathscr{Q}$-modules. We say that N is a *$\mathscr{Q}$-submodule* of M if there is a module morphism $N \to M$ whose components are (set-wise) inclusions; and we write $N \preceq M$. As submodules of M are ordered ("by inclusion"), we can express the ascending (descending) chain condition on submodules of M, and thus speak of a noetherian (artinian) $\mathscr{Q}$-module in the obvious way.

Examples of submodules are the *image* and the *kernel* of a $\mathscr{Q}$-morphism $\alpha\colon N \to M$: indeed $Im(\alpha) \preceq M$ is determined by $Im(\alpha)(A)\colon = \{\alpha_A(x) \mid x \in NA\}$, whereas $Ker(\alpha) \preceq N$ is determined by $Ker(\alpha)(A)\colon = \{x \in NA \mid \alpha_A(x) = \perp_{MA}\}$. In particular, any monomorphism $\alpha\colon N \rightarrowtail M$ in $\mathscr{Q}$-$\mathsf{M}od$ determines an isomorphism of N with $Im(\alpha)$, so we can think of N as a submodule of M (up to this isomorphism).

Furthermore, the zero object in the category $\mathsf{S}l$ being the sup-lattice with just one element, it follows easily that the zero $\mathscr{Q}$-module (i.e. the zero object in $\mathsf{M}od$-$\mathscr{Q}$), which we shall write as $\mathbf{1}\colon \mathscr{Q} \to \mathsf{S}l$, is defined by mapping every object $A \in \mathscr{Q}$ to the zero object of $\mathsf{S}l$. We shall say that a $\mathscr{Q}$-submodule $E \preceq M$ is *essential*, denoted by $E \preceq_e M$, if for any $\mathscr{Q}$-submodule $N \preceq M$ we have that $E \cap N = \mathbf{1}$ implies $N = \mathbf{1}$. To the contrary, a $\mathscr{Q}$-submodule $S \preceq M$ is *superfluous*, written $S \preceq_s M$, if for any $N \preceq M$ we have that $S + N = M$ implies $N = M$. The *radical* $Rad(M) \preceq M$ is the sum of all superfluous submodules of M; and the *socle* $Soc(M) \preceq M$ is the intersection of all the essential submodules of M.

For our study of simple and semimple modules, we will also need a stronger kind of substructure of a $\mathscr{Q}$-module M. Extending the definition given for quantale-modules by Paseka and Kruml in [9], we say that a submodule $I \preceq M$ is a *$\mathscr{Q}$-ideal* if, for any $A \in \mathscr{Q}$, if $m \in IA$ and $n \in MA$ such that $n \leq m$, then $n \in IA$; so each IA is a down-closed subset of MA.

In fact, the kernel of any $\mathscr{Q}$-module morphism $\alpha\colon M \to N$ is a $\mathscr{Q}$-ideal in M (but its image need not be $\mathscr{Q}$-ideal of N). We will use the term 'weak-ideal of $\mathscr{Q}$' for the $\mathscr{Q}$-submodules of the representable $\mathscr{Q}$-modules, and use the term 'ideal of $\mathscr{Q}$' for their $\mathscr{Q}$-ideals.

Finally, given a $\mathscr{Q}$-module M, a *congruence* $\equiv$ on M is defined to be a $\mathscr{Q}$-submodule of the product $M \times M$ which is also an equivalence relation: explicitly, for each $A \in \mathscr{Q}$ there is a sup-lattice congruence $\equiv^A$ on MA, and for each $f\colon A \to B$ in $\mathscr{Q}$ the sup-morphism $Mf\colon MA \to MB$ maps $\equiv^A$-equivalent elements of MA to $\equiv^B$-equivalent elements of MB. Therefore, any congruence $\equiv$ on M determines a quotient module $M \twoheadrightarrow M/\!\equiv$, whereby $M/\!\equiv$ sends $A \in \mathscr{Q}$ to the quotient sup-lattice $MA/\equiv^A$. In particular, any submodule $N \preceq M$ induces a congruence $\equiv_N$ on M by setting, for each $A \in \mathscr{Q}$,

$$m \equiv_N^A m' \text{ in } MA \quad \stackrel{\text{def.}}{\iff} \quad \exists n, n' \in NA\colon m \vee n = m' \vee n'.$$

We usually write M/N for the *quotient* $\mathscr{Q}$-module $M/\equiv_N$. (When no confusion can arise, we shall drop the super- and subscripts for better readability.)

3. On simple $\mathcal{Q}$-modules

In ring theory, a non-zero module is said to be simple if it has no non-zero proper submodules; this definition can easily be transposed to the quantaloidal case. In [9], a quantale is said to be simple if it is non-zero and only have the trivial congruences (i.e. the diagonal and the total congruence); this definition too can easily be generalized to quantaloid-modules. However, for quantaloid-modules, these two properties do not necessarily coincide (as we shall see in the examples below).

3.1 *Definitions and basic properties*

Remark that in the category of modules over a quantaloid, constant homomorphisms are exactly the zero morphism. In fact, let $\alpha\colon M \to N$ be a constant homomorpism of $\mathcal{Q}$-modules, then $\alpha(m) = \alpha(0_M) = 0_N$, for all $m \in M$. Conversly, a zero homomorphism is trivially constant. Indeed, we can assert the following result:

Proposition 1 For a $\mathcal{Q}$-module M, the following assertions are equivalent:

(1) M has only two congruences (the diagonal and the total congruence);

(2) every surjective $\mathcal{Q}$-module-morphism from M is either an isomorphism or constant.

Proof. $(2) \to (1)$ Let $\equiv$ be a congruence on M. Then, we have the following quotient morphism $\pi\colon M \twoheadrightarrow M/_{\equiv}$ which is either an isomorphism or constant, by assumption. It is easy to see that the congruence $\equiv$ is either diagonal or total.

$(1) \to (2)$ Let $\alpha\colon M \twoheadrightarrow H$ be a quotient morphism. Define the following congruence, for $A \in \mathcal{Q}$, $m, n \in MA$ $m \equiv_\alpha n$ if and only if $\alpha_A(m) = \alpha_A(n)$. It is clear that $\equiv_\alpha$ is a congruence on M, hence it is either total or diagonal. Totality of $\equiv_\alpha$ leads to α being constant. Diagonality of $\equiv_\alpha$ leads to α being injective, which in addition to its surjectivity gives the desired result. □

Hence, we have two kinds of simplicity in $\mathcal{Q}$-Mod:

Definition 2 A non-trivial $\mathcal{Q}$-module M is weakly-simple if it has only two congruences (the diagonal and the total congruence).

A non-trivial $\mathcal{Q}$-module M is simple if it has no proper $\mathcal{Q}$-submodules, or equivalenty, if the only closure operators on M are the identity and the zero closure operator.

Example 3 • The quantale $\mathbf{2}{:} = \{0, 1\}$ is weakly-simple and also simple.

• $End(\mathcal{L})$, for an integral linear sup-lattice $\mathcal{L}$, is weakly-simple but not simple. For similar argument, see [22].

• Lawvere's quantale $(\mathcal{Q} = [0, \infty], +, \vee, 0)$ is neither simple nor weakly simple. It is easy to see that the intervals $[a, \infty]$, for $a \in \mathcal{Q}$, are indeed submodules and $\mathcal{Q}$-ideals of $\mathcal{Q}$.

• If we consider the quantale of Lawvere $(\mathcal{Q} = [0, \infty], +, \vee, 0)$, the $\mathcal{Q}$-module of integers $(\mathbb{Z}, +, \leq)$ is simple under the action $r.z = E[r].z$ with $r \in \mathcal{Q}$ and $z \in \mathbb{Z}$.

• Consider the quantaloid of the powerset quantaloid over a set X, i.e;, for a set X consider the quantaloid $(\mathscr{P}(X), \subseteq)$. Consider the module of natural numbers over this quantaloid with:

For $A \in \mathscr{P}(X)$, $n \in \mathbb{N}$, $A.n = n$ if $n \in A$, and $A.n = 0$ otherwise.

An easy computation using the properties of congruences will show that this module is both simple and weakly simple.

The following result provides a characterization of weakly-simple Q-modules through their ideals:

Lemma 4 M is weakly-simple then it has only two $\mathcal{Q}$-ideals (zero and M itself).

Proof. Let N be a non-zero $\mathcal{Q}$-ideal of M. And consider the congruence $\equiv_N$ on M associated to N, then $\equiv_N$ is either the diagonal congruence or the total one. Remark that $\equiv_N$ can not be diagonal. Take $A \in \mathcal{Q}_0$, and let $n \in NA$, then $n \equiv_N \perp$ since $n = \perp \vee n$. Hence, $\equiv_N$ is the total congruence on M, i.e., for all $A \in \mathcal{Q}_0$, and all $m \in MA$, $m \equiv_N \perp$ and so there exist $n \in NA$ such that $m \vee n \in NA$, since N is a $\mathcal{Q}$-ideal, we get that $m \in NA$ and so $M = N$. □

Remark 1 For $\mathcal{Q}$-modules, congruences and $\mathcal{Q}$-ideals are strongly related. Indeed, for any congruence, the class zero also called the kernel of the congruence is a $\mathcal{Q}$-ideal, conversly for any $\mathcal{Q}$-ideal I we have a universal congruence whose kernel is the ideal I. Hence, having only two $\mathcal{Q}$-ideals ($\mathbf{1}$ and I)can not imply having only two congruences. Still, we can assure that the only congruences we may have are those whose kernel is either $\mathbf{1}$ or I.

In the following, we will characterize homomorphisms on simple modules.

Lemma 5 • A module $M\colon \mathscr{Q} \to \mathsf{S}l$ is simple if and only if every morphism to M is either constant or surjective.

• If $M\colon \mathscr{Q} \to \mathsf{S}l$ is simple then every morphism from M is either constant or injective. Consequently, simple implies weakly simple

Proof. • Let $M\colon \mathscr{Q} \to \mathsf{S}l$ be a simple module and $\alpha\colon N \to M$ is a module homomorphism. Then, $Im(\alpha)$ as a submodule, of a simple module, is either zero or M itself; which leads to α being either constant or surjective.

Conversely, let $N\colon \mathscr{Q} \to \mathsf{S}l$ be a submodule of M and consider the inclusion homomorphism $i\colon N \to M$. By assumption, i is either constant or surjective. Hence, N is either zero of M; which proves the simplicity of M.

• The second statement is followed immediately by examining the correspondence between submodules and congruences. □

In ring theory, we characterize simple modules via their quotient modules. In the following, we will give a characterization of simple modules over a quantaloid via its quotient by a closure operator, which will lead us to the statement that every simple module is cyclic.

Proposition 6 A $\mathscr{Q}$-module $M\colon \mathscr{Q} \to \mathsf{S}l$ is simple if and only if $M \cong (h_A)_\Gamma$ for some $A \in \mathscr{Q}$ and a maximal closure operator Γ.

Proof. Suppose that $M\colon \mathscr{Q} \to \mathsf{S}l$ is simple. Then, there exist $A \in \mathscr{Q}$ such that MA is not trivial. Consider $h_A\colon = \mathscr{Q}(A, -)$ the $\mathscr{Q}$-module representable by $A \in \mathscr{Q}$. Since, M is simple then $\beta\colon h_A \to M$ is surjective. And so, there exist a closure operator Γ on h_A such that $M \cong (h_A)_\Gamma$.
Now, suppose that there exist another operator ω on h_A such that $\Gamma \leq \omega < id$, hence $h_A \preceq (h_A)_\omega \preceq (h_A)_\Gamma$, where $(h_A)_\omega$ is a submodule of $M \cong (h_A)_\Gamma$, which is simple and so $(h_A)_\omega \cong (h_A)_\gamma$ which implies that $\gamma = \omega$. And so, γ is maximal.

Conversely, suppose that $M \cong (h_A)_\gamma$ for some maximal closure operator γ and let N be a non zero submodule of M, hence there exist β a closure operator on h_A such that $N = (h_A)_\beta$ with $\gamma \leq \beta < id$. Then $\gamma = \beta$ and so M is simple. □

And, so it is immediate that:

Corollary 7 Every simple $\mathscr{Q}$-module is cyclic.

Another very known result for ring modules (see Corollary 2.17, [3]) holds again for simple $\mathscr{Q}$-modules:

Lemma 8 If U is a maximal $\mathscr{Q}$-ideal of L, then $L/U\colon = L/\cong_U$ is weakly-simple.

Proof. Let U be a maximal $\mathscr{Q}$-ideal of L. Suppose their exist a submodule N of L such that $L/N \preceq L/U$; then by definition $\cong_U \leq \cong_N$ which implies that $\cong_N$ is either equal to $\cong_U$ or equal to the total congruence. Consequently, $L/N = L/U$ or $L/N = L(= L/frm[o])$. Hence, L/U is simple which implies weakly-simple. □

As a consequence, we get the following property of simple modules:

Corollary 9 If every submodule is a direct summand of a $\mathscr{Q}$-moule M then every non-zero submodule of M contains a weakly simple module.

Proof. Let N be a non-zero submodule of M, hence there exist $A \in \mathscr{Q}$ and $n \in NA$ such that $n \neq \perp$. The finitely generated submodule L generated by $\langle A, n\rangle$ contains a maximal submodule U. Hence, by assumption, there exists $M_1 \preceq M$ such that $M = U \oplus M_1$.

L being a $\mathscr{Q}$-ideal [All direct summands of M are Q-ideals of M.], it is easy to conclude that $L = (M_1 \cap L) \oplus U$. And so $L/U \cong M_1 \cap L \subseteq M$ which is weakly simple. □

In the following, we will characterize weakly simple modules via its universe and then via the annihilator of its elements. For this purpose, we will define the annihilator congruence:

$$m \equiv_{Ann} n \text{ if and only if } Ann(m) = Ann(n)$$

where $Ann(m) = \{f \in \mathscr{Q}(A, B) | M(f)(m) = \perp\}$ for $m \in MA$.

Lemma 10 $\equiv_{Ann}$ is a congruence on any module $M\colon \mathscr{Q} \to \mathsf{S}l$. If M is weakly simple then it is the diagonal congruence.

Proof. The proof of the previous lemma is very straightforwarrd from the properties of the module M and the bottom element of each lattice MA for $A \in \mathscr{Q}$. □

Using this lemma, we get two characterizations of simple $\mathscr{Q}$-modules. First, simple $\mathscr{Q}$-modules are either isomorphic to some simple right ideal of $\mathscr{Q}$ or the only possible morphism between them is the zero morphism.

3.2 *Structure properties of simple $\mathscr{Q}$-modules*

Lemma 11 Let $\mathscr{Q}$ be an integral quantaloid, M be a simple $\mathscr{Q}$-module and N a weakly-simple cyclic $\mathscr{Q}$-ideal of $\mathscr{Q}$, we have either $M \cong N$ or $Hom_Q(M, N) = 0$.

Proof. Assume there exists a non-zero morphism $\alpha \colon M \to N$. By Lemma 5, α is injective. It remains to prove that it is surjective. Indeed, let $\langle A, m\rangle$ be a generator of M (since it is cyclic).

By weak-simplicity of N, the congruence $\equiv_{\alpha(M)}$ on N is the total congruence. And so, all the element of N are in congruence via $\equiv_{\alpha(M)}$.

Let $\langle B, a\rangle$ be a generator of N. Using the assertion [for more details see definition 10.1 in [7]]: for every element x in NE, for some $E \in \mathscr{Q}$, $x = N(\gamma)(a)$ where $\gamma \colon B \to E$, a simple computation leads to: $a \equiv_{\alpha(M)} \perp$ in NB. Hence, there exist n and n' in $\alpha_B(M)(= \alpha(MB) \subseteq NB)$ such that $\alpha = n'$. Consequently,

$$N(u)(a) \vee N(v)(\alpha_A(m)) = N(v')(\alpha_A(m))$$

where $u \colon B \to B$, v, $v' \colon A \to B$.

A simple computation leads to $Ann(a) = Ann(N(v')(\alpha(m))$. And so,

$$a = N(v')(\alpha_A(m)) = \alpha_B(M(v')(m)).$$

Hence α is surjective, which completes the proof. □

The second result is the following generalization of Corollary 8.7 in [6] to non-commutative unital quantales. For that, let$\mathscr{L}$ be a unital quantale [The definition stands for quantales, since quantales are just quantaloids with one object], $M \colon \mathscr{L} \to \mathsf{S}l$ be an $\mathscr{L}$-module and $f \in \mathscr{L}$,. We define the set $M_f \colon = \{m \in M \colon M(f)(m) = \perp\}$. It is easy to conclude that M_f is a submodule of M.

Proposition 12 For a quantale $\mathscr{L}$, an $\mathscr{L}$-module $M \colon \mathscr{L} \to \mathsf{S}l$ is weakly simple if and only if the underlying lattice M is either zero or contains exactly one non-trivial element.

Proof. Let $M \colon \mathscr{L} \to \mathsf{S}l$ be a weakly simple module. And let $f \in \mathscr{L}$. Consider $\equiv_f$ to be the relation on M associated to M_f. By simplicity, either $\equiv_f$ is diagonal or total. Hence, either $M_f = \mathbf{1}$ or $M_f = M$.

Now, if $m, n \in M \setminus \{\perp\}$ then $m \equiv_{Ann} n$ and so $m = n$. Consequently, either $M = \{\perp\}$ or $M = \{\perp, \top\}$. □

By a strong ideal of $\mathscr{Q}$, we mean an ideal I such that, for all $f, g \in \mathscr{Q}(A, B)$, $f \vee g \in I$ implies that $f, g \in I$. And by a prime ideal of $\mathscr{Q}$, we mean an ideal I of $\mathscr{Q}$ such that $f \circ g \in I$ implies that either $f \in I$ or $g \in I$ for all $f \in \mathscr{Q}(A, B)$ and $g \in \mathscr{Q}(C, A)$ where $A, B, C \in \mathscr{Q}_0$.

Now, we will characterize weakly simple module via the annihilators of its universe element.

Proposition 13 Every $\mathscr{Q}$-module $M \colon \mathscr{Q} \to \mathsf{S}l$ such that, for every $A \in \mathscr{Q}_0$, MA contains exactly one non-trivial element m_A, is weakly-simple.

Conversly, let $\mathscr{M} \colon = \{(A, M_A) | A \in \mathscr{Q}, M_A \in \mathsf{S}l\}$ such that each MA contains exaclty one non-trivial element (m_A). If $Ann(m_A)$ is a strong prime ideal of Q, then a weakly-simple $\mathscr{Q}$-module can be defined by the family $\mathscr{M}$.

Proof. Suppose that $M \colon \mathscr{Q} \to \mathsf{S}l$ as defined in the proposition is weakly simple. And prove that $Ann(m_A)$ is a strong prime ideal of $\mathscr{Q}$. To prove the primness, let $f \in \mathscr{Q}(B, A)$ and $g \in \mathscr{Q}(C, B)$ such that $f \circ g \in Ann(m_A)$, i.e., $M(f \circ g)(m_A) = \perp$ so $M(g)(M(f)(m_A)) = \perp$. Hence, $M(f)(m_A) \in M_f$. By weak simplicity of M, M_f is either trivial or equal to MA. Hence either $M(f)(m_A) = m_A$ or $M(f)(m_A) = \bot$ and either $f \in Ann(m_A)$ or $g \in Ann(m_A)$. As for $Ann(m_A)$ being strong, it follows directly from the fact that $M(f \vee g)(m_A) = M(f)(m_A) \vee M(g)(m_A)$ for $f, g(B, A)$.

Now, suppose that $Ann(m)$ is a strong prime ideal, and prove that M can be endowed with a structure of a weakly-simple $\mathcal{Q}$-module. In fact, let $A, B \in \mathcal{Q}$, and let $f \in \mathcal{Q}(B, A)$. We define $M(f)(m_A) = \bot$ if $f \in Ann(m_A)$ and $M(f)(m_A) = m_A$ otherwise. And $M(f)(\bot) = \bot$ for all $f \in \mathcal{Q}(B, A)$. It is very easy to check that via this modulation, M is a weakly simple $\mathcal{Q}$-module. □

We end this section on simple $\mathcal{Q}$-modules by characterizing the lattice $End_{\mathcal{Q}}(M)$ for a simple $\mathcal{Q}$-module M.

Proposition 14 Let $\mathcal{Q}$ be a quantaloid and M be either a simple $\mathcal{Q}$-module or a weakly-simple cyclic $\mathcal{Q}$-module, then the endomorphism quantaloid $End_Q(M)$ is isomorphic to the Boolean quantale **2**.

Proof. For this, we need to prove that every non-zero endomorphism on M is the identity on M. For that, let $\alpha \colon M \to M$ be a endomorphism on M. Remark that the simplicity of M implies that it is cyclic, and so proving that α is the identity on M is equivalent to prove that $\alpha_A(u) = u$ where $\langle A, u\rangle$ is a generator of M. Since α is injective (by simplicity of M), then $Ann(n) = Ann(\alpha_A(u))$ and so again using the simplicity of M, we get that $n = \alpha(u)$.

Hence, for a simple $\mathcal{Q}$-module M, $End(M) = \{0, id_M\}$ which completes the proof. □

4. On semisimple $\mathcal{Q}$-modules

As in ring theory, we will define a semisimple $\mathcal{Q}$-module M to be a direct sum of simple $\mathcal{Q}$-modules. This section is dedicated to give several algebraic and homological characterizations of these modules, getting full inspiration of such results from ring theory.

Firstly, we establish that semisimple $\mathcal{Q}$-modules can be expressed as a finite direct sum of their simple Q-ideals. Additionally, we will provide characterizations of semisimple modules through projective and injective modules, subtly introducing the notion of homological dimension. Finally, we will conclude this section by examining the radical structure of such modules.

We begin by recalling the definition of a semisimple $\mathcal{Q}$-module

Definition 15 A $\mathcal{Q}$-module $M \colon \mathcal{Q} \to \mathsf{S}l$ is said to be (weakly-)semisimple if it is a direct sum of its (weakly-)simple submodules.

4.1 *Properties of semisimple $\mathcal{Q}$-modules*

Remark that in the definition there is no condition of finitness on the direct sum that generates semisimple modules beacause for this special kind of modules all finitness conditions are equivalent, see Proposition 32. But, for the very special case of the $\mathcal{Q}$-module $\mathcal{Q}$, the finiteness of the direct sum is a result as shown in the following:

Lemma 16 If $\mathcal{Q}$ is a direct sum of non-zero weak-ideals, then it is a direct sum of a finite number of these weak-ideals.

Proof. Let $\mathcal{Q} = \oplus_{i\in\gamma} I_i$ where I_i is a weak-ideal of $\mathcal{Q}$ for every $i \in \gamma$ and suppose that γ is infinite.
For $A \in \mathcal{Q}$, we have $1_A = \bigvee_{j=1}^{k} e_{i_j}$ for $k \in \mathbb{N}$, $i_j \in \gamma$ and $e_{i_j} \in I_{i_j}$.

Let $i \in \gamma\{i_1, \cdots, i_k\}$ and consider an element $n_i \in I_i \setminus \{\bot\}$. Then, we have $n_i = n_i \circ 1_A = \bigvee_{j=1}^{k} n_i \circ e_{i_j}$, which the uniqueness of the decomposition of n_i. Consequently, γ is finite. □

Remark that for a $\mathcal{Q}$-module M, every direct summand N of M is a $\mathcal{Q}$-ideal: the only property that needs a check-up is: If $a \vee b \in N$ and $b \in N$ then $a \in N$. In fact, suppose that $M = N \oplus L$, then $a = n \vee l$ for some $n \in N$, $l \in L$. and so $n \vee l \vee b \in N$ Hence $n \vee (l \vee b) \in N$, since the sum is direct, then $l \vee b = \bot$ and so $a \in N$.

As in ring theory, ideals of semisimple module carry some of its strucutre such as being a direct sum of simple modules. In quantaloid modules, we have a similar statement, that is ideals of semisimple modules are also semisimple.

Lemma 17 Let $\mathcal{Q}$ be a (weakly) semisimple quantaloid, where $\mathcal{Q} = \mathcal{Q}_1 \oplus \cdots \mathcal{Q}_n$. Then, every ideal of $\mathcal{Q}$ is a direct summand and moreover $I = \oplus_{j\in J} \mathcal{Q}_j$, for some set $J \subseteq \{1, \cdots, n\}$.

Proof. Let I be an ideal of $\mathscr{Q}$, $J{:}=\{j\in\{1,\cdots,n\}|I\cap\mathscr{Q}_j\neq\mathbf{1}\}$, $\mathscr{Q}_J{:}=\oplus_{j\in J}\mathscr{Q}_j$ and $\mathscr{Q}_{\bar{J}}{:}=\oplus_{j\in\bar{J}}\mathscr{Q}_j$ where $\bar{J}=\{1,\cdots,k\}\setminus J$.

For $j\in J$, $\mathbf{1}\neq I\cap\mathscr{Q}_j\subseteq\mathscr{Q}_j$ and so $I\cap\mathscr{Q}_j=\mathscr{Q}_j$ consequently $I=\mathscr{Q}_J\oplus(\mathscr{Q}_{\bar{J}}\cap I)$.

Now, suppose that $\mathscr{Q}_{\bar{J}}\cap I=\mathbf{1}$. Let $A\in\mathscr{Q}$, so $1_A=a_1\vee\cdots\vee a_n$ for some $a_i\in\mathscr{Q}_i$. Then, for every $q_i\in\mathscr{Q}_i$, we have, $q_i=q_i\circ 1_A=q_i\circ a_1\vee\cdots\vee q_i\circ a_n$. Since each $q_i\circ a_j\in\mathscr{Q}_j$ and the sum is direct we get, for $i\neq j$ $q_i\circ a_j=\perp$ and $q_i\circ a_i=q_i$.

Now, let $\alpha=\bigvee_{j\in\bar{J}}\alpha_i\in I\cap\mathscr{Q}_{\bar{J}}$. For every $j\in\bar{J}$, $\alpha_j=\alpha\circ a_j$ and so $\alpha_j\in I\cap\mathscr{Q}_j$ and so $\alpha_j=\perp$ and consequently $\alpha=0$. Which completes the proof. □

Slightly modifying the proof of Lemma 17, we get the same result but for any $\mathscr{Q}$-module:

Proposition 18 Let M: $\mathscr{Q}\to\mathsf{S}l$ be a (weakly-)semisimple module and I a submodule (or particularly, a $\mathscr{Q}$-ideal) of M. Then, I is direct summand of M and it is also direct sum of (weakly-) simple modules.

Proof. Let M: $\mathscr{Q}\to\mathsf{S}l$ be a (weakly-)semisimple module and I a submodule (or particularly, a $\mathscr{Q}$-ideal) of M. Let $J{:}=\{j\in\{1,\cdots,n\}|I\cap M_j\neq\mathbf{1}\}$, $M_J{:}=\oplus_{j\in J}M_j$ and $M_{\bar{J}}{:}=\oplus_{j\in\bar{J}}\mathscr{Q}_j$ where $\bar{J}=\{1,\cdots,k\}\setminus J$.
For $j\in J$ and $A\in\mathscr{Q}_0$, $\mathbf{1_A}\neq IA\cap MA_j\subseteq MA_j$ and so $I\cap M_j=M_j$ consequently $I=M_J\oplus(M_{\bar{J}}\cap I)$.

Now, suppose that $M_{\bar{J}}\cap I=\mathbf{1}$. Let $A\in\mathscr{Q}_0$, so $1_A=a_1\vee\cdots\vee a_n$ for some $a_i\in M_i$. Then, for every $m_i\in M_i$, we have, $m_i=m_i\circ 1_A=m_i\circ a_1\vee\cdots\vee m_i\circ a_n$. Since each $m_i\circ a_j\in M_j$ and the sum is direct we get, for $i\neq j$ $m_i\circ a_j=\perp$ and $m_i\circ a_i=m_i$.

Now, let $\alpha=\bigvee_{j\in\bar{J}}\alpha_i\in I\cap M_{\bar{J}}$. For every $j\in\bar{J}$, $\alpha_j=\alpha\circ a_j$ and so $\alpha_j\in I\cap M_j$ and so $\alpha_j=\perp$ and consequently $\alpha=0$. Which completes the proof. □

We also have the following result concerning modules spanned by its simple submodules:

Lemma 19 Let $(M_\alpha)_{\alpha\in\Gamma}$ be a set of simple submodules of the $\mathscr{Q}$-module M. If $M=\sum_\Gamma M_\alpha$ then for $K\preceq M$ there is $\Lambda\subseteq\Gamma$ such that $(M_i)_{i\in\Lambda}$ is a family of independent [A family $(M_a)_{a\in A}$ of $\mathscr{Q}$-modules is said to be independent if, for any $a\in A$, we have $M_a\bigcap(\sum_{b\in A,\,b\neq a}M_b)=\mathbf{1}$] $\mathscr{Q}$-modules such that $M=K\oplus(\oplus_\Lambda M_i)$.

Proof. Let $K\preceq M$. Applying Zorn's lemma, there exist $\Lambda\subseteq\Gamma$ maximal with respect to $(M_i)_{i\in\Lambda}$ is independent and $K\cap(\sum_{i\in\Lambda}M_i)=\mathbf{1}$. Then, the sum $N=K+(\sum_{i\in\Lambda}M_i)$ is direct.

Remains to prove that $N=M$. For that, let $j\in\Gamma$. Since M_j is simple then $M_j\cap N$ is either equal to M_j or $\mathbf{1}$. The Latter contradicts the maximality of Λ. □

We recall that a short exact sequence $\mathbf{1}\to M\to N\to L\to\mathbf{1}$ is split if the surjective or the injective morphism in the sequence have a retraction or equivalently M is a direct summand of M or again equivalenty L is a direct summand of N. As an immediate sequence we have the following result:

Corollary 20 M is a (weakly-)semisimple $\mathscr{Q}$-module then any short exact sequences of the form

$$\mathbf{1}\longrightarrow L\stackrel{f}{\longrightarrow}M\stackrel{g}{\longrightarrow}N\longrightarrow\mathbf{1}$$

splits and consequently N and L are also semisimple.

Proof. We only prove the result for weakly semisimple. The proof for semisimple is done similarly using kernels instead of images. $(\to)$ Let $M=\oplus_{\alpha\in\Gamma}M_\alpha$. Since $Im(f)$ is an ideal of M, by Lemma 17, $Im(f)$ is a direct summand of M and so the sequence splits. Now, $N\cong M/Im(f)$ And so $N\cong\oplus_\Lambda M_\lambda$ where $\Lambda\subseteq\Gamma$.But $M=\oplus_\Lambda M_\lambda\oplus(\oplus_{\Gamma\setminus\Lambda}M_\alpha)$ so that $\oplus_{\Gamma\setminus\Lambda}M_\alpha\cong Im(f)$. □

Remark 2 From the previous result, one can rem that the notions of weak-semisimplicity and semisimplicity coincide for $\mathscr{Q}$-modules. And so, as long as a $\mathscr{Q}$-module is a direct sum of its simple-like module, we get all the possible characterization we have done previously, weather it is weak semiplicity or not.

4.2 *Homology of semisimple $\mathscr{Q}$-modules*

The last result of the previous subsequence states that every submodule and every epimorphic image of semisimple module are also semisimple. And every submodule is a direct summand. As we will see later on, this charcterize only semisimple modules.

Therefore, we present the following fundamental characterizations of semisimple modules which is an exact counterpart of Theorem 9.6 in [23].

Proposition 21 The following assertions are equivalent:

(i) M: $\mathscr{Q} \to \mathsf{S}l$ is semisimple;

(ii) M is the sum of its simple modules;

(iii) M is the sum of some set of simple modules;

(iv) M is generated by simple modules;

(v) Every submodule of M is a direct summand and consequently direct summands are exactly $\mathscr{Q}$-ideals;

(vi) Every short exact sequence $\mathbf{1} \to N \to M \to L \to \mathbf{1}$ of $\mathscr{Q}$-modules splits.

Proof. The implication $(vi) \to (v)$ is immediate from the properties of split sequences, $(iv) \to (i)$ is by Lemma 19 for $K = \mathbf{1}$. The equivalences $(ii) \Leftrightarrow (iii) \Leftrightarrow (iv)$ are trivial.

Finally, $(v) \to (ii)$ Let N be the sum of all simple submodules of M. Then $M = N \oplus L$ for some $L \preceq M$. By Corollary 9, $L = \mathbf{1}$ and so $M = N$ as desired. □

Now, we give a homological characterization of semisimple quantaloids; which is a counterpart of Corollary 8.2.2 in [24].

Theorem 22 For an integral quantaloid $\mathscr{Q}$, the following assertions are equivalent:

(i) $\mathscr{Q}$ is semisimple;

(ii) All $\mathscr{Q}$-modules are projective;

(iii) All $\mathscr{Q}$-modules are injective;

(iv) All $\mathscr{Q}$-modules are semisimple;

(v) All short exact sequences of Q-modules are split;

(vi) All weak ideals are direct summand and consequently all direct summands are exactly ideals of $\mathscr{Q}$ and so ideals and weak ideals of $\mathscr{Q}$ coincide;

(vii) $\mathscr{Q}$ is weakly-semisimple.

(viii) $\mathscr{Q}$ has a projective semisimple generator.

Proof. The implication $(iv) \to (i)$ is trivial. And $(ii) \Leftrightarrow (iii) \Leftrightarrow (v)$ are easy to see from the mutual definition of split short exact sequences, projectivity and injectivity.

For $(i) \to (iv)$, it suffices to remark that:

• For any non-zero element m of a non trivial lattice MA, for some $A \in \mathscr{Q}$, the submodule $L_{A,\,m}$ generated by $\langle A,\, m\rangle$ are semisimple as an epimorphic image of a semisimple module;

• M is a sum of all the submodule $L_{A,\,m}$, $A \in \mathscr{Q}$ and $m \in MA$. And so it is semisimple as a sum of semisimple modules.

Now, $(iv) \Leftrightarrow (v) \Leftrightarrow (vii)$ and $(i) \Leftrightarrow (vi)$ follows from Lemma 17.

Finally, for $(vii) \Leftrightarrow (i)$, suppose that $\mathscr{Q}$ has a semisimple left generator A. Then A generates Q and it is a sum of simple submodules. Hence,$\mathscr{Q}$ is also a sum of simple submodules which implies semisimplicity.

For the converse, suppose that $\mathscr{Q}$ is semisimple. By Lemma 11, we can consider $\mathscr{Q}_1, \cdots, \mathscr{Q}_n$ to be a complete set of representatives of simple modules $I_1, \cdots, I_m$ of $\mathscr{Q}$ (that is, each I_j is isomorphic to one and only one of the $\mathscr{Q}_i$). Let $\mathscr{R}$ be the (co-)product of $\mathscr{Q}_1, \cdots, \mathscr{Q}_n$. Then $\mathscr{R}$ is certainly finitely generated projective and it generates each I_j , so it generates $\mathscr{Q}$. And so, it is a progenerator, which is semisimple as a module over a semisimple $\mathscr{Q}$-module. □

In order to prove a weak version of the well-know Artin-Wedderburn theorem, we need the following result, which follows directly from Theorem 22.

Corollary 23 Let $\mathscr{Q}$ be an integral quantale. A $\mathscr{Q}$-module is semisimple if and only if its quantale of matrices $Mat(\mathscr{Q})$ is semisimple.

Hence, using all the result in the previous setions, we can give a decomposition of semisimple quantaloids via the quantaloid of matrices over some special quantales which is a weak counterpart of the Artin-Wedderburn theorem (for rings, see [25]).

Proposition 24 For any integral quantale $\mathscr{Q}$, the following statements are equivalent:

(i) $\mathscr{Q}$ is a left semisimple quantale;

(ii) $\mathscr{Q} \cong M_{n_1}(\mathbb{B}) \times \cdots \times M_{n_k}(\mathbb{B})$ where $\mathbb{B}$ is a **2**-module, and $k \geq 0$ and n_i, $i \in \{1, \cdots, k\}$ are positive integers;

(iii) $\mathscr{Q}$ is a right semisimple quantale.

Proof. $(i) \to (ii)$ Let $\mathscr{Q}$ be a left semisimple quantaloid, then $\mathscr{Q}$ is a finite direct sum of its simple left ideals. Applying Lemma 11 and taking classes of summands according to their isomorphism types, we obtain

$$\mathscr{Q} \cong I_1^{n_1} \oplus \cdots \oplus I_k^{n_k}$$

where the right ideals $I_1, \cdots, I_k$ of $\mathscr{Q}$ are mutually nonisomorphic simple.

Remark that each I_i for $i \in \{1, \cdots, k\}$ is a direct summand of $\mathscr{Q}$, and consequently it is cyclic and projective and so isomorphic to $h_A u$ for some idempotent $u \in \mathscr{Q}$. And so, $End_{\mathscr{Q}}(I_i) \cong \mathbb{B}$ for $i \in \{1, \cdots, k\}$,

Since $Hom_{\mathscr{Q}}(I_i, I_j) = \{\bot\}$ for $i \neq j$, we have

$$End(\mathscr{Q}) \cong End(I_1^{n_1} \oplus \cdots \oplus I_k^{n_k})$$

$$\cong End(I_1^{n_1}) \times \cdots \times End(I_k^{n_k})$$

Finally, by noticing that $End(I^n) \cong M_n(I)$ for any $\mathscr{Q}$-module I, we conclude that

$$\mathscr{Q} \cong M_{n_1}(\mathbb{B}) \times \cdots \times M_{n_k}(\mathbb{B})$$

$(ii) \to (i)$ It suffices to show the semisimpleness of a matrix quantaloid $Mat(\mathbb{B})$, which follows directly from proposition 14 and proposition 23.

The equivalence $(ii) \Leftrightarrow (iii)$. is immediate using the properties of the symmetry. □

4.3 *Radical structure of semisimple $\mathscr{Q}$-modules*

In the theory of modules, the radical and its dual, the socle, serves as a fundamental element in the classification of structures. Hence, we define the socle of a module M, as in ring theory, to be the sum of all its simple submodules. We present the following extension of the socle characterization of semisimple modules.

Proposition 25 For a $\mathscr{Q}$-module M, the following assertions are equivalent:

(i) $M = Soc(M)$;

(ii) M is a semisimple $\mathscr{Q}$-module;

(iii) M has no proper essential $\mathscr{Q}$-ideals.

Proof. $(i) \to (ii)$ is trivial.

$(ii) \to (i)$ If $Soc(M) \subsetneq M$, there exists a nonzero submodule $N \preceq M$ such that $M = Soc(M) \oplus N$ and $Soc(M) \cap N = \mathbf{1}$. However, the latter contradicts the fact that the submodule N contains a simple submodule; therefore, $Soc(M) = M$.

$(ii) \to (iii)$ Let E be an essentiel submodule of M. Then there exists $N \preceq M$ such that $M = E \oplus N$ and $E \cap N = \mathbf{1}$; and therefore, $N = \mathbf{1}$ and $M = E$.

$(iii) \to (ii)$ For a submodule $K \preceq M$, consider thefamily $\mathscr{F} = \{N \preceq M \mid K \cap N = \mathbf{1}\}$. By Zorn's lemma, there exists a maximal element $N \in \mathscr{F}$. We prove that $K \oplus N$ is an essentiel submodule of M, and therefore, $K \oplus N = M$. Indeed, if $(K \oplus N) \cap L = \mathbf{1}$ for a submodule $L \preceq M$, then $K \cap (N + L) = \mathbf{1}$. By maximality of N, we conclude that $L = \mathbf{1}$, as desired. □

Corollary 26 Let M be a $\mathscr{Q}$-module and $N \subseteq M$, then $Soc(N) = N \cap Soc(M)$, in particular, $Soc(Soc(M)) = Soc(M)$.

Proof. For a submodule K of a $\mathscr{Q}$-module of M, it is easy to see that $Soc(K) \preceq K \cap Soc(M)$. Since $K \cap Soc(M)$ is a $\mathscr{Q}$-ideal, there exists a submodule $N \subseteq K \cap Soc(M)$ such that $K \subseteq Soc(M) = Soc(K) \oplus N$ and $Soc(K) \cap N = \mathbf{1}$ Now, if $N \neq \mathbf{1}$ then, by Corollary 9, N would contain a simple submodule, and consequntly, $Soc(K) \cap N \neq \mathbf{1}$. Thus, $N = \mathbf{1}$, and $K \cap Soc(M) = Soc(K)$. □

Corollary 27 Let M be a $\mathscr{Q}$-module. Then, $Soc(M)$ is the intersection of all essential $\mathscr{Q}$-ideals of M.

Proof. The inclusion $\cap\{A|A \text{ is essential in } M\} \preceq Soc(M)$ is clear. From Corollary 26, we have

$$Soc(M)(\cap\{A|A \text{ is essential in } M\})$$

$$= \cap\{A|A \text{ is essential in } M\} \cap Soc(M)$$

$$= \cap\{A|A \text{ is essential in } M\}.$$

□

Similarly to modules over rings, we call a module M finitely cogenerated if for every set $\mathscr{A}$ of submodules of M, $\cap\mathscr{A} = \mathbf{1}$ if and only if $\cap\mathscr{F} = \mathbf{1}$ for some finite $\mathscr{F} \subseteq \mathscr{A}$, we also have:

Corollary 28 Let M be a $\mathscr{Q}$-module. Then M is finitely cogenerated if and only if $Soc(M)$ is finitely cogenerated and $Soc(M)$ is an essential submodule of M.

Proof. Since $Soc(M)$ is finitely cogenerated once M is finitely cogenerated. The second part follows from Corollary 27 immediately since

$$Soc(M) \cap L = \mathbf{1} \text{ if and only if } \cap\{A|A \text{ is essential in } M\} \cap L = \mathbf{1}$$

for any submodule L of M. The converse follows again for the same reason. □

4.4 *Disscussion*

The definition of quantaloids reveals their inherent structure as 2-categories, where relations serve as 2-homomorphisms. In the study of semisimple 2-categories, in [26], the author established an intresting result: every semisimple 2-category is Cauchy-complete. This finding extends the notion of Cauchy completeness from quantaloids to 2-categories, forming a bridge between purely categorical/algebraic concepts and the analytical Cauchy completeness of metric spaces. Specifically, considering the Lawvere quantale of positive numbers, it was showen, in [12], that a Cauchy complete $\mathscr{Q}$-category corresponds to a Cauchy complete generalized metric space.

The equivalence between the semisimplicity of quantaloids and their Cauchy completeness is not merely a theoretical insight. It holds practical significance, motivating our forthcoming exploration into the homology of quantaloids and their modules. Beyond the scope of comparison and classification, our objective is to establish meaningful connections between quantaloids and various contexts using suitable categories.

5. Noetherian and artinian $\mathcal{Q}$-modules

In ring theory, the notions of Noetherian and Artinian rings are very important, due to their role in simplifying the ideal structure of a ring. Also, these two notions are very related to the notion of semisimplicity, since all semisimple rings are Artinian and Noetherian. Here, in this section, we investigate if this property stands true for quantaloids. In other words, whether semisimple rings are Artinian and Noetherian.

Definition 29 A quantaloid $\mathcal{Q}$ is said to satisfy the the ascending chain condition (ACC) (resp. descending chain condition (DCC)) on a family $\mathscr{I}$ of ideals of $\mathcal{Q}$ if for every ascending sequence

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots (resp.\ I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots)$$

of ideals in $\mathscr{I}$, there exists a positive integer $n \in \mathbb{N}$ such that

$$I_n = I_{n+1} = I_{n+2} = \cdots.$$

We say that a quantaloid $\mathcal{Q}$ is left *Noetherian (resp. left Artinian)* if it satisfies the ACC (resp. DCC) on its left ideals.

A very important example of Noetherian and Artinian modules are semisimple modules. Recall from Proposition 21 that all ideals in a semisimple quantaloids are direct summands. So we give the following proposition on quantaloids satisfying the ACC on direct summand.

Proposition 30 A quantaloid $\mathcal{Q}$ satisfies the ACC on direct summands if and only if it satisfies the DCC on direct summands.

Proof. Suppose that M satisfies the DCC on direct summands. Let $I_1 \subseteq \cdots \subseteq I_i \subseteq \cdots$ be an ascending chain of direct summands of M. For every $i \in \mathbb{N}$, there exists a submodule N_i of M such that $M = I_i \oplus N_i$.

We have $M = I_1 \oplus N_1$. Since $I_1 \subseteq U_2$, we have $I_2 = I_1 \oplus (I_2 \cap N_1)$, where

$$M = I_2 \oplus N_2 = I_1 \oplus (I_2 \cap N_1) \oplus N_2.$$

Since $I_2 \cap N_1 \subseteq N_1$, we get that $N_1 = (I_2 \cap N_1) \oplus (N_1 \cap (I_1 \oplus N_2))$, hence

$$M = I_1 \oplus N_1 = I_1 \oplus (I_2 \cap N_1) \oplus (N_1 \cap (I_1 \oplus N_2)).$$

Let $L_1 := N_1$ and $L_2 := N_1 \cap (I_1 \oplus N_2)$, we have $I_1 \oplus L_1 = M = I_2 \oplus L_2$ where $L_2 \subseteq L_1$. Since $M = I_2 \oplus L_2$ and $I_2 \subseteq I_3$, it follows that $I_3 = I_2 \oplus (I_3 \cap L_2)$, hence

$$M = I_3 \oplus N_3 = I_2 \oplus (I_3 \cap L_2) \oplus N_3.$$

Since $I_3 \cap L_2 \subseteq L_2$, we have then $L_2 = (I_3 \cap L_2) \oplus (L_2 \cap (I_2 \oplus N_3))$. Putting $L_3 := L_2 \cap (I_2 \oplus N_3)$, we get that $L_3 \subseteq L_2$ and

$$M = I_2 \oplus L_2 = I_2 \oplus (I_3 \cap L_2) \oplus L_3 = I_3 \oplus L_3.$$

Repeating this reasonning, we obtain a descending chain

$$L_1 \supseteq \cdots \supseteq L_i \supseteq \cdots$$

of direct summands of M such that $M = I_i \oplus L_i$ for every $i \in \mathbb{N}$. By our assumption, the descending chain must be finite, i.e. there exists some $k \in \mathbb{N}$ such that $L_i = L_k$ for every $i \geq k$.

Now, for every $i \geq k$, we have $I_k \subseteq I_i$, $M = I_k \oplus L_k$ and so $I_i \cap L_i = \mathbf{1}$ consequently

$$I_i = I_k \oplus (I_i \cap L_k) = I_k \oplus (I_i \cap L_i) = I_k.$$

Thus the ascending chain $I_1 \subseteq \cdots \subseteq I_i \subseteq \cdots$ is also finite. □

This observation can be stated in a more general form:

Corollary 31 A quantaloid $\mathscr{Q}$ is Noetherian if and only if it is Artinian.

Hence, to prove that semisimple modules are Artinian and Noetherian, it suffices to prove either Noetherniannity or Artiniannity. Indeed:

Proposition 32 If $\mathscr{Q}$ is a semisimple, then it is Noetherian.

Proof. Let $I_1 \supsetneqq I_2 \supsetneqq \cdots \supsetneqq I_k \supsetneqq \cdots$ be an infinite strictly descending chain of $\mathscr{Q}$-ideals of $\mathscr{Q}$. By assumption, there exists some ideal $J_k \preceq \mathscr{Q}$ such that $\mathscr{Q} = J_k \oplus I_k$, for every $k \in \mathbb{N}$. The ideals I_k and J_k are non-zero as the chain is infinite. Since $I_1 \supseteq I_2$ and $\mathscr{Q} = I_2 \oplus J_2$, we have then $I_1 = I_2 \oplus (I_1 \cap J_2)$. And so, $L_1 := I_1 \cap J_2$ is a non-zero $\mathscr{Q}$-ideal of $\mathscr{Q}$ and $I_1 = I_2 \oplus L_1$.

Repeating this process, we obtain at the k^{th} step, a non-zero ideal $L_k \preceq \mathscr{Q}$ such that $I_k = I_{k+1} \oplus L_k$ and $I_1 = L_1 \oplus \cdots \oplus L_k \oplus I_{k+1}$. Let $\tilde{L}_i := K_1 \oplus \cdots \oplus K_i$ for each $i \in \mathbb{N}$. Then, we have $\mathscr{Q} = \tilde{L}_i \oplus I_{i+1} \oplus J_1$ where $\tilde{L}$ is an ideal. And so, $L := \cup_{i \in \mathbb{N}} \tilde{L}_i$ is also an ideal.

By assumption, $\mathscr{Q} = L \oplus J$ for some ideal J of $\mathscr{Q}$. Thus, for $A \in \mathscr{Q}$, $1_A = l \vee j$ for some unique $l \in LA$ and $j \in JA$. Since $l \in \tilde{L}_i A$ for some $i \in \mathbb{N}$, it can be written in a unique way as $l = l_1 \vee \cdots \vee l_i$ for some unique $l_k \in \tilde{L}_k A$, for $k \in \{1, 2, \cdots, i\}$.

Now, since, for any $A \in \mathscr{Q}$, $\tilde{L}_{i+1} A \subseteq LA$, then $\tilde{L}_{i+1} + J$ is a direct sum and so $L_1 + \cdots L_{i+1} + J$ is also direct. Whence, putting $U := L_1 \oplus \cdots \oplus L_i \oplus J$, we get that the sum $U + L_{i+1}$ is direct. Finally, let $\alpha_{i+1} \in L_{i+1} A \setminus \{\perp\}$, then we have

$$\alpha_{i+1} = \alpha_{i+1} \circ 1_A = \alpha_{i+1} \circ (l_1 \vee \cdots \vee l_i \vee j) = \alpha_{i+1} \circ l_1 \vee \cdots \vee \alpha_{i+1} \circ l_i \vee \alpha_{i+1} \circ j$$

with $\alpha_{i+1} \circ l_k \in L_k A$ for $k \in \{1, 2, \cdots, i\}$ and $\alpha_{i+1} \circ j \in JA$. Consequently, $\alpha_{i+1} \in L_{i+1} A \cap JA = \{\perp\}$ which contradicts the assumption on α_{i+1}. So, the descending chain $\{I_i\}_{(i \in \mathbb{N})}$ must be finite. Hence $\mathscr{Q}$ is Artinian. □

Hence, we can only focus on using either the ACC or DCC to get the desired results. First, we have the following result on the structure of quantaloids satisfying the ACC on direct summands. For that, we say that a submodule N of a $\mathscr{Q}$-module M is a maximal direct summand of M if and only if N is a direct summand of M with $N \neq M$ and for any ther direct summand L of M such that $NA \subseteq LA \subseteq MA$, for all $A \in \mathscr{Q}$, then either $L = N$ or $L = M$. And a direct summand N of M is called an indecomposable summand if and only if $\mathbf{1}$ is a maximal direct summand of N.

Proposition 33 If $\mathscr{Q}$ satisfies the ACC on direct summands then $\mathscr{Q} = I_1 \oplus I_2 \oplus \cdots \oplus I_n$ where each I_i is an indecomposable summand of $\mathscr{Q}$, $i \in \{1, \cdots, n\}$, for some $n \in \mathbb{N}$.

Proof. If $\mathscr{Q}$ has no non-trivial direct summand, then $\mathbf{1}$ is the maximal summand of $\mathscr{Q}$, thus $\mathscr{Q}$ is an indecomposable summand. If not, let S_0 be a non-trivial direct summand of $\mathscr{Q}$. Then consider the non-empty family

$$\mathscr{D} := \{S_0 \subsetneqq S | S \text{ is a direct summand of } \mathscr{Q}\}.$$

Using our asumption on $\mathscr{Q}$, every descending chain $(S_k)_{k\in J}$ $\mathscr{D}$ terminates and has a lower bound Hence, it follows by Zorn's Lemma, that $\mathscr{D}$ has a minimal element, say D_1. Since there is no direct summand between S_0 and D_1, we see that S_0 is a maximal summand of D_1.

The same reasoning applied to the family

$$\mathscr{A} := \{S \subsetneqq S_0 | S \text{ is a direct summand of } \mathscr{Q}\}$$

leads to proving the existence of a maximal element of $\mathscr{A}$ say A_1 and A_1 is a maximal summand of S_0. We proved that every non-trivial direct summand is a maximal summand of a direct summand and has a maximal summand.

Next, let A_0 be a non-trivial direct summand of $\mathscr{Q}$ hence there exists a direct summand A_1, of $\mathscr{Q}$, such that A_0 is a maximal summand of A_1. If A_1 is non-trivial, then there exists A_2, a direct summand of $\mathscr{Q}$, such that A_1 is a maximal summand of A_2. Repeating this reasonning again and again, we obtain an ascending chain $A_0 \subsetneqq A_1 \subsetneqq A_2 \subsetneqq \cdots$ of direct summands of $\mathscr{Q}$, which should terminate. Hence, there exists $n \in \mathbb{N}$ such that $A_0 \subsetneqq A_1 \subsetneqq A_2 \subsetneqq \cdots A_n = \mathscr{Q}$ and A_i is maximal summand of A_{i+1} for $i \in \{0, 1, \cdots, n-1\}$. Since A_0 is a non-trivial direct summand of $\mathscr{Q}$, A_0 has maximal summand A_{-1}. If A_{-1} is non-trivial, then A_{-1} has maximal summand A_{-2}. Again, we obtain a descending chain $A_0 \supsetneqq A_{-1} \supsetneqq \cdots$ of direct summands of $\mathscr{Q}$. Thus, there exists $m \in \mathbb{N}$ such that $A_0 \supsetneqq A_{-1} \supsetneqq \cdots A_{-m} = \mathbf{1}$ and A_{-i} is maximal summand of A_{-i+1} for $i \in \{1, 2, \cdots, m\}$. Combining the two sequences obtained, we get

$$A_n = \mathscr{Q} \supsetneqq A_{n-1} \supsetneqq \cdots \supsetneqq A_1 \supsetneqq A_0 \supsetneqq A_{-1} \supsetneqq \cdots \supsetneqq A_{-m+1} \supsetneqq \mathbf{1} = A_{-m}$$

is an ascending chain of direct summands of $\mathscr{Q}$ with each A_i is a maximal summand of A_{i+1} for $i \in \{-m, -m+1, \cdots, -1, 0, 1, \cdots, n-1\}$. Let$\mathscr{Q} = A_i \oplus D_i$, for $i \in \{-m, -m+1, \cdots, -1, 0, 1, \cdots, n-1\}$. Then, putting $L_i := A_i \cap D_{i-1}$, we get

$$\mathscr{Q} = A_n = L_{-m+1} \oplus L_{-m+2} \oplus \cdots \oplus L_n.$$

Suppose that L_i is a reducible summand for some $i \in \{-m+1, -m+2, \cdots, n\}$. Then, there exists a direct summand L of L_i such that $\mathbf{1} \neq L \subsetneqq L_i$. Let $L_i := L \oplus N$, then $\mathscr{Q} = A_i \oplus D_i = A_{i-1} \oplus L_i \oplus D_i = A_{i-1} \oplus L \oplus N \oplus D_i$, it follows that $A_{i-1} \oplus L$ is a direct summand of $\mathscr{Q}$ such that $A_{i-1} \subsetneqq A_{i-1} \oplus L \subsetneqq A_i$, which contradicts the maximality of A_{i-1}. And so, the proof is complete. □

The previous result provides an other way to prove the spatiality of semisimple quantaloids. In fact, in [27], Theorem 6 gives a suffisiant condition for a $\mathscr{Q}$-module to admit a decomposition by indecomposable ideals which is being a finitely spatial $\mathscr{Q}$-module. In Proposition 33, we provide a necessary and a suffisiant condition to admit such decomposition which is satisfying the ACC or DCC on direct summands. Notice that semisimple $\mathscr{Q}$-modules satisfy such condition and so are a good example of such result.

We also have the following result concerning Noetherian quantaloids.

Proposition 34 If $\mathscr{Q}$ is a quantaloid such that every short exact sequence of modules $\mathbf{1} \to N \to \mathscr{Q} \to I \to \mathbf{1}$ is splitting, then $\mathscr{Q}$ is Noetherian.

Proof. Let $J_0 \subsetneqq J_1 \subsetneqq \cdots \subsetneqq J_k \subsetneqq \cdots$ be a non-terminating ascending chain of $\mathscr{Q}$-ideals of $\mathscr{Q}$. Notice that $J := \cup_{n\in\mathbb{N}} J_n$ is a $\mathscr{Q}$-ideal of $\mathscr{Q}$, hence (by assumption) the following short exact sequence

$$\mathbf{1} \longrightarrow J \xrightarrow{\ i\ } \mathscr{Q} \xrightarrow{\ \pi\ } \mathscr{Q}/J \longrightarrow \mathbf{1}$$

of modules is splitting. Let α: $\mathscr{Q} \to J$ be a $\mathscr{Q}$-homomorphism such that $\alpha \circ i = id_J$. Then, for $A \in \mathscr{Q}$, $\alpha_A(1_A) \in J$, that is $\alpha_A(1_A) \in J_i$ for some $i \in J$. If we take $x \in J_{i+1} \setminus J_i$, then

$$x = (\alpha_A \circ i_A)(x) = \alpha_A(x) = \alpha_A(x \circ 1_A) = x \circ \alpha(1_A) \in J_i.$$

Which is absurd and consequently $\mathscr{Q}$ is Noetherian. □

Corollary 35 If $\mathscr{Q}$ is a quantaloid such that every $\mathscr{Q}$-ideal is injective, then $\mathscr{Q}$ is a Noetherian quantaloid.

Using the previous two results, it is easy to see that semisimple quantaloids are indeed Noetherian.

6. Conclusion

As we have seen through this paper, the category of modules over a quantaloid is very rich in structure and it reaches various subjects of mathematics. Which motivates a deep study of this aspect of quantaloids.

As established earlier, the connection between semisimplicity and Cauchy completeness is significant. Cauchy completeness plays a pivotal role in the exploration of fixed-point theorems in both analytical and categorical contexts (a categorical fixed-point theorem is outlined in [28]). Showing that the homological classification of quantaloids opens up to extend the study of fixpoint theorems. Homological dimensions, with their capacity to determine the existence of resolutions and facilitate diagrams that transfer algebraic or analytic properties, become instrumental in this context. The interplay between homological classifications and fixpoint theorems becomes a focal point for our future investigations.

Applications of $\mathbb{Q}$-modules extend into the field of image processing (for further details, refer to [29]). Specifically, homomorphisms between modules over a quantale can be interpreted as transformations between distinct image representations. These morphisms effectively capture the relationships between images, enabling the formulation of image processing functions that preserve specific algebraic properties encoded by the quantale. Consequently, the homological study of modules over a quantale proves applicable in image analysis. Through the definition of homological invariants associated with image representations, tools can be developed for characterizing and classifying images based on their underlying algebraic properties. This has the potential to advance methods in image recognition, classification, and retrieval, showcasing the versatility and relevance of $\mathbb{Q}$-modules in the context of image processing.

In conclusion, the homological study of quantaloid modules transcends disciplinary boundaries, offering a versatile toolkit for exploring the intricacies of quantaloids and their interactions with diverse mathematical theories. This research not only enhances our theoretical understanding but also paves the way for practical applications in quantum computing, computer science, and other emerging fields. This study is a promising road not only to deepen our understanding of the structural aspects of quantaloids but also to elucidate their relationships with diverse mathematical landscapes, providing a rich tapestry of interconnected insights.

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Conflict of intrest

The authors declare no competing financial interest.

References

[1] Cartan H, Eilenberg S. *Homological algebra*. Princeton, NJ, USA: Princeton University Press; 1956.

[2] Enochs EE, Jenda OMG. *Relative homological algebra*. Berlin, Germany: Walter de Gruyter; 2000.

[3] Rotman J. *An introduction to homological algebra*. Orlando, FL, USA: Academic Press; 1979.

[4] Kilp M, Knauer U, Mikhalev AV. *Monoids, acts and categories.* Berlin, Germany: Walter de Gruyter; 2000.

[5] Fofanova TS. Polygons over distributive lattices. *Colloquia Mathematica Societatis Janos Bolyai*. 1982; 29: 289-292.

[6] Elliott J. Prequantales and applications to semistar operations and module systems. *arXiv e-prints:1101.2462*. 2010.

[7] Galatos N, Gil-Férez J. Modules over quantaloids: Applications to the isomorphism problem in algebraic logic and π-institutions. *Journal of Pure and Applied Algebra*. 2017; 221(1): 1-24.

[8] Eklund P, García JG, Höhle U, Kortelainen J. *Semigroups in complete lattices: Quantales, modules and related topics*. Berlin, Germany: Springer Nature; 2018.

[9] Kruml D, Paseka J. On simple and semisimple quantales. *In topology atlas invited contributions.* Toronto, ON, Canada: Topology Atlas; 2002. p.1-13.

[10] Paseka K. Spatial quantales. *Applied Categorical Structures*. 2002; 10: 49-62.

[11] Russo C. Quantales and their modules: Projective objects, ideals, and congruences. *South American Journal of Logic*. 2016; 2(2): 405-424.

[12] Stubbe I. Towards "Dynamic Domains": Totally Continuous Cocomplete $\mathcal{Q}$-categories. *Electronic Notes in Theoretical Computer Science*. 2006; 155: 617-634.

[13] Abramsky S, Vickers S. Quantales, observational logic and process semantics. *Mathematical Structures in Computer Science.* 1993; 3: 161-227.

[14] Borceux F. *Handbook of categorical algebra*. London, UK: Cambridge University Press; 1994.

[15] Kelly GM. *Basic concepts of enriched category theory*. London, UK: Cambridge University Press; 1982.

[16] Borceux F, Stubbe I. Short introduction to enriched categories. In: Coecke B, Moore D, Wilce A. (eds.) *Current research in operational quantum logic: algebras, categories, languages, fundamental theories of physics*. Cambridge, MA, USA; Academic Publishers; 2000. p.167-194.

[17] Heymans H, Stubbe I. Grothendieck quantales for allegories of enriched categories. *Bulletin of the Belgian Mathematical Society*. 2012; 19: 861-890.

[18] Niefield S. Projectivity, continuity and adjointness: quantales, $\mathcal{Q}$-posets and $\mathcal{Q}$-modules. *Theory and Applications of Categories.* 2016; 31(30): 839-851.

[19] Niefield S, Wood RJ. Coexponentiability and projectivity: Rigs, rings, and quantales. *Theory and Applications of Categories*. 2017; 32(36): 222-1228.

[20] Slesinger R. *Projective quantales and quantale modules*. Master's thesis under the direction of K. Paseka. University of Masaryk; 2008. Available from: https://is.muni.cz/th/ridwl/masterthesis.pdf.

[21] Stubbe I. An introduction to quantaloid-enriched categories. *Fuzzy Sets and Systems*. 2014; 256: 95-116.

[22] Zumbrägel J. Classification of finite congruence-simple semirings with zero. *Journal of Algebra and Its Applications* 2008; 7: 363-377.

[23] Anderson FW, Fuller KR. *Rings and categories of modules*. New-York, NY, USA: Springer-Verlag; 1992.

[24] Kasch F. *Modules and Rings*. New York, NY, USA: Academic Press Inc.; 1982.

[25] Nicholson WK. A short proof of the Wedderburn-Artin theorem. *New Zealand Journal of Mathematics*. 1993; 22(1): 83-86.

[26] Decoppet TD. Compact semisimple 2-categories. *Transcations of the American Mathematical Society*. 2023; 376: 8309-8336. Available from: https://doi.org/10.1090/tram/9044.

[27] Slesinger R. Decomposition and projectivity of quantale modules. *Mathematics*. 2009; 16: 81-89.

[28] Benkhadra A, Stubbe I. A logical analysis of fixpoint theorems. *Categorical Topology and Differential Geometry Notebooks*. 2023; 64(1): 97-121.

[29] Russo C. *Quantale modules, with applications to logic and image processing*. Doctoral dissertation. University of Salerno; 2009. Available from: https://doi.org/10.13140/RG.2.1.3155.8889.