Estimates for the Bounds of the Essential Spectrum of a 2 × 2 Operator Matrix

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Abstract: We consider a 2 × 2 operator matrix $\mathcal{A}_\mu$, $\mu > 0$, related to the lattice systems describing three particles in interaction, without conservation of the number of particles on a d-dimensional lattice. We obtain an analogue of the Faddeev type integral equation for the eigenfunctions of $\mathcal{A}_\mu$. We describe the two- and three-particle branches of the essential spectrum of $\mathcal{A}_\mu$ via the spectrum of a family of generalized Friedrichs models. It is shown that the essential spectrum of $\mathcal{A}_\mu$ consists of the union of at most three bounded closed intervals. We estimate the lower and upper bounds of the essential spectrum of $\mathcal{A}_\mu$ with respect to the dimension d $\in \mathbb{N}$ of the torus $\mathbb{T}^d$, and the coupling constant $\mu > 0$.
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1. Introduction and statement of the problem

In this paper we study the essential spectrum of the 2 × 2 operator matrix of the form

$$\mathcal{A}_\mu := \begin{pmatrix} A_{11} & \mu A_{12} \\ \mu A_{12}^* & A_{22} \end{pmatrix}, \mu > 0$$

acting in the Hilbert space

$$\mathcal{H} := \mathcal{H}_1 \oplus \mathcal{H}_2$$

with $\mathcal{H}_1 := L^2(\mathbb{T}^d)$ and $\mathcal{H}_2 := L^2_{\mathrm{sym}}((\mathbb{T}^d)^2)$. Here $\mathbb{T}^d$ is the d-dimensional torus, the cube $(-\pi, \pi]^d$ with appropriately identified sides equipped with its Haar measure and $L^2_{\mathrm{sym}}((\mathbb{T}^d)^2)$ stands for the subspace of $L^2((\mathbb{T}^d)^2)$ consisting of symmetric functions (with respect to the two variables). The matrix entries $A_{ij} : \mathcal{H}_j \to \mathcal{H}$, $i \le j$, $i, j = 1, 2$, are given by

$$(A_{11} f_1)(k_1) = w_1(k_1) f_1(k_1), (A_{12} f_2)(k_1) = \int_{\mathbb{T}^d} f_2(k_1, t) dt,$$

$$(A_{22} f_2)(k_1, k_2) = w_2(k_1, k_2) f_2(k_1, k_2), \ f_i \in \mathcal{H}_i, i = 1, 2.$$

Here $\mu > 0$ is a coupling constant, the functions $w_1(\cdot)$ and $w_2(\cdot, \cdot)$ have the form

$$w_1(k_1) := \varepsilon(k_1) + \gamma, \ w_2(k_1, k_2) := \varepsilon(k_1) + \varepsilon(\frac{1}{2}(k_1 + k_2)) + \varepsilon(k_2)$$

with $\gamma \in \mathbb{R}$ and the dispersion function $\varepsilon(\cdot)$ is defined by

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$$\varepsilon(k_1):=\sum_{i=1}^{\mathrm{d}}(1-\cos k_1^{(i)}),\ k_1=(k_1^{(1)},\ldots,k_1^{(\mathrm{d})})\in\mathbb{T}^{\mathrm{d}}, \tag{1}$$

A_{12}^* denotes the adjoint operator to A_{12} and

$$(A_{12}^* f_1)(k_1,k_2)=\frac{1}{2}(f_1(k_1)+f_1(k_2)),\ f_1\in\mathcal{H}_1.$$

Under these assumptions, the operator $\mathcal{A}_\mu$ is bounded and self-adjoint. An important problem of the spectral theory of such operator matrices is the study of the number of eigenvalues located outside the essential spectrum, i.e. the finiteness or infiniteness of the discrete spectrum. In the analysis of the discrete spectrum to know more detailed information on the bounds of the two-particle and three-particle branches of the essential spectrum of $\mathcal{A}_\mu$ is important.

We point out that the rank 2 perturbation of $\mathcal{A}_\mu$ has been considered before in [4, 5, 10, 18], which were studied its essential and discrete spectrum. Non compact perturbation of $\mathcal{A}_\mu$ was considered in [15, 16] and the structure of its essential spectrum was described; in addition, conditions for the infiniteness of the number of eigenvalues located inside, in the gap and below of the bottom of the essential spectrum were found.

It is remarkable that, the results about the essential spectrum and the number of the eigenvalues of $\mathcal{A}_\mu$ were announced without proofs in [19], and this paper is devoted to the detailed discussions of the results related to the essential spectrum of $\mathcal{A}_\mu$.

In the present paper we obtain the following results:

(i) We derive an analogue of the Faddeev type integral equation for eigenfunctions of $\mathcal{A}_\mu$;

(ii) We describe the location of the essential spectrum of $\mathcal{A}_\mu$, via the spectrum of a family of generalized Friedrichs models $\mathcal{A}_\mu(k)$, $k\in\mathbb{T}^{\mathrm{d}}$;

(iii) We introduce a new branches of $\sigma_{\mathrm{ess}}(\mathcal{A}_\mu)$ and show that it consists of the union of at most 3 bounded closed intervals;

(iv) We estimate the lower and upper bounds of the essential spectrum of the operator matrix $\mathcal{A}_\mu$ for all dimensions $\mathrm{d}\in\mathbb{N}$ of the torus $\mathbb{T}^{\mathrm{d}}$ and all values of $\mu>0$.

The next sections are devoted to the discussion of these problems. For the convenience of the reader, we have added an appendix on the relation of $\mathcal{A}_\mu$ with the lattice systems describing three particles in interaction, without conservation of the number of particles.

The abstract results on the essential spectrum in [2] do not apply since the required compactness assumptions on certain auxiliary operators are violated mainly due to the non-compactness of partial-integral operators. For the present approach, since the last diagonal entry is a multiplication operator, it turned out to be natural to use singular sequence (the Weyl criterion) to describe one part of the essential spectrum and to employ a Faddeev equation approach to describe the second part.

It is important that in [19] it was found the critical value μ_0 of the coupling constant μ, to establish the existence of infinitely many eigenvalues lying in both sides of essential spectrum of $\mathcal{A}_{\mu 0}$ and to obtain an asymptotics for the number of these eigenvalues. The latter assertion seems to be quite new for the discrete models and similar result has not been obtained yet for the three-particle discrete Schrödinger operators and operator matrices in Fock space. In this sense the results obtained here plays a crucial role in the next investigations of the spectrum of $\mathcal{A}_\mu$.

We mention that the study of the systems of non-conserved number of quasi-particles is reduced to the investigation of the spectral properties of self-adjoint operators acting in the *cut subspace* of the bosonic Fock space [13]. The operator matrix $\mathcal{A}_\mu$ is related to such systems arising, for example, in the theory of solid-state physics [13], quantum field theory [8] and statistical physics [11].

Throughout this paper, we use the following notations. If A is a linear bounded self-adjoint operator from a Hilbert space to another, then $\sigma(A)$ denotes its spectrum, $\sigma_{\mathrm{ess}}(A)$ its essential spectrum and $\sigma_{\mathrm{disc}}(A)$ its discrete spectrum.

2. Family of generalized Friedrichs models and its spectrum

In this section we study some spectral properties of the family of generalized Friedrichs models. We notice that its spectrum, threshold eigenvalues and virtual levels have been studied in detail in [20, 21].

Let $H_0 := \mathbb{C}$. To study the spectral properties of the operator $\mathcal{A}_\mu$, we introduce the following auxiliary family of bounded self-adjoint operators (generalized Friedrichs models) $\mathcal{A}_\mu(k)$, $k \in \mathbb{T}^d$, which acts in $\mathcal{H}_0 \oplus \mathcal{H}_1$ as 2×2 operator matrices

$$\mathcal{A}_\mu(k) := \begin{pmatrix} A_{00} & \frac{\mu}{\sqrt{2}} A_{01} \\ \frac{\mu}{\sqrt{2}} A_{01}^* & A_{11}(k) \end{pmatrix},$$

with matrix elements

$$A_{00} f_0 = \gamma f_0, (A_{01} f_1) = \int_{\mathbb{T}^d} f_1(t) dt,$$

$$(A_{11}(k) f_1)(k_1) = E_k(k_1) f_1(k_1),\ f_i \in \mathcal{H}_i, i = 0,1,$$

where the function $E_k(\cdot)$ is defined by

$$E_k(k_1) := \varepsilon(\frac{1}{2}(k + k_1)) + \varepsilon(k_1).$$

Let the operator $\mathcal{A}^0(k)$, $k \in \mathbb{T}^d$ act in $\mathcal{H}_0 \oplus \mathcal{H}_1$ as

$$\mathcal{A}^0(k) := \begin{pmatrix} 0 & 0 \\ 0 & A_{11}(k) \end{pmatrix}.$$

The perturbation $\mathcal{A}_\mu(k)$-$\mathcal{A}^0(k)$ of the operator $\mathcal{A}^0(k)$ is a self-adjoint operator of rank 2. Therefore, in accordance with the Weyl theorem about the invariance of the essential spectrum under finite rank perturbations, the essential spectrum of the operator $\mathcal{A}_\mu(k)$ coincides with the essential spectrum of $\mathcal{A}^0(k)$. It is evident that

$$\sigma_{\text{ess}}(\mathcal{A}^0(k)) = [m(k); M(k)],$$

where the numbers $m(k)$ and $M(k)$ are defined by

$$m(k) := \min_{k_1 \in \mathbb{T}^d} E_k(k_1) \text{ and } M(k) := \max_{k_1 \in \mathbb{T}^d} E_k(k_1).$$

This yields $\sigma_{\text{ess}}(\mathcal{A}_\mu(k)) = [m(k); M(k)]$.

For any fixed $\mu > 0$ and $k \in \mathbb{T}^d$, we define an analytic function $\Delta\mu(k;\cdot)$ in $\mathbb{C} \setminus [m(k); M(k)]$ by

$$\Delta_\mu(k; z) := \gamma - z - \frac{\mu^2}{2} \int_{\mathbb{T}^d} \frac{dt}{E_k(t) - z}.$$

Usually the function $\Delta_\mu(k;\cdot)$ is called the Fredholm determinant associated with the operator $\mathcal{A}_\mu(k)$.

The following lemma[4] is a simple consequence of the Birman-Schwinger principle and the Fredholm theorem.

Lemma 2.1 For any $\mu > 0$ and $k \in \mathbb{T}^d$ the operator $\mathcal{A}_\mu(k)$ has an eigenvalue $z_\mu(k) \in \mathbb{C} \setminus [m(k); M(k)]$ if and only if $\Delta_\mu(k; z_\mu(k)) = 0$.

From Lemma 2.1 it follows that for the discrete spectrum of $\mathcal{A}_\mu(k)$ the equality

$$\sigma_{\text{disc}}(\mathcal{A}_\mu(k)) = \{z \in \mathbb{C} \setminus [m(k); M(k)] : \Delta_\mu(k; z) = 0\}$$

holds.

The following lemma describes the number and location of the eigenvalues of $\mathcal{A}_\mu(k)$.

Lemma 2.2 For any fixed $\mu > 0$ and $k \in \mathbb{T}^{\mathrm{d}}$ the operator $\mathcal{A}_\mu(k)$ has no more than one simple eigenvalue lying on the l.h.s. of $m(k)$ (resp. r.h.s. of $M(k)$).

The proof of Lemma 2.2 is elementary and it follows from the fact that for any fixed $\mu > 0$ and $k \in \mathbb{T}^{\mathrm{d}}$ the function $\Delta_\mu(k;\cdot)$ is a monotonically decreasing on $(-\infty; m(k))$ and $(M(k); +\infty)$.

3. The Faddeev equation and its symmetric version: main properties

In this section we derive an analogue of the Faddeev type integral equation and its symmetric version for eigenfunctions corresponding to the discrete eigenvalues (isolated eigenvalues with finite multiplicity) of $\mathcal{A}_\mu$, which plays a crucial role in our analysis of the spectrum of $\mathcal{A}_\mu$.

We recall that for $\lambda \in \mathbb{R}$ and $\Omega \subset \mathbb{R}$, their arithmetic sum is defined as

$$\lambda + \Omega := \{\lambda + \omega : \omega \in \Omega\}.$$

To simplify the notation we set

$$\Lambda_\mu := \bigcup_{k \in \mathbb{T}^{\mathrm{d}}} (\varepsilon(k) + \sigma_{\text{disc}}(\mathcal{A}_\mu(k))), \Sigma_\mu := [0; 6\mathrm{d}] \cup \Lambda_\mu.$$

Here by Lemma 2.1 we may define the set Λ_μ as the set of all complex numbers $z \in \mathbb{C} \setminus [m(k); M(k)]$ such that the equality $\Delta_\mu(k; z-\varepsilon(k)) = 0$ holds for some $k \in \mathbb{T}^{\mathrm{d}}$.

For each $\mu > 0$ and $z \in \mathbb{C} \setminus \Sigma_\mu$ we introduce the operator $T_\mu(z)$ acting in $\mathcal{H}_1$ as

$$(T_\mu(z)g)(k_1) = \frac{\mu^2}{2\Delta_\mu(k_1; z - \varepsilon(k_1))} \int_{\mathbb{T}^{\mathrm{d}}} \frac{g(t)dt}{w_2(k_1, t) - z}.$$

We note that $T_\mu(z)$ is the Hilbert-Schmidt operator for each $\mu > 0$ and $z \in \mathbb{C} \setminus \Sigma_\mu$.

The following theorem is an analog of the well-known Faddeev's result for the operator $\mathcal{A}_\mu$ and establishes a connection between eigenvalues of $\mathcal{A}_\mu$ and $T_\mu(z)$.

Theorem 3.1 For any fixed $\mu > 0$ the number $z \in \mathbb{C} \setminus \Sigma_\mu$ is an eigenvalue of the operator $\mathcal{A}_\mu$ if and only if the number $\lambda = 1$ is an eigenvalue of the operator $T_\mu(z)$. Moreover the eigenvalues z and 1 have the same multiplicities.

Proof. Let $\mu > 0$ be a fixed, $z \in \mathbb{C} \setminus \Sigma_\mu$ be an eigenvalue of the operator $\mathcal{A}_\mu$ and $f = (f_1, f_2) \in \mathcal{H}$ be the corresponding eigenvector. Then the functions f_1 and f_2 satisfy the system of equations

$$\begin{aligned} &(w_1(k_1) - z) f_1(k_1) + \mu \int_{\mathbb{T}^{\mathrm{d}}} f_2(k_1, t) dt = 0; \\ &\frac{\mu}{2}(f_1(k_1) + f_1(k_2)) + (w_2(k_1, k_2) - z) f_2(k_1, k_2) = 0. \end{aligned} \tag{2}$$

The condition $z \notin [0; 6\mathrm{d}]$ yields that the inequality $w_2(k_1, k_2) - z \neq 0$. Then from the second equation of the system (2) for f_2 we have

$$f_2(k_1, k_2) = -\frac{\mu(f_1(k_1) + f_1(k_2))}{2(w_2(k_1, k_2) - z)}. \tag{3}$$

Substituting the expression (3) for f_2 into the first equation in the system (2), we obtain

$$\Delta_\mu(k_1; z-\varepsilon(k_1)) f_1(k_1) - \frac{\mu^2}{2}\int_{\mathbb{T}^d} \frac{f_1(t)dt}{w_2(k_1,t)-z} = 0. \tag{4}$$

By definition of Λ_μ we have $\Delta_\mu(k_1; z-\varepsilon(k_1)) \neq 0$ for all $z \notin \Lambda_\mu$ and $k_1 \in \mathbb{T}^d$. Therefore, the equation (4) has a nontrivial solution if and only if the following equation

$$f_1(k_1) = \frac{\mu^2}{2\Delta_\mu(k_1; z-\varepsilon(k_1))}\int_{\mathbb{T}^d} \frac{f_1(t)dt}{w_2(k_1,t)-z}$$

or the operator equation

$$f_1 = T_\mu(z) f_1,\ f_1 \in \mathcal{H}_1 \tag{5}$$

has a nontrivial solution.

Now we prove that moreover the eigenvalues z and 1 have the same multiplicities, that is, the linear subspaces of solutions of (2) and (5) have the same dimension.

Let $z \in \mathbb{C} \setminus \Sigma_\mu$ be an eigenvalue of $\mathcal{A}_\mu$ with multiplicity n and the number 1 be an eigenvalue of $T_\mu(z)$ with multiplicity m. Our next aim is to prove $n = m$.

Assume $n < m$. Then there exist the linearly independent eigenfunctions φ_i, $i = 1, \ldots, m$ of the operator $T_\mu(z)$ corresponding to the eigenvalue 1. If we define $f^{(i)}$ as $f^{(i)} = (f_1^{(i)}, f_2^{(i)})$, where $f_1^{(i)} = \varphi_i$ and the function $f_2^{(i)}, i = 1, \ldots, m$ is defined by (3), then it satisfies the equation $\mathcal{A}_\mu f^{(i)} = z f^{(i)}$ for $i = 1, \ldots, m$. Since $n < m$, there exists a non-trivial vector $(c_1, \ldots, c_m) \in \mathbb{C}^m$ such that $\sum_{i=1}^m c_i\varphi_i \neq 0$ and $\sum_{i=1}^m c_i f^{(i)} = 0$. We have

$$0 = \sum_{i=1}^m c_i f^{(i)} = \begin{pmatrix} \sum_{i=1}^m c_i f_1^{(i)}(k_1) \\ \sum_{i=1}^m c_i f_2^{(i)}(k_1,k_2) \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{i=1}^m c_i\varphi_i(k_1) \\ -\mu(2(w_2(k_1,k_2)-z)^{-1}\sum_{i=1}^m (c_i\varphi_i(k_1) + c_i\varphi_i(k_2)) \end{pmatrix} \neq 0.$$

This contradicts the fact that $n < m$.

Let now $n > m$. In this case there exist linearly independent eigenvectors $f^{(i)} = (f_1^{(i)}, f_2^{(i)})$, $i = 1, \ldots, n$ of the operator $\mathcal{A}_\mu$ corresponding to the eigenvalue z. It is obvious that the function $\varphi_i = f_1^{(i)}, i = 1, \ldots, n$, is an eigenfunction of $T_\mu(z)$ corresponding to the eigenvalue 1. By the assumption $n > m$ there exists non-trivial vector $(d_1, \ldots, d_n) \in \mathbb{C}^n$ such that $\sum_{i=1}^n d_i\varphi_i = 0$. But in this case due to the linearly independence of eigenvectors $f^{(i)}$, $i = 1, \ldots, n$ we obtain $\sum_{i=1}^n d_i f^{(i)} \neq 0$ Hence

$$0 \neq \sum_{i=1}^{n} d_i f^{(i)} = \begin{pmatrix} \sum_{i=1}^{n} d_i f_1^{(i)}(k_1) \\ \sum_{i=1}^{n} d_i f_2^{(i)}(k_1, k_2) \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{i=1}^{n} d_i f_1^{(i)}(k_1) \\ -\mu(2(w_2(k_1,k_2)-z)^{-1} \sum_{i=1}^{n} d_i (f_1^{(i)}(k_1) + f_1^{(i)}(k_2)) \end{pmatrix} = 0.$$

This contradicts the fact that $n > m$. Therefore, $n = m$.

Remark 3.2 We point out that the matrix equation (5) is an analogue of the Faddeev type integral equation for eigenfunctions of the operator $\mathcal{A}_\mu$ and it plays a crucial role in the analysis of the spectrum of $\mathcal{A}_\mu$.

Let

$$a_\mu^{\min} := \min \Sigma_\mu, a_\mu^{\max} := \max \Sigma_\mu.$$

We notice that by the definition of the quantity $a_\mu^{\min}$ for any $\mu > 0$, $k \in \mathbb{T}^d$ and $z < a_\mu^{\min}$ (resp. $z > a_\mu^{\max}$)), the function $\Delta_\mu(k; z-\varepsilon(k))$ (resp. $-\Delta_\mu(k; z-\varepsilon(k))$) is positive and hence, its positive square root exists.

In our analysis of the discrete spectrum of $\mathcal{A}_\mu$ the crucial role is played in the following compact operator $\hat{T}_\mu(z), z \in \mathbb{R} \setminus [a_\mu^{\min}; a_\mu^{\max}]$, acting in the space $L^2(\mathbb{T}^d)$ as integral operator

$$(\hat{T}_\mu(z)g)(k_1) = \frac{\mu^2}{2\sqrt{\Delta_\mu(k_1; z-\varepsilon(k_1))}} \int_{\mathbb{T}^d} \frac{g(t)dt}{\sqrt{\Delta_\mu(t; z-\varepsilon(t))}(w_2(k_1,t)-z)},$$

for $z < a_\mu^{\min}$, and

$$(\hat{T}_\mu(z)g)(k_1) = -\frac{\mu^2}{2\sqrt{-\Delta_\mu(k_1; z-\varepsilon(k_1))}} \int_{\mathbb{T}^d} \frac{g(t)dt}{\sqrt{-\Delta_\mu(t; z-\varepsilon(t))}(w_2(k_1,t)-z)},$$

for $z > a_\mu^{\max}$.

The following theorem establishes a connection between the eigenvalues of $\mathcal{A}_\mu$ and $\hat{T}_\mu(z)$.

Theorem 3.3 For any fixed $\mu > 0$ the number $z \in \mathbb{R} \setminus [a_\mu^{\min}; a_\mu^{\max}]$ is an eigenvalue of the operator $\mathcal{A}_\mu$ if and only if the number $\lambda = 1$ is an eigenvalue of the operator $\hat{T}_\mu(z)$. Moreover the eigenvalues *z and* 1 have the same multiplicities.

Proof. In the proof of Theorem 3.1 we show that the number $z \in \mathbb{C} \setminus \Sigma_\mu$ is an eigenvalue of the operator $\mathcal{A}_\mu$ if and only if the equation (4) has a nontrivial solution, and the linear subspaces of solutions of $\mathcal{A}_\mu f = zf$ and (4) have the same dimension.

By the definition of Λ_μ the inequality $\Delta_\mu(k; z-\varepsilon(k)) > 0$ (resp. $-\Delta_\mu(k; z-\varepsilon(k)) > 0$) holds for any $\mu > 0$, $k \in \mathbb{T}^d$ and $z < a_\mu^{\min}$ (resp. $z > a_\mu^{\max}$).

Therefore, the following equation

$$g(k_1) = \frac{\mu^2}{2\sqrt{\Delta_\mu(k_1; z-\varepsilon(k_1))}} \int_{\mathbb{T}^d} \frac{(\Delta_\mu(t; z-\varepsilon(t)))^{-1/2} g(t)dt}{w_2(k_1,t)-z}, \; z < a_\mu^{\min}, \qquad (6)$$

resp.

$$g(k_1) = -\frac{\mu^2}{2\sqrt{-\Delta_\mu(k_1; z-\varepsilon(k_1))}}\int_{\mathbb{T}^d}\frac{(-\Delta_\mu(t; z-\varepsilon(t)))^{-1/2} g(t)dt}{w_2(k_1,t)-z}, \; z > a_\mu^{\max}, \quad (7)$$

has a nontrivial solution if the system of equations (4) has a nontrivial solution. Moreover, the linear subspaces of solutions of (6) (resp. (7)) and (4) have the same dimension.

4. Essential spectrum of $\mathcal{A}_\mu$ and its new branches

In this section applying the statements of sections 2 and 3, the Weyl criterion [22] we describe the location of the essential spectrum of $\mathcal{A}_\mu$, then we define its two- and three-particle branches.

Denote by $\|\cdot\|$ and $(\cdot,\cdot)$ the norm and scalar product in the corresponding Hilbert spaces.

For the convenience of the reader we formulate Weyl's criterion for the essential spectrum of $\mathcal{A}_\mu$ as follows. First, a number λ is in the spectrum of $\mathcal{A}_\mu$ if and only if there exists a sequence $\{F_n\}$ in the space $\mathcal{H}$ such that $\| F_n \| = 1$ and

$$\lim_{n\to\infty}\|(\mathcal{A}_\mu - \lambda E)F_n = 0. \quad (8)$$

Here E is an identity operator on $\mathcal{H}$. Furthermore, λ is in the essential spectrum if there is a sequence satisfying this condition, but such that it contains no convergent subsequence (this is the case if, for example $\{F_n\}$ is an orthonormal sequence); such a sequence is called a singular sequence.

The following theorem describes the location of the essential spectrum of the operator $\mathcal{A}_\mu$ by the spectrum of the family $\mathcal{A}_\mu(k)$ of generalized Friedrichs models.

Theorem 4.1 For the essential spectrum of $\mathcal{A}_\mu$ the equality $\sigma_{\text{ess}}(\mathcal{A}_\mu) = \Sigma_\mu$ holds. Moreover, the set Σ_μ consists of no more than three bounded closed intervals.

Proof. We starts the proof with the inclusion $\Sigma_\mu \subset \sigma_{\text{ess}}(\mathcal{A}_\mu)$. Since the set Σ_μ has form $\Sigma_\mu = \Lambda_\mu \cup [0; 6d]$ first we show that $[0; 6d] \subset \sigma_{\text{ess}}(\mathcal{A}_\mu)$. Let $\lambda_0 \in [0; 6d]$ be an arbitrary point. We prove that $\lambda_0 \in \sigma_{\text{ess}}(\mathcal{A}_\mu)$. To this end it is convenient to use Weyl criterion[22], i.e. it is suffces to construct a sequence of orthonormal vector-functions $\{F_n\} \subset \mathcal{H}$ satisfying (8).

From continuity of the function $w_2(\cdot,\cdot)$ on the compact set $(\mathbb{T}^d)^2$ it follows that there exists some points $(k_1^0, k_2^0) \in (\mathbb{T}^d)^2$ such that $\lambda_0 = w_2(k_1^0, k_2^0)$.

For $n \in \mathbb{N}$ we consider the following neighborhood of the point $(k_1^0, k_2^0) \in (\mathbb{T}^d)^2$:

$$W_n := V_n(k_1^0) \times V_n(k_2^0),$$

where

$$V_n(k_1^0) := \left\{k_1 \in \mathbb{T}^d : \frac{1}{n+n_0+1} < | k_1 - k_1^0 | < \frac{1}{n+n_0}\right\}$$

is the punctured neighborhood of the point $k_1^0 \in \mathbb{T}^d$ and $n_0 \in \mathbb{N}$ is chosen in such way that $V_n(k_1^0) \cap V_n(k_2^0) = \phi$ for all $n \in \mathbb{N}$ (provided that $k_1^0 \neq k_2^0$).

Let mes(Ω) be the Lebesgue measure of the set Ω and $\chi_\Omega(\cdot)$ be the characteristic function of the set Ω. We choose the sequence of functions $\{F_n\} \subset \mathcal{H}$ as follows:

$$F_n := \frac{1}{\sqrt{2\text{mes}(W_n)}}\begin{pmatrix} 0 \\ \chi_{W_n}(k_1,k_2) + \chi_{W_n}(k_2,k_1) \end{pmatrix}.$$

It is clear that $\{F_n\}$ is an orthonormal sequence.

For any $n \in \mathbb{N}$ let us consider $(\mathcal{A}_\mu - \lambda_0 E)F_n$ and estimate its norm:

$$\| \mathcal{A}_\mu - \lambda_0 E)F_n \ ^2 \leq \sup_{(k_1,k_2)\in W_n} | w_2(k_1,k_2) - \lambda_0 |^2 + 2\mathrm{mes}(V_n(k_1^0)) \max_{k_1 \in \mathbb{T}^{\mathrm{d}}} | v(k_1) |^2 .$$

From the construction of the set $V_n(k_1^0)$ and from the continuity of the function $w_2(\cdot,\cdot)$ it follows $\|(\mathcal{A}_\mu - \lambda_0 E)F_n\| \to 0$. as n → ∞, i.e. $\lambda_0 \in \sigma_{\mathrm{ess}}(\mathcal{A}_\mu)$. Since the point λ_0 is an arbitrary we have $[0; 6d] \subset \sigma_{\mathrm{ess}}(\mathcal{A}_\mu)$.

Now let us prove that $\Lambda_\mu \subset \sigma_{\mathrm{ess}}(\mathcal{A}_\mu)$. Taking an arbitrary point $z_\mu \in \Lambda_\mu$ we show that $z_\mu \in \sigma_{\mathrm{ess}}(\mathcal{A}_\mu)$. Two cases are possible: $z_\mu \in [0; 6d]$ or $z_\mu \notin [0; 6d]$. If $z_\mu \in [0; 6d]$, then it is already proven above that $z_\mu \in \sigma_{\mathrm{ess}}(\mathcal{A}_\mu)$. Let $z_\mu \in \Lambda_\mu \setminus [0; 6d]$. Definition of the set Λ_μ and Lemma 2.1 imply that there exists a point $k_1^1 \in \mathbb{T}^{\mathrm{d}}$ such that $\Delta_\mu(k_1^1; z_\mu - \varepsilon(k_1^1)) = 0$.

We choose the sequence of orthogonal vector-functions $\{\Phi_n\}$ in the following form

$$\Phi_n := \begin{pmatrix} \phi_1^{(n)}(k_1) \\ \phi_2^{(n)}(k_1,k_2) \end{pmatrix},$$

where

$$\phi_1^{(n)}(k_1) := \frac{c_n(k_1)\chi_{V_n(k_1^1)}(k_1)}{\sqrt{\mathrm{mes}(V_n(k_1^1))}};$$

$$\phi_2^{(n)}(k_1,k_2) := -\frac{\mu(\phi_1^{(n)}(k_1) + \phi_1^{(n)}(k_2))}{2(w_2(k_1,k_2) - z_\mu)}.$$

Here $c_n(\cdot) \in L^2(\mathbb{T}^{\mathrm{d}})$ is chosen from the orthonormality condition for $\{\Phi_n\}$, that is, from the condition

$$(\Phi_n, \Phi_m) = \frac{\mu^2}{2\sqrt{\mathrm{mes}(V_n(k_1^1))}\sqrt{\mathrm{mes}(V_m(k_1^1))}} \int_{V_n(k_1^1)} \int_{V_m(k_1^1)} \frac{c_n(s)c_m(t)}{(w_2(s,t) - z_\mu)^2} ds dt = 0 \tag{9}$$

for $n \neq m$ and $\|\Phi_n\| = 1$.

The existence of $\{c_n(\cdot)\}$ is a consequence of the following proposition.

Proposition 4.2 There exists an orthonormal system $\{c_n(\cdot)\} \in L^2(\mathbb{T}^{\mathrm{d}})$ satisfying the conditions supp $cn(\cdot) \subset V_n(k_1^1)$ and (9).

Proof of Proposition 4.2. We construct the sequence $\{c_n(\cdot)\}$ by the induction method. Suppose that $c_1(k_1) := \chi_{V_1(k_1^1)}(k_1)\left(\sqrt{\mathrm{mes}(V_1(k_1^1))}\right)^{-1}$. Now we choose the function $\tilde{c}_2(\cdot) \in L^2(V_2(k_1^1))$ so that $\| \tilde{c}_2(\cdot) \| = 1$ and $(\tilde{c}_2(\cdot), \xi_1^{(2)}(\cdot)) = 0$, where

$$\xi_1^{(2)}(k_1) := \chi_{V_2(k_1^1)}(k_1) \int_{\mathbb{T}^{\mathrm{d}}} \frac{c_1(t)dt}{(w_2(k_1,t) - z_\mu)^2}.$$

Set $c_2(k_1) := \tilde{c}_2(k_2)\chi_{V_1(k_1^1)}(k_1)$. We continue this process. Suppose that $c_1(k_1), \ldots, c_n(k_1)$ are constructed. Then the function $\tilde{c}_{n+1}(\cdot) \in L^2(V_{n+1}(k_1^1))$ is chosen so that it is orthogonal to all functions

$$\xi_m^{(n+1)}(k_1) := \chi_{V_{n+1}(k_1^1)}(k_1) \int_{\mathbb{T}^{\mathrm{d}}} \frac{c_m(t)dt}{(w_2(k_1,t) - z_\mu)^2}, \ m = 1, \ldots, n$$

and $\|\tilde{c}_{n+1}(\cdot)\|=1$. Let $c_{n+1}(k_1):=\tilde{c}_{n+1}(k_1)\chi_{V_{n+1}(k_1^1)}(k_1)$. Thus, we have constructed the orthonormal system of functions $\{c_n(\cdot)\}$ satisfying the assumptions of the proposition. Proposition 4.2 is proved.

We continue the proof of Theorem 4.1. Now we are in a position to show

$$\lim_{n\to\infty}\|(\mathcal{A}_\mu - z_\mu E)\Phi_n \quad = 0.$$

To this end for $n \in \mathbb{N}$ we consider $(\mathcal{A}_\mu - z_\mu E)\Phi_n$ and estimate its norm as

$$\|(\mathcal{A}_\mu - z_\mu E)\Phi_n\|^2 \le C(\mu)\,\mathrm{mes}\,(V_n(k_1^1)) + 2\sup_{k_1\in V_n(k_1^1)} |\Delta_\mu(k_1;z_\mu-\varepsilon(k_1))|^2 \tag{10}$$

for some $C(\mu) > 0$.

Since $\mathrm{mes}\,(V_n(k_1^1))\to 0$ and $\sup_{k_1\in V_n(k_1^1)} |\Delta_\mu(k_1;z_\mu-\varepsilon(k_1))|^2\to 0$ as $n\to\infty$, from the estimate (10) it follows that $\|(\mathcal{A}_\mu - z_\mu E)\Phi_n\| \to 0$ as $n\to\infty$. This implies $z_\mu \in \sigma_{\mathrm{ess}}(\mathcal{A}_\mu)$. Since the point z_μ is an arbitrary, we have $\Lambda_\mu \subset \sigma_{\mathrm{ess}}(\mathcal{A}_\mu)$. Therefore, we have proved that $\Sigma_\mu \subset \sigma_{\mathrm{ess}}(H)$.

Now we prove the inverse inclusion, i.e. $\sigma_{\mathrm{ess}}(\mathcal{A}_\mu) \subset \Sigma_\mu$. For each $z\in\mathbb{C}\setminus\Sigma_\mu$ the operator $T_\mu(z)$ is a compact-operator-valued function on $\mathbb{C}\setminus\Sigma_\mu$. Then from the self-adjointness of $\mathcal{A}_\mu$ and Theorem 3.1 it follows that the operator $(I-T_\mu(z))^{-1}$ exists if z is real and has a large absolute value. The analytic Fredholm theorem (see, e.g., Theorem VI.14 in [22]) implies that there is a discrete set $S_\mu\subset\mathbb{C}\setminus\Sigma_\mu$ such that the operator-valued function $(I-T_\mu(z))^{-1}$ exists and is analytic on $\mathbb{C}\setminus(S_\mu \cup \Sigma_\mu)$ and is meromorphic on $\mathbb{C}\setminus\Sigma_\mu$ with finite-rank residues. This implies that the set $\sigma(\mathcal{A}_\mu)\setminus\Sigma_\mu$ consists of isolated points, and the only possible accumulation points of Σ_μ can be on the boundary. Thus

$$\sigma(\mathcal{A}_\mu)\setminus\Sigma_\mu \subset \sigma_{\mathrm{disc}}(\mathcal{A}_\mu) = \sigma(\mathcal{A}_\mu)\setminus\sigma_{\mathrm{ess}}(\mathcal{A}_\mu).$$

Therefore, the inclusion $\sigma_{\mathrm{ess}}(\mathcal{A}_\mu)\subset\Sigma_\mu$ holds. Finally we obtain the equality $\sigma_{\mathrm{ess}}(\mathcal{A}_\mu)=\Sigma_\mu$.

By Lemma 2.2 for any $k\in\mathbb{T}^{\mathrm{d}}$ the operator $\mathcal{A}_\mu(k)$ has no more than one simple eigenvalue on the l.h.s. (resp. r.h.s) of 0 (resp. 6d). Then by the theorem on the spectrum of decomposable operators [22] and by the definition of the set Λ_μ it follows that the set Λ_μ consists of the union of no more than two bounded closed intervals. Therefore, the set Σ_μ consists of the union of no more than three bounded closed intervals. Theorem 4.1 is completely proved.

In the following we introduce the new subsets (branches) of the essential spectrum of $\mathcal{A}_\mu$.

Definition 4.3 The sets $\sigma_{two}(\mathcal{A}_\mu)$: $= \Lambda_\mu$ and $\sigma_{three}(\mathcal{A}_\mu)$: $= [0; 6\mathrm{d}]$ are called two- and three-particle branches of the essential spectrum of $\mathcal{A}_\mu$, respectively.

The definition of the set Λ_μ and the equality

$$\bigcup_{k\in\mathbb{T}^{\mathrm{d}}}\left[\varepsilon(k)+m(k);\varepsilon(k)+M(k)\right]=[0;6\mathrm{d}]$$

together with Theorem 4.1 give the equality

$$\sigma_{\mathrm{ess}}(\mathcal{A}_\mu)=\bigcup_{k\in\mathbb{T}^{\mathrm{d}}}\left\{\varepsilon(k)+\sigma(\mathcal{A}_\mu(k))\right\}. \tag{11}$$

Here the family of operators $\mathcal{A}_\mu(k)$ have a simpler structure than the operator $\mathcal{A}_\mu$. Hence, in many instance, (11) provides an effective tool for the description of the essential spectrum.

Since for any $z\in\mathbb{C}\setminus\Sigma_\mu$ the kernel of the integral operator $T_\mu(z)$ is a continuous function on $(\mathbb{T}^{\mathrm{d}})^2$, the Fredholm determinant $\Omega_\mu(z)$ of the operator $I-T_\mu(z)$, where I is the identity operator in $\mathcal{H}_1$, exists and is a real-analytic function on $\mathbb{C}\setminus\Sigma_\mu$.

According to Fredholm's theorem [22] and Theorem 3.1 the number $z\in\mathbb{C}\setminus\Sigma_\mu$ is an eigenvalue of $\mathcal{A}_\mu$ if and only if $\Omega_\mu(z)=0$, that is,

$$\sigma_{\text{disc}}(\mathcal{A}_\mu) = \{z \in \mathbb{C} \setminus \Sigma_\mu : \Omega_\mu(z) = 0\}.$$

5. Estimates for the bounds of the essential spectrum of $\mathcal{A}_\mu$

In this section, we estimate the lower and upper bounds of the essential spectrum of $\mathcal{A}_\mu$ for all dimensions $\mathrm{d} \in \mathbb{N}$ of the torus $\mathbb{T}^{\mathrm{d}}$ and for all values of a coupling constant $\mu > 0$.

It is easy to show that the function $w_2(\cdot,\cdot)$ has a unique non-degenerate minimum (resp. maximum) at the point $(\overline{0},\overline{0}) \in (\mathbb{T}^{\mathrm{d}})^2$ (resp. $(\overline{\pi},\overline{\pi}) \in (\mathbb{T}^{\mathrm{d}})^2$) and

$$\min_{k_1,k_2 \in \mathbb{T}^{\mathrm{d}}} w_2(k_1,k_2) = w_2(\overline{0},\overline{0}) = 0, \quad \max_{k_1,k_2 \in \mathbb{T}^{\mathrm{d}}} w_2(k_1,k_2) = w_2(\overline{\pi},\overline{\pi}) = 6\mathrm{d},$$

where $\overline{0} := (0,\ldots,0),\ \overline{\pi} := (\pi,\ldots,\pi) \in \mathbb{T}^{\mathrm{d}}$. A trivial verification shows that

$$\min \sigma_{\text{ess}}(\mathcal{A}_\mu(\overline{0})) = m(\overline{0}) = 0, \ \ \max \sigma_{\text{ess}}(\mathcal{A}_\mu(\overline{\pi})) = M(\overline{\pi}) = 4\mathrm{d}.$$

Therefore, in order to study the bounds of the essential spectrum of $\mathcal{A}_\mu$ it is necessary to study the eigenvalues of $\mathcal{A}_\mu(\overline{0})$ (resp. $\mathcal{A}_\mu(\overline{\pi})$) smaller than 0 (resp. bigger than 4d). So, in the remainder of this section we work with the functions $E_{\overline{0}}(\cdot)$ and $E_{\overline{\pi}}(\cdot)$:

$$E_{\overline{0}}(k_1) = \varepsilon(k_1/2) + \varepsilon(k_1), \ E_{\overline{\pi}}(k_1) = \varepsilon((\overline{\pi}+k_1)/2) + \varepsilon(k_1).$$

By the construction of the function $\varepsilon(\cdot)$ there exist positive numbers δ, C_1, C_2 such that

$$C_1 |t|^2 \le \varepsilon(t/2) + \varepsilon(t) \le C_2 |t|^2, \ t \in U_\delta(\overline{0}). \tag{12}$$

Using the Lebesgue dominated convergence theorem we obtain that there exist the positive (finite or infinite) limits

$$\lim_{z \to -0} \int_{\mathbb{T}^{\mathrm{d}}} \frac{dt}{\varepsilon(t/2) + \varepsilon(t) - z} = \int_{\mathbb{T}^{\mathrm{d}}} \frac{dt}{\varepsilon(t/2) + \varepsilon(t)};$$

$$\lim_{z \to 4\mathrm{d}+0} \int_{\mathbb{T}^{\mathrm{d}}} \frac{dt}{z - (\varepsilon((\overline{\pi}+t)/2) + \varepsilon(t))} = \int_{\mathbb{T}^{\mathrm{d}}} \frac{dt}{4\mathrm{d} - (\varepsilon((\overline{\pi}+t)/2) + \varepsilon(t))}.$$

An easy computation shows that

$$\int_{\mathbb{T}^{\mathrm{d}}} \frac{dt}{4\mathrm{d} - (\varepsilon((\overline{\pi}+t)/2) + \varepsilon(t))} = \int_{\mathbb{T}^{\mathrm{d}}} \frac{dt}{\varepsilon(t/2) + \varepsilon(t)}.$$

For the cases d = 1, 2 we show that the latter integral is not finite.

Let d = 1. From the additivity of the integral it follows that

$$\int_{\mathbb{T}^1} \frac{dt}{\varepsilon(t/2) + \varepsilon(t)} = \int_{\mathbb{T}^1 \setminus U_\delta(0)} \frac{dt}{\varepsilon(t/2) + \varepsilon(t)} + \int_{-\delta}^{\delta} \frac{dt}{\varepsilon(t/2) + \varepsilon(t)} \ge \int_{-\delta}^{\delta} \frac{dt}{\varepsilon(t/2) + \varepsilon(t)}.$$

Applying (12) we deduce that

$$\int_{-\delta}^{\delta} \frac{dt}{\varepsilon(t/2) + \varepsilon(t)} \ge \frac{1}{C_2} \int_{-\delta}^{\delta} \frac{dt}{t^2} = +\infty.$$

Taking into account above mentioned facts we get

$$\lim_{z\to -0}\Delta_\mu(0;z)=-\infty,\quad \lim_{z\to 4+0}\Delta_\mu(\pi;z)=+\infty.$$

Then from monotonicity and continuity property of $\Delta_\mu(k;\cdot)$, and from the equalities

$$\lim_{z\to -\infty}\Delta_\mu(0;z)=+\infty,\quad \lim_{z\to +\infty}\Delta_\mu(\pi;z)=-\infty$$

it follows that for any $\mu > 0$ there exist two points $E_\mu^{(l)}\in(-\infty;0)$ and $E_\mu^{(r)}\in(4;+\infty)$ such $\Delta_\mu(0;E_\mu^{(l)})=0$ and $\Delta_\mu(\pi;\ E_\mu^{(r)})=0$. So, by Lemma 2.1 for any $\mu > 0$ the operator $\mathcal{A}_\mu(0)$ has a unique negative eigenvalue $E_\mu^{(l)}$ and the operator $\mathcal{A}_\mu(\pi)$ has a unique eigenvalue $E_\mu^{(r)}$ bigger than 4.

By the definitions of the quantities $a_\mu^{\min}$ and $a_\mu^{\max}$ introduced in Section 3 we have

$$a_\mu^{\min}=\min\{\lambda:\lambda\in\sigma_{\mathrm{ess}}(\mathcal{A}_\mu)\},\ a_\mu^{\max}=\max\{\lambda:\lambda\in\sigma_{\mathrm{ess}}(\mathcal{A}_\mu)\}.$$

Then, $a_\mu^{\min}$, $a_\mu^{\max}\in\sigma_{\mathrm{ess}}(\mathcal{A}_\mu)$ and they are called the lower and upper bounds of the essential spectrum of $\mathcal{A}_\mu$, respectively.

From the construction of the sets $\sigma_{\mathrm{two}}(\mathcal{A}_\mu)$ and $\sigma_{\mathrm{three}}(\mathcal{A}_\mu)$ it follows that

$$a_\mu^{\min}=\min\sigma_{\mathrm{two}}(\mathcal{A}_\mu)\le\varepsilon(0)+E_\mu^{(l)}<\min\sigma_{\mathrm{three}}(\mathcal{A}_\mu)=0;$$

$$a_\mu^{\max}=\max\sigma_{\mathrm{two}}(\mathcal{A}_\mu)\ge\varepsilon(\pi)+E_\mu^{(r)}>\max\sigma_{\mathrm{three}}(\mathcal{A}_\mu)=6.$$

We now turn to the case d = 2. In the same manner we can see that

$$\int_{\mathbb{T}^2}\frac{dt}{\varepsilon(t/2)+\varepsilon(t)}\ge\int_{U_\delta(\bar{0})}\frac{dt}{\varepsilon(t/2)+\varepsilon(t)}\ge\frac{1}{C_2}\int_{U_\delta(\bar{0})}\frac{dt}{|t|^2}.$$

Now, passing to the polar coordinate system

$$t_1=r\sin\alpha,\ t_2=r\cos\alpha,\ 0\le r\le\delta,\ 0\le\alpha\le 2\pi,$$

we can assert that

$$\int_{U_\delta(\bar{0})}\frac{dt}{|t|^2}=+\infty.$$

Therefore, for the lower and upper bounds of the essential spectrum of $\mathcal{A}_\mu$ we get

$$a_\mu^{\min}=\min\sigma_{\mathrm{two}}(\mathcal{A}_\mu)\le\varepsilon(\bar{0})+E_\mu^{(l)}<\min\sigma_{\mathrm{three}}(\mathcal{A}_\mu)=0;$$

$$a_\mu^{\max}=\max\sigma_{\mathrm{two}}(\mathcal{A}_\mu)\ge\varepsilon(\bar{\pi})+E_\mu^{(r)}>\max\sigma_{\mathrm{three}}(\mathcal{A}_\mu)=12.$$

Summarizing, we have the result about bounds of the essential spectrum of $\mathcal{A}_\mu$.

Theorem 5.1 Assume $d = 1, 2$. Then for any $\mu > 0$ we have the following estimates for the bounds of the two- and three-particle branches of the essential spectrum of $\mathcal{A}_\mu$:

$$a_\mu^{\min} = \min \sigma_{\text{two}}(\mathcal{A}_\mu) \le \varepsilon(\overline{0}) + E_\mu^{(l)} < \min \sigma_{\text{three}}(\mathcal{A}_\mu) = 0;$$

$$a_\mu^{\max} = \max \sigma_{\text{two}}(\mathcal{A}_\mu) \ge \varepsilon(\overline{\pi}) + E_\mu^{(r)} > \max \sigma_{\text{three}}(\mathcal{A}_\mu) = 6\text{d}.$$

This result is important in the proof of the finiteness of the number of eigenvalues of $\mathcal{A}_\mu$ in the cases d = 1, 2.

Suppose d ≥ 3. The additivity property of the integral implies

$$\int_{\mathbb{T}^\text{d}} \frac{dt}{\varepsilon(t/2)+\varepsilon(t)} = \int_{\mathbb{T}^\text{d}\setminus U_\delta(\overline{0})} \frac{dt}{\varepsilon(t/2)+\varepsilon(t)} + \int_{U_\delta(\overline{0})} \frac{dt}{\varepsilon(t/2)+\varepsilon(t)}. \tag{13}$$

Since the integrand of the first summand on the r.h.s. of (13) is continuous function on a compact set $\mathbb{T}^\text{d}\setminus U_\delta(\overline{0})$, it is finite. Applying (12) we deduce that

$$\int_{U_\delta(\overline{0})} \frac{dt}{\varepsilon(t/2)+\varepsilon(t)} \le \frac{1}{C_1}\int_{U_\delta(\overline{0})} \frac{dt}{|t|^2} \le \frac{C_2}{C_1}\int_{\{k_1^2+k_2^2+k_3^2<\delta^2\}} \frac{dt_1 dt_2 dt_3}{t_1^2+t_2^2+t_3^2}.$$

Now, passing to the spherical coordinate system

$$t_1 = r\sin\psi\cos\varphi,$$

$$t_2 = r\sin\psi\sin\varphi,$$

$$t_3 = r\cos\psi,\ 0\le r\le\delta,\ 0\le\varphi\le 2\pi,\ 0\le\psi\le\pi,$$

we can assert that

$$\int_{\{k_1^2+k_2^2+k_3^2<\delta^2\}} \frac{dt_1 dt_2 dt_3}{t_1^2+t_2^2+t_3^2} = 4\pi\delta < \infty.$$

We introduce the following quantities

$$\mu_l^0(\gamma) := \sqrt{\gamma}\left(\int_{\mathbb{T}^\text{d}} \frac{dt}{\varepsilon(t/2)+\varepsilon(t)}\right)^{-1/2} \text{ for } \gamma > 0;$$

$$\mu_r^0(\gamma) := \sqrt{4\text{d}-\gamma}\left(\int_{\mathbb{T}^\text{d}} \frac{dt}{\varepsilon(t/2)+\varepsilon(t)}\right)^{-1/2} \text{ for } \gamma < 4\text{d}.$$

By the definition of $\mu_l^0(\gamma)$ and $\mu_r^0(\gamma)$ one can conclude that
if $\gamma \in (0; 2\text{d})$, then $\mu_l^0(\gamma) < \mu_r^0(\gamma)$;
if $\gamma = 2\text{d}$, then $\mu_l^0(\gamma) = \mu_r^0(\gamma)$;
if $\gamma \in (2\text{d}; 4\text{d})$, then $\mu_l^0(\gamma) > \mu_r^0(\gamma)$.

The modification of the following two Theorems are proved in [20] for the case d = 3, but they still hold for the case d > 3 due to the finiteness of the integral on the l.h.s. of (13).

Theorem 5.2 (i) If $\gamma \le 0$, then for any $\mu > 0$ the operator $\mathcal{A}_\mu(\overline{0})$ has a unique negative eigenvalue.
(ii) Let $\gamma > 0$. Then

(ii$_1$) for any $\mu \in (0;\ \mu_l^0(\gamma)]$ the operator $\mathcal{A}_\mu(\overline{0})$ has no negative eigenvalues;
(ii$_2$) for any $\mu > \mu_l^0(\gamma)$ the operator $\mathcal{A}_\mu(\overline{0})$ has a unique negative eigenvalue.

Theorem 5.3 (i) If $\gamma \geq 4\mathrm{d}$, then for any $\mu > 0$ the operator $\mathcal{A}_\mu(\overline{\pi})$ has no eigenvalues bigger than 4d.
(ii) Let $\gamma < 4\mathrm{d}$. Then
(ii$_1$) for any $\mu \in (0;\ \mu_r^0(\gamma)]$ the operator $\mathcal{A}_\mu(\overline{\pi})$ has no eigenvalues bigger than 4d;
(ii$_2$) for any $\mu > \mu_r^0(\gamma)$ the operator $\mathcal{A}_\mu(\overline{\pi})$ has a unique eigenvalue in $(4\mathrm{d}; +\infty)$.

Since $\mu_l^0(2\mathrm{d}) = \mu_r^0(2\mathrm{d})$, setting $\mu_0 := \mu_l^0(2\mathrm{d})$ let us mention an important consequence of Theorems 5.2 and 5.3.

Corollary 5.4 (i) If $\gamma \in (0; 2\mathrm{d}]$, then for $\mu = \mu_l^0(\gamma)$ the operator $\mathcal{A}_\mu(0)$ has no negative eigenvalues and the operator $\mathcal{A}_\mu(\overline{\pi})$ has no eigenvalues, bigger than 4d;
(ii) If $\gamma \in (2\mathrm{d}; 4\mathrm{d})$, then for $\mu = \mu_r^0(\gamma)$ the operator $\mathcal{A}_\mu(\overline{0})$ has an unique negative eigenvalue and the operator $\mathcal{A}_\mu(\overline{\pi})$ has no eigenvalues, bigger than 4d.

First we recall that the equalities $\min\sigma_{\text{three}}(\mathcal{A}_\mu) = 0$ and $\max\sigma_{\text{three}}(\mathcal{A}_\mu) = 6\mathrm{d}$ hold for any $\mu > 0$. We can now state the detailed information on bounds of the essential spectrum of $\mathcal{A}_\mu$ for the case $\mathrm{d} \geq 3$ with respect to the spectral parameters $\gamma \in \mathbb{R}$ and $\mu > 0$:

Case I. Let $\gamma \leq 0$. Then for any $\mu > 0$ we have

$$a_\mu^{\min} = \min \sigma_{\text{two}}(\mathcal{A}_\mu) \leq \varepsilon(\overline{0}) + E_\mu^{(l)} < 0;$$

moreover,

- $a_\mu^{\max} = 6\mathrm{d}$, if $\mu \in (0;\ \mu_r^0(\gamma)]$;
- $a_\mu^{\max} = \max \sigma_{\text{two}}(\mathcal{A}_\mu) \geq \varepsilon(\overline{\pi}) + E_\mu^{(r)} > 6\mathrm{d}$, if $\mu > \mu_r^0(\gamma)$.

Case II. Let $\gamma \in (0; 2\mathrm{d}]$. Then

- $a_\mu^{\min} = 0$ and $a_\mu^{\max} = 6\mathrm{d}$, if $\mu \in (-\infty; \mu_l^0(\gamma)]$;
- $a_\mu^{\min} = \min \sigma_{\text{two}}(\mathcal{A}_\mu) \leq \varepsilon(\overline{0}) + E_\mu^{(l)} < 0$ and $a_\mu^{\max} = 6\mathrm{d}$, if $\mu \in (\mu_l^0(\gamma); \mu_r^0(\gamma)]$;
- $a_\mu^{\min} = \min \sigma_{\text{two}}(\mathcal{A}_\mu) \leq \varepsilon(\overline{0}) + E_\mu^{(l)} < 0$ and $a_\mu^{\max} = \max \sigma_{\text{two}}(\mathcal{A}_\mu) \geq \varepsilon(\overline{\pi}) + E_\mu^{(r)} > 6\mathrm{d}$, if $\mu \in (\mu_r^0(\gamma); +\infty)$.

Case III. Let $\gamma \in (2\mathrm{d}; 4\mathrm{d})$. Then

- $a_\mu^{\min} = 0$ and $a_\mu^{\max} = 6\mathrm{d}$, if $\mu \in (-\infty; \mu_r^0(\gamma)]$;
- $a_\mu^{\min} = 0$ and $a_\mu^{\max} = \max \sigma_{\text{two}}(\mathcal{A}_\mu) \geq \varepsilon(\overline{\pi}) + E_\mu^{(r)} > 6\mathrm{d}$, if $\mu \in (\mu_r^0(\gamma); \mu_l^0(\gamma)]$;
- $a_\mu^{\min} = \min \sigma_{\text{two}}(\mathcal{A}_\mu) \leq \varepsilon(\overline{0}) + E_\mu^{(l)} < 0$ and $a_\mu^{\max} = \max \sigma_{\text{two}}(\mathcal{A}_\mu) \geq \varepsilon(\overline{\pi}) + E_\mu^{(r)} > 6\mathrm{d}$, if $\mu \in (\mu_l^0(\gamma); +\infty)$.

Case IV. Let $\gamma \geq 4\mathrm{d}$. Then for any $\mu > 0$ we have $a_\mu^{\max} = 6\mathrm{d}$; moreover,

- $a_\mu^{\min} = 0$, if $\mu \in (0; \mu_l^0(\gamma)]$;
- $a_\mu^{\min} = \min \sigma_{\text{two}}(\mathcal{A}_\mu) \leq \varepsilon(\overline{0}) + E_\mu^{(l)} < 0$, if $\mu > \mu_l^0(\gamma)$.

All assertions mentioned above play crucial role in the study of the number of discrete eigenvalues of $\mathcal{A}_\mu$ lying outside of its essential spectrum.

6. Appendix: lattice systems with non-conserved number of particles

The quantum systems with variable but finite number of particles occur naturally in quantum field theory, condensed matter physics, statistical physics and the theory of chemical reactions. It is remarkable that the study of such lattice systems with at most three particles is reduced to the investigation of the spectral properties of self-adjoint 3 × 3 operator matrix of the form

$$H_\mu := \begin{pmatrix} A_{00} & \mu A_{01} & \mu A_{02} \\ \mu A_{01}^* & A_{11} & \mu A_{12} \\ \mu A_{02}^* & \mu A_{12}^* & A_{22} \end{pmatrix},$$

acting in the Hilbert space $\mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2$. Here the matrix elements $A_{ij} : \mathcal{H}_j \to \mathcal{H}_i,\ i \leq j,\ i, j = 0,1,2$ are defined by

$$A_{00}f_0 = w_0 f_0,\ A_{01}f_1 = \int_{\mathbb{T}^d} v(t) f_1(t)dt,\ A_{02} = 0;$$

$$(A_{11}f_1)(k_1) = w_1(k_1) f_1(k_1),\ (A_{12}f_2)(k_1) = \int_{\mathbb{T}^d} v(t) f_2(k_1,t)dt;$$

$$(A_{22}f_2)(k_1,k_2) = w_2(k_1,k_2) f_2(k_1,k_2),\ f_i \in \mathcal{H}_i,\ i = 0,1,2,$$

where $w_0 \in \mathbb{R}$; $\mu > 0$ is a coupling constant, $v(\cdot)$, $w_1(\cdot)$ are real-valued continuous functions on $\mathbb{T}^d$ and $w_2(\cdot,\cdot)$ is a real-valued continuous symmetric function on $(\mathbb{T}^d)^2$.

We remark that the operators A_{01} and A_{12} resp. A_{01}^* and A_{12}^* are called annihilation resp. creation operators, respectively. A trivial verification shows that

$$A_{01}^* : \mathcal{H}_0 \to \mathcal{H}_1,\ (A_{01}^* f_0)(k_1) = v(k_1) f_0,\ f_0 \in \mathcal{H}_0;$$

$$A_{12}^* : \mathcal{H}_1 \to \mathcal{H}_2,\ (A_{12}^* f_1)(k_1,k_2) = \frac{1}{2}(v(k_2) f_1(k_1) + v(k_1) f_1(k_2)),\ f_1 \in \mathcal{H}_1.$$

These operators have widespread applications in quantum mechanics, notably in the study of quantum harmonic oscillators and many-particle systems [7]. An annihilation operator lowers the number of particles in a given state by one. A creation operator increases the number of particles in a given state by one, and it is the adjoint of the annihilation operator. In many subfields of physics and chemistry, the use of these operators instead of wave-functions is known as second quantization. In this paper we consider the case, where the number of annihilations and creations of the particles of the considering system is equal to 1. It means that $A_{ij} \equiv 0$ for all $|i - j| > 1$.

We should note that if the parameter functions of the operator H_μ are defined as

$$w_0 = \varepsilon \mathrm{s},\ w_1(k_1) = -\varepsilon \mathrm{s} + \omega(k_1),\ w_2(k_1,k_2) = -\varepsilon \mathrm{s} + \omega(k_1) + \omega(k_2),$$

then we can use this operator to study in detail the spectral properties of the lattice model radiation with a fixed atom and at most two photons [14, 17]. Here $s = \pm$, $\varepsilon > 0$; $\omega(k_1)$ is the energy of photon with momentum k_1 (the free field dispersion), $v(\cdot)$ is a continuous function related with the interaction between the atom and photons, and $\mu > 0$ is a coupling constant.

It is well-known that the three-particle discrete Schrödinger operator $\hat{H}$ in the momentum representation is the bounded self-adjoint operator on the Hilbert space $L^2((\mathbb{T}^d)^3)$. Introducing the total quasimomentum $K \in \mathbb{T}^d$ of the system, it is easy to see that the operator $\hat{H}$ can be decomposed into the direct integral of the family $\{\hat{H}(K),\ K \in \mathbb{T}^d\}$ of self-adjoint operators [1, 3, 9]:

$$\hat{H} = \int_{\mathbb{T}^d} \oplus H(K) dK,$$

where the operator $\hat{H}(K)$ acts on the Hilbert space $L_2(\Gamma_K)$ ($\Gamma_K \subset (\mathbb{T}^d)^2$ is some manifold).

Observe that H_μ enjoys the main spectral properties of the three-particle discrete Schrödinger operator $\hat{H}(0)$ (see [1, 3, 9]), and the generalized Friedrichs model plays the role of the two-particle discrete Schrödinger operator. For this reason the Hilbert space $\mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2$ is called the *three-particle cut subspace* of the bosonic Fock space $\mathcal{F}_\mathrm{b}(L^2(\mathbb{T}^d))$ over $L^2(\mathbb{T}^d)$, while the operator matrix H_μ is called the Hamiltonian of the system with at most three particles on a lattice. Here the space $\mathcal{F}_\mathrm{b}(L^2(\mathbb{T}^d))$ is defined by

$$\mathcal{F}_\mathrm{b}(L^2(\mathbb{T}^d)) := \mathbb{C} \oplus L^2(\mathbb{T}^d) \oplus L^2_{\mathrm{sym}}((\mathbb{T}^d)^2) \oplus L^2_{\mathrm{sym}}((\mathbb{T}^d)^3) \oplus \ldots$$

We write elements F of the space $\mathcal{F}_{\mathrm{b}}(L^2(\mathbb{T}^{\mathrm{d}}))$ in the form

$$F = \{f_0, f_1(k_1), f_2(k_1,k_2), \ldots, f_n(k_1,k_2,\ldots,k_n), \ldots\}$$

of (equivalence class of) functions of an increasing number of variables ($k_1, \ldots, k_n$), $k_i \in \mathbb{T}^{\mathrm{d}}$; the functions are symmetric with respect to the variables k_i, $i = 1, \ldots, n$, $n \in \mathbb{N}$. The norm in $\mathcal{F}_{\mathrm{b}}(L^2(\mathbb{T}^{\mathrm{d}}))$ is given by

$$\| F \|^2 := | f_0 |^2 + \sum_{n=1}^{\infty} \int_{(\mathbb{T}^{\mathrm{d}})^n} | f_n(k_1,\ldots,k_n) |^2 \, dk_1 \ldots dk_n.$$

Sometimes it is useful to consider the partition of the matrices to blocks. We consider the operator H_μ with respect to the decomposition $\mathcal{H}_0 \oplus \{\mathcal{H}_1 \oplus \mathcal{H}_2\}$ and rewrite it as sum of two operator matrices

$$H_\mu = \begin{pmatrix} \widehat{A}_{00} & \mu \widehat{A}_{01} \\ \mu \widehat{A}_{01}^* & \widehat{\mathcal{A}}_\mu \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & \widehat{\mathcal{A}}_\mu \end{pmatrix}}_{=:H_\mu^0} + \underbrace{\begin{pmatrix} \widehat{A}_{00} & \mu \widehat{A}_{01} \\ \mu \widehat{A}_{01}^* & 0 \end{pmatrix}}_{=:\mathcal{V}_\mu}$$

with matrix entries

$$\widehat{A}_{00} := A_{00}, \ \widehat{A}_{01} := (A_{01} \ \ 0), \ \widehat{\mathcal{A}}_\mu := \begin{pmatrix} A_{11} & \mu A_{12} \\ \mu A_{12}^* & A_{22} \end{pmatrix}.$$

From the definitions of the operators $\mathcal{A}_\mu$ and $\widehat{\mathcal{A}}_\mu$ one can see that if

$$v(k_1) \equiv 1, \ w_1(k_1) = \varepsilon(k_1) + \gamma, \ w_2(k_1,k_2) = \varepsilon(k_1) + \varepsilon(\frac{1}{2}(k_1+k_2)) + \varepsilon(k_2),$$

and the dispersion function $\varepsilon(\cdot)$ has form (1), then we obtain $\mathcal{A}_\mu \equiv \widehat{\mathcal{A}}_\mu$.

Since the operators A_{00} and A_{01} are of rank 1, the operator $\mathcal{V}_\mu$ is a bounded self-adjoint operator of rank 2. Therefore, in accordance with the invariance of the essential spectrum under the finite rank perturbations the essential spectrum of H_μ coincides with the essential spectrum of $\mathcal{A}_\mu$.

If $z \in \rho(H_\mu) \cap \rho(H_\mu^0)$ then it is easy to see that

$$\mathcal{K}_\mu := (H_\mu - zE)^{-1} \mathcal{V}_\mu (H_\mu^0 - zE)^{-1}$$

and $(H_\mu^0 - zE)^{-1}$, $(H_\mu - zE)^{-1}$ are bounded operators. Now using the relation for rank of product of bounded operators

$$\text{rank } (A_1 A_2 \cdots A_n) \leq \min \{ \text{rank } A_1, \ \text{rank } A_2, \ \ldots, \ \text{rank } A_n \},$$

we have $\text{rank}\,\mathcal{K}_\mu \leq \text{rank}\,\mathcal{V}_\mu$.

Since $\text{rank}\,\mathcal{V}_\mu = 2$, for all $z \in \rho(H_\mu) \cap \rho(H_\mu^0)$ for the rank of the difference of resolvents we have

$$\text{rank } ((H_\mu - zE)^{-1} - (H_\mu^0 - zE)^{-1}) \leq 2.$$

For an interval $\Delta \subset \mathbb{R}$, let $E_\Delta(H_\mu)$ stands for the spectral subspace of H_μ corresponding to Δ. Let us denote by $N_{(a;\,b)}(H_\mu)$ the number of eigenvalues of the operator H_μ, including multiplicities, lying in $(a; b) \subset \mathbb{R} \setminus \sigma_{\text{ess}}(H_\mu)$, that is,

$$N_{(a;b)}(H_\mu) = \dim E_{(a;b)}(H_\mu)\mathcal{H}.$$

Then taking into account $\sigma_{\text{disc}}(H_\mu^0) = \{0\} \cup \sigma_{\text{disc}}(\mathcal{A}_\mu)$ from Theorem 9.3.3 in [6] we obtain

$$\lim_{z \nearrow a_\mu^{\min}} \frac{N_{(-\infty;z)}(H_\mu)}{N_{(-\infty;z)}(\mathcal{A}_\mu)} = \lim_{z \searrow a_\mu^{\max}} \frac{N_{(z;+\infty)}(H_\mu)}{N_{(z;+\infty)}(\mathcal{A}_\mu)} = 1.$$

As a conclusion we notice that, the results obtained for $\mathcal{A}_\mu$ plays crucial role in the investigations of the essential and discrete spectrum of the energy operator of the lattice systems describing three particles in interaction, without conservation of the number of particles on a d-dimensional lattice and the lattice spin-boson model with at most two photons.

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